CS6533/CS4533 Lecture 3 Slides/Notes

HW1 Discussion; 3D Transformations (Notes, Ch 2,3,4)

By Prof. Yi-Jen Chiang
CSE Dept., Tandon School of Engineering
New York University

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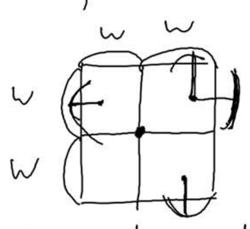
HW1 Part (d)

Input file circles (c_{x_i}, c_{y_i}, r_i) for each circle i.
in world coord. system.

Sol. Window width & height are each $2w$
Graphics



From input, find

$max\_x = \max_i \{ c_{x_i} + r_i, \ |c_{x_i} - r_i| \}$

$max\_y = \max_i \{ c_{y_i} + r_i, \ |c_{y_i} - r_i| \}$

$max\_abs\_coord = \max\{ max\_x, max\_y \}$

* check for values < 0 to see if the scaling is as desired.

If $max\_abs\_coord > w$, then we need to scale down (c_x, c_y, r) for all circles:

$c_x \leftarrow (c_x \cdot w) / max\_abs\_coord$. $\quad r \leftarrow (r \cdot w) / max\_abs\_coord$.

$c_y \leftarrow (c_y \cdot w) / max\_abs\_coord$.

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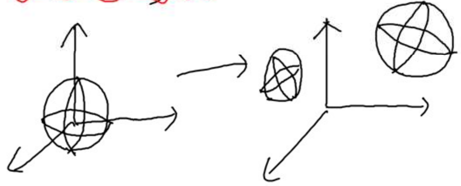
3D Transformations

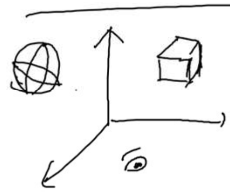
Motivations: Why do we study transformations?

① Modeling Transformation



Scaling
Rotation (change orientation)
Translation (" position)

② Coordinate System change.



World { Frame / coord. system

"what should be the final graphics image when seen from the viewer?"

Word Frame $\xrightarrow{\text{transf.}}$ Eye Frame

3

I. Vector Space, Basis & Linear Transformations.

(1) Basis Vectors

Let V be a vector space.

A set of vectors $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n$ $\in V$ is a basis for the vector space V if

① $\vec{b}_1, \vec{b}_2, \cdots \vec{b}_n$ are linearly indept. i.e. $a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n = 0$ for scalars $a_1, \cdots a_n$ $\Longleftrightarrow a_1 = a_2 = \cdots = a_n = 0$.

[meaning: any $\vec{b}_i$ can NOT be expressed as a linear combination of the other $n-1$ vectors $\vec{b}_j$, $j \neq i$]

($\vec{b}_1, \vec{b}_2, \cdots \vec{b}_n$ span V)

② Any vector $\vec{v} \in V$ can be expressed as a linear combination of $\vec{b}_1, \cdots, \vec{b}_n$

i.e. $\vec{v} = \sum_{i=1}^{n} c_i \vec{b}_i$ for some scalars $c_1, c_2, \cdots, c_n$

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* $\vec{b}_1, \ldots \vec{b}_n$ are called the basis vectors.
* The number of basis vectors, n, is called the dimension of the vector space V. In graphics, $n = 3$.
* Consider dimension $n = 3$.
  For any vector $\vec{v} \in V$, $\exists$ scalars c_1, c_2, c_3 s.t. $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$

$$= \underbrace{\left(\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3\right)}_{\text{row matrix}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\text{column matrix}}$$

$\Rightarrow$ $(\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3)$: row of basis vectors

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$: (column) coordinate vector

eg. $\vec{v} = (\underset{c_1}{1}, \underset{c_2}{2}, \underset{c_3}{3})$ basis: $\vec{b}_1 = e_1 = (1,0,0)$ x
$\vec{b}_2 = e_2 = (0,1,0)$ y
$\vec{b}_3 = e_3 = (0,0,1)$ z

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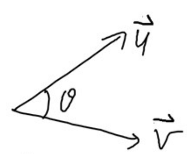
(2) Inner Product & Cross Product.

A. Inner Product. $\vec{u}, \vec{v}$ are vectors



inner product : $\vec{u} \cdot \vec{v}$

$\vec{u} \cdot \vec{v}$ returns a scalar value (real number)

* It allows to define the square length (or length) of a vector $\vec{v}$.
  $|\vec{v}|^2 \overset{\text{def}}{=} \vec{v} \cdot \vec{v}$ (i.e. $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$)
* $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta$. $\quad \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$

Two vector $\vec{u}, \vec{v}$ are orthogonal ($\vec{u} \perp \vec{v}$) if $\theta = 90°$
$\Longleftrightarrow \cos\theta = 0 = \vec{u} \cdot \vec{v}$

* A basis is orthonormal if ① all basis vectors are of length 1. and
  ② all " " are pairwise orthogonal

eg.

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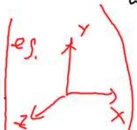
* In orthonormal basis: $\vec{u} = \sum_i c_i \vec{b}_i$. $\vec{v} = \sum_i d_i \vec{b}_i$ for vectors $\vec{u}, \vec{v}$

where $|\vec{b}_i| = 1 \quad \forall i$; $\vec{b}_i \cdot \vec{b}_j = 0 \ \forall i \neq j$.

Usually we use such basis

e.g.



$\left(\vec{b}_i \cdot \vec{b}_i = |\vec{b}_i|^2 = 1\right)$

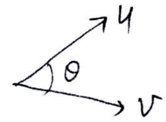
$$\vec{u} \cdot \vec{v} = \left(\sum_i c_i \vec{b}_i\right) \cdot \left(\sum_j d_j \vec{b}_j\right) = \sum_i \sum_j (c_i d_j)(\vec{b}_i \cdot \vec{b}_j) = \sum_i c_i d_i \cdot 1$$

In 3D $\vec{u} = (c_1, c_2, c_3)$
$\vec{v} = (d_1, d_2, d_3)$

$$\vec{u} \cdot \vec{v} = c_1 d_1 + c_2 d_2 + c_3 d_3$$

(2) Inner Product & Cross Product.

A. Inner Product : u, v are vectors



$\vec{u}\cdot\vec{v}$ returns a scalar value (real number)

* It allows to define the (squared length) (or simply (length)) of a vector:

$|v|^2 := v\cdot v$ (ie $|v| = \sqrt{v\cdot v}$)

* $\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta$. $\quad \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$

Two vectors $\vec{u},\vec{v}$ are (orthogonal) if $\theta = 90°$ $\Leftrightarrow$ $\cos\theta = 0 = \vec{u}\cdot\vec{v}$

* A basis is (orthonormal) if all the basis vectors are (unit length) and (pairwise orthogonal).   length = 1

Note: Usually we use orthonormal basis

* In orthonormal basis: $\vec{u} = \sum_i c_i\vec{b}_i$ $\quad \vec{v} = \sum_i d_i\vec{b}_i$ $\quad$ where $\vec{b}_i\cdot\vec{b}_i = |\vec{b}_i|^2 = 1$

$\vec{b}_i\cdot\vec{b}_j = 0$ when $i \neq j$

$$\vec{u}\cdot\vec{v} = \Big(\sum_i c_i\vec{b}_i\Big)\Big(\sum_j d_j\vec{b}_j\Big) = \sum_i\sum_j (c_i d_j)(\vec{b}_i\cdot\vec{b}_j)$$

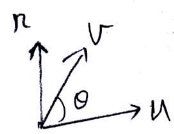
$$= \sum_i c_i d_i$$

In 3D. $\vec{u} = (c_1\ c_2\ c_3)$, $\vec{v} = (d_1\ d_2\ d_3)$ $\quad \vec{u}\cdot\vec{v} = c_1d_1 + c_2d_2 + c_3d_3$

Geometric Meaning:

* $\vec{u}\times\vec{v}$ gives a third vector $\perp$ to both $\vec{u}$ and $\vec{v}$

B. Cross Product : In 3D two vectors $\vec{u} = \sum_i c_i\vec{b}_i$ $\quad \vec{v} = \sum_i d_i\vec{b}_i$



$\vec{u}\times\vec{v} = (|\vec{u}|\,|\vec{v}|\sin\theta)\,\vec{n}$. $\quad \vec{n}$ is the (unit vector) perpendicular to the plane spanned by $\vec{u}$ & $\vec{v}$ in the direction by right-hand rule

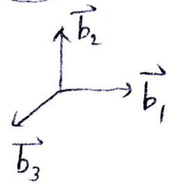
(4 fingers curling from $\vec{u}$ to $\vec{v}$. the thum points to the direction of $\vec{n}$)

In a (right-handed orthonormal) basis

$$\vec{u}\times\vec{v} = \Big(\sum_i c_i\vec{b}_i\Big)\times\Big(\sum_j d_j\vec{b}_j\Big)$$

$$= \sum_i\sum_j c_i d_j\,(\vec{b}_i\times\vec{b}_j)$$

$\vec{b}_1\times\vec{b}_2 = \vec{b}_3$
$\vec{b}_2\times\vec{b}_3 = \vec{b}_1$
$\vec{b}_3\times\vec{b}_1 = \vec{b}_2$
$\vec{b}_i\times\vec{b}_i = 0 \quad \forall i$



$$\begin{array}{rccc} & (c_1 & c_2 & c_3) \\ \times) & (d_1 & d_2 & d_3) \\ \hline & (c_2d_3 - c_3d_2, & c_3d_1 - c_1d_3, & c_1d_2 - c_2d_1) \end{array}$$

\#

(3) Linear Transformations

A linear transformation $\mathcal{L}$ is a transformation from V to V satisfying:

① $\mathcal{L}(\vec{u}+\vec{v}) = \mathcal{L}(\vec{u}) + \mathcal{L}(\vec{v})$ $\forall$ vectors $\vec{u}, \vec{v} \in V$, and

② $\mathcal{L}(a\vec{u}) = a\,\mathcal{L}(\vec{u})$ $\forall$ vector $\vec{u} \in V$ and real number (scalar) a.

Now let $[\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ be a basis of the 3D vector space V.

For any $\vec{v} \in V$, we have $\vec{v} = \sum_i c_i \vec{b}_i = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for some c_1, c_2, c_3 scalar values (real numbers)

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b}_i\right) = \sum_i c_i \mathcal{L}(\vec{b}_i) = [\mathcal{L}(\vec{b}_1)\ \mathcal{L}(\vec{b}_2)\ \mathcal{L}(\vec{b}_3)]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

But $\mathcal{L}(\vec{b}_1)$ is still a vector in V, thus $\mathcal{L}(\vec{b}_1) = \sum_j M_{j1} \vec{b}_j = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$

Similarly for $\mathcal{L}(\vec{b}_2)$ and $\mathcal{L}(\vec{b}_3)$ for some real numbers M_{11}, M_{21}, M_{31}.

$$\therefore \mathcal{L}(\vec{v}) = [\mathcal{L}(\vec{b}_1)\ \mathcal{L}(\vec{b}_2)\ \mathcal{L}(\vec{b}_3)]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

ie Linear transformation

$\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ can be expressed by matrix multiplication:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}}_{3\times 3 \text{ matrix } M}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(4) Standar Transformations:

Scaling, Rotation, Translation

We like to express each of them by matrix multiplication so that they can be easily supported.

★ Plan:

① Define the transformation

② Derive $P \longrightarrow Q$ using def. and simple geometry

③ Derive matrix M s.t $Q = MP$.

① Scaling: $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $Q = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ where $\begin{cases} x' = \beta_x x \\ y' = \beta_y y \\ z' = \beta_z z \end{cases}$ $\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} \beta_x & 0 & 0 \\ 0 & \beta_y & 0 \\ 0 & 0 & \beta_z \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{P}$

$S = S(\beta_x, \beta_y, \beta_z) =$ #

$S^{-1}(\beta_x, \beta_y, \beta_z) = S\left(\frac{1}{\beta_x}, \frac{1}{\beta_y}, \frac{1}{\beta_z}\right)$

Now let $[\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ be a basis of the 3D vector space V

For any $\vec{v} \in V$, we have $\boxed{\vec{v}} = \sum_i c_i \vec{b}_i = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for some c_1 scalar value (real number)

$\mathcal{L}(\vec{v}) = \mathcal{L}(\sum_i c_i \vec{b}_i) = \sum_i c_i \mathcal{L}(\vec{b}_i) = [\mathcal{L}(\vec{b}_1)\ \mathcal{L}(\vec{b}_2)\ \mathcal{L}(\vec{b}_3)]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

But $\mathcal{L}(\vec{b}_1)$ is still a vector in V, thus $\mathcal{L}(\vec{b}_1) = \sum_j M_{j1} \vec{b}_j = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$ for some real numbers M_{11}, M_{21}, M_{31}

Similarly for $\mathcal{L}(\vec{b}_2)$ and $\mathcal{L}(\vec{b}_3)$

$\therefore \mathcal{L}(\vec{v}) = [\mathcal{L}(\vec{b}_1)\ \mathcal{L}(\vec{b}_2)\ \mathcal{L}(\vec{b}_3)]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

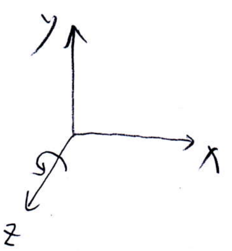
ie Linear transformation

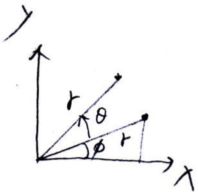
$\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ can be expressed by matrix multiplication:

② Rotation:

(i) Rotation about the z-axis by an angle of θ

$R_z(\theta)$:





$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \qquad \begin{cases} x' = r\cos(\phi+\theta) \\ y' = r\sin(\phi+\theta) \\ z' = z \end{cases}$$

$$x' = r\cos(\phi+\theta) = r\left[\cos\phi\cos\theta - \sin\phi\sin\theta\right] = (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$
$$= (\cos\theta)x - (\sin\theta)y$$

$$y' = r\sin(\phi+\theta) = r\left[\sin\phi\cos\theta + \cos\phi\sin\theta\right] = (r\sin\phi)\cos\theta + (r\cos\phi)\sin\theta$$
$$= (\sin\theta)x + (\cos\theta)y$$

$$\therefore \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} (\cos\theta)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta)y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= R_z(\theta)} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \#$$

* Similarly, we can derive $R_x(\theta)$ (rotation about x-axis by an angle θ)
$R_y(\theta)$ (" " y-axis " ")

See next page for details
can skip

* We can undo the rotation by rotating $-\theta$:

$R^{-1}(\theta) = R(-\theta)$ [R = any of R_x, R_y, R_z, and in fact any rotation]

* All $\cos\theta$'s are on the diagonal. $\sin\theta$'s are off diagonal

$\cos(-\theta) = \cos\theta \quad \sin(-\theta) = -\sin\theta \quad \Rightarrow \underline{R^{-1}(\theta) = R(-\theta)} = \underline{R^T(\theta)}$ → transpose.

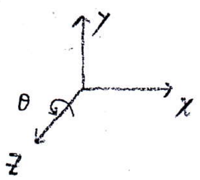
* Any rotation about the origin can be expressed as $R = R_z R_y R_x$

$\underline{R^{-1}} = (R_z R_y R_x)^{-1} = R_x^{-1} R_y^{-1} R_z^{-1} = R_x^T R_y^T R_z^T = (R_z R_y R_x)^T = \underline{R^T}$

$\therefore$ For any rotation matrix $\underline{R^{-1} = R^T}$

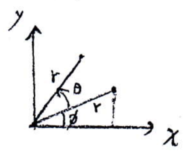
* Not commutative: $R_x R_y \neq R_y R_x$ (use textbook as an example)

Derivation of $R_x(\theta)$, $R_y(\theta)$, $R_z(\theta)$:



Supplementary skip !

(1) $R_z(\theta)$:



$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\begin{cases} x' = r\cos(\phi+\theta) \\ y' = r\sin(\phi+\theta) \\ z' = z \end{cases}$$

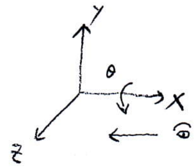
$$x' = r[\cos\phi\cos\theta - \sin\phi\sin\theta]$$
$$= (r\cos\phi)\cos\theta - (\sin\theta)(r\sin\phi)$$
$$= (\cos\theta)x - (\sin\theta)y$$

$$y' = r[\sin\phi\cos\theta + \cos\phi\sin\theta]$$
$$= (\cos\theta)y + (\sin\theta)x$$
$$= (\sin\theta)x + (\cos\theta)y$$

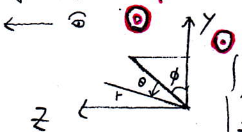
$$\therefore R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#

(2) $R_x(\theta)$:



(From X axis at infinity look at y-z plane.



$$\begin{cases} y = r\cos\phi \\ z = r\sin\phi \end{cases}$$

$$y' = r\cos(\phi+\theta) = r[\cos\phi\cos\theta - \sin\phi\sin\theta]$$
$$= (r\cos\phi)\cos\theta - (\sin\theta)(r\sin\phi)$$
$$= (\cos\theta)y - (\sin\theta)z$$

same as before

$$z' = r\sin(\phi+\theta) = r[\sin\phi\cos\theta + \cos\phi\sin\theta]$$
$$= (r\sin\phi)(\cos\theta) + (r\cos\phi)\sin\theta$$
$$= (\sin\theta)y + (\cos\theta)z$$

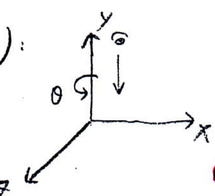
same as before

$$x' = x$$

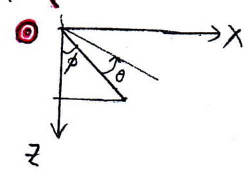
$$\therefore R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#

(3) $R_y(\theta)$:



(From y axis at infinity look at the x-z plane :



$$\begin{cases} x = r\sin\phi \\ z = r\cos\phi \end{cases}$$

$$x' = r\sin(\phi+\theta) = r[\sin\phi\cos\theta + \cos\phi\sin\theta]$$
$$= (r\sin\phi)\cos\theta + (r\cos\phi)\sin\theta$$
$$= (\cos\theta)x + (\sin\theta)z$$

same as before

$$z' = r\cos(\phi+\theta) = r[\cos\phi\cos\theta - \sin\phi\sin\theta]$$
$$= (r\cos\phi)\cos\theta - (\sin\theta)(r\sin\phi)$$
$$= (-\sin\theta)x + (\cos\theta)z$$

same as before.

$$y' = y$$

$$\therefore R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#

③ Translation : $p \xrightarrow{\quad} q$ $\quad d = (d_x, d_y, d_z)$ $\qquad p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad q = p + d = \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

Matrix representation : $q = Mp$: $\begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\Rightarrow$ $x + d_x = ax + by + cz$

$a = 1 \quad b = c = 0$

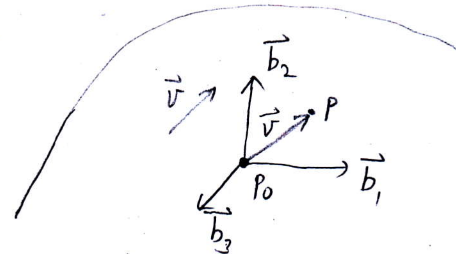
How about d_x ?

$\Rightarrow$ There is NOT enough expressive power to make it work

We need new concepts : Affine space & Homogeneous coordinate System

II. Affine Space and Homogeneous Coordinate System

* Affine Space : vector space + points



Frame (or Coordinate System) : Basis vectors + origin point P_0 $\quad \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$: basis vectors

* Vectors are NOT fixed w.r.t origin. But each point p is fixed relative to the origin P_0

$P = P_0 + \vec{v} \quad (\vec{v} = P - P_0)$

Vector $\vec{v}$ is uniquely expressed by : $\vec{v} = x\vec{b}_1 + y\vec{b}_2 + z\vec{b}_3 \; (+\, 0 \cdot P_0)$

Point P : $P = P_0 + \vec{v} = x\vec{b}_1 + y\vec{b}_2 + z\vec{b}_3 + 1 \cdot P_0$

$\vec{v} = (x, y, z)$ the
$P = (x, y, z)$ same ? (origin point P_0 is NOT shown)

Sol : Use Homogeneous Coord. System : $P = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \; P_0] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ $\qquad \vec{v} = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \; P_0] \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$

i.e. Extending to 4D 4th component : $\begin{cases} 0 & \text{if vector} \\ 1 & \text{if point} \end{cases}$

Consistency check : vector + vector = vector $\quad$ vector + pt = pt $\quad$ pt + pt = ? (Never happens!) $\quad$ pt − pt = vector

4th component : $0 + 0 = 0$ ✓ $\quad 0 + 1 = 1$ ✓ $\quad 1 + 1 = 2$ $\quad 1 - 1 = 0$ ✓

* Translation : $q = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=T} \overbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}^{P} = \underbrace{\begin{bmatrix} \begin{bmatrix} & 3\times3 & \\ & I & \end{bmatrix} & \begin{matrix} d_x \\ d_y \\ d_z \end{matrix} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}}_{=T} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ $\quad$ (I : identity matrix)

$p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

* $T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$

Note: * Using Homogeneous Coord. System and 4×4 matrix multiplication to perform a 3D transformation is called an affine transformation

* Translation is NOT a linear transformation, but is an affine transformation

* We can use affine transformation to perform linear transformation (eg. scaling, rotation) on points:

Let l be a 3×3 matrix for linear transformation (such as scaling, rotation)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \overset{3\times3,\ l}{\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \overset{q}{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}} = \begin{bmatrix} \overset{3\times3,\ l}{\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \overset{p}{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}} = \begin{bmatrix} x'+0 \\ y'+0 \\ z'+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

* Scaling
$S(\beta_x, \beta_y, \beta_z)$:
Fixed pt is at the origin.

* Rotation
$R(\theta, \vec{v})$:
rotation about axis (whose vector is $\vec{v}$) going thru origin:
Fixed pt is the origin.

$\therefore$ The corresponding 4×4 matrix L is

$$L = \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix} \quad ※$$

* If l is rotation (resp. scaling), the properties of rotation (resp. scaling) stay the same for L.

cf: The 4×4 matrix T for translation is:

$$T = \begin{bmatrix} \overset{3\times3,\ i}{\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}} & \overset{t}{\begin{matrix} d_x \\ d_y \\ d_z \end{matrix}} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} i & t \\ 0 & 1 \end{bmatrix} \quad ※$$

where i is a 3×3 identity matrix

t is a column vector $\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$

* In general, a 4×4 matrix M for affine transformation is

$$M = \begin{bmatrix} \overset{3\times3,\ l}{\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}} & \begin{matrix} d_x \\ d_y \\ d_z \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} l & t \\ 0 & 1 \end{bmatrix} = \overset{i}{\begin{bmatrix} \begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix} & \begin{bmatrix} t \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}} \overset{l}{\begin{bmatrix} \begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}} = T \cdot L \quad (\neq LT)$$

i.e. Decomposed into ① L (scaling and/or rotation with fixed pt at the origin) then
② T (translation) ※

* Concatenation of Transformations

e.g. $M \leftarrow M_1 M_2 M_3$. Applying M to obj: ⊙ * $M \cdot obj = (M_1 M_2 M_3) \cdot obj$ ① ② ③

Order is very important, since $M_1 M_2 \neq M_2 M_1$

* Standard Transformation functions:

glRotatef(angle, V_x, V_y, V_z) ⟷ Rotate(angle, V_x, V_y, V_z) → in degrees

O θ right hand

The rotation axis goes thru the origin
∴ Origin is the fixed pt.

specifies the vector of the rotation axis

glScalef(β_x, β_y, β_z) ⟷ Scale(β_x, β_y, β_z)

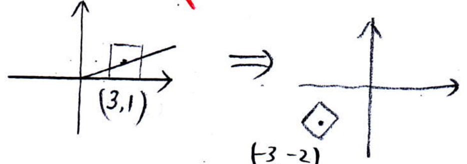
Fixed pt ~~at~~ is the origin

glTranslatef(d_x, d_y, d_z) ⟷ Translate(d_x, d_y, d_z)

O * Helper functions Rotate( ), Scale( ), Translate( ) each returns a 4×4 matrix
* They can be multiplied together

* In general, first translate so that the object center is at the origin,
⊙ perform rotations / scalings (with center at the origin)
then translate the obj center to the final location.

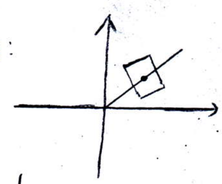
Eg 1.



R: rotation about the origin (fixed at origin)

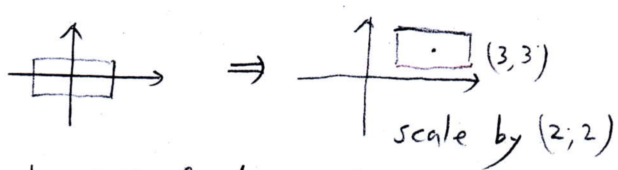
Correct:
① T(-3,-1)
② R(45°)
③ T(-3,-2)

Common mistake:
① R(45°)



center also moves

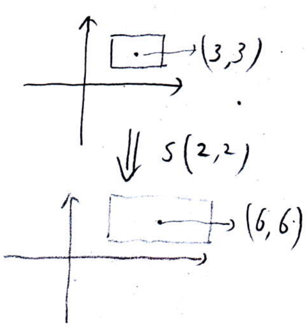
Eg 2



S: scaling with fixed pt at the origin

Correct:
① S(2,2)
② T(3,3)

Wrong:
① T(3,3)
② S(2,2)



center also moves

O Note: Textbook Sec. 3.6 (Transformation for Normal Vectors) is skipped here.
Cover it later when discussing Shading.