Geometric Markov Renewal Processes and Its Application

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1 Introduction

Motivated by the geometric compound Poisson processes, Swishchuk and Islam [1, 2, 3] studied the geometric Markov renewal processes and its application to the security market. Specifically, they presented the martingale properties, infinitesimal operators of geometric Markov renewal processes and derived the Markov renewal equation for expectation. In [1, 2], Swishchuk and Islam considered the geometric Markov renewal processes in averaging scheme and diffusion approximation scheme, found the approximations in the forms of ergodic and merged geometric Markov renewal processes in these two schemes, and presented weak convergence analysis and rates of convergence of ergodic geometric Markov renewal processes. In [3], Swishchuk and Islam stated normal deviations and Poisson approximation results for geometric Markov renewal processes.

This paper is based on the research in [1, 2, 3] and is organized as follows. Section 2 is the statement of the problem. In Section 3 we give a brief introduction about the general geometric markov renewal processes, present the main results of ergodic and merged geometric Markov Renewal processes in averaging scheme and diffusion scheme, and discuss normal deviations and Poisson approximation. In section 4, we apply the Black-Scholes formula to the GMRP with diffusion and normal deviations, and then use a numerical example to illustrate it.

2 Statement of the Problem

The mathematical model of the evolution of a system placed in a random medium consists of a switching process which describes the changes of the medium, and a switched process which describes the evolution of the system[1]. In the case of geometric Markov renewal processes (GMRP) for a security market, we treat Markov renewal processes and semi-Markov processes as switched-switching process. Averaging and diffusion approximation methods are important approximation methods for a switched-switching system. We are interested in the following problem: if we consider the geometric Markov Renewal Processes in series scheme, *i.e*, we consider a new series of processes S_t^T which depend on a parameter $T > 0$, will S_t^T converge to any S_t as $T \to \infty$? To satisfy the assumption, we consider the counting process in new faster time $v_T(t) := v(tT)$ and $v_T(t) := v(tT^2)$, and these two different condition will lead us to averaging and diffusion approximation separately.

3 Description of the Process

3.1 The Geometric Markov Renewal Processes

The geometric Markov renewal process(GMRP)$\{S_t\}_{t\in\mathbb{R}_+}$ is defined as a stochastic functional

$$S_t := S_0 \prod_{k=1}^{v(t)} (1+\rho(x_k)), \quad t \in \mathbb{R}_+, \tag{1}$$

where $S_0 > 0$ is the initial value of S_t, $(x_n, \theta_n)_{n\in\mathbb{Z}_+}$ is a Markov renewal process on the phase space $X \times \mathbb{R}_+$ with the semi-Markov kernel

$$Q(x, A, t) = \mathbb{P}\{x_{n+1} \in A, \theta_{n+1} \leq t | x_n = x\}, \ \forall x \in X, \ A \in \mathcal{L}, \ t \in \mathbb{R}_+ \tag{2}$$

$\rho(x)$ is a bounded continuous function on X such that $\rho(x) > -1$, and $v(t)$ is the counting process

$$v(t) := \sup\{k : \tau_k \leq t\}, \tag{3}$$

where the random variables τ_k are called renewal times and is defined by

$$\tau_k := \sum_{n=0}^{k} \theta_n. \tag{4}$$

Now we consider the martingale property of the geometric Markov renewal process $\{S_t\}_{t\in\mathbb{R}_+}$. Define

$$L_t := L_0 \prod_{k=1}^{v(t)} h(x_k), \ EL_0 = 1, \ t \in [0, T] \tag{5}$$

where $h(x)$ is a bounded continuous function such that

$$\int_X h(y)P(x, dy) = 1, \ \int_X h(y)P(x, dy)\rho(y) = 0 \tag{6}$$

If $EL_T = 1$, then the geometric Markov renewal process $\{S_t\}_{t\in\mathbb{R}_+}$ defined by (1) is an $(\mathcal{F}_t, P^*)$-martingale, where the measure P^* is defined as

$$\frac{dP^*}{dP} = L_T,$$

and

$$\mathcal{F}_t := \sigma(x(s); 0 \leq s \leq t).$$

3.2 Averaged Geometric Markov Renewal Processes

Consider a series of processes S_t^T(which depend on a parameter $T > 0$) with small random perturbations $T^{-1}\rho(x_k)$ on a large time interval $[0, tT)$, the averaged geometric Markov renewal processes S_t^T will be obtained as ergodic and merged geometric Markov renewal processes under different conditions. We are interested in the convergence of S_t^T as $T \to \infty$ in these two situation.

3.2.1 Ergodic Geometric Markov Renewal Processes

Let $(x_n)_{n\in\mathbb{Z}_+}$ be a Markov chain on the phase space $(X,\mathcal{L})$. Suppose the Markov chain $(x_n)_{n\in\mathbb{Z}_+}$ has a stationary distribution $p(A), A\in\mathcal{L}$. The evolution of the geometric Markov renewal process S_t takes place in a stationary regime if the effect of ergodicity of $(x_n)_{n\in\mathbb{Z}_+}$ is sufficiently influenced. This can be done if we consider the counting process $v(t)$ in a new faster time, *i.e*, define $v_T(t) := v(tT), T>0$. In order to avoid infinite changes of S_t for finite time under increasing T, we consider

$$\rho_T(x) = \frac{\rho(x)}{T},$$

In this way, S_t in (1) has the following form

$$S_t^T = S_0 \prod_{k=1}^{v(tT)} (1 + T^{-1}\rho(x_k)). \tag{7}$$

For the ergodic geometric Markov renewal process S_t defined in (7), we have following important theorem

Theorem 1 *Let $m := \int_X p(dx)\bar{m}(x), \bar{m}(x) := \int_0^\infty (1-G_x(t))dt, G_x(t) = \mathbb{P}(\tau_{n+1}-\tau_n < t|x_n = x), \hat{p} := \int_X p(dx)\rho(x)/m$. If $\hat{p} := \int_X p(dx)(\rho(x))^2 < \infty$, then the ergodic geometric Markov renewal process $\hat{S}_t$ (where $\hat{S}_t = \lim_{T\to+\infty} S_t^T$) has following form*

$$\hat{S}_t = S_0 e^{\hat{\rho}t} \tag{8}$$

for all $t\in\mathbb{R}_+$, and $S_0>0$.

This theorem can be proved by using Taylor expansion and showing that

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{v(Tt)} \rho(x_k) = \hat{\rho}t$$

3.2.2 Merged Geometric Markov Renewal Processes

A merged Matkov process can be thought of as a Markov chain embedded within a Markov chain and it is used to approximate the behavior of the larger state-space Markov chain. If the Markov state space X consists of r ergodic classes $X_i (i = 1,2,...,r)$ with stationary distributions $p_i(dx)(i = 1,2,...,r)$ in each class. Then we say that the Markov chain $(x_k)_{k\in\mathbb{Z}_+}$ is merged to the Markov chain $(\hat{x}_s)_{s\in\mathbb{Z}_+}$ in the merged phase space $\hat{X} = \{1,2,...,r\}$. We also have an important theorem for the merged geometric Markov renewal processes as follow

Theorem 2 *Let $\hat{\rho}_k := \int_{X_k} p_k(dx)\rho(x)/m(k)$, $m(k) := \int_{X_k} p_k(dx)m(x)$, $m(x) := \int_0^\infty (1 - G_x(t))dt$ and $\hat{x}(s)$ is a merged Markov process in the merged phase space $\hat{X}$. Then the merged geometric Markov renewal processes has the form*

$$\tilde{S}_t = S_0 e^{\int_0^t \hat{\rho}(\hat{x}(s))ds} \tag{9}$$

where $t\in\mathbb{R}_+$ and $S_0>0$.

This theorem can be proved by showing that $\frac{1}{T}\sum_{k=0}^{v(Tt)} \rho(x_k)$ is merged to the integral functional

$$\tilde{\rho}(t) := \int_0^t \hat{\rho}(\hat{x}(s))ds.$$

3.3 Diffusion Approximation of the Geometric Markov Renewal Processes

If we consider the counting process $v(t)$ in (3) in a new accelerated scale of time tT^2, then the more rapid jumps of $v(tT^2)$ will make the process $S_T(t)$ fluctuate near the point S_0 as $T \to \infty$. Thus, we will use a diffusion process to describe the fluctuations.

3.3.1 Ergodic Diffusion Approxiamation

Suppose that following balance condition is fulfilled for functional $S_t^T = S_0 \prod_{k=1}^{v(tT)}(1 + \rho_T(x_k))$

$$\hat{\rho} = \frac{\int_X p(dx) \int_X P(x, dy)\rho(y)}{m} = 0$$

where $p(x)$ is ergodic distribution of Markov chain $(x_k)_{k\in\mathbb{Z}_+}$. Similar to the ergodic GMRP in averaged scheme, here we consider $v_T(t) := v(tT^2)$ in the new faster time, and denote $\rho_T(x) = T^{-1}\rho(x)$ for all $x \in X$.In this way, S_t in (1) has the following form

$$S_T(t) := S_{tT^2}^T = S_0 \prod_{k=1}^{v(tT^2)} (1 + T^{-1}\rho(x_k)). \tag{10}$$

By imitating the proof of Theorem 4.1 in [1], we have following expression

$$\ln \frac{S_T(t)}{S_0} = T^{-1} \sum_{k=1}^{v(tT^2)} \rho(x_k) - \frac{1}{2}T^{-2} \sum_{k=1}^{v(tT^2)} \rho^2(x_k) + T^{-2} \sum_{k=1}^{v(tT^2)} r(T^{-1}\rho(x_k))\rho^2(x_k), \tag{11}$$

However, the difference between model (10) and model (4.1) in [1] is that, both the last two terms of the right side in (4.5) in [1] tend to zero as $T \to \infty$, here the last term of the right side in (11) will also goes to zero as $T \to \infty$ but we can show that the second term will goes to $\frac{1}{2}t\hat{\rho}_2$ (where $\hat{\rho_2} := \int_X p(dx) \int_X P(x, dy)\rho^2(y)/m$) as $T \to \infty$ by using the algorithms of ergodic averaging. We can also find that the first term in (11) will goes to $\sigma_\rho \omega(t)$ by using the algorithms of diffusion approximation, where $\sigma_\rho^2 := \int_X p(dx)[1/2P(x, dy)\rho^2(y) + \int_X P(x, dy)\rho(y)R_0P(x, dy)\rho(y)]/m$, R_0 is a potential of $(x_n)_{n\in\mathbb{Z}_+}$ and $\omega(t)$ is a standard Wiener process. Therefore, taking limit on both hand sides of (11) we obtain

$$\lim_{T\to+\infty} \ln \frac{S_T(t)}{S_0} = \ln \frac{\hat{S}(t)}{S_0} = \sigma_\rho \omega(t) - \frac{1}{2}t\hat{\rho}_2, \tag{12}$$

which gives us following theorem about the ergodic diffusion GMRP

Theorem 3 *The ergodic diffusion geometric Markov renewal process $\hat{S}(t)$ (where $\hat{S}(t) = \lim_{T\to+\infty} S_T(t)$) has following form*

$$\hat{S}(t) = S_0 e^{-\frac{1}{2}t\hat{\rho}_2} e^{\sigma_\rho \omega(t)}, \tag{13}$$

and it satisfies the following SDE

$$\frac{d\hat{S}(t)}{\hat{S}(t)} = \frac{1}{2}(\sigma_\rho - \hat{\rho}_2)dt + \sigma_\rho d\omega(t). \tag{14}$$

3.3.2 Merged Diffusion Approxiamation

As for the merged diffusion geometric Markov renewal processes, suppose that the balance condition satisfies

$$\hat{\rho}(k) := \frac{\int_{X_k} p_k(dx) \int_{X_k} P(x,dy)\rho(y)}{m(k)} = 0, \tag{15}$$

for all $k = 1, 2, ...r$ where $(x_n)_{n\in\mathbb{Z}_+}$ is the supporting embedded Markov chain, p_k is the stationary density for the ergodic component X_k and $m(k)$ is defined in the Theorem 2.If we use the algorithms of merged averaging and algorithms of merged diffusion approximation, then take limit on both hand sides of (11) we obtain

$$\lim_{T\to+\infty} \ln \frac{S_T(t)}{S_0} = \ln \frac{\tilde{S}(t)}{S_0} = \int_0^t \hat{\sigma}_\rho(\hat{x}(s))d\omega(s) - \frac{1}{2}\int_0^t \hat{\rho}_2(\hat{x}(s))ds, \tag{16}$$

where

$$\hat{\rho}_2(k) := \frac{\int_{X_k} p_k(dx) \int_{X_k} P(x,dy)\rho^2(y)}{m(k)},$$

Therefore, we have following theorem for merged diffusion GMRP

Theorem 4 *The merged diffusion geometric Markov renewal process $\tilde{S}(t)$ has following form*

$$\tilde{S}(t) = S_0 e^{-\frac{1}{2}\int_0^t \hat{\rho}^2(\hat{x}(s))ds + \int_0^t \hat{\sigma}_\rho(\hat{x}(s))d\omega(s)}, \tag{17}$$

and it satisfies the following SDE

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2}(\hat{\sigma}_\rho(\hat{x}(t)) - \hat{\rho}_2(\hat{x}(t)))dt + \hat{\sigma}_\rho(\hat{x}(t))d\omega(t), \tag{18}$$

where $\hat{x}(t)$ is a merged Markov process.

3.4 Normal Deviations of the Geometric Markov Renewal Processes

In this section, the forms of ergodic and merged normal deviated GMRP will be considered.

3.4.1 Ergodic Normal Deviations

Let $\alpha_T(t) := \ln S_t^T / \ln S_0 \approx T^{-1}\sum_{k=1}^{v(tT)} \rho(x_k)$, S_t^T is defined in (7), then the normal deviated process

$$\omega_T(t) := \sqrt{T}(\alpha_T(t) - \hat{\rho}t) = T^{-\frac{1}{2}} \sum_{k=1}^{v(tT)} [\rho(x_k) - \hat{\rho}] - \hat{\rho}T^{-\frac{1}{2}}(tT - v(tT)), \tag{19}$$

where ρ is defined in Theorem 1. It can be seen that the second term tends to zero as $T \to \infty$. As for the first term, since $\int_X p(dx)\int_X P(x,dy)(\rho(y) - \hat{\rho}) = 0$, which means that it satisfies the balance condition and we can use algorithms of diffusion approximation to show that the first term tends to $\hat{\sigma}\hat{\omega}_t$ as $T \to \infty$, where $\hat{\sigma} := \int_X p(dx)[1/2P(x,dy)(\rho - \hat{\rho})^2 + \int_X P(x,dy)(\rho - \hat{\rho})R_0 P(x,dy)(\rho - \hat{\rho})]/m$, R_0 is a potential of $(x_n)_{n\in\mathbb{Z}_+}$ and $\hat{\omega}_t$ is a standard Wiener process. Therefore, we have that

$$\frac{\ln S_t^T}{\ln S_0} = \alpha_T(t) = \hat{\rho}t + T^{-\frac{1}{2}}\omega_T(t) \simeq \hat{\rho}t + T^{-\frac{1}{2}}\hat{\sigma}\hat{\omega}_t. \tag{20}$$

Therefore, we have following theorem for ergodic normal deviated GMRP

Theorem 5 *The ergodic normal deviated GMRP has following form*

$$S_t^T \simeq S_0 e^{\hat{\rho} t + T^{-\frac{1}{2}} \hat{\sigma} \hat{\omega}(t)}, \tag{21}$$

and it satisfies the following SDE

$$\frac{dS_t^T}{S_t^T} \simeq (\hat{\rho} + \frac{1}{2} T^{-1} \hat{\sigma}^2) dt + T^{-\frac{1}{2}} \hat{\sigma} d\hat{\omega}(t). \tag{22}$$

3.4.2 Merged Normal Deviations

Suppose that the balance condition in (15) is not fulfilled, *i.e*, $\hat{\rho}(k) \neq 0$, then consider the normal deviated process

$$\tilde{\omega}_T(t) := \sqrt{T}(\alpha_T(t) - \tilde{\rho}(t)), \tag{23}$$

where $\tilde{\rho}(t) = \int_0^t \hat{\rho}(\hat{x}(s)) ds$. In this case, $\tilde{\omega}_T(t)$ is a stochastic Ito integral ([4]) under large T:

$$\tilde{\omega}_T(t) \simeq \int_0^t \tilde{\sigma}(\hat{x}(s)) d\omega(s), \tag{24}$$

where

$$\tilde{\sigma}(k) := \int_{X_k} p_k(dx)[1/2P(x,dy)(\rho - \hat{\rho}(k))^2 + \int_{X_k} P(x,dy)(\rho - \hat{\rho}(k)) R_0 P(x,dy)(\rho - \hat{\rho}(k))]/m(k)$$

combined (23) and (24) we have that

$$\frac{\ln S_t^T}{\ln S_0} = \alpha_T(t) \simeq \tilde{\rho}(t) + T^{-\frac{1}{2}} \int_0^t \tilde{\sigma}(\hat{x}(s)) d\omega(s), \tag{25}$$

which gives us following theorem for merged normal deviated GMRP

Theorem 6 *The merged normal deviated GMRP has following form*

$$S_t^T \simeq S_0 e^{\tilde{\rho}(t) + T^{-\frac{1}{2}} \int_0^t \tilde{\sigma}(\hat{x}(s)) d\omega(s)}, \tag{26}$$

and it satisfies the following SDE

$$\frac{dS_t^T}{S_t^T} \simeq (\hat{\rho}(\hat{x}(t)) + \frac{1}{2} T^{-1} \tilde{\sigma}^2(\hat{x}(t)) dt + T^{-\frac{1}{2}} \tilde{\sigma}(\hat{x}(t)) d\omega(t). \tag{27}$$

3.5 Averaging in Poisson Scheme

As for the averaging of GMRP in Poisson scheme, we consider a more general case: let $\rho_k^T(x_k) \equiv \rho_k^T(x_k, \omega)$ be a sequence of random variables for all $x \in X$ and for all $T > 0$, and consider the process S_t^T in following series scheme:

$$S_t^T = S_0 \prod_{k=1}^{v(tT)} (1 + T^{-1} \rho_k(x_k)). \tag{28}$$

Here, $x(t) := x_{v(t)}$ is a Markov process with the transition kernel

$$Q(x, A, t) := \mathbb{P}(x_{k+1} \in A | x_k = x)\mathbb{P}(\theta_{k+1} \le t | x_k = x), \tag{29}$$

where $\mathbb{P}(\theta_{k+1} \le t | x_k = x) := 1 - e^{-q(x)t}$. We suppose that $x(t)$ is ergodic Markov process with stationary distribution $\pi(dx)$, and $(x_n)_{n\in Z_+}$ is ergodic Markov chain with stationary distribution $p(dx)$, where

$$\pi(dx)q(x) = qp(dx), \quad q := \int_X \pi(dx)q(x),$$

and we define

$$F_x^T(z) = \mathbb{P}(\rho_k^T(x, \omega) \le z)$$

for all $z \in R$. We also suppose that the sequence $\rho_k^T(x_k)$ are independent random variables for fixed sequence x_k and for $k \in Z_+$. Then, using the result in [5] we obtain

$$\lim_{T\to\infty} \sum_{k=1}^{v(tT)} \rho_k^T(x_k) = P(t) := \sum_{k=1}^{N_0(t)} \alpha_k^0 + a_0 qt, \tag{30}$$

where α_k^0 are i.i.d.r.v with distribution function $F^0(z)$

$$Eg(\alpha_k^0) = \int_R g(z)F^0(dz) = \hat{F}(g)/\hat{F}(1),$$

$$\hat{F}(q) := \int_X p(dx)F_x(g), \quad \hat{F}(1) := \int_X p(dx)F_x(R).$$

Here $F_x(g)$ is the function such that $\int_X g(z)F_x(dz) = T^{-1}[F_x(g) + o_T(x, g)]$, where $\sup_x |F_x(g)| \le +\infty$, $g(z)$ is the function such that $g(z)/z^2 \to_{|z|\to 0} 0$, and $F_x(g) = \int_R g(z)F_x(dz)$. The compound Poisson process $N_0(t)$ is defined by the intensity $q_0 := q\hat{F}(1)$. Value a_0 is defined as $a_0 := \hat{a} - \hat{F}(1)E\alpha_1^0$, where $\hat{a} := \int_X p(dx)a(x)$ and $a(x)$ is defined asymptotically

$$\int_R zF_x^T(dz) = T^{-1}[a(x) + o_T(x)]$$

with $\sup_x |a(x)| \le a < +\infty$ and $\lim_{T\to\infty} o_T(x) = 0$.

Therefore, from (30) we have that

$$S_P(t) := \lim_{T\to+\infty} S_t^T = S_0 \exp\{\sum_{k=1}^{N_0(t)} \alpha_k^0 + a_0 qt\}. \tag{31}$$

4 Application

4.1 European Call Option Pricing Formulas for Diffusion GMRP

Since We already proved that the ergodic diffusion GMRP $\hat{S}(t)$ can be obtained by a stochastic differential equation, we can approximated $\hat{S}(t)$ in the form of a SDE for a geometric Brownian motion under a risk neutral measure, and then use the Black-Scholes formula to find the price for European call option directly.

As the Theorem 3 states, The ergodic diffusion GMRP $\hat{S}(t)$ satisfies the following SDE

$$\frac{d\hat{S}(t)}{\hat{S}(t)} = \frac{1}{2}(\sigma_\rho - \hat{\rho}_2)dt + \sigma_\rho d\omega(t). \tag{32}$$

Using the Girsanov's theorem, we can find the risk-neutral probability measure P^* as following

$$\frac{dP^*}{dP} = \exp\{-\theta t - \frac{1}{2}\theta^2\omega(t)\}, \tag{33}$$

where

$$\theta = \frac{\frac{1}{2}(\sigma_\rho - \hat{\rho}_2) - r}{\sigma_\rho}.$$

Under the risk-neutral probability measure P^*, the process $e^{-rt}\hat{S}_t$ is a martingale and the process $\omega^*(t) = \omega(t) + \theta t$ is a Brownian motion. Moreover, in the risk neutral world, the process $\hat{S}_t$ has following form

$$\frac{d\hat{S}_t}{\hat{S}_t} = rdt + \sigma_\rho d\omega^*(t). \tag{34}$$

Then using the Black-Scholes formula directly and we can obtain the European call option pricing formula for the model (34)

$$C = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \tag{35}$$

where

$$d_+ = \frac{\ln(S_0/K) + (r + \sigma_\rho/2)t}{\sigma_\rho\sqrt{t}},$$

$$d_- = \frac{\ln(S_0/K) + (r - \sigma_\rho/2)t}{\sigma_\rho\sqrt{t}},$$

where $\Phi(x)$ is a normal distribution.

As for the merged diffusion GMRP which has the following form

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2}(\hat{\sigma}_\rho(\hat{x}(t)) - \hat{\rho}_2(\hat{x}(t)))dt + \hat{\sigma}_\rho(\hat{x}(t))d\omega(t), \tag{36}$$

taking into account the result on European call option pricing formula for regime-switching geometric Brownian motion, we obtain the option pricing formula for the merged diffusion GMRP

$$C = \int C_T^{BS}((\frac{z}{T})^{\frac{1}{2}}, T, S_0)F_T^x(dz), \tag{37}$$

where C_T^{BS} is a Black-Scholes value and $F_T^x(dz)$ is a distribution of the random variable

$$Z_T^x = \int_0^T \hat{\sigma}_\rho^2(\hat{x}(t))ds,$$

where $\hat{x}$ is a merged Markov process.

4.2 European Call Option Pricing Formulas for Normal Deviated GMRP

From Theorem 5 we know that the ergodic normal deviated GMRP satisfies the following SDE

$$\frac{dS_t^T}{S_t^T} \simeq (\hat{\rho} + \frac{1}{2}T^{-1}\hat{\sigma}^2)dt + T^{-\frac{1}{2}}\hat{\sigma}d\hat{\omega}(t). \tag{38}$$

Similar to the section 4.1, we can find the risk-neutral probability measure P^* by using the Girsanov's theorem

$$\frac{dP^*}{dP} = \exp\{-\theta t - \frac{1}{2}\theta^2\omega(t)\}, \tag{39}$$

where

$$\theta = \frac{\hat{\rho} + \frac{1}{2}T^{-1}\hat{\sigma}^2 - r}{T^{-\frac{1}{2}}\hat{\sigma}}.$$

Under the risk-neutral probability measure P^*, the process $e^{-rt}S_t^T$ is a martingale and the process $\omega^*(t) = \omega(t) + \theta t$ is a Brownian motion. The process S_t^T has following form

$$\frac{dS_t^T}{S_t^T} = rdt + T^{-\frac{1}{2}}\hat{\sigma}d\omega^*(t) \tag{40}$$

in the risk neutral world. Then we can use $T^{-\frac{1}{2}}\hat{\sigma}$ to take place of σ_ρ in (35) and obtain the European call option pricing formula for model (38), which gives us

$$C = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \tag{41}$$

where

$$d_+ = \frac{\ln(S_0/K) + (r + T^{-\frac{1}{2}}\hat{\sigma}/2)t}{T^{-\frac{1}{2}}\hat{\sigma}\sqrt{t}},$$

$$d_- = \frac{\ln(S_0/K) + (r - T^{-\frac{1}{2}}\hat{\sigma}/2)t}{T^{-\frac{1}{2}}\hat{\sigma}\sqrt{t}},$$

where $\Phi(x)$ is a normal distribution.

4.3 Numerical Example for Ergodic Diffusion Approximation

Let us assume that the stock price at time t is S_t, and S_t is a two-state geometric Markov renewal process. At each step, if the stock price goes up compared to the last step, then it will goes up with the probability 0.4 and goes down with the probability 0.6; if the stock price goes down compared to the last step, then it will goes up with the probability 0.6 and goes down with the probability 0.4. The returns of goes up and goes down are 0.01 and -0.01 respectively. We can use following Markov transition matrix and vector to describe the process

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0.01 & -0.01 \end{pmatrix}.$$

We use following equation to find the stationary distribution π

$$\begin{pmatrix} p_0 & p_1 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} p_0 & p_1 \end{pmatrix},$$

and we obtain that

$$\pi = \begin{pmatrix} p_0 & p_1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix},$$

$$\Pi := \begin{pmatrix} \pi \\ \pi \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

The balance condition is fulfilled since

$$\int_X p(dx) \int_X P(x,dy)\rho(y) = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.01 \\ -0.01 \end{pmatrix} = 0.$$

Assume that the waiting time between two jumps satisfy following distribution

$$G_{x_0}(t) = \mathbb{P}(\tau_{n+1} - \tau_n \leq t | x_n = x_0) = 1 - e^{-5t},$$

$$G_{x_1}(t) = \mathbb{P}(\tau_{n+1} - \tau_n \leq t | x_n = x_1) = 1 - e^{-6t},$$

where x_0 and x_1 are two state of S_t. Therefore, we obtain

$$\bar{m}(x_0) = \int_0^\infty (1 - G_{x_0}(t))dt = \int_0^\infty e^{-5t} dt = \frac{1}{5},$$

$$\bar{m}(x_1) = \int_0^\infty (1 - G_{x_1}(t))dt = \int_0^\infty e^{-6t} dt = \frac{1}{6},$$

and

$$m = \int_X p(dx)\bar{m}(x) = \begin{pmatrix} p_0 & p_1 \end{pmatrix} \begin{pmatrix} \bar{m}_{x_0} \\ \bar{m}_{x_1} \end{pmatrix} = \frac{11}{60}.$$

We can obtain

$$\begin{aligned} \hat{\rho}_2 := \int_X p(dx) \int_X P(x,dy)\rho^2(y)/m &= \frac{\begin{pmatrix} p_0 & p_1 \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} \begin{pmatrix} \rho_0^2 \\ \rho_1^2 \end{pmatrix}}{m} \\ &= \frac{\begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.01^2 \\ (-0.01)^2 \end{pmatrix}}{\frac{11}{60}} \\ &= 5.5 \times 10^{-4}. \end{aligned}$$

Now we are going to find the potential R_0 of $(x_n)_{n \in Z_+}$ before calculating the σ_ρ^2. We know that the potential R_0 of a Markov chain has following form[6]

$$R_0 = \sum_{n=0}^{\infty} (P^n - \Pi),$$

where P is the Markov transition matrix and Π is the matrix of limiting probabilities. We diagonalize the transition matrix P and write it in following form

$$P = EDE^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -0.2 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix},$$

and we obtain that

$$P^n = ED^nE^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-0.2)^n \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(-0.2)^n & \frac{1}{2}-\frac{1}{2}(-0.2)^n \\ \frac{1}{2}-\frac{1}{2}(-0.2)^n & \frac{1}{2}+\frac{1}{2}(-0.2)^n \end{pmatrix},$$

therefore we have

$$R_0 = \sum_{n=0}^{\infty}(P^n - \Pi) = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{1}{2}(-0.2)^n & \frac{1}{2}(-0.2)^n \\ \frac{1}{2}(-0.2)^n & \frac{1}{2}(-0.2)^n \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\sum_{n=0}^{\infty}(-0.2)^n = \begin{pmatrix} \frac{5}{12} & -\frac{5}{12} \\ -\frac{5}{12} & \frac{5}{12} \end{pmatrix},$$

and

$$\sigma_\rho^2 = \int_X p(dx)[1/2P(x,dy)\rho^2(y) + \int_X P(x,dy)\rho(y)R_0P(x,dy)\rho(y)]/m$$

$$= \frac{\begin{pmatrix} p_0 & p_1 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}\begin{pmatrix} \rho_0^2 \\ \rho_1^2 \end{pmatrix} + \sum_{n=0}^{\infty}(P^n - \Pi)\begin{pmatrix} p_{00}\rho_0 + p_{01}\rho_1 \\ p_{10}\rho_0 + p_{11}\rho_1 \end{pmatrix}^2\right)}{m}$$

$$= 2.7 \times 10^{-4}.$$

In this case, we assume that the initial stock price S_0 is \$50, the strike price K is \$40, the annuak interest rate r is 0.01, and the maturity T is 0.5 year. Then, according to (35), we have that

$$d_+ = \frac{\ln(S_0/K) + (r + \sigma_\rho/2)t}{\sigma_\rho\sqrt{t}} = \frac{\ln(50/40) + (0.01 + \sqrt{2.7 \times 10^4}/2) \times 0.5}{\sqrt{2.7 \times 10^4}\sqrt{0.5}} \approx 0.35552,$$

$$d_- = \frac{\ln(S_0/K) + (r - \sigma_\rho/2)t}{\sigma_\rho\sqrt{t}} = \frac{\ln(50/40) + (0.01 - \sqrt{2.7 \times 10^4}/2) \times 0.5}{\sqrt{2.7 \times 10^4}\sqrt{0.5}} \approx -0.35159,$$

thus we have

$$C = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-) \approx 17.51,$$

$i.e$ the European call option price in this case is \$17.51.

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