

## BOUNDARY VALUE PROBLEMS FOR A FAMILY OF DOMAINS IN THE SIERPINSKI GASKET

ZIJIAN GUO, RACHEL KOGAN, HUA QIU AND ROBERT S. STRICHARTZ

ABSTRACT. For a family of domains in the Sierpinski gasket, we study harmonic functions of finite energy, characterizing them in terms of their boundary values, and study their normal derivatives on the boundary. We characterize those domains for which there is an extension operator for functions of finite energy. We give an explicit construction of the Green's function for these domains.

### 1. Introduction

Consider the domain  $\Omega_x$  in the Sierpinski Gasket ( $\mathcal{SG}$ ) consisting of all points above the horizontal line  $L_x$  at the distance  $x$  from the top vertex  $q_0$ , for  $0 < x \leq 1$ .

Let  $S(x) = \mathcal{SG} \cap L_x$ . For  $x$  not a dyadic rational, this is a Cantor set. The boundary of  $\Omega_x$  consists of  $S(x)$  together with  $q_0$ . By general principles, harmonic functions on  $\Omega_x$  are determined by their boundary values, where harmonic functions are defined to be solutions of  $\Delta h = 0$  on the interior of  $\Omega_x$ , where  $\Delta$  is the Kigami Laplacian on  $\mathcal{SG}$ . The study of such harmonic functions was initiated in [S1], and continued in [OS] for the special case  $x = 1$ . In this paper, we extend the results in [OS] to the general case. In Section 2, we give an explicit description of the analog of the Poisson kernel

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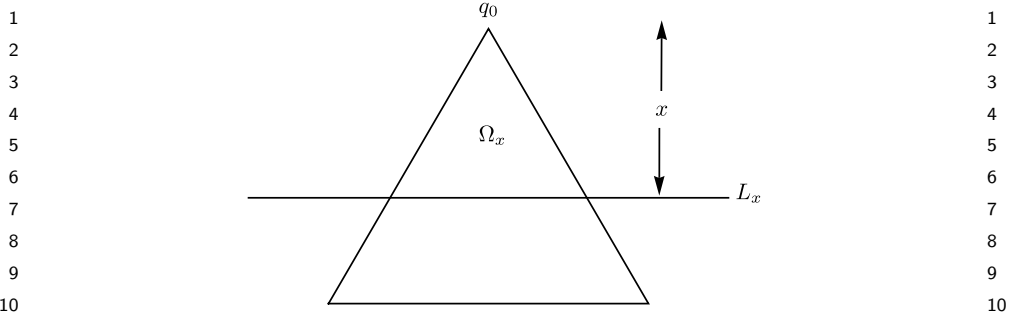


FIGURE 1

to recover the harmonic function from its boundary values, in terms of the Haar series expansion of the boundary values on  $S(x)$ , and we characterize the boundary values that correspond to harmonic functions of finite energy. In Section 3, we define normal derivatives on the boundary and give a description of the Dirichlet-to-Neumann map as a multiplier transform on the Haar series expansion.

In Section 4, we study the extension problem for functions of finite energy on  $\Omega_x$  to functions of finite energy on  $\mathcal{SG}$ . We are able to characterize the values of  $x$  for which such extensions are possible. In particular, the value  $x = 1$  studied in [OS] does not admit such extensions. This may be regarded as the first of a family of Sobolev extension problems, based on Sobolev spaces on  $\mathcal{SG}$  discussed in [S2]. We leave these as open problems for future research. Related problems are studied in [LS] and [LRSU].

In Section 5, we give a construction of a Green's function on  $\Omega_x$  to solve the Dirichlet problem  $-\Delta u = F$  on  $\Omega_x$ ,  $u|_{\partial\Omega_x} = 0$  via an integral transform of  $F$ . The construction of the Green's function is analogous to Kigami's construction on  $\mathcal{SG}$ .

The reader is referred to the books [Ki] and [S3] for a description of the theory of the Laplacian on  $\mathcal{SG}$ , and related fractals. It would be interesting to extend the results of this paper to other domains in  $\mathcal{SG}$ , and to domains in other fractals. In this regard, we offer the following cautionary tale. Consider the fractal  $\mathcal{SG}_3$ , defined similarly to  $\mathcal{SG}$  but by subdivisions of the sides of triangles into three rather than two pieces (see Figure 2).

We may consider domains  $\Omega_x$  defined as before, with the boundary  $S(x)$  modeled as a Cantor set with divisions into three pieces. There is a natural analog of Haar functions on  $S(x)$ , with two generators as shown in Figure 3.

Because the second generator is symmetric rather than skew-symmetric, we cannot glue to zero at the top, so the analog of Lemma 2.3 does not hold. It is not clear how to overcome this difficulty.

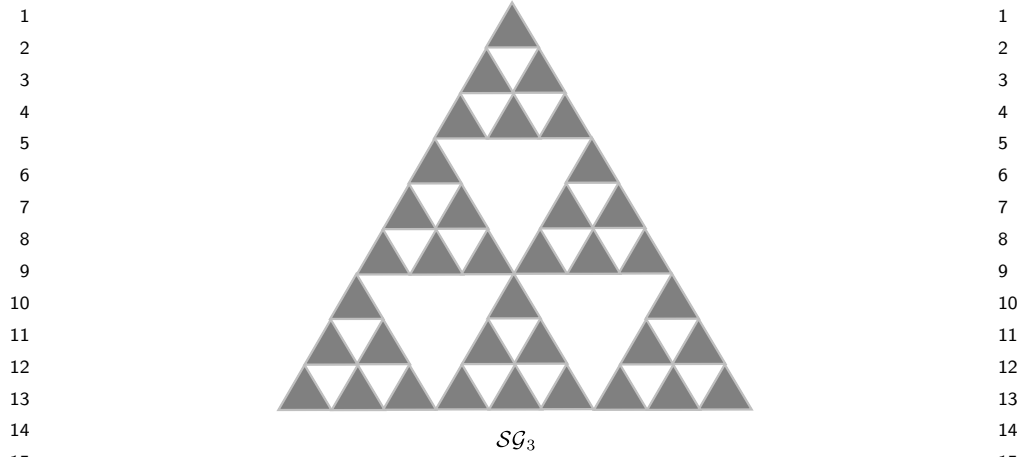


FIGURE 2

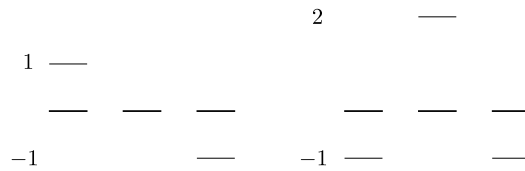


FIGURE 3. Haar generators.

## 2. Harmonic functions on $\Omega_x$

For  $0 < x \leq 1$ , there is a unique representation

$$(2.1) \quad x = \sum_{k=1}^{\infty} 2^{-n_k}$$

for a sequence

$$(2.2) \quad 0 < n_1 < n_2 < \dots$$

of increasing positive integers. We will approximate  $\Omega_x$  by the increasing sequence of domains  $\Omega_x^{(m)}$  where each  $\Omega_x^{(m)}$  is the closure of  $\Omega_{x_{[m]}}$  where

$$(2.3) \quad x_{[m]} = \sum_{k=1}^m 2^{-n_k}$$

is the partial sum of (2.1). (Note that (2.3) is not the representation of  $x_{[m]}$  of the form (2.1) since it is a finite binary representation.) The domain  $\Omega_x^{(m)}$  is a finite union of cells, specifically 1  $n_1$ -cell, 2  $n_2$ -cells, 4  $n_3$ -cells,  $\dots$ ,  $2^{m-1}$



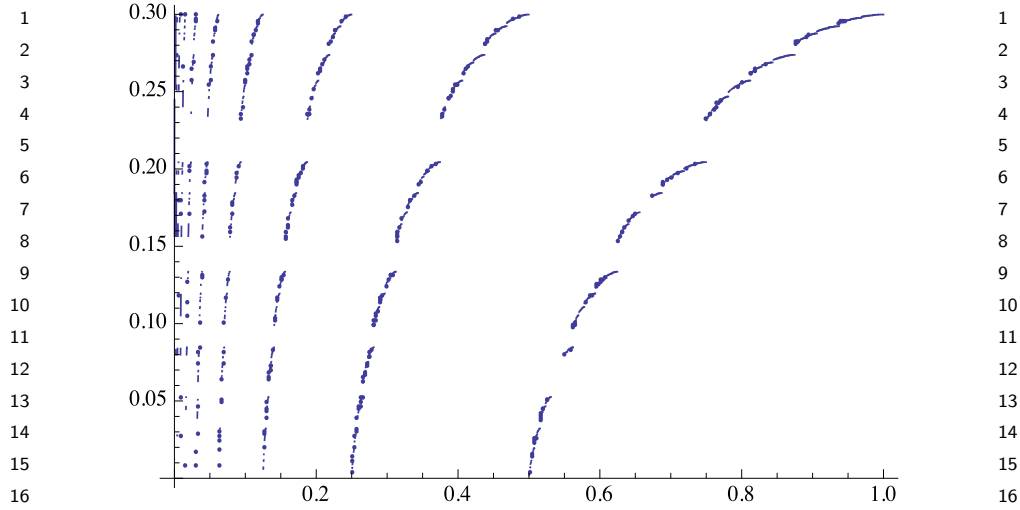
Following [S1], we define

and the function  $\alpha_0(x)$  by the identity

which is easily solved to obtain a variant of a continued fraction representation

for

See Figure 5 for the graph of  $\alpha_0(x)$  on  $(0, 1]$ .

FIGURE 5. The graph of  $\alpha_0(x)$ .

We also define

$$(2.8) \quad \alpha_1(x) = \frac{1 - \alpha_0(x)^2}{2\alpha_0(x) + 1}, \quad \alpha_2(x) = \frac{\alpha_0(x) - \alpha_0(x)^2}{2\alpha_0(x) + 1}.$$

Note that

$$(2.9) \quad \alpha_0(x) + \alpha_1(x) + \alpha_2(x) = 1.$$

These functions enable us to describe harmonic functions in  $\Omega_x$ . The boundary of  $\Omega_x$  consists of the top vertex  $q_0$  and  $S(x) = L_x \cap \mathcal{SG}$ . If  $x$  is not a dyadic rational, then  $S(x)$  is a Cantor set. We will assume this holds. Then a harmonic function is determined by the value  $h(q_0)$  and the expansion of  $h|_{\mathcal{SG}}$  in a Haar basis.

DEFINITION 2.1. The harmonic function  $h_0$  satisfies

$$(2.10) \quad h_0(q_0) = 1, \quad h_0|_{S(x)} = 0.$$

The harmonic function  $h_1$  satisfies

$$(2.11) \quad \begin{aligned} h_1(q_0) &= 0, & h_1|_{S(x) \cap F_0^{n_1-1} F_1(\mathcal{SG})} &= 1, \\ h_1|_{S(x) \cap F_0^{n_1-1} F_2(\mathcal{SG})} &= -1. \end{aligned}$$

We write  $h_0^x$  and  $h_1^x$  when we need to explicitly show the dependence on  $x$ .

Note that  $1 - h_0$  satisfies

$$(2.12) \quad (1 - h_0)(q_0) = 0, \quad (1 - h_0)|_{S(x)} = 1,$$

so that  $1 - h_0$  and  $h_1$  vanish at  $q_0$  and give the first two Haar functions when restricted to  $S(x)$ . Also it is shown in [S1] that

$$(2.13) \quad h_0(F_0^{n_1-1}F_1q_0) = h_0(F_0^{n_1-1}F_2q_0) = \alpha_0(x)$$

and

$$(2.14) \quad h_1(F_0^{n_1-1}F_1q_0) = -h_1(F_0^{n_1-1}F_2q_0) = \alpha_1(x) - \alpha_2(x).$$

LEMMA 2.2. *Let  $y = 2^{n_1}Rx$ . Then*

$$(2.15) \quad h_0^x \circ (F_0^{n_1-1}F_1) = h_0^x \circ (F_0^{n_1-1}F_2) = \alpha_0(x)h_0^y$$

and

$$(2.16) \quad h_1^x \circ (F_0^{n_1-1}F_1) = -h_1^x \circ (F_0^{n_1-1}F_2) = 1 + (\alpha_1(x) - \alpha_2(x) - 1)h_0^y.$$

*Proof.* The function  $\alpha_0(x)h_0^y$  is a harmonic function on  $\Omega_y$  with boundary values  $\alpha_0(x)$  at  $q_0$  and zero on  $S(y)$ . Note that  $F_0^{n_1-1}F_1(S(y)) = S(x)$ , so  $h_0^x \circ (F_0^{n_1-1}F_1)$  is also a harmonic function on  $\Omega_y$  vanishing on  $S(y)$ , and it assume the value  $\alpha_0(x)$  at  $q_0$  by (2.13). Thus, (2.15) holds. A similar argument shows that (2.14) implies (2.16).  $\square$

Next, we consider the general Haar basis functions on  $L^2(S(x))$ . Let  $\omega = (\omega_1, \dots, \omega_m)$  be a word of length  $|\omega| = m$ , with each  $\omega_j = 1$  or  $2$ . Then

$$(2.17) \quad S_\omega(x) = S(x) \cap F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}(S\mathcal{G})$$

describe the dyadic pieces of  $S(x)$ . In particular,

$$(2.18) \quad S(x) = \bigcup_{|\omega|=m} S_\omega(x).$$

The Cantor measure  $\mu$  on  $S(x)$  assigns measure  $2^{-m}$  to each piece  $S_\omega(x)$ . The Haar function  $\psi_\omega$  is supported on  $S_\omega(x)$  and satisfies

$$(2.19) \quad \psi_\omega|_{S_{\omega_1}(x)} = 2^{m/2} \quad \text{and} \quad \psi_\omega|_{S_{\omega_2}(x)} = -2^{m/2}.$$

Then  $1 \cup \{\psi_\omega\}$  is an orthonormal basis for  $L^2(S(x), d\mu)$ . We define  $h_\omega^x$  to be the harmonic function on  $\Omega_x$  with boundary values  $h_\omega^x(q_0) = 0$  and  $h_\omega^x|_{S(x)} = \psi_\omega$ .

LEMMA 2.3. *Let  $y_m = 2^{n_m}R^m x$ . Then  $h_\omega^x$  is supported in*

$$\Omega_x \cap F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}(S\mathcal{G})$$

and

$$(2.20) \quad h_\omega^x \circ (F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}) = 2^{m/2}h_1^{y_m}.$$

*Proof.* The key observation is that, because of skew-symmetry, the function  $h_1$  not only vanishes at  $q_0$  but also has normal derivative vanishing at  $q_0$ . Thus, we may glue the function defined by (2.20) to zero outside this cell and still have a harmonic function. This function clearly has the required boundary values for  $h_\omega^x$ .  $\square$

THEOREM 2.4. *The energies are given by*

$$(2.21) \quad \mathcal{E}(h_0^x) = (1 - \alpha_0(x))^2 \sum_{j=1}^{\infty} 2^{2-j} \left(\frac{5}{3}\right)^{2n_1-n_j},$$

$$(2.22) \quad \mathcal{E}(h_1^x) = 6 \left(\frac{1 - \alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} + 2 \left(\frac{3\alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} \mathcal{E}(h_0^y),$$

and

$$(2.23) \quad \mathcal{E}(h_\omega^x) = 2^m \left(\frac{5}{3}\right)^{n_m} \mathcal{E}(h_1^{y_m}),$$

where  $m = |\omega|$ . Moreover, there exist positive constants  $C_1$  and  $C_2$ , independent of  $x$ , such that

$$(2.24) \quad C_1 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \leq \mathcal{E}(h_\omega^x) \leq C_2 2^m \left(\frac{5}{3}\right)^{n_{m+1}}.$$

*Proof.* We compute the energy of  $h_0^x$  on the top cell  $F_0^{n_1}(\mathcal{SG})$  using (2.13) to be  $(\frac{5}{3})^{n_1} 2(\alpha_0(x) - 1)^2$ , since there are two edges where the difference of  $h_0^x$  is  $\alpha_0(x) - 1$ . On the remaining cells  $F_0^{n_1-1}F_1(\mathcal{SG})$  and  $F_0^{n_1-1}F_2(\mathcal{SG})$  the function  $h_0^x$  is equal to  $\alpha_0(x)h_0^y \circ (F_0^{n_1-1}F_1)^{-1}$  and  $\alpha_0(x)h_0^y \circ (F_0^{n_1-1}F_2)^{-1}$  by (2.15). These each have energy  $\alpha_0(x)^2(\frac{5}{3})^{n_1}\mathcal{E}(h_0^y)$ , so

$$(2.25) \quad \mathcal{E}(h_0^x) = 2 \left(\frac{5}{3}\right)^{n_1} ((\alpha_0(x) - 1)^2 + \alpha_0(x)^2 \mathcal{E}(h_0^y)).$$

Before iterating this identity, we observe that

$$(2.26) \quad (1 - \alpha_0(y))\alpha_0(x) = \frac{1}{2} \left(\frac{5}{3}\right)^{n_1-n_2} (1 - \alpha_0(x)).$$

This follows from (2.5) and the observation, from (2.7), that  $\alpha_0(x)$  depends only on the sequence of differences  $n_k - n_{k-1}$  and therefore  $\alpha_0(y) = \alpha_0(2^{n_1}Rx) = \alpha_0(Rx)$ . Thus,

$$\begin{aligned} \mathcal{E}(h_0^x) &= (1 - \alpha_0(x))^2 \left( 2 \left(\frac{5}{3}\right)^{n_1} + \left(\frac{5}{3}\right)^{2n_1-n_2} \right) \\ &\quad + 4 \left(\frac{5}{3}\right)^{n_2} \alpha_0(x)^2 \alpha_0(y)^2 \mathcal{E}(h_0^{y_2}) \end{aligned}$$

and by iterating we obtain (2.21).

Similarly, we use (2.14) to compute the energy of  $h_1^x$  on the top cell  $F_0^{n_1}(\mathcal{SG})$  to be  $(\frac{5}{3})^{n_1} 6(\alpha_2(x) - \alpha_1(x))^2 = (\frac{5}{3})^{n_1} 6(\frac{1-\alpha_0(x)}{2\alpha_0(x)+1})^2$  by (2.8). Then by using (2.16) we compute the energy in each of the other cells to be  $(\frac{5}{3})^{n_1}(\alpha_1(x) - \alpha_2(x) - 1)^2 \mathcal{E}(h_0^y) = (\frac{5}{3})^{n_1}(\frac{3\alpha_0(x)}{2\alpha_0(x)+1})^2 \mathcal{E}(h_0^y)$ , and by adding we obtain (2.22). Then (2.23) follows by Lemma 2.3.

To obtain the estimate (2.24), we observe that since  $0 \leq \alpha_0(x) \leq \frac{3}{10}$  it follows from (2.7) that  $\alpha_0(x)$  is bounded above and below by multiples of  $(\frac{5}{3})^{n_1-n_2}$ . It follows from (2.21) that  $\mathcal{E}(h_0^x)$  is bounded above and below by multiples of  $(\frac{5}{3})^{n_1}$  since the infinite series is dominated by its first term. We get the same estimate for  $\mathcal{E}(h_1^x)$  using (2.22) since the second summand is bounded by a multiple of  $(\frac{5}{3})^{2(n_1-n_2)}(\frac{5}{3})^{n_1}(\frac{5}{3})^{n_2-n_1}$ . Then (2.24) follows from this estimate and (2.23).  $\square$

COROLLARY 2.5. *Let  $h$  be the harmonic function on  $\Omega_x$  with boundary values  $h(q_0) = a$  and  $h|_{S(x)} = f$ , where*

$$(2.27) \quad f = b + \sum_{\omega} c_{\omega} \psi_{\omega}$$

for

$$(2.28) \quad c_{\omega} = \int_{S(x)} f \psi_{\omega} d\mu.$$

Then  $\mathcal{E}(h)$  is bounded above and below by multiples of

$$(2.29) \quad \left(\frac{5}{3}\right)^{n_1} (a-b)^2 + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^m \left(\frac{5}{3}\right)^{n_{m+1}} |c_{\omega}|^2.$$

In particular,  $h$  has finite energy if and only if (2.29) is finite.

*Proof.* By subtracting a constant we may assume without loss of generality that  $a = 0$  (this does not change  $c_{\omega}$ ). Then from (2.27) we have

$$(2.30) \quad h = b(1 - h_0) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} c_{\omega} h_{\omega},$$

and the functions  $h_0 \cup \{h_{\omega}\}$  are orthogonal in energy by symmetry considerations. Thus,

$$(2.31) \quad \mathcal{E}(h) = b^2 \mathcal{E}(1 - h_0) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} |c_{\omega}|^2 \mathcal{E}(h_{\omega})$$

and the result follows by the estimates (2.24).  $\square$

We are also interested in the corresponding result for the  $L^2$  norm of  $h$ . Using similar reasoning, we can show that  $\|h\|_2^2$  is bounded above and below by multiples of

$$(2.32) \quad \left(\frac{1}{3}\right)^{n_1} (a^2 + b^2) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^m \left(\frac{1}{3}\right)^{n_{m+1}} |c_{\omega}|^2.$$

Of course this allows the coefficients to grow so that  $\sum_{\omega} |c_{\omega}|^2$  is infinite, meaning that the boundary values  $f$  on  $S(x)$  may not be in  $L^2(S(x))$ .



### 3. Normal derivatives

We follow the general outline from [OS] to define a normal derivative on  $S(x)$ . We define

$$(3.1) \quad \partial_n u|_{S(x)} = \lim_{m \rightarrow \infty} 2^m \sum_{|\omega|=m} (-\partial_n u(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(\mathcal{SG})}$$

if the limit exists, where

$$(3.2) \quad \tilde{F}_\omega = F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1} F_{\omega_m}.$$

The cells  $\tilde{F}_\omega(\mathcal{SG})$  for  $|\omega| = m$  cover  $S(x)$ , and  $\tilde{F}_\omega q_0$  is the top vertex. Since  $\partial_n u(\tilde{F}_\omega q_0)$  is outer directed, upward, we insert the minus sign to get an outer directed normal across  $S(x)$ .

LEMMA 3.1.  $\partial_n h_0^x$  is the constant function on  $S(x)$  with value  $-2(\frac{5}{3})^{n_1}(1 - \alpha_0(x))$ .

*Proof.* We compute  $\partial_n h_0^x(q_0) = 2(\frac{5}{3})^{n_1}(1 - \alpha_0(x))$  from the cell  $F_0^{n_1}(\mathcal{SG})$ . Next, consider the cell  $F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1}(\mathcal{SG})$ . The top vertex is  $F_0^{n_1-1} F_{\omega_1} q_0$ , and by symmetry (on the cell  $F_0^{n_1}(\mathcal{SG})$ ),  $\partial_n h_0^x(F_0^{n_1-1} F_{\omega_1} q_0) = \frac{1}{2} \partial_n h_0^x(q_0)$  for  $\omega_1 = 1, 2$ . Thus  $2 \sum_{|\omega|=1} (-\partial_n h_0^x(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(\mathcal{SG})} = -\partial_n h_0^x(q_0) \chi_{S(x)}$ . By similar reasoning, there is no change on the right side of (3.1) as  $m$  increases.  $\square$

LEMMA 3.2.  $\partial_n h_\omega^x = 6 \cdot 2^m (\frac{5}{3})^{n_{m+1}} (\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}) \psi_\omega$ .

*Proof.* On the cell  $F_0^{n_1}(\mathcal{SG})$  we compute (using (2.14))

$$\partial_n h_1^x(F_0^{n_1-1} F_1 q_0) = -\partial_n h_1^x(F_0^{n_1-1} F_2 q_0) = 3 \left(\frac{5}{3}\right)^{n_1} (\alpha_1(x) - \alpha_2(x)),$$

so we have

$$\sum_{|\omega|=1} 2(-\partial_n h_1^x(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(\mathcal{SG})} = 6 \left(\frac{5}{3}\right)^{n_1} (\alpha_1(x) - \alpha_2(x)) \psi_\emptyset,$$

and by the same reasoning as in Lemma 3.1, this does not change if we instead sum over  $|\omega| = m$  for any  $m \geq 2$ . So this gives the correct result for  $\omega = \emptyset$ . We then use Lemma 2.3 to scale the result for general  $\omega$ .  $\square$

THEOREM 3.3. Suppose  $h$  and  $f$  are given as in Corollary 2.5. Then  $\partial_n h$  is given by

$$(3.3) \quad 2(b-a) \left(\frac{5}{3}\right)^{n_1} (1 - \alpha_0(x)) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 6 \cdot 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \left(\frac{1 - \alpha_0(y_m)}{2\alpha_0(y_m) + 1}\right) c_\omega \psi_\omega$$

provided the series converges. In other words, the Dirichlet-to-Neumann map  $f \rightarrow \partial_n h$  is a Haar series multiplier map with multiplier  $6 \cdot 2^m (\frac{5}{3})^{n_{m+1}} \times (\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1})$ .

COROLLARY 3.4. Suppose  $f$  satisfies

$$(3.4) \quad \sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^{2m} \left(\frac{5}{3}\right)^{2n_{m+1}} |c_{\omega}|^2 < \infty.$$

Then  $\partial_n h$  is well-defined in  $L^2(S(x))$  and  $\|\partial_n h\|_2^2$  is bounded above and below by a multiple of (3.4).

*Proof.* The theorem follows from Lemma 3.2, and the corollary follows from the fact that  $\frac{1-\alpha_0(x)}{2\alpha_0(x)+1}$  is uniformly bounded above and below independent of  $x$ .  $\square$

Note that the finiteness of (3.4) is a stronger condition than the finiteness of (2.29), so harmonic functions of finite energy do not necessarily satisfy (3.4), but functions  $h$  satisfying the conditions of Corollary 3.4 automatically have finite energy.

COROLLARY 3.5. Suppose  $h$  is as in Corollary 2.5 with coefficients that satisfy (3.4), and  $v$  is any function of finite energy of  $\Omega_x$ , then the following Gauss–Green formula holds:

$$(3.5) \quad \mathcal{E}(h, v) = v(q_0) \partial_n h(q_0) + \int_{S(x)} v \partial_n h d\mu.$$

*Proof.*  $v$  is continuous since  $v$  is of finite energy, hence has a well-defined restriction to  $S(x)$  that is bounded and thus in  $L^2(\mu)$ . Apply the standard Gauss–Green formula on the domain  $\bigcup_{|\omega| \leq m} \tilde{F}_{\omega}(S\mathcal{G})$  and take the limit as  $m \rightarrow \infty$ .  $\square$

#### 4. Extending functions of finite energy

In this section, we will write  $\Omega_x^+$  for the region above  $L(x)$  that was previously denoted  $\Omega_x$ , and  $\Omega_x^-$  for the region below  $L(x)$ . Under the assumption that  $x$  is not a dyadic rational,  $S(x)$  is the common boundary of  $\Omega_x^+$  and  $\Omega_x^-$ . For functions  $u^{\pm}$  defined on  $\Omega_x^{\pm}$ , we use  $\mathcal{E}_{\Omega_x^{\pm}}(u^{\pm})$  to denote the energies of  $u^{\pm}$ , which are naturally defined by taking the graph energy sum with edges restricted to lie in  $\Omega_x^{\pm}$ , and then computing the usual renormalized limit. Let  $\text{dom } \mathcal{E}_{\Omega_x^{\pm}}$  to denote the collections of functions of finite energy on  $\Omega_x^{\pm}$ , respectively.

The first issue that we address is under what conditions can we glue together functions  $u^{\pm}$  of finite energy on  $\Omega_x^{\pm}$  to obtain a function of finite energy on  $S\mathcal{G}$ . Since functions of finite energy are continuous,  $u^{\pm}$  must have boundary values on  $S(x)$  that agree. It turns out that this is the only condition that

we need to impose. This is not surprising since the same is true for gluing functions of finite energy on domains that intersect at a finite set of points.

THEOREM 4.1. Let  $u^\pm \in \text{dom } \mathcal{E}_{\Omega_x^\pm}$ , and suppose

$$(4.1) \quad u^+|_{S(x)} = u^-|_{S(x)},$$

the values being defined by continuity. Then

$$(4.2) \quad u = \begin{cases} u^+ & \text{on } \overline{\Omega}_x^+, \\ u^- & \text{on } \overline{\Omega}_x^-, \end{cases}$$

belongs to  $\text{dom } \mathcal{E}$  in  $\mathcal{SG}$  and

$$(4.3) \quad \mathcal{E}(u) = \mathcal{E}_{\Omega_x^+}(u^+) + \mathcal{E}_{\Omega_x^-}(u^-).$$

*Proof.* Let  $S_m$  denote the strip of  $2^m$  cells of order  $n_m$  containing  $S(x)$ , and let  $B_m^\pm$  denote the unions of the cells of order  $n_m$  contained in  $\Omega_x^\pm$ . Then

$$\mathcal{E}^{(n_m)}(u) = \mathcal{E}_{B_m^+}^{(n_m)}(u) + \mathcal{E}_{B_m^-}^{(n_m)}(u) + \mathcal{E}_{S_m}^{(n_m)}(u).$$

Since  $\mathcal{E}_{B_m^\pm}^{(n_m)}(u) \rightarrow \mathcal{E}_{\Omega_x^\pm}(u^\pm)$  as  $m \rightarrow \infty$ , it suffices to show

$$(4.4) \quad \mathcal{E}_{S_m}^{(n_m)}(u) \rightarrow 0.$$

Let  $C$  denote one of the  $n_m$ -cells in  $S_m$  with boundary points  $x_m \in \Omega_x^+$  and  $y_m, z_m \in \Omega_x^-$ . We need to estimate

$$(4.5) \quad \left(\frac{5}{3}\right)^{n_m} [(u^+(x_m) - u^-(y_m))^2 + (u^+(x_m) - u^-(z_m))^2 + (u^-(y_m) - u^-(z_m))^2].$$

It suffices to estimate the first two terms in (4.5) since  $u^-(y_m) - u^-(z_m) = (u^+(x_m) - u^-(z_m)) - (u^+(x_m) - u^-(y_m))$ , and by symmetry it suffices to estimate the first term. Let  $S_m^\pm$  be the portion of  $S_m$  above or below  $S(x)$ . There will be an infinite sequence of points  $\{x_m, x_{m+1}, \dots\}$  in  $S_m^+$  and  $\{y_m, y_{m+1}, \dots\}$  in  $S_m^-$ , both converging to the same point  $p \in S(x)$ . Since  $u^+(p) = u^-(p)$  by (4.1), we may write

$$(4.6) \quad u^+(x_m) - u^-(y_m) = \sum_{j=m}^{\infty} (u^+(x_j) - u^+(x_{j+1})) - \sum_{j=m}^{\infty} (u^-(y_j) - u^-(y_{j+1})).$$

Now each pair  $(x_j, x_{j+1})$  are vertices of a cell  $C_j$  of order  $n_{j+1}$  in  $\Omega_x^+$ . Note that all these cells are essentially disjoint, and  $C = \bigcup_j C_j$ .

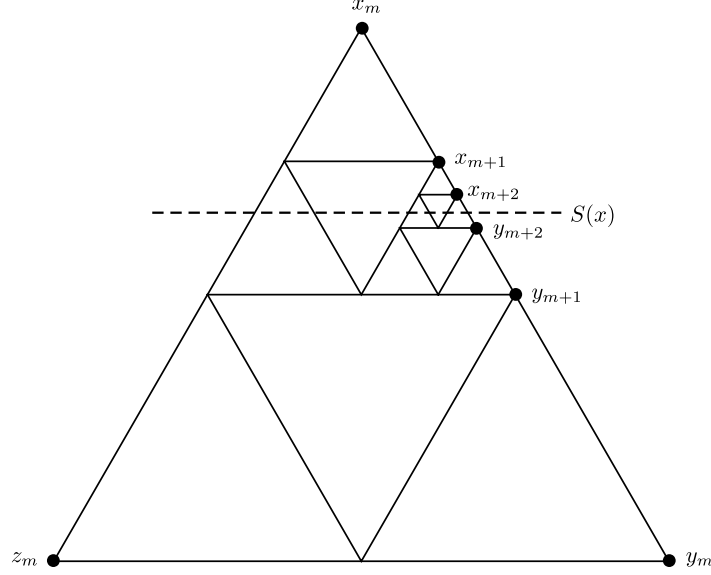


FIGURE 6

So we have the estimate

$$(4.7) \quad |u^+(x_j) - u^+(x_{j+1})| \leq \left(\frac{3}{5}\right)^{n_{j+1}/2} \mathcal{E}_{C_j}(u^+)^{1/2}.$$

By the Cauchy–Schwarz inequality, we obtain

$$(4.8) \quad \sum_{j=m}^{\infty} |u^+(x_j) - u^+(x_{j+1})| \leq \left( \sum_{j=m}^{\infty} \left(\frac{3}{5}\right)^{n_{j+1}} \right)^{1/2} \left( \sum_{j=m}^{\infty} \mathcal{E}_{C_j}(u^+) \right)^{1/2} \\ \leq c \left(\frac{3}{5}\right)^{n_m/2} \mathcal{E}_{C \cap S_m^+}(u^+)^{1/2}.$$

By similar reasoning, we obtain the same estimate with  $|u^-(y_j) - u^-(y_{j+1})|$ , so by (4.6) we have

$$(4.9) \quad \left(\frac{5}{3}\right)^{n_m} |u^+(x_m) - u^-(y_m)|^2 \leq c \mathcal{E}_{C \cap S_m^+}(u^+) + c \mathcal{E}_{C \cap S_m^-}(u^-).$$

Summing (4.9) over all the  $2^m$  cells  $C$  yields

$$(4.10) \quad \mathcal{E}_{S_m}^{(n_m)}(u) \leq c \mathcal{E}_{S_m^+}(u^+) + c \mathcal{E}_{S_m^-}(u^-)$$

and  $\mathcal{E}_{S_m^\pm}(u^\pm) \rightarrow 0$  because  $\bigcap_m S_m^\pm = S(x)$  and  $S(x)$  has measure zero in the Kusuoka measure (this follows easily from Theorem 5.1 of [AHS]).  $\square$

It is easy to characterize the restrictions to  $S(x)$  of functions of finite energy on  $\Omega_x^+$ .

**THEOREM 4.2.** *A function  $f$  on  $S(x)$  is the restriction to  $S(x)$  of a function  $u^+$  of finite energy on  $\Omega_x^+$  if and only if  $f$  has a Haar series expansion (2.27) with (2.29) finite (here  $a = 0$ ), and (2.29) is bounded by a multiple of  $\mathcal{E}_{\Omega_x^+}(u^+)$ .*

*Proof.* Let  $h$  be the harmonic function on  $\Omega_x^+$  with the same boundary values  $f$ . Since harmonic functions minimize energy,  $\mathcal{E}_{\Omega_x^+}(h) \leq \mathcal{E}_{\Omega_x^+}(u^+)$ , and the result follows from Corollary 2.5.  $\square$

However, there is no such simple result for  $\Omega_x^-$ . We pose the following extension problem.

**PROBLEM 4.3.** Does there exist a bounded linear extension operator (meaning  $Tu|_{\Omega_x^+} = u$ )  $T : \text{dom}_{\Omega_x^+}(\mathcal{E}) \rightarrow \text{dom}_{SG}(\mathcal{E})$ ?

There is a simple obstruction to solving this problem.

**DEFINITION 4.4.**  $x$  satisfies the *nonconsecutive condition with bound  $N$*  if there are no  $N$  consecutive integers in the sequence  $\{n_m\}$ . If there is some  $N$  for which this holds then  $x$  is said to satisfy the *nonconsecutive condition*.

Note that a generic value of  $x$  will not satisfy this condition. However, there are uncountably many (of Hausdorff dimension 1) values of  $x$  that do satisfy the condition. Perhaps the simplest choice has  $n_m = 2m - 1$ , with  $N = 2$ .

**THEOREM 4.5.** *Let  $E$  denote the collection of  $x$  satisfying the nonconsecutive condition. Then the Hausdorff dimension of  $E$  is 1.*

*Proof.* Let  $E_N$  denote the collection of  $x$  satisfying the nonconsecutive condition with bound  $N$ . Then  $E = \bigcup_{N \geq 2} E_N$  and

$$(4.11) \quad E_2 \subset E_3 \subset \cdots \subset E_N \subset \cdots.$$

We will first prove that the Hausdorff dimension of  $E_N$  is the unique positive root of the equation

$$(4.12) \quad 2 - 2^s - 2^{-Ns} = 0.$$

Consider the set  $E_N$ . We divide it into the disjoint union  $E_N = \bigcup_{k \geq 1} E_{N,k}$  where  $E_{N,k}$  is the set of  $x$  in  $E_N$  whose  $n_1$ -digit is  $k$ . Obviously, for each  $k$ ,  $E_{N,k}$  is a similar copy of  $E_{N,1}$  with contraction ratio  $2^{1-k}$ . Since the Hausdorff dimension is stable for countable unions, we just need to compute the dimension of  $E_{N,1}$ . For this set, by the nonconsecutive condition, we can

write

$$(4.13) \quad E_{N,1} = \left( \bigcup_{j \geq 3} (2^{-1} + E_{N,j}) \right) \cup \dots \cup \left( \bigcup_{j \geq N+1} (2^{-1} + \dots + 2^{-(N-1)} + E_{N,j}) \right).$$

Since  $|E_{N,j}| \leq 1/2^j$ , it is easy to check that the above union is disjoint. Moreover, (4.13) is essentially a self-similar identity for the set  $E_{N,1}$  with contraction ratios,

$$2^{-2}, 2^{-3}, \dots, 2^{-3}, 2^{-4}, \dots, 2^{-N}, 2^{-(N+1)}, \dots,$$

satisfying the open set condition (with the open set  $(2^{-1}, 1)$ ). (See [M] for the theory of infinitely generated self-similar sets.) Hence the Hausdorff dimension of  $E_{N,1}$  is the solution of the equation

$$(4.14) \quad 1 = \sum_{k=2}^N \sum_{j \geq k} (2^{-s})^j = \sum_{k=2}^N \frac{(2^{-s})^k}{1 - 2^{-s}} = \frac{2^{-2s} - 2^{-s(N+1)}}{(1 - 2^{-s})^2},$$

which simplifies to (4.12). So we get the Hausdorff dimension of  $E_N$ .

Using (4.11), an easy calculation will show that the Hausdorff dimension of  $E$  is 1.  $\square$

If  $x$  fails to satisfy the nonconsecutive condition, then there are pairs of points in  $\Omega_x^+$  that are much closer to each other in  $\mathcal{SG}$  than in  $\Omega_x^+$ . For example, if  $n_j = j$  for  $j \leq N$  then the points  $F_1 F_2^{N-1} q_0$  and  $F_2 F_1^{N-1} q_0$  in  $\Omega_x^+$  are distance on the order of  $(\frac{3}{5})^N$  apart in the resistance metric on  $\mathcal{SG}$ , but are far apart in  $\Omega_x^+$ . Note that  $h_1^x(F_1 F_2^{N-1} q_0) - h_1^x(F_2 F_1^{N-1} q_0) = 2h_1^x(F_1 F_0^{N-1} q_0)$  and  $\mathcal{E}(h_1^x)$  is bounded. The estimate analogous to (4.7) shows

$$c \leq \left( \frac{3}{5} \right)^{N/2} \mathcal{E}(u)^{1/2}$$

for any extension  $u$  of  $h_1^x$  to  $\mathcal{SG}$ , hence  $\mathcal{E}(u) \geq c(\frac{5}{3})^N$ . This means that the bound on the operator  $T$ , if it exists, would be bounded below by a multiple of  $(\frac{5}{3})^{N/2}$ .

The same reasoning applies locally if  $\{n_m\}$  has a consecutive string of  $N$  integers. Thus if such strings exist for all  $N$  then  $T$  cannot be bounded. On the other hand, it is easy to see that if the nonconsecutive condition holds for  $x$  then distances in  $\Omega_x^+$  and  $\mathcal{SG}$  are comparable. Note that this is very reminiscent of the type of condition that appears in the work of Peter Jones in the Sobolev extension problem in domains in Euclidean space ([J], [R]).

**THEOREM 4.6.** *The extension Problem 4.3 has a positive solution if and only if  $x$  satisfies the nonconsecutive condition, in which case the bound on  $T$  is  $O((\frac{10}{3})^{N/2})$ .*

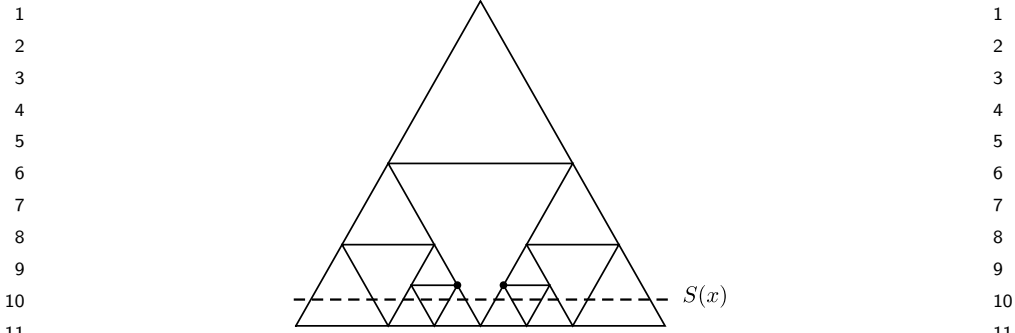


FIGURE 7

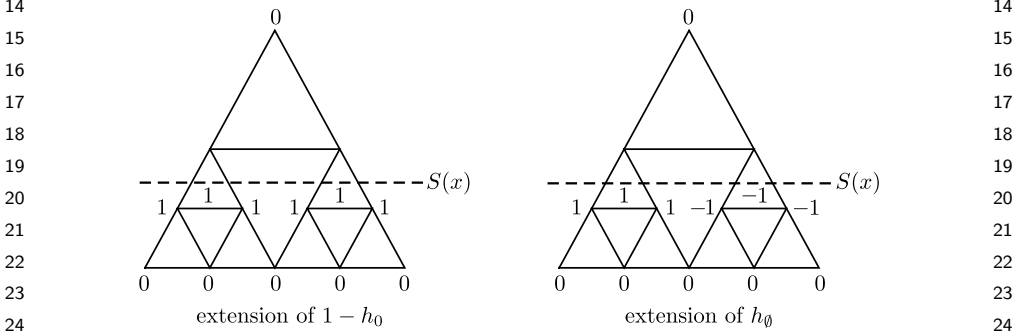


FIGURE 8

*Proof.* We need to construct an extension operator  $T$  under the assumption that  $x$  satisfies the nonconsecutive condition. In view of our previous results, it suffices to solve the extension problem for the functions  $h_\omega$  (and also  $1 - h_0$ ), say  $Th_\omega = \tilde{h}_\omega$  where the functions  $\tilde{h}_\omega$  are orthogonal in energy and

$$(4.15) \quad \mathcal{E}(\tilde{h}_\omega) \leq C(N)2^m \left(\frac{5}{3}\right)^{n_m+1}.$$

Suppose first that  $N = 2$ . Consider first  $1 - h_0$  and  $h_0$ . Assume for simplicity that  $n_1 = 1$ . Then  $n_2 \geq 3$ . Then  $S(x)$  passes through the cells  $F_1F_0(\mathcal{SG})$  and  $F_2F_0(\mathcal{SG})$ . We will extend  $1 - h_0$  to be identically 1 on the bottom portions of these cells, and  $h_0$  to be 1 on  $F_1F_0(\mathcal{SG})$  and  $-1$  on  $F_2F_0(\mathcal{SG})$ . On the remaining four cells of level 2, we make the extension harmonic with boundary values 0 on the bottom vertices (see Figure 8).

Note that the added energy of these extensions is exactly  $8(\frac{5}{3})^2$ . Also, since one extension is symmetric and one is skew-symmetric with respect to the vertical reflection, they are orthogonal in energy. If  $n_1 > 1$ , we may

repeat the same process on  $F_0^{n_1-1}(\mathcal{SG})$  and then continue the extension to be identically zero on the complement of  $F_0^{n_1-1}(\mathcal{SG})$ . The added energy is exactly  $8(\frac{5}{3})^{n_1+1}$ , but the energy of the original functions was also a multiple of  $(\frac{5}{3})^{n_1}$ , so this is consistent with (4.15) with  $m = 0$  and gives a uniform bound on the extension operator.

For the extension of  $h_\omega$ , we just have to repeat the same procedure miniaturized. If  $|\omega| = m$  then  $h_\omega$  is supported on a cell of order  $n_{m+1} - 1$  and since  $n_{m+2} \geq n_{m+1} + 2$  the right side of Figure 8 describes  $h_\omega$  and its extension (except for a factor of  $2^{m/2}$ ) to that cell, and then we may glue this to zero in the complement of the cell. Thus, we get an extension with the same energy bound. For words  $\omega$  with  $|\omega| = m$ , the extended functions have disjoint support, so the energies are orthogonal. Comparing extensions for words of different length with overlapping support, we again have a symmetry/skew-symmetry dichotomy with respect to the local reflection in the vertical axis of the smaller supporting cell (this is the overlap of the supports) and so we again have energy orthogonality. This completes the proof for  $N = 2$ .

For general  $N$ , the argument is similar. In Figure 9, we show the extension of  $h_\emptyset$  when  $N = 3$  and  $n_1 = 1, n_2 = 2, n_3 \geq 4$ .

Here we have  $2^N$  cells of order  $N$  contributing to the energy, and this multiplies the energy by  $O((\frac{10}{3})^N)$ . Since the norm of the extension is measured in terms of the square root of the energy, we obtain the  $O((\frac{10}{3})^{N/2})$  bound.  $\square$

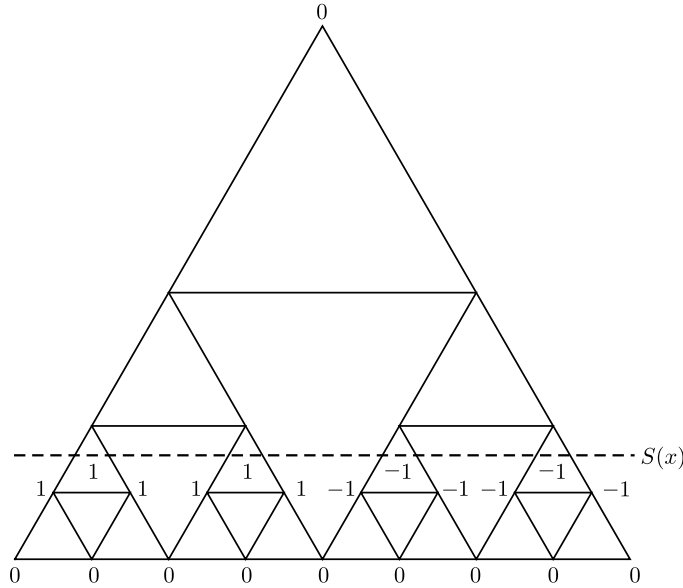


FIGURE 9



The optimal extension operator would produce functions that are harmonic on  $\Omega_x^-$ . In particular, it would be interesting to have an explicit description of the functions  $h_\omega^-$  that are harmonic on  $\Omega_x^-$  and are equal to  $\psi_\omega$  on  $S(x)$ , again under the nonconsecutive condition.

We may regard Theorem 4.2 as a trace theorem and Theorem 4.6 as an extension theorem for  $\text{dom } \mathcal{E}$  regarded as a Sobolev space, and then we should ask if there are analogous results for other Sobolev spaces. In [S2], the spaces  $\text{dom}_{L^2}(\Delta^k)$  on  $\mathcal{SG}$  are considered as Sobolev spaces ( $\text{dom}_{L^2}(\Delta^k) = \{u \in L^2(\mathcal{SG}) : \Delta^j u \in L^2(\mathcal{SG}) \text{ for all } j \leq k\}$ ). Similarly for the space  $\{u \in \text{dom}_{L^2}(\Delta^k) : \mathcal{E}(\Delta^k u) < \infty\}$ . These spaces are easily characterized in terms of expansions in eigenfunctions of the Laplacian. A complete theory of the eigenspaces of the Laplacian on  $\Omega_1$  is given in [Q].

**PROBLEM 4.7.** For each of these Sobolev spaces, characterize the space of traces on  $S(x)$  and restrictions to  $\Omega_x^+$ , for  $x$  satisfying the nonconsecutive condition.

It seems plausible that the trace problem may have a solution with a condition similar to (2.29) for the Haar expansion (2.27) with different multiples of  $|c_\omega|^2$  depending on the Sobolev space. The restriction problem is likely to be more challenging. It is clear that restrictions of functions in  $\text{dom}_{L^2}(\Delta^k)$  must satisfy  $\Delta^j u \in L^2(\Omega_x^+)$  for  $j \leq k$ , but that is not sufficient because all harmonic functions automatically have  $\Delta^j u = 0$ . It would seem that the characterization of restriction Sobolev spaces would also have to involve conditions on traces on  $S(x)$ . Related problems are discussed in [LS] and [LRSU].

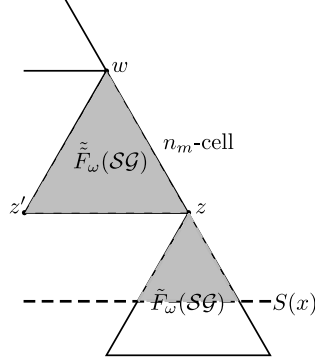
## 5. Green's function

For a given  $k$ , let  $V_k$  denote the set of vertices on the  $k$ -level graph approximation of  $\mathcal{SG}$ . For a point  $z \in V_k \setminus V_0$ , let  $\phi_z^k$  denote the piecewise harmonic spline of level  $k$  satisfying  $\phi_z^k(t) = \delta_{zt}$  for  $t \in V_k$  and extended harmonically on  $\mathcal{SG}$ . Notice that  $\phi_z^k \in \text{dom}_0 \mathcal{E}$  because  $z \notin V_0$ , and it is supported in the two  $k$ -cells meeting at  $z$ . Recall that in the standard theory (see the books [Ki] and [S3]), the Green's function  $G(s, t)$  to solve the Dirichlet problem  $-\Delta u = F$  on  $\mathcal{SG}$ , subject to the boundary condition  $u|_{V_0} = 0$  via an integral transform  $\int_{\mathcal{SG}} G(s, t) F(t) dt$ , has the following explicit formula,

$$(5.1) \quad G(s, t) = \lim_{M \rightarrow \infty} G^M(s, t) \quad (\text{uniform limit})$$

with

$$(5.2) \quad G^M(s, t) = \sum_{k=1}^M \sum_{z, z' \in V_k \setminus V_{k-1}} g(z, z') \phi_z^k(s) \phi_{z'}^k(t),$$

FIGURE 10. The support of  $\phi_z^{n_m}$ .

where

$$(5.3) \quad g(z, z') = \begin{cases} \frac{3}{10} \left(\frac{3}{5}\right)^k & \text{for } z = z' \in V_k \setminus V_{k-1}, \\ \frac{1}{10} \left(\frac{3}{5}\right)^k & \text{for } z \neq z' \in V_k \setminus V_{k-1}, \\ & \text{contained in the same } (k-1)\text{-cell,} \\ 0, & \text{otherwise.} \end{cases}$$

To get an analogous Green's function on  $\Omega_x$ , we should first modify the definition of those piecewise harmonic splines  $\phi_z^k$  whose support intersects the boundary  $S(x)$  of the domain  $\Omega_x$ . More specially, let  $\omega$  be a word of symbols  $\{1, 2\}$  with  $|\omega| = m$  and  $z = \tilde{F}_\omega(q_0)$ . We redefine  $\phi_z^{n_m}$  to be the piecewise harmonic spline with value 1 on  $z$ , 0 on  $V_{n_m} \cap \Omega_x$  and  $S(x)$ , and extended harmonically on  $\Omega_x$ . Obviously the support of  $\phi_z^{n_m}$  is contained in two  $n_m$ -cells meeting at  $z$ , with  $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$  on the cell  $\tilde{F}_\omega(S\mathcal{G})$  and with values unchanged on the other cell, denoted by  $\tilde{\tilde{F}}_\omega(S\mathcal{G})$ , where

$$(5.4) \quad \tilde{\tilde{F}}_\omega = \begin{cases} F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \dots & \\ F_0^{n_{m-1}-n_{m-2}-1} F_{\omega_{m-1}} F_0^{n_m-n_{m-1}} & \text{for } m \geq 2, \\ F_0^{n_1} & \text{for } m = 1. \end{cases}$$

LEMMA 5.1. Let  $z = \tilde{F}_\omega(q_0)$ , then

$$(5.5) \quad \mathcal{E}_{\Omega_x}(\phi_z^{n_m}, v) = \left(\frac{5}{3}\right)^{n_m} \left( \frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} v(z) - v(z') - v(w) \right)$$

for any  $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$ , where  $z' = \tilde{F}_{\omega_1 \dots \omega_{m-1}(3-\omega_m)}(q_0)$  and  $w = \tilde{F}_{\omega_1 \dots \omega_{m-1}}(q_0)$  are the two  $n_m$ -neighbors of  $z$  (see Figure 10).

*Proof.* On the cell  $\tilde{F}_\omega(\mathcal{SG})$ , by using the localized Gauss–Green formula (see (3.5)),

$$(5.6) \quad \begin{aligned} \mathcal{E}_{\Omega_x \cap \tilde{F}_\omega(\mathcal{SG})}(\phi_z^{n_m}, v) &= v(z) \partial_n \phi_z^{n_m}(z) \\ &= 2 \left( \frac{5}{3} \right)^{n_m+1} (1 - \alpha_0(y_m)) v(z). \end{aligned}$$

The last equality follows from the same argument as the proof of Lemma 3.1 with suitable scaling.

On the other cell  $\tilde{\tilde{F}}_\omega(\mathcal{SG})$ , by using the standard theory,

$$(5.7) \quad \mathcal{E}_{\Omega_x \cap \tilde{\tilde{F}}_\omega(\mathcal{SG})}(\phi_z^{n_m}, v) = \left( \frac{5}{3} \right)^{n_m} (2v(z) - v(z') - v(w)).$$

Summing the energies on the two cells, we get the desired result by using (2.5).  $\square$

Let  $T_x^m$  be the set of vertices in  $V_{n_m} \cap \Omega_x$  which can be expressed as  $\tilde{F}_\omega(q_0)$  for some word  $\omega = \omega_1, \dots, \omega_m$  of symbols  $\{1, 2\}$ , and  $T_x = \bigcup_{m \geq 1} T_x^m$ .

DEFINITION 5.2. For fixed  $m$ , let

$$(5.8) \quad G_{\Omega_x}^m(s, t) = \sum_{k=1}^{n_m} \sum_{z, z' \in (V_k \setminus V_{k-1}) \cap \Omega_x} g_x(z, z') \phi_z^k(s) \phi_{z'}^k(t),$$

with

$$(5.9) \quad g_x(z, z') = \begin{cases} \frac{\alpha_0(y_{l-1}) + \alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1}) + 1} \left( \frac{3}{5} \right)^{n_l} & \text{for } z = z' \in T_x^l \text{ with } l \leq m, \\ \frac{\alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1}) + 1} \left( \frac{3}{5} \right)^{n_l} & \text{for } z \neq z' \in T_x^l, \text{ being } n_l\text{-neighbors,} \\ & \text{with } l \leq m, \\ g(z, z') & \text{for } z, z' \in V_k \setminus V_{k-1} \text{ contained in} \\ & \text{a } (k-1)\text{-cell in } \Omega_x, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is obvious that  $G_{\Omega_x}^m(s, t)$  converges uniformly to a function  $G_{\Omega_x}(s, t)$  as  $m$  goes to infinity.

THEOREM 5.3.  $G_{\Omega_x}$  is the Green's function for  $\Omega_x$ , namely

$$(5.10) \quad u(s) = \int_{\Omega_x} G_{\Omega_x}(s, t) F(t) dt$$

solves the Dirichlet problem  $-\Delta u = F$  on  $\Omega_x$  with  $u|_{\partial\Omega_x} = 0$ , for any continuous  $F$ .

*Proof.* Similar to the  $\mathcal{SG}$  case, suppose we could prove

$$(5.11) \quad \mathcal{E}_{\Omega_x}(G_{\Omega_x}^m(\cdot, t), v) = \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t)$$

for any  $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$ .

Then just multiply (5.11) by  $F(t)$  and integrate, using the standard arguments to interchange the energy and integral, to obtain

$$(5.12) \quad \mathcal{E}_{\Omega_x}(u_m, v) = \int_{\Omega_x} F(t) \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t) dt$$

for

$$(5.13) \quad u_m(s) = \int_{\Omega_x} G_{\Omega_x}^m(s, t) F(t) dt.$$

Since

$$(5.14) \quad \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t) \rightarrow v(t)$$

uniformly as  $m \rightarrow \infty$ , the right side of (5.12) converges to  $\int_{\Omega_x} F(t) v(t) dt$ , and the left-hand side converges to  $\mathcal{E}_{\Omega_x}(u, v)$  as  $m$  goes to  $\infty$ . Thus, we have

$$(5.15) \quad \mathcal{E}_{\Omega_x}(u, v) = \int_{\Omega_x} Fv dt$$

for any  $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$ , which yields that  $-\Delta u = F$  with  $u|_{\partial\Omega_x} = 0$ .

Hence, our goal is to prove (5.11). The function  $G_{\Omega_x}^m(s, t)$ , which we regard as a function of the single variable  $s$ , could be viewed as a linear combination of terms  $\phi_z^k(s)$ . Then it is clear that  $\mathcal{E}_{\Omega_x}(G_{\Omega_x}^m(\cdot, t), v)$  is a linear combination of  $v(z)$  for  $z \in V_{n_m} \cap \Omega_x$ . So we need to compute the combination coefficient of  $v(z)$  for each  $z$ .

Let  $z_0 \in V_{n_m} \cap \Omega_x$ . If  $z_0 \notin T_x$ , it is easy to observe that there exists an  $n_m$ -cell containing  $z_0$  as an interior point. The terms in  $G_{\Omega_x}^m$  that contribute to the coefficient of  $v(z_0)$  all have supports away from  $S(x)$ . Thus, the standard argument for the  $\mathcal{SG}$  case shows that the coefficient of  $v(z_0)$  should be  $\phi_{z_0}^{n_m}(t)$ .

Hence, we only need to consider the case that  $z_0 \in T_x$ . We first do this when  $z_0 \in T_x^m$ . Let  $z'_0$  denote the unique  $n_m$ -neighbor of  $z_0$  in the same level. Then the only terms in  $G_{\Omega_x}^m$  that contribute to the coefficient of  $v(z_0)$  are

$$\begin{aligned} g_x(z_0, z_0) \phi_{z_0}^{n_m}(s) \phi_{z_0}^{n_m}(t), & \quad g_x(z_0, z'_0) \phi_{z_0}^{n_m}(s) \phi_{z'_0}^{n_m}(t), \\ g_x(z'_0, z_0) \phi_{z'_0}^{n_m}(s) \phi_{z_0}^{n_m}(t), & \quad g_x(z'_0, z'_0) \phi_{z'_0}^{n_m}(s) \phi_{z'_0}^{n_m}(t). \end{aligned}$$

By Lemma 5.1, the total contribution is

$$(5.16) \quad \left(\frac{5}{3}\right)^{n_m} \left( \frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z_0) - g_x(z'_0, z_0) \right) \phi_{z_0}^{n_m}(t) \\ + \left(\frac{5}{3}\right)^{n_m} \left( \frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z'_0) - g_x(z'_0, z'_0) \right) \phi_{z'_0}^{n_m}(t).$$

By substituting the value of  $g_x(z_0, z_0) = g_x(z'_0, z'_0) = \frac{\alpha_0(y_{m-1}) + \alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1}) + 1} \left(\frac{3}{5}\right)^{n_m}$  and  $g_x(z_0, z'_0) = g_x(z'_0, z_0) = \frac{\alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1}) + 1} \left(\frac{3}{5}\right)^{n_m}$  into (5.16), it is easy to verify that

$$(5.17) \quad \left(\frac{5}{3}\right)^{n_m} \left( \frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z_0) - g_x(z'_0, z_0) \right) = 1,$$

and

$$(5.18) \quad \left(\frac{5}{3}\right)^{n_m} \left( \frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z'_0) - g_x(z'_0, z'_0) \right) = 0.$$

So the coefficient of  $v(z_0)$  is  $\phi_{z_0}^{n_m}(t)$ .

Next, we consider the general case. Suppose  $z_0 = \tilde{F}_\omega(q_0) \in T_x^l$  with  $1 \leq l < m$ . We need to compute the coefficient of  $v(z_0)$ . The previous discussion immediately shows that the contribution of terms in  $G_{\Omega_x}^l$  to  $v(z_0)$  is  $\phi_{z_0}^{n_l}(t)$ . Now we consider the terms in  $G_{\Omega_x}^m - G_{\Omega_x}^l$ . Let  $z_1 = \tilde{F}_{\omega 1}(q_0)$  and  $z_2 = \tilde{F}_{\omega 2}(q_0)$ . Notice that in all the terms in  $G_{\Omega_x}^m - G_{\Omega_x}^l$  that contribute to  $v(z_0)$ , only those which contain  $\phi_{z_1}^{n_{l+1}}(s)$  or  $\phi_{z_2}^{n_{l+1}}(s)$  have supports intersecting the boundary  $S(x)$ . Moreover, in calculating the energy  $\mathcal{E}_{\Omega_x}(\phi_{z_i}^{n_{l+1}}, v)$ , only the part  $\phi_{z_i}^{n_{l+1}}|_{\tilde{F}_{\omega i}(\mathcal{SG})}$  is involved in contributing to the coefficient of  $v(z_0)$ , for  $i = 1, 2$ . Comparing to the standard  $\mathcal{SG}$  case, the function  $\phi_{z_i}^{n_{l+1}}(s)$  has been redefined, but the restriction of it to  $\tilde{F}_{\omega i}(\mathcal{SG})$  is unchanged. So the total contribution of  $G_{\Omega_x}^m - G_{\Omega_x}^l$  to  $v(z_0)$  is as same as the standard case, namely  $\phi_{z_0}^{n_m}(t) - \phi_{z_0}^{n_l}(t)$ . Thus, we get that in  $\mathcal{E}_{\Omega_x}(G_{\Omega_x}^m, v)$ , the coefficient of  $v(z_0)$  is  $\phi_{z_0}^{n_m}(t)$ , as required.

Thus, we have proved (5.11).  $\square$

**THEOREM 5.4.** *For continuous  $F$ , the normal derivative of the solution  $u$  given by (5.10) is continuous on  $S(x)$ .*

*Proof.* From Theorem 5.3,

$$(5.19) \quad \partial_n u|_{S(x)} = \sum_{m \geq 1} \sum_{z, z' \in T_x^m} g_x(z, z') \partial_n \phi_z^{n_m}|_{S(x)} \int_{\Omega_x} \phi_{z'}^{n_m}(t) F(t) dt,$$

since only those terms containing  $\phi_z^k$  whose supports intersect  $S(x)$  contribute to the value of  $\partial_n u|_{S(x)}$ .

For fixed  $m$ , let  $z = \tilde{F}_\omega(q_0) \in T_x^m$ . Note that on the cell  $\tilde{F}_\omega(\mathcal{SG})$ ,  $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$ . By Lemma 3.1, we have

$$(5.20) \quad \partial_n \phi_z^{n_m}|_{S(x)} = -2 \left(\frac{5}{3}\right)^{n_{m+1}} (1 - \alpha_0(y_m)) 2^m \chi_{S_\omega(x)}.$$

On the other hand, for  $z, z' \in T_x^m$ ,  $g_x(z, z')$  is bounded above by a multiple of  $\alpha_0(y_{m-1}) \left(\frac{3}{5}\right)^{n_m}$ , hence by a multiple of  $\left(\frac{3}{5}\right)^{n_{m+1}}$  using (2.7). It is also easy to see that  $\int_{\Omega_x} \phi_{z'}^{n_m}(t) F(t) dt$  is bounded above by a multiple of  $\frac{1}{3^{n_m}} \|F\|_\infty$ . Combing these estimates with (5.20), we conclude that  $|\partial_n u|_{S(x)}$  is bounded above by a multiple of

$$(5.21) \quad \sum_{m \geq 1} \sum_{|\omega|=m} \frac{2^m}{3^{n_m}} \|F\|_\infty \chi_{S_\omega(x)}.$$

From (5.21), one can easily verify that  $\partial_n u$  is continuous on  $S(x)$ .  $\square$

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## BOUNDARY VALUE PROBLEMS

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