SURF 2023: Nilpotence Variety of the Orthosymplectic Group $\mathfrak{osp}(k|4, \mathbb{C})$

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Abstract

This research project, done under the guidance of Professor Chris Elliott of the Math Department at Amherst College, lies at an interesting intersection of a few fields. Its applications and motivations lie in the field of physics, specifically Quantum Theory and field transformations (unfortunately, as the name would suggest, to truly understand the applications would go much beyond the scope of a single summer). The methodology and the bulk of the research took place in the world of Linear Algebra.

More specficially, this project revolves around a mathematical approach to a physics topic of describing transformations in 3D space. Transformations called **Lie Algebras** and specifically the algebra $\mathfrak{osp}(k|4,\mathbb{C})$ and important sub-spaces shows an interesting triviality in translation or special conformal transformations as well as a way to describe the **twist**.

Contents

1 Lie Algebras

1.1 Defintions

Normal Lie Algebra

The most prominent element in this paper centers around bilinear transformations called **Lie Algebras**. There are three features that define a Lie Algebra and a *bracket operation* ($[x, y]$):

1. The bracket operation is bilinear.
2. Any lie algebra bracketed with itself goes to zero:
$$[xx] = 0, \forall x \in L$$
3. The bracket satisfies the *Jacobi Identity*:
$$[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad (x, y, z \in L)$$
where L is a Lie Algebra.

Note that condition 1 and 2 implies *anti-commutavity*:

$$\begin{aligned} [x+y, x+y] &= [x, x+y] + [y, x+y] \\ 0 &= [x, x] + [x, y] + [y, x] + [y, y] \\ [x, y] &= -[y, x] \end{aligned}$$

The notation of a Lie Algebra is written with its name followed by its dimension and the field it is working over, for example the orthogonal group $\mathfrak{g}(5, \mathbb{C})$.

Super Lie Algebras

One special form of Lie Algebras are **Super Lie Algebras**. Think of Super Lie Algebras as $\mathbb{Z}/2$-graded Lie Algebras, where it contains an even Lie Algebra and an odd Lie Algebra. And, as with any even and odd elements, even times even yields even, odd times even yields odd, and odd times odd yields even. One way to write this more formally is to use the **parity** (evenness and oddness of a homogeneous element) of an element:

$$|a| = \begin{cases} 0, & \text{if } a \in W^0 \\ 1, & \text{if } a \in W^1 \end{cases}$$

Now, a Super Lie Algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ contains modified versions of the same conditions with the sign rule in mind:

1. The skew symmetry of the bracket is:
$$[a, b] = -(-1)^{|a||b|}[b, a]$$
2. As such, the bracket changes depending on parity:
$$[a, b] = ab - (-1)^{|a||b|}ba$$

3. The Super Jacobi Identity:

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$$

One example of a Super Lie Algebra is the Orthosymplectic: $\mathfrak{osp}(k|4,\mathbb{C}) = \mathfrak{so}(k) \oplus \mathfrak{sp}(4)$. We will talk about what the componenents mean in section 1.2.

Matrix Representation and Adjoint

Note that as Lie Algebras are bilinear operators, after choosing a basis for the vector space each can be represented using a matrix. One related function is called the **adjoint map** of a Lie Algebra:

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, \text{ where } \mathrm{ad}_x(y) = [x,y], \quad x,y \in \mathfrak{g}$$

1.2 Special Categorizations of Lie Algebras

There are a few unique categorizations of Lie Algebras that we work with in this paper: the symplectic, othrogonal, and special linear Lie Algebras. For each I will show its definition and work out the matrix representation which is much easier to work and compute with.

Inner Product

Before we move on, I just wanna define an **inner product** on a real vector space V

$$(\cdot,\cdot) : V \times V \to \mathbb{R}$$

With the following conditions:

1. Bilinearity
2. Symmetric
3. Non-Degenerative: $\text{if}(v,w) = 0, \forall w \in V$ if and only if $v = 0$

1.2.1 Special Linear Lie Algebra

$\mathfrak{sl}(l,F)$ or the **special linear lie algebra** is the set of all $l \times l$ matrices with Trace zero. As such, its dimension is $l^2 - 1$

1.2.2 Orthogonal Lie Algebra

The **orthogonal lie algebra** satisfies the following form on an inner product:

$$\langle x(v), w\rangle = -\langle v, x(w)\rangle$$

Which, we can work out to be the following in matrix representation after choosing a basis:

$$\begin{aligned}\langle x(v), w\rangle &= -\langle v, x(w)\rangle \\ \langle x(v), w\rangle &= -\langle x^\dagger(v), w\rangle \\ \langle x + x^\dagger(v), w\rangle &= 0 \\ x + x^\dagger &= 0 \\ A &= -A^T\end{aligned}$$

Thus we see that $\mathfrak{so}(2l, F)$ or $\mathfrak{so}(2l+1, F)$ dependingo on if it's even or odd consists of all $l \times l$ matrices such that it is equal to its negative transpose.

1.2.3 Symplectic Lie Algebra

Lastly, the **symplectic lie algebra** $\mathfrak{sp}(2l, F)$ follows a similar condition to the orthogonal one, but instead of an inner product we are working with a *symplectic form* w which is anti-symmetric:

$$w(x(v), w) = -w(v, x(w))$$

We can use $S = -S^\dagger$ to represent the symplectic form using an inner product:

$$\begin{aligned}w(w, v) &= \langle Sw, v\rangle \\ \langle v, S^\dagger w\rangle &= \\ -\langle v, Sw\rangle &= -w(v, w)\end{aligned}$$

Thus we can now work out a matrix represention while using $J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$:

$$\begin{aligned}w(x(v), w) &= -w(v, x(w)) \\ \langle x(v), Sw\rangle &= \langle v, Sx(w)\rangle \\ \langle Sx(v), w\rangle &= -\langle x^\dagger S(v), w\rangle \\ \langle Sx + x^\dagger S(v), w\rangle &= 0 \\ Sx + x^\dagger S &= 0 \\ JA + A^T J &= 0 \\ JA &= A^T J\end{aligned}$$

The matrix ends up looking like the such:

$$\begin{pmatrix} \alpha & \beta \\ \delta & -\alpha^T \end{pmatrix}$$

Where $\alpha = l \times l$ matrix, $\delta = \delta^T$, $\beta = \beta^T$, all arbitrary $l \times l$ matrices.

Whithin these lie algebras are some interesting isomorphisms. For example $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. More importantly, the isomorphism $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$ shows up later in this paper with important physics applications, and so it's nice to see the exact proof of the isomorphism in the next section.

1.3 Isomorphism Proof of $\mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$

We will run through the proof of the isomorphism of $\mathfrak{sp}(4,\mathbb{C})$ and $\mathfrak{so}(5,\mathbb{C})$ centered around the 1st Isomorphism Theorem. While this proof isn't terribly long or complicated, it is still very helpful to see the roadmap of each step of the proof and its role in the overall process.

The **1st Isomorphism Theorem** states:

$$\text{Given an algebra homomorphism } \phi : V \to W, \quad V/ker(\phi) \cong Im(\phi)$$

Our goal is then to construct a homomorphism that sends $\mathfrak{sp}(4,\mathbb{C})$ whose image is $\mathfrak{so}(5,\mathbb{C})$ with a trivial kernel. The following are the steps that will take us there:

1. Construct the vector space W and show that it has dimension 5.
2. Create a homomorphism ϕ from $\mathfrak{sp}(4,\mathbb{C})$ to $\mathfrak{gl}(W)$.
3. Show the injectivity of ϕ by proving the kernel is trivial.
4. Define an inner product on $\mathfrak{sp}(4,\mathbb{C})$.
5. Prove the image of ϕ is in $\mathfrak{so}(5,\mathbb{C})$.

Proof

Dimension of vector space W Define a vector space W as the set of all 4×4 matrices A such that
$Tr(A) = 0$ and $JA = A^T J$. Now to show that the dimension of W is 5:

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \text{ each are 2 by 2 matrices}$$
$$JA = A^T J$$
$$\begin{pmatrix} \delta & \gamma \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} -\delta^T & \alpha^T \\ -\gamma^T & \beta^T \end{pmatrix}$$
$$\to \gamma = \alpha^T, \beta = -\beta^T, \delta = -\delta^T$$

From here we can see that β and δ each have one free variable: $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$. Further with the Tr(A) = zero condition, we can see that α has three free variables: $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, we can then construct a basis for W with dimension 5:

$$\{e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\}$$

Injection $\mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{gl}(W)$

Define $\phi : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{gl}(W)$ given by $\phi(X) = [X, A], \forall A \in W$. To show injectivity, we need to prove that the $ker(\psi) = \{0\}$.

$$\begin{aligned} X \in ker(\psi) \iff [X, A] = 0 \qquad &\forall A \in \mathfrak{sp}(4,\mathbb{C}) \\ XA - AX = 0& \\ XA = AX& \end{aligned}$$

To find all elements of $\mathfrak{sl}(4,\mathbb{C})$ that commute, we use the basis to form a system of equations

$$X = \begin{pmatrix} a & c & e & g \\ d & -b & g & f \\ h & j & -a & -c \\ j & i & -d & -b \end{pmatrix}$$

$$Xe_1 - e_1X = \begin{pmatrix} 0 & -2c & 0 & -2g \\ 2d & 0 & 2g & 0 \\ 0 & -2j & 0 & 2d \\ 2j & 0 & -2c & 0 \end{pmatrix} = 0 \to c, d, g, j = 0$$

$$[\begin{pmatrix} a & 0 & e & 0 \\ 0 & -b & 0 & f \\ h & 0 & -a & 0 \\ 0 & i & 0 & -b \end{pmatrix}, e_2] = \begin{pmatrix} 0 & a-b & 0 & f \\ 0 & 0 & -f & 0 \\ 0 & -h & 0 & 0 \\ -h & 0 & -b+a & 0 \end{pmatrix} = 0 \to h, f = 0, a = b$$

$$[\begin{pmatrix} a & 0 & e & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & i & 0 & -a \end{pmatrix}, e_3] = \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = 0 \to i, e = 0, \text{since Tr(A) = 0, a = 0}$$

$$\Rightarrow A = 0$$

Thus we can see that the only element in $\mathfrak{sp}(4,\mathbb{C})$ that commutes is 0, showing that $ker(\phi) = \{0\}$, proving ψ is injective.

Inner Product on W

Define an inner product on $W \times W : \langle A, B\rangle := Tr(AB)$. To show it is a valid inner product, we need to show it is 1.bilinear 2.symmetric 3.non-degenerate.

1. Bilinearity $a_1, a_2 \in \mathbb{C}, A_1, A_2, B \in W$

$$\begin{aligned} \langle a_1A_1 + a_2A_2, B\rangle &= Tr((a_1A_1 + a_2A_2)B) \\ &= Tr(a_1A_1B + a_2A_2B) \\ &= a_1Tr(A_1B) + a_2Tr(A_2B) \\ &= a_1\langle A_1, B\rangle + a_2\langle A_2, B\rangle \end{aligned}$$

2. Symmetry

$$\langle A, B\rangle = Tr(AB) = Tr(BA) = \langle B, A\rangle$$

3. Non-Degeneracy

$$\langle A, B\rangle = 0 \qquad \forall B \in W$$
$$\Rightarrow Tr(Ae_i) = 0 \qquad \forall 1 \leq i \leq 5$$
$$A = \begin{pmatrix} a & b & 0 & d \\ c & -a & -d & 0 \\ 0 & e & a & c \\ -e & 0 & b & -a \end{pmatrix}$$
$$Tr(Ae_2 = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}) = 0 \to c = 0$$
$$Tr(Ae_3 = \begin{pmatrix} b & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & -a \end{pmatrix}) = 0 \to -b = a \to a = 0$$
$$Tr(Ae_4 = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & -e & \cdot \\ \cdot & \cdot & \cdot & -e \end{pmatrix}) = 0 \to e = 0$$
$$Tr(AE_{22} = \begin{pmatrix} -d & \cdot & \cdot & \cdot \\ \cdot & -d & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}) = 0 \to d = 0$$
$$A = 0$$

Thus we show that $\langle A, B\rangle = Tr(AB)$ is a valid inner product.

Showing $\phi(X) \in \mathfrak{so}(W)$

Lastly, we need to show that with our inner product, $\phi(X)$ is in $\mathfrak{so}(W)$. For simplicity, I use the notation $ad(X) := [X, A] \forall A \in W$

$$ad(X)^\dagger = -ad(X) \qquad X \in \mathfrak{sp}(4, \mathbb{C})$$
$$\Rightarrow \langle v, ad(X)w\rangle + \langle ad(X)v, w\rangle = 0$$
$$Tr(vad(X)w) + Tr(ad(X)vw) = 0$$
$$Tr(v(Xw - wX)) + Tr((Xv - vX)w) = 0$$
$$Tr(vXw) - Tr(vwX) + Tr(Xvw) - Tr(vXw) = 0$$
$$Tr(vXw) - Tr(vXw) + (Tr(Xvw) - Tr(vwX)) = 0 - 0 = 0$$

Thus by the 1st Isomorphism Theorem we have proved $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$.

Q.E.D.

1.4 The Orthosymplectic Algebra and the Nilpotence Variety

Finally we're able to define and look at the **Orthosymplectic Algebra** or $\mathfrak{osp}(k|4, \mathbb{C})$. It is a *super lie algebra* where the even part is $\mathfrak{so}(k, \mathbb{C}) \oplus \mathfrak{sp}(4, \mathbb{C})$. In matrix form the even part occupies the diagonal, looking like:

$$\begin{pmatrix} \mathfrak{so}(k,\mathbb{C}) & 0 \\ 0 & \mathfrak{sp}(4,\mathbb{C}) \end{pmatrix}$$

Now to find the matrix representation of the odd component, we use a general form knowing that two odd elements multiply into an even element:

$$\beta \text{ is a } 4 \times k \text{ matrix }, \alpha \text{ is a } k \times 4 \text{ matrix}$$

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \begin{pmatrix} \mathfrak{so}(k,\mathbb{C}) & 0 \\ 0 & \mathfrak{sp}(4,\mathbb{C}) \end{pmatrix}$$
$$\begin{pmatrix} \alpha\beta & 0 \\ 0 & \beta\alpha \end{pmatrix} \in \begin{pmatrix} \mathfrak{so}(k,\mathbb{C}) & 0 \\ 0 & \mathfrak{sp}(4,\mathbb{C}) \end{pmatrix}$$
$$\alpha\beta \in \mathfrak{so}(k,\mathbb{C}), \beta\alpha \in \mathfrak{sp}(4,\mathbb{C})$$

Thus we can see that $\alpha = -\beta^T J$ and thus the odd component takes the form:

$$\begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix}$$

As such, we usually use just δ to describe an odd element of $\mathfrak{osp}$.

Nilpotent Elements

Within the Orthosymplectic Algebra lie a few interesting subalgebras. One of which is are **Nilpotent Elements** where $Q \in \mathfrak{osp}_1, [Q, Q] = 0$. The **Nilpotence variety** looks at the forms of nilpotent elements.

Twists: $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$

The last componenet of $\mathfrak{osp}$ that is of note for this paper are the twists, $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$. There are a few way to understand these subspaces:

$$\mathfrak{z}_Q = \{A \in \mathfrak{osp}_0, [A, Q] = 0\}$$

$$\mathfrak{b}_Q = \{A \in \mathfrak{osp}_0, A = [Q, Q'] \text{ for some } Q' \in osp_1\}$$

Another way is through the image of the following morphisms:

$$\mathfrak{osp}_0 \xrightarrow{\phi} \mathfrak{osp}_1 \xrightarrow{\psi} \mathfrak{osp}_0$$

Where $\mathfrak{b}_Q$ is the image of ψ and $\mathfrak{z}_Q$ is the kernel of ϕ. These two subspaces represent transformations in field mechanics, and so $\mathfrak{b}_Q$ and $\mathfrak{z}_Q$ for specific nilpotent elements pose interesting results for us as.

Note: $\mathfrak{b}_Q \subseteq \mathfrak{z}_Q$ and is in fact an ideal, as such, the quotient $\mathfrak{z}_Q/\mathfrak{b}_Q$ is of particular interest for us as they represent non-trivial transformations.

2 Calculations and Results

There's two lemmas that help us understand the Nilpotence Variety and the twists:

- **Lemma 1:** All $Q \in$ Nilp have rank 1 or 2.
- **Lemma 2:** Nilp is acted on by the Lie group $G = SO(k,\mathbb{C}) \times SP(4,\mathbb{C})$ via conjugation, and $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$ are preserved up to isomrphism within the orbits of G.

All rank 1 Nilp elements are in the orbit of the following Q:

$$Q = \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix}, \delta = \begin{pmatrix} 1 & i & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

More interestingly, we hold an assumption that all rank 2 Nilp elements orbits the following Q with choice of x:

$$Q = \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix}, \delta = \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

As such, the calculations of the $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$ for this Q is of particular interest. First, to confirm Q is in fact nilpotent, we see that

$$\begin{aligned} [Q,Q] &= QQ - (-1)^{1*1} QQ \\ &= QQ + QQ \\ &= 2 \begin{pmatrix} -\delta^T J \delta & 0 \\ 0 & -\delta\delta^T J \end{pmatrix} \end{aligned}$$

$$-\delta\delta^T J = -\begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & x & 0 & 0 \\ i & ix & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} J = \begin{pmatrix} 1-1 & x-x \\ x-x & x-x+1-1 \end{pmatrix} J = 0$$

$$-\delta^T J \delta = -\begin{pmatrix} 0 & 0 & 1 & x \\ 0 & 0 & i & ix \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = 0$$

$$\Rightarrow Q \in Nilp$$

Thus we can now examine z_Q and b_Q for our specific Q.

Calculating $\mathfrak{z}_Q$

To find a general form of $\mathfrak{z}_Q$ we start with a general $A \in osp_0$ where

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}, \alpha \in \mathfrak{so}(k, \mathbb{C}), \gamma \in \mathfrak{sp}(4, \mathbb{C})$$

Since $A \in z_Q$, we see that

$$\begin{aligned} [A, Q] = AQ - QA &= 0 \\ \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} &= 0 \\ \begin{pmatrix} 0 & -\alpha\delta^T J + \delta^T J \gamma \\ \gamma\delta - \delta\alpha & 0 \end{pmatrix} &= 0 \end{aligned}$$

Which gives us the following conditions for A:

$$\alpha\delta^T J = \delta^T J \gamma \text{ and } \gamma\delta = \delta\alpha$$

Lets start with the first condition, using

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^T & \alpha_3 \end{pmatrix}, \gamma = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_1 \\ c_1 & c_2 & -a_1 & -a_3 \\ c_3 & c_1 & -a_2 & -a_4 \end{pmatrix}$$

Where $\alpha_1 = \begin{pmatrix} 0 & m_1 & m_2 & m_3 \\ -m_1 & 0 & m_4 & m_5 \\ -m_2 & -m_4 & 0 & m_6 \\ -m_3 & -m_5 & -m_6 & 0 \end{pmatrix}$, and $\alpha_3 \in \mathfrak{so}(k-4, \mathbb{C})$.

Now, we can expand the first condition into

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^T & \alpha_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & x \\ 0 & 0 & i & ix \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & x \\ 0 & 0 & i & ix \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_1 \\ c_1 & c_2 & -a_1 & -a_3 \\ c_3 & c_1 & -a_2 & -a_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & m_1 i & im_1x + m_2 + im_3 \\ 0 & 0 & -m_1 & -m_1x + m_4 + im_5 \\ 0 & 0 & -m_2 - im_4 & -m_2x - im_4x + im_6 \\ 0 & 0 & -m_3 - im_5 & -m_3x - im_5x - m_6 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_3x & c_2 + c_1x & -a_1 - a_2x & -a_3 - a_4x \\ ic_1 + ic_3x & ic_2 + ic_1x & -ia_1 - ia_2x & -ia_3 - ia_4x \\ c_3 & c_1 & -a_2 & -a_4 \\ ic_3 & ic_1 & -ia_2 & -ia_4 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What a handful... However, we see very quickly that:

$$c_1 = c_2 = c_3 = 0$$

Now we look at rest of the equation, giving us the following system:

$$im_1 = -a_1 - a_2x$$

$$\begin{aligned} -m_2 - im_4 &= -a_2 \\ -m_3 - im_5 &= -ia_2 \\ \Rightarrow -im_2 + m_4 &= -m_3 - im_5 \end{aligned}$$

$$\begin{aligned} im_1x + m_2 + im_3 &= -a_3 - a_4x \\ -m_1x + m_4 + im_5 &= -ia_3 - ia_4x \\ \Rightarrow -m_1x + im_2 - m_3 &= -m_1x + m_4 + im_5 \end{aligned}$$

$$\begin{aligned} -m_2x - im_4x + im_6 &= -a_4 \\ -m_3x - im_5x - m_6 &= -ia_4 \\ \Rightarrow -im_2x + m_4x - m_6 &= -m_3x - im_5x - m_6 \end{aligned}$$

Before we try to make sense of this system, lets look at the second condition $\gamma\delta = \delta\alpha$. Expanding it out using a general matrix of each algebra gives us:

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_1 \\ 0 & 0 & -a_1 & -a_3 \\ 0 & 0 & -a_4 & -a_4 \end{pmatrix} \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^T & \alpha_3 \end{pmatrix}$$

We can now write $\alpha_2 = \begin{pmatrix} R_1 & \cdots \\ R_2 & \cdots \\ R_3 & \cdots \\ R_4 & \cdots \end{pmatrix}$, allowing us to multiply the above equality:

$$\begin{pmatrix} a_1 + a_2x & ia_1 + ia_2x & a_2 & ia_2 & \cdots & 0 \\ a_3 + a_4x & ia_3 + ia_4x & a_4 & ia_4 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -im_1 & m_1 & m_2 + im_4 & m_3 + im_5 & R_1 + iR_2 & \\ -im_1x - m_2 - im_3 & m_1x - m_4 - im_5 & m_2x + im_4x - im_6 & m_3x + im_5x + m_6 & R_1x + iR_2x + R_3 + iR_4 & \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Right away we see that

$$R_2 = iR_1, R_4 = iR_3$$

Now compiling the rest of the equations:

$$im_1 = -a_1 - a_2x$$

$$\begin{aligned} -im_1x - m_2 - im_3 &= a_3 + a_4x \\ m_1x - m_4 - im_5 &= ia_3 + ia_4x \\ \Rightarrow -im_2 + m_4 &= -m_3 - im_5 \\ &\cdots \end{aligned}$$

We can see that the conditions are the same as for the first condition. Finalizing into the following system:

$$m_1 = -ia_1 - ia_2x \tag{1}$$
$$m_4 = im_2 - m_3 - im_5 \tag{2}$$
$$m_2 + im_4 = a_2 = im_3 - m_5 \tag{3}$$
$$-m_1x + im_2 - m_3 = -ia_3 - ia_4x = -m_1x + m_4 + im_5 \tag{4}$$
$$-m_2x - im_4x + im_6 = -a_4 = -im_3x + m_5x - im_6 \tag{5}$$

What does this say?

1. m_1 has one free variable, the rest are determined by values in γ.
2. m_4 is determined by other elements of m.
3. Equations 3-4 show that m_3, m_4, m_5 are determined by a_2, a_3, a_4, x.
4. And lastly m_6 is determined by other values of a as well.

Thus we can see that α_1 is dependent on γ.

Using the general form of $\mathfrak{z}_Q$: $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ this is what our $\mathfrak{z}_Q$ looks like:

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^T & \alpha_3 \end{pmatrix}, \delta = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_1 \\ 0 & 0 & -a_1 & -a_3 \\ 0 & 0 & -a_4 & -a_4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} R_1 & \cdots \\ iR_1 & \cdots \\ R_2 & \cdots \\ iR_2 & \cdots \end{pmatrix}$$

Calculating $\mathfrak{b}_Q$

Let's take an general element $Q' \in \mathfrak{osp}_1$

$$
\begin{aligned}
[Q, Q'] &= \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix} \begin{pmatrix} 0 & -\delta'^T J \\ \delta' & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\delta'^T J \\ \delta' & 0 \end{pmatrix} \begin{pmatrix} 0 & -\delta^T J \\ \delta & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\delta^T J \delta' & 0 \\ 0 & -\delta\delta'^T J \end{pmatrix} + \begin{pmatrix} -\delta'^T J \delta & 0 \\ 0 & -\delta'\delta^T J \end{pmatrix} \\
&= \begin{pmatrix} -\delta^T J \delta' - \delta'^T J \delta & 0 \\ 0 & -\delta\delta'^T J - \delta'\delta^T J \end{pmatrix}
\end{aligned}
$$

Using $\delta' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & R_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & R_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & R_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & R_4 \end{pmatrix}$ we can work out work what $\alpha \in \mathfrak{so}(k)$ and $\beta \in \mathfrak{sp}(4)$ looks like:

$\beta = -\delta\delta'^T J - \delta'\delta^T J$

$$
= \begin{pmatrix} a_{31} + ia_{32} & a_{41} + ia_{42} & -a_{11} - ia_{12} & -a_{21} - ia_{22} \\ xa_{31} + ixa_{32} + a_{33} + ia_{34} & xa_{41} + ixa_{42} + a_{43} + ia_{44} & -xa_{11} - ixa_{12} - a_{13} - ia_{14} & -xa_{21} - ixa_{22} - a_{23} - ia_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
- \begin{pmatrix} 0 & 0 & -a_{11} - ia_{12} & -xa_{11} - ixa_{12} - a_{13} - ia_{14} \\ 0 & 0 & -a_{21} - ia_{22} & -xa_{21} - ixa_{22} - a_{23} - ia_{24} \\ 0 & 0 & -a_{31} - ia_{32} & -xa_{31} - ixa_{32} - a_{33} - ia_{34} \\ 0 & 0 & -a_{41} - ia_{42} & -xa_{41} - ixa_{42} - a_{43} - ia_{44} \end{pmatrix} =
$$

$$
\begin{pmatrix} a_{31} + ia_{32} & a_{41} + ia_{42} & 0 & -a_{21} - ia_{22} - xa_{11} - ixa_{12} - a_{13} - ia \\ xa_{31} + ixa_{32} + a_{33} + ia_{34} & xa_{41} + ixa_{42} + a_{43} + ia_{44} & -xa_{11} - ixa_{12} - a_{13} - ia_{14} - a_{21} - ia_{22} & 0 \\ 0 & 0 & -a_{31} - ia_{32} & -xa_{31} - ixa_{32} - a_{33} - ia_{34} \\ 0 & 0 & -a_{41} - ia_{42} & -xa_{41} - ixa_{42} - a_{43} - ia_{44} \end{pmatrix}
$$

Pretty similar to $\mathfrak{z}_Q$ other than two more slots of zeroes. No for α:

$$
\begin{pmatrix} 0 & 0 & -1 & -x \\ 0 & 0 & -i & -xi \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & R_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & R_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & R_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & R_4 \end{pmatrix} - \begin{pmatrix} -a31 & -a_{41} & a_{11} & a_{21} \\ -a32 & -a_{42} & a_{12} & a_{22} \\ -a33 & -a_{43} & a_{13} & a_{23} \\ -a34 & -a_{44} & a_{14} & a_{24} \\ R_1 & R_2 & R_3 & R_4 \end{pmatrix} \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ x & ix & 1 & i & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} -a_{31} - xa_{41} & -a_{32} - xa_{42} & -a_{33} - xa_{43} & -a_{34} - xa_{44} & 0 & \cdots & 0 \\ -ia_{31} - ixa_{41} & -ia_{32} - xa_{42} & -ia_{33} - ixa_{43} & -ia_{34} - ixa_{44} & 0 & \cdots & 0 \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} & 0 & \cdots & 0 \\ -ia_{41} & -ia_{42} & -ia_{43} & -ia_{44} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} -
$$

3 Further Computations

4 Conclusion