Twisting Quantum Field Theories: Nilp Variety in $\mathfrak{osp}(k|4,\mathbb{C})$

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Abstract

This project revolves around a mathematical approach to a physics topic of describing transformations in 3D space. Transformations called **Lie Algebras** and specifically the algebra $\mathfrak{osp}(k|4,\mathbb{C})$ and important sub-spaces shows an interesting triviality in translation or special conformal transformations as well as a way to describe the **twist**.

Definitions

The primary object of interest in this project were a class of algebraic structures called *Lie Algebras*—as well as a generalization of these structures.

- **Def.** A Lie algebra $\mathfrak{g}$ is a vector space equipped with a bilinear map called the bracket operation

$$[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},$$

that is both **antisymmetric**

$$[A,B]=-[B,A]$$

and satisfies the **Jacobi Identity**. [1]

$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=0$$

An example of a bracket is the **commutator**:

$$[A,B]=AB-BA$$

An important generalization of this idea is that of a $\mathbb{Z}/2$-graded Lie algebra(a *Super Lie Algebra*).

- **Def.** A super Lie algebra is a direct sum

$$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$$

of two vector spaces, where $\mathfrak{g}_0$ is considered the "even" part and $\mathfrak{g}_1$ the "odd" part of $\mathfrak{g}$. We can formalize this by defining the **parity** of a homogeneous element $v\in\mathfrak{g}$ as

$$|v|=\begin{cases}0 & \text{if } v\in\mathfrak{g}_0\\ 1 & \text{if } v\in\mathfrak{g}_1\end{cases}$$

where $|v|\in\mathbb{Z}/2$. We then equip $\mathfrak{g}$ with a bilinear super bracket operation

$$[-,-]:[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

that satisfies a generalization of the conditions on a standard Lie bracket. [2]

Special Lie Algebra $\mathfrak{osp}(k|4,\mathbb{C})$

The Orthosymplectic Super Lie algebra, written $\mathfrak{osp}(k|4,\mathbb{C})$, is the space of all $(k+4)\times(k+4)$ complex matrices A of the form

$$A=\left(\begin{array}{c|c}\alpha & 0\\ \hline 0 & \delta\end{array}\right)_0\oplus\left(\begin{array}{c|c}0 & -\gamma^TJ\\ \hline \gamma & 0\end{array}\right)_1$$

where α is a $k\times k$ matrix such that $\alpha=-\alpha^T$ (called $\mathfrak{so}(k)$), where δ is a 4×4 matrix such that $J\delta=-\delta^TJ$ (called $\mathfrak{sp}(4)$) for the 4×4 matrix J which denotes a **symplectic** structure

$$J=\left(\begin{array}{c|c}0 & I_2\\ \hline -I_2 & 0\end{array}\right),$$

and where γ is a general $4\times k$ matrix. The 0 subscript denote the even component and 1 the odd. Additionally, $\mathfrak{osp}(k|4)$ has the bracket:

$$[A,B]:=AB-(-1)^{|A||B|}BA$$

for homogeneous elements $A,B\in\mathfrak{osp}(k|4)$.

Nilpotence Variety

Within $\mathfrak{osp}(k|4)$, we are particularly interested in *nilpotent* elements:

- **Def.** The nilpotence variety of $\mathfrak{osp}(k|4)$ is defined as

$$\mathrm{Nilp}:=\{Q\in\mathfrak{osp}(k|4)_1:[Q,Q]=0\}.$$

Once we have found some element $Q\in\mathrm{Nilp}$, we wish to compute kernel and image of the function

$$\mathrm{ad}(Q)(X):=[Q,X].$$

In particular we wish to describe the spaces:

$$\mathfrak{z}_Q=\{a\in\mathfrak{osp}(k|4)_0,[Q,a]=0\}$$

and subsequently for $\mathfrak{b}_Q$:

$$\mathfrak{b}_Q=\{a\in\mathfrak{osp}(k|4)_0,[Q,Q']=a,Q'\in osp(k|4)_1\}.$$

Note: the space $\mathfrak{b}_Q$ is an ideal of $\mathfrak{z}_Q$, and as such we are also interested in studying the quotient $\mathfrak{z}_Q/\mathfrak{b}_Q$.

Orbits of Nilp

Two facts about Nilp allowed us to fully describe this variety:

- **Lemma 1:** All $Q\in\mathrm{Nilp}$ have rank 1 or 2.
- **Lemma 2:** Nilp is acted on by the Lie group $G=SO(k,\mathbb{C})\times SP(4,\mathbb{C})$ via conjugation, and $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$ are preserved up to isomrphism within the orbits of G.

As a consequence of these lemmas, we determined that there is one Nilp containing all rank 1 elements Q, and a family of orbits parametrized by $x\in\mathbb{C}$ corresponding to all rank 2 elements $Q_x\in\mathrm{Nilp}$.

Calculations

We calculated the results for $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$ for the following Q_x:

$$Q_x=\left(\begin{array}{c|c}0 & -\gamma^TJ\\ \hline \gamma & 0\end{array}\right),\ \text{where } \delta=\begin{pmatrix}1 & i & 0 & 0 & \cdots & 0\\ x & ix & 1 & i & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & 0\end{pmatrix}$$

as with choice of x, this Q_x parameterizes the rank 2 family of orbits. **Note:** there is an exceptional isomorphism

$$\mathfrak{so}(5,\mathbb{C})\cong\mathfrak{so}(5,\mathbb{C})$$

This isomorphism is why we work in osp(k|4,C).

Results

Using the general form of $\mathfrak{z}_Q$ we look at α and δ elements.

$$\alpha=\begin{pmatrix}\alpha_1 & \alpha_2\\ -\alpha_2^T & \alpha_3\end{pmatrix},\delta=\begin{pmatrix}a_1 & a_2 & b_1 & b_2\\ a_3 & a_4 & b_3 & b_1\\ 0 & 0 & -a_1 & -a_3\\ 0 & 0 & -a_4 & -a_4\end{pmatrix}$$

Where $\alpha_3\in\mathfrak{so}(k-4)$ and $\alpha_1\in\mathfrak{so}(4)$ with only one free variable. $\mathfrak{b}_Q$ holds the exact same form except where $\alpha_3=0$. Thus the quotient is:

$$\mathfrak{z}_Q/\mathfrak{b}_Q=\mathfrak{so}(k-4)$$

Which represents the **non-trivial** transformations.

Conclusions

What do these results mean? We see that the Lie Algebra $\mathfrak{so}(5)$ can be written as:

$$\left[\begin{array}{c|c}\mathfrak{so}(3) & v_1|v_2\\ \hline \frac{v_1^T}{v_2^T} & \mathfrak{so}(2)\end{array}\right]$$

Where $\mathfrak{so}(3)$ represents the rotations in 3D space, $\mathfrak{so}(2)$ the dilations, v_1 the translations, and v_2 the special conformal transformations. When we use the isomorphism $\mathfrak{sp}(4)\to\mathfrak{so}(5)$ one of v_1 or v_2 becomes zero.

This shows that in these family of orbits either special conformal transformations or translations become trivial.

Further Areas of Study

This project only touches a small area of a much larger field of study. One specific area of further exploration with more applicable physics implications is studying the orbits formed by the conjugation of the same group but over the Real field: $G=SO(k,\mathbb{R})\times SP(4,\mathbb{R})$. Additionally, generalizing this process for higher dimensions could likely reveal patterns that grant further insight.

References

[1] Humphreys, James E. *Introduction to Lie Algebras and Representation Theory.* 3rd ed., Springer, 1980.

[2] Freed, Daniel S. *Five Lectures on Supersymmetry.* American Mathematical Society, 1999.

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