

PARTIAL DIFFERENTIAL EQUATIONS

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CONTENTS

1 Theory of (Unbounded) Operators	2
1.1 Preliminaries on Operators	2
1.1.1 Resolvent Operator	2
1.1.2 Dual Operators	4
1.2 Compact Operators	4
1.3 Adjoint, Symmetric and Self-adjoint Operators	9
1.4 Dissipative Operator and Numerical range	11
2 Introduction to the theory of Semi-groups	15
2.1 Intro to the Introduction	15
2.2 Uniformly Continuous SG–BLO	15
2.3 Strongly Continuous Semigroups (C_0 -Semigroups)	18
2.4 Hille-Yosida Theorem	22
2.4.1 The Yosida Approximation	23
2.4.2 The General Hille-Yosida Theorem	27
2.4.3 Reduction to the Case $\omega = 0$	28
2.4.4 Renorming Lemma	30
2.5 Inverse Laplace Transform and Resolvent Formula	35
2.6 Spectral Mapping Theorem	36
3 Applications to Partial Differential Equations	42
3.1 The Cauchy Problem	42
3.2 Homogeneous Cauchy Problem	44

CHAPTER
ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. An operator $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathcal{A})$ is a subspace of X and $\sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|} = +\infty$

Exercise 1

Let $\mathcal{A} : H^1 \rightarrow L^2$, such that $f \mapsto f'$ and $D(\mathcal{A}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f, f' \in L^2\}$. Show \mathcal{A} is an unbounded linear operator.

Notation: If \mathcal{A} and \mathcal{B} are unbounded linear operators, then $\mathcal{A} \supset \mathcal{B}$ if and only if $D(\mathcal{A}) \supset D(\mathcal{B})$ and for all $x \in D(\mathcal{B})$, $\mathcal{A}x = \mathcal{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. Let $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ be a UBLO.

$$\rho(\mathcal{A}) = \text{Resolvent of } \mathcal{A} = \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} (\lambda I - \mathcal{A}) : D(\mathcal{A}) \rightarrow X \text{ is bijective, and} \\ (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X) \end{array} \right\}$$

$$\sigma(\mathcal{A}) = \text{Spectrum of } \mathcal{A} = \mathbb{C} \setminus \rho(\mathcal{A}).$$

Definition 1.1.3. \mathcal{A} is closed if and only if the graph of \mathcal{A} , denoted as $G(\mathcal{A})$ is closed. Also \mathcal{A} is closable if and only if there exists $\tilde{\mathcal{A}} \supset \mathcal{A}$ such that $G(\tilde{\mathcal{A}}) = \overline{G(\mathcal{A})}$.

Note that $G(\mathcal{A}) = \{(x, \mathcal{A}x) \mid x \in D(\mathcal{A})\}$.

Exercise 2

1. Prove that if it exists, $\tilde{\mathcal{A}}$ is unique, it then denoted by $\overline{\mathcal{A}}$ called closure of \mathcal{A} .
2. Let $\mathcal{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a, b], \mathbb{R}), \|\cdot\|_\infty = \sup |f(x)|)$ and $D(\mathcal{A}_\ell) = C^\ell([a, b], \mathbb{R})$.
Prove $\overline{\mathcal{A}_\ell} = \mathcal{A}_1$.

Lemma 1.1.1. If \mathcal{A} an unbounded linear operator is closable, then $\rho(\overline{\mathcal{A}}) = \rho(\mathcal{A})$. If \mathcal{A} is closed then $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$.

Hints (Exercise): If $\rho(\mathcal{A}) \neq \emptyset$ implies \mathcal{A} is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathcal{L}(X)$ implies T is closed).

Corollary 1.1.1. Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closed UBLO then $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$ where

1. $\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) \neq \{0\}\}$ (punctual spectrum and λ 's are the eigenvalue).
2. $\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) = \{0\}, \overline{\text{Rg}(\lambda I - \mathcal{A})} \subsetneq X \right\}$ (continuous spectrum).
3. $\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is injective, } \overline{\text{Rg}(\lambda I - \mathcal{A})} = X, \text{Rg}(\lambda I - \mathcal{A}) \neq X \right\}$ (residual spectrum).

Exercise 3

Let

$$X = \ell^2(\mathbb{C}) = \left\{ (x_n)_{n \geq 0} : \sum_n |x_n|^2 < \infty \right\},$$

with $(\mathcal{A}x_n)_{n \geq 0} = \left(\frac{x_n}{1+n} \right)_{n \geq 0}$. Prove that \mathcal{A} is a BLO, injective, $\overline{\text{Rg}(\mathcal{A})} = X$ and $\text{Rg}(\mathcal{A}) \subsetneq X$.

Theorem 1.1.1. If \mathcal{A} is a closed UBLO then $\rho(\mathcal{A})$ is open. If $\mu \in \rho(\mathcal{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathcal{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathcal{A})$ and

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathcal{A})^{-(n+1)}$$

To do Question: do you need \mathcal{A} closed?

Theorem 1.1.2 (Resolvent Identity). Let \mathcal{A} be a UBLO. For $\lambda \in \rho(\mathcal{A})$, define the resolvent operator

$$R(\lambda) := (\lambda I - \mathcal{A})^{-1}.$$

Then for all $\lambda, \mu \in \rho(\mathcal{A})$,

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

Corollary 1.1.2. The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathcal{A})$ into $\mathcal{L}(X)$ is analytic. Moreover,

$$\frac{d^n}{d\lambda^n} (\lambda I - \mathcal{A})^{-1} = (-1)^n n! [(\lambda I - \mathcal{A})^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and \mathcal{A} a closed UBLO with $\overline{D(\mathcal{A})} = X$ a dense UBLO.

If X and Y are Banach spaces with duals X^* and Y^* , then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x \rangle$.

Definition 1.1.4 (Dual Operator of \mathcal{A}). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow Y$ (UBLO), be such that $\overline{D(\mathcal{A})} = X$. The dual operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y^* \rightarrow X^*$ is a UBLO defined as follows:*

$$D(\mathcal{A}^*) := \{y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x \rangle \forall x \in D(\mathcal{A})\}.$$

and $y^* \in D(\mathcal{A}^*)$, the element z^* is unique and we define $A^*y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathcal{A} \in \mathcal{L}(X, Y)$. Then $\mathcal{A}^* \in \mathcal{L}(Y^*, X^*)$ and*

$$\|\mathcal{A}^*\|_{\mathcal{L}(Y^*, X^*)} = \|\mathcal{A}\|_{\mathcal{L}(X, Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closedly dense UBLO. Then $\overline{D(\mathcal{A}^*)} = X^*(\cong X)$, and \mathcal{A}^* is closed.*

Theorem 1.1.3. *Let \mathcal{A} be a closedly dense UBLO. Then $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$[(\lambda I - \mathcal{A})^{-1}]^* = (\lambda I - \mathcal{A}^*)^{-1}.$$

Exercise 4

1. Let $\mathcal{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathcal{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:
 - a. $\rho(\mathcal{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathcal{A}) = i\mathbb{R}$.
 - b. \mathcal{A} is a closed unbounded linear operator.
 - c. If $\lambda \in \rho(\mathcal{A})$ then $(\lambda I - \mathcal{A})^{-1} : X \rightarrow D(\mathcal{A})$ is bounded.

For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathcal{A})$ such that $(\lambda I - \mathcal{A})f = g$.

For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathcal{A})f_n \rightarrow 0$.

2. Do same for $\mathcal{A} = -i\frac{d}{dx}$.

1.2 Compact Operators

Let X and Y be Banach spaces on \mathbb{K} .

Definition 1.2.1. *Let $K : X \rightarrow Y$ be a BLO (in $\mathcal{L}(X, Y)$), then K compact iff $K(B_1^X(0))$ is relatively compact in Y (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) \mid K \text{ is compact}\}.$$

Exercise 5

Let $X = C([a, b], \mathbb{C})$ and $k \in C^0([a, b] \times [c, d], \mathbb{C})$

Define $K \in \mathcal{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t, s)x(s) ds.$$

Show $K \in \mathcal{K}(X)$.

Theorem 1.2.1. $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.

Proof. We shall show this in two steps

1. $\mathcal{K}(X, Y)$ is a vector space (Do it).
2. Closed: $K_n \rightarrow K$ ([Prof. Yacine said he would send a different proof](#)).

□

Exercise 6

Let $X = \ell^2(\mathbb{C})$,

$$\mathcal{A}((x_n)_{n \geq 0}) = \left(\frac{x_n}{n+1} \right)_{n \geq 0}.$$

Show that \mathcal{A} is compact.

Theorem 1.2.2. Let X, Y and Z be Banach spaces on \mathbb{K} .

$$X \xrightarrow{\mathcal{A}} Y \xrightarrow{\mathcal{B}} Z, \quad \mathcal{A} \in \mathcal{L}(X, Y), \mathcal{B} \in \mathcal{L}(Y, Z).$$

1. If \mathcal{A} is compact or \mathcal{B} is compact, then $\mathcal{B}\mathcal{A}$ is compact.
2. If \mathcal{A} is compact then $\mathcal{A}^* \in \mathcal{K}(Y^*, X^*)$.
3. If \mathcal{A} is compact and $\text{Rg}(\mathcal{A})$ is closed (in Y), then it is finite dimensional.

To proceed with further results on compact operators, we need the following lemma

Lemma 1.2.1 (Riesz Lemma). Let E be a normed vector space, $F = \overline{F} \subset E$. Then $\forall r \in (0, 1)$, $\exists x_r \in E$, such that

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

Proof. Since $F \neq E$ then this implies $\exists z \in E \setminus F$. Let $d = d(z, F) > 0$.

For $0 < r < 1$, $\exists y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|} \|z - (y_r + \|z - y_r\| y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

□

Proposition 1.2.1. Let $\mathcal{A} \in \mathcal{K}(X)$, such that X is a Banach space on \mathbb{C} . If $\lambda \in \mathbb{C}^*$, then $\ker((\lambda I - \mathcal{A})^n)$ has finite dimension.

Proof. Only for $n = 1$. (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathcal{A}) = \{x \in X : \mathcal{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda} \mathcal{A}x\right\} \subset \text{Rg}(\mathcal{A}).$$

So \tilde{K} is closed in $\text{Rg}(\mathcal{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists (x_n)$ in \tilde{K} , such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|} \|\mathcal{A}x_n - \mathcal{A}x_m\| \geq \frac{1}{2}, \quad \forall n \neq m$$

and so we have $\|\mathcal{A}x_n\| \leq \|\mathcal{A}\|$. So $(\mathcal{A}x_n)$ is not Cauchy, hence a contradiction.

□

Exercise 7

Let X be a Banach space on \mathbb{K} . If $\mathcal{A} \in \mathcal{L}(X)$, assume $\exists n_0$ s.t. $\ker(\mathcal{A}^{n_0}) = \ker(\mathcal{A}^{n_0+1})$.

Then $\forall n \geq n_0$,

$$\ker(\mathcal{A}^n) = \ker(\mathcal{A}^{n_0}).$$

Proposition 1.2.2. Let $\mathcal{A} \in \mathcal{K}(X)$ and X be a Banach space on \mathbb{C} , $\lambda \neq 0$. Then $\exists n_0$ such that

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathcal{A})^n) = \ker((\lambda I - \mathcal{A})^{n_0}).$$

Proof. Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathcal{A})^n) \subset \ker((\lambda I - \mathcal{A})^{n+1})$ and each of them is closed.

RL: with $r = \frac{1}{2}$ with $(x_n)_{n \geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathcal{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathcal{A})^n)) \geq \frac{1}{2}.$$

For $n = 1, x \in \ker(\lambda I - \mathcal{A}) \Rightarrow x = \frac{\mathcal{A}}{\lambda} x$. For all $1 \leq m < n$,

$$\begin{aligned} \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} &= x_n - x_m + \frac{\mathcal{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathcal{A}x_m}{\lambda} \right) \\ &= x_n - \left[\frac{(\lambda I - \mathcal{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathcal{A})x_m}{\lambda} \right]. \end{aligned}$$

So,

$$\left\| \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} \right\| \geq d(x_n, \ker(\lambda I - \mathcal{A})^n) \geq \frac{1}{2}.$$

which is a contradiction. \square

Notice that if $\ker(\lambda I - \mathcal{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathcal{A})$. Notice,

$$\dim \ker(\lambda I - \mathcal{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2 $\Rightarrow \exists n_0$ (smallest one) such that

$$\ker((\lambda I - \mathcal{A})^{n_0}) = \ker((\lambda I - \mathcal{A})^n), \quad \forall n \geq n_0.$$

Note that,

$$\ker((\lambda I - \mathcal{A})^{n_0}) := \text{generalized eigenspace}.$$

$$\dim \ker((\lambda I - \mathcal{A})^{n_0}) := \text{algebraic multiplicity of } \lambda.$$

Proposition 1.2.3 (Fredholm alternative). *Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} .*

$$\text{Rg}(\lambda I - \mathcal{A}) = X \iff \ker(\lambda I - \mathcal{A}) = \{0\}.$$

Proposition 1.2.4. *Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} , $\dim X = \infty$. If $\lambda_n \rightarrow \lambda$, $\lambda_n \in \sigma(\mathcal{A}) \setminus \{0\}$, pairwise distinct, then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathcal{A}) \setminus \{0\}$ is isolated.*

Proof. Let $\lambda_n \in \sigma_p(\mathcal{A})$, $\exists \|x_n\| = 1$ such that $\mathcal{A}x_n = \lambda_n x_n$. Let

$$X_n = \text{span}(x_1, \dots, x_n), \quad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \Rightarrow \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^n \alpha_i x_i, \text{ which implies } \lambda_{n+1} x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathcal{A}x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^n \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since (x_i) are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \quad 1 \leq i \leq n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathcal{A})X_n \subset X_{n-1}, \quad \forall n \geq 2.$$

Recall:

$$\|y_n\| = 1, \quad y_n \in X_n,$$

$$d(y_n, X_{n-1}) \geq \frac{1}{2}.$$

For $2 \leq m < n$,

$$\begin{aligned} \left\| \frac{\mathcal{A}y_n}{\lambda_n} - \frac{\mathcal{A}y_m}{\lambda_m} \right\| &= \left\| y_n - \left[\frac{\lambda_n I - \mathcal{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathcal{A}}{\lambda_m} y_m \right] \right\| \\ &\geq d(y_n, X_{n-1}) \geq \frac{1}{2}. \end{aligned}$$

Assume that

$$\lambda_n \rightarrow \lambda \quad (n \rightarrow \infty).$$

Suppose $\lambda \neq 0$, then

$$\left| \frac{1}{\lambda_n} \right| \leq C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathcal{A}y_n}{\lambda_n} \right)_{n \geq 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathcal{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. \square

Theorem 1.2.3. Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} . Then $\sigma(\mathcal{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.

Main use of compact operators (in PDEs)

They appear as “inverse” of UBLO.

Definition 1.2.2. Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ UBLO, closed, $\rho(\mathcal{A}) \neq \{0\}$. \mathcal{A} is said to have compact resolvent if

$$(\lambda I - \mathcal{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathcal{A}).$$

Main Example: $\mathcal{A} = -\Delta$ on Ω with $\mathcal{A}u = -u_{xx}$.

1.3 Adjoints, Symmetric and Self-adjoint Operators

Let \mathcal{H} be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}.$$

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ UBLO}, \overline{D(\mathcal{A})} = \mathcal{H}.$$

Definition 1.3.1 (Adjoint Operator \mathcal{A}°).

$$D(\mathcal{A}^\circ) = \{x \in \mathcal{H} : v \mapsto \langle \mathcal{A}v, x \rangle_{\mathcal{H}} : D(\mathcal{A}) \rightarrow \mathbb{C} \text{ bdd operator}\}.$$

If $x \in D(\mathcal{A}^\circ)$, then there exists uniquely $z \in \mathcal{H}$ such that $\langle v, z \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$ for all $v \in D(\mathcal{A})$.

Observe, with Riesz representation and the fact that $\overline{D(\mathcal{A})} = \mathcal{H}$, we have that $z := \mathcal{A}^\circ x$ and $\langle v, \mathcal{A}^\circ x \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$ for all $v \in D(\mathcal{A})$.

Remark 1.3.1. Let \mathcal{H} be a Hilbert space, $E : \mathcal{H} \rightarrow \mathcal{H}^*$, $x \mapsto \langle x, \cdot \rangle$. Linear isometry between \mathcal{H} and \mathcal{H}^* . (One can identify \mathcal{H} and \mathcal{H}^*). Now, we define the Dual operator as the following:

$$\mathcal{A}^* : D(\mathcal{A}^*) \subset \mathcal{H}^* \rightarrow \mathcal{H}, \quad \mathcal{A}^\circ = E^{-1} \mathcal{A}^* E.$$

Definition 1.3.2 (Symmetric and Self-adjoint Operator). Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a UBLO, then

1. **Symmetric:** If $\overline{D(\mathcal{A})} = \mathcal{H}$ and $\mathcal{A}^\circ \supset \mathcal{A}$ with $D(\mathcal{A}^\circ) \supset D(\mathcal{A})$ and for all $x, y \in D(\mathcal{A})$, $\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle$.
2. **Self-adjoint:** If $\overline{D(\mathcal{A})} = \mathcal{H}$ and $\mathcal{A}^\circ = \mathcal{A}$.

Exercise 8

1. Let \mathcal{H} be a Hilbert space, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, $\overline{D(\mathcal{A})} = \mathcal{H}$. If \mathcal{A} is closed, then $\overline{D(\mathcal{A}^\circ)} = \mathcal{H}$.
2. Let \mathcal{H} be a Hilbert space, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, $\overline{D(\mathcal{A})} = \mathcal{H}$. Suppose \mathcal{A} is symmetric and if $\lambda \in \sigma_p(\mathcal{A})$, then prove that $\lambda \in \mathbb{R}$ and

$$\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle.$$

Proposition 1.3.1. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an Hilbert space over \mathbb{C} , If \mathcal{A} is self-adjoint, injective and $\overline{D(\mathcal{A})} = \mathcal{H}$. Then $\mathcal{A}^{-1} : \text{Rg}(\mathcal{A}) \rightarrow \mathcal{H}$ is self-adjoint.

Proof. Since \mathcal{A} is injective then \mathcal{A}^{-1} is well defined, and since $\mathcal{A} = \mathcal{A}^\circ$ then \mathcal{A} is closed.

Now assume $(x_n) \subset D(\mathcal{A})$ such that $x_n \rightarrow x \in D(\mathcal{A})$ (because \mathcal{A} is closed) and $\mathcal{A}x_n \rightarrow y$ then for all $z \in D(\mathcal{A})$, $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}^\circ z \rangle$ which implies $\langle y, z \rangle = \langle x, \mathcal{A}^\circ z \rangle$ and so we have

$x \in D(\mathcal{A}^\circ) = D(\mathcal{A})$ and $y = \mathcal{A}^\circ$. Notice $\overline{\text{Rg}(\mathcal{A})} = \ker(\mathcal{A})^\perp$ (because of self-adjointness). Injectivity implies $\overline{\text{Rg}(\mathcal{A})} = \mathcal{H}$ which implies $\overline{D(\mathcal{A}^{-1})} = \mathcal{H}$. So \mathcal{A}^{-1} is densely defined. Now, observe for all $u, v \in D(\mathcal{A}^{-1})$, $u = \mathcal{A}^\circ x$ and $v = \mathcal{A} y$ with $x, y \in D(\mathcal{A})$. Hence,

$$\langle \mathcal{A}^{-1}u, v \rangle = \langle x, \mathcal{A}y \rangle = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}^{-1}y \rangle.$$

To this end, $(\mathcal{A}^{-1})^\circ \subset \mathcal{A}^{-1}$, $\forall z \in D((\mathcal{A}^{-1})^\circ) \exists w, \forall u \in D(\mathcal{A}^{-1}) = R(\mathcal{A})$ (i.e. $u = \mathcal{A}x$)

$$\langle \mathcal{A}^{-1}u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathcal{A}) \quad \langle x, z \rangle = \langle \mathcal{A}x, w \rangle$$

By definition $w \in D(\mathcal{A}^\circ)$ and $\mathcal{A}^\circ w = z$. $\mathcal{A}w = z \Rightarrow z \in \text{Rg}(\mathcal{A}) = D(\mathcal{A}^{-1})$. \square

Theorem 1.3.1. Let \mathcal{H} be a Hilbert space, suppose $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is symmetric and surjective then \mathcal{A} is self-adjoint.

Proof. \mathcal{A} and \mathcal{A}° are injective. Do it only for \mathcal{A} , let $x \in D(\mathcal{A})$ and $\mathcal{A}x = 0$.

$$\forall y \in D(\mathcal{A}), \quad 0 = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle \Rightarrow x \perp \text{Rg}(\mathcal{A}) = \mathcal{H}.$$

which implies $x = 0$.

Next, we show \mathcal{A} closed.

$$(x_n)_{n \geq 1} \subset D(\mathcal{A}), \quad x_n \rightarrow x \text{ in } \mathcal{H}, \quad \mathcal{A}x_n \rightarrow y \text{ in } \mathcal{H}$$

We shall show $y = \mathcal{A}x$. Now, $\forall z \in D(\mathcal{A})$ then $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}z \rangle$, which implies $\langle y, z \rangle = \langle x, \mathcal{A}z \rangle$ which implies $x \in D(\mathcal{A}^\circ)$ and $y = \mathcal{A}^\circ x$. Since \mathcal{A} surjective $\Rightarrow \exists w \in D(\mathcal{A})$ s.t. $\mathcal{A}w = y$ and $\mathcal{A}^\circ x = y$.

Since \mathcal{A} is symmetric: $\mathcal{A}^\circ w = \mathcal{A}w$. Then $\mathcal{A}^\circ w = \mathcal{A}^\circ x$, \mathcal{A} is injective $\Rightarrow w = x$. Hence $\mathcal{A}x = \mathcal{A}w = y \Rightarrow y = \mathcal{A}x \Rightarrow \mathcal{A}$ is closed.

By closed graph theorem both \mathcal{A} and $\mathcal{A}^{-1} \in \mathcal{L}(X)$. We can conclude that \mathcal{A} is a self-adjoint operator. \square

Exercise 9

Let $\mathcal{H} = L^2(0, \pi)$ with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ and

$$\mathcal{A}f = -f''$$

If $D(\mathcal{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is \mathcal{A} a self-adjoint operator?

Similarly, if $D(\mathcal{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is \mathcal{A} a self-adjoint operator?

Theorem 1.3.2 (Fredrich's Extension). Let \mathcal{H} be a Hilbert space on \mathbb{C} with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, symmetric then \mathcal{A} admits a unique self adjoint extension. If either

- a. $\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle > -\infty$

$$b. \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle < +\infty$$

such that, $\mathcal{A} \subset \mathcal{A}^\circ \subset (\mathcal{A}^\circ)^\circ \subset \dots$. If (a) or (b) holds, then; $\mathcal{A} \subset \mathcal{A}^\circ = (\mathcal{A}^\circ)^\circ$.

1.4 Dissipative Operator and Numerical range

Definition 1.4.1 (Duality Map). Let X be a Banach space on \mathbb{K} . The duality map is defined as $J : X \rightarrow 2^{X^*}$, $x \mapsto J(x) = \{x^* \in X^* \mid \operatorname{Re} \langle x^*, x \rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$. By the Hahn-Banach theorem, $J(x) \neq \emptyset$.

Question: What can you say about $J(X)$ when X is an Hilbert space or Reflexive?

Definition 1.4.2. A map $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ (UBLO) is dissipative iff for all $x \in D(\mathcal{A})$, there exists $x^* \in J(X)$ such that $\operatorname{Re} \langle x^*, \mathcal{A}x \rangle \leq 0$.

Lemma 1.4.1. \mathcal{A} is dissipative if and only iff for all $\lambda > 0$, $x \in D(\mathcal{A})$ we have that

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

Proof. let $x^* \in J(x)$. Then

$$\begin{aligned} \|(\lambda I - \mathcal{A})x\| \|x^*\| &\geq |\langle x^*, (\lambda I - \mathcal{A})x \rangle| \geq \Re \langle x^*, (\lambda I - \mathcal{A})x \rangle, \\ &= \lambda \Re \langle x^*, x \rangle - \Re \langle x^*, \mathcal{A}x \rangle \geq \lambda \|x\|^2. \end{aligned}$$

Hence, if $\|x\| \neq 0$, then

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

(\Leftarrow) Let $x \in D(\mathcal{A})$, $x \neq 0$, and $\lambda > 0$. Let $y_\lambda^* \in J((\lambda I - \mathcal{A})x)$ and set $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$. Then

$$\|(\lambda I - \mathcal{A})x\|^2 = \|(\lambda I - \mathcal{A})x\| \|y_\lambda^*\| = \Re \langle y_\lambda^*, (\lambda I - \mathcal{A})x \rangle.$$

Since $y_\lambda^* \neq 0$, we have

$$\lambda \|x\| \leq \|(\lambda I - \mathcal{A})x\| = \Re \langle g_\lambda^*, (\lambda I - \mathcal{A})x \rangle = \lambda \langle g_\lambda^*, x \rangle - \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Hence,

$$\Re \langle g_\lambda^*, \mathcal{A}x \rangle \leq \lambda \langle g_\lambda^*, x \rangle - \lambda \|x\| \leq \|g_\lambda^*\| \|x\| = \|x\|.$$

Therefore,

$$\Re \langle g_\lambda^*, \mathcal{A}x \rangle \leq 0. \tag{**}$$

Idea: Let $\lambda \rightarrow +\infty$.

Unit ball in X^* is compact for weak* topology (Banach–Alaoglu).

(Up to subsequence)

$$g_\lambda^* \rightharpoonup g^* \in X^*, \quad \|g^*\| \leq 1.$$

Then from (**),

$$\Re \langle g^*, \mathcal{A}x \rangle \leq 0.$$

$$(*) \quad \|x\| \leq \langle g_\lambda^*, x \rangle - \frac{1}{\lambda} \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Let $\lambda \rightarrow +\infty$. Then $\|x\| \leq \langle g^*, x \rangle$. Hence, $\|g^*\| = 1$ and $\langle g^*, x \rangle = \|x\|$. Set $x^* = \|x\|g^*$. Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x \rangle = \|x\|^2,$$

that is, $x^* \in J(x)$. □

Theorem 1.4.1 (Lumer-Phillips). *Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO. Assume that \mathcal{A} is dissipative and that there exists $\lambda_0 > 0$ such that $\text{Rg}(\lambda_0 I - \mathcal{A}) = X$.*

Then \mathcal{A} is closed, $\rho(\mathcal{A}) \supset \mathbb{R}_+^$, and for all $\lambda > 0$,*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}. \quad (1.4)$$

Proof. Let $\lambda_0 > 0$.

(1) To prove: $(\lambda_0 I - A)$ is bijective.

- Surjective: Assumption,
- Injective: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathcal{A})^{-1} : X \rightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any $y \in X$, there exists a unique $x \in X$ such that

$$x = (\lambda_0 I - \mathcal{A})^{-1}y, \quad (\lambda_0 I - \mathcal{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0} \|y\| \geq \|(\lambda_0 I - \mathcal{A})^{-1}y\|.$$

Hence,

$$\|(\lambda_0 I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_0}, \quad \lambda_0 \in \rho(\mathcal{A}).$$

(2) \mathcal{A} is closed.

Let $x_n \rightarrow x$, $x_n \in D(\mathcal{A})$, and $Ax_n \rightarrow y$.

Then

$$(\lambda_0 I - \mathcal{A})x_n \rightarrow \lambda_0 x - y.$$

Since $(\lambda_0 I - \mathcal{A})^{-1} \in \mathcal{L}(X)$, we have

$$x_n \rightarrow (\lambda_0 I - \mathcal{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathcal{A})x \iff y = \mathcal{A}x.$$

Therefore, \mathcal{A} is closed.

(3) $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$ and (1.4).

Since \mathcal{A} is closed and $\rho(\mathcal{A}) \neq \emptyset$, we know that $\rho(\mathcal{A})$ is open.

Let $\Lambda = \rho(\mathcal{A}) \cap \mathbb{R}_+^*$, which is open in \mathbb{R}_+^* . We show that it is closed.

Let $(\lambda_n)_{n \in \mathbb{N}} \subset \Lambda$ such that $\lambda_n \rightarrow \lambda \in \mathbb{R}_+^*$. Note that since $\lambda_n \in \Lambda$, we have

$$\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$\begin{aligned} (\lambda I - \mathcal{A}) &= [I + u_n](\lambda_n I - \mathcal{A}) \implies \lambda I - \mathcal{A} = \lambda_n I - \mathcal{A} + u_n(\lambda_n I - \mathcal{A}), \\ &\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathcal{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathcal{A})^{-1} = u_n. \end{aligned}$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For n large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that $\lambda \in \rho(\mathcal{A})$. Hence, Λ is closed, and therefore $\Lambda = \mathbb{R}_+^*$.

□

Corollary 1.4.1. *Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO, closed, with $\overline{D(\mathcal{A})} = X$. Assume that \mathcal{A} and \mathcal{A}^* are dissipative. Then*

$$\rho(\mathcal{A}) \supset \mathbb{R}_+^*, \quad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathcal{A})^{-1}\| \leq 1.$$

Proof. It is easy to show that $\text{Rg}(I - A) = X$ (i.e. $\lambda_0 = 1$ + Theorem 1.4.1).

\mathcal{A} dissipative and closed implies

$\text{Rg}(I - \mathcal{A})$ is a closed subspace of X .

(give details!!!)

Let $x^* \in X^*$ such that

$$\langle x^*, (I - \mathcal{A})x \rangle = 0, \quad \forall x \in D(\mathcal{A}). \tag{**}$$

Let us prove that $x^* = 0$.

Then $x^* \in D(\mathcal{A}^*)$ and

$$(I - \mathcal{A}^*)x^* = 0.$$

Since \mathcal{A}^* is dissipative, by Lemma 1.4.1, we have $x^* = 0$. This implies that

$$\overline{\text{Rg}(I - \mathcal{A})} = X.$$

Since $\text{Rg}(I - \mathcal{A})$ is closed, we obtain

$$\text{Rg}(I - \mathcal{A}) = X.$$

By contradiction and using Hahn–Banach. \square

Definition 1.4.3 (Numerical Range). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be UBLO. The numerical range of \mathcal{A} , denoted by $W(\mathcal{A})$,*

$$W(\mathcal{A}) = \{\langle x^*, \mathcal{A}x \rangle \mid x^* \in J(x), x \in D(\mathcal{A}), \|x\| = \|x^*\| = 1, \langle x^*, x \rangle = 1\}.$$

In case of a Hilbert space, we have that $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid x \in D(\mathcal{A}), \|x\| = 1\}$.

Linear algebra in finite dimension $\mathcal{A} \in \mathcal{M}_n(\mathbb{K})$, we have that $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid \|x\| = 1\}$.

Theorem 1.4.2 (Home-work). *Let $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ be closed, with $\overline{D(\mathcal{A})} = X$.*

1) If $\lambda \notin \overline{W(\mathcal{A})}$, then $(\lambda I - \mathcal{A})$ is injective, has closed image, and for all $x \in D(\mathcal{A})$,

$$\|(\lambda I - \mathcal{A})x\| \geq d(\lambda, W(\mathcal{A})) \|x\|.$$

Moreover, if $\lambda \in \rho(\mathcal{A})$, then

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{d(\lambda, W(\mathcal{A}))}. \quad (**)$$

2) If Λ is a connected open subset of $\mathbb{C} \setminus W(\mathcal{A})$ such that $\rho(\mathcal{A}) \cap \Lambda \neq \emptyset$, then $\rho(\mathcal{A}) \supset \Lambda$ and $(**)$ holds true.

CHAPTER
TWO

INTRODUCTION TO THE THEORY OF SEMI-GROUPS

2.1 Intro to the Introduction

Definition 2.1.1. Let X be a Banach space over \mathbb{K} . A one-parameter family of bounded linear operators on X , $(T(t))_{t \geq 0}$, is a semigroup (SG) of bounded linear operators on X if:

1. $T(0) = Id_X$,
2. $\forall (t, s) \in \mathbb{R}_+^2 : T(t + s) = T(t) \cdot T(s)$ (SG property).

Remark 2.1.1. $T(t)$ and $T(s)$ commute.

2. Infinitesimal generator of SG-LO $(T(t))_{t \geq 0}$

Let $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ be an unbounded linear operator with

$$D(\mathcal{A}) = \left\{ x \in X \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \mathcal{A}x, \quad x \in D(\mathcal{A}).$$

where $D(\mathcal{A}) = \text{domain of } \mathcal{A}$.

2.2 Uniformly Continuous SG-BLO

Definition 2.2.1. A SG-BLO on X , $(T(t))_{t \geq 0}$ is uniformly continuous if

$$\|T(t) - Id\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as } t \rightarrow 0^+$$

Lemma 2.2.1. Let $(T(t))_{t \geq 0}$ be a SG-BLO which is uniformly continuous. Then, $\forall t > 0$,

$$\|T(s) - T(t)\| \xrightarrow{s \rightarrow t} 0$$

(continuity everywhere).

Proof. Let t be fixed. $T(s) = T(s - t + t)$, $s - t \geq 0$.

$$s \geq t \Rightarrow T(s) = T(s - t)T(t) \Rightarrow T(s) - T(t) = T(t)[T(s - t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\| \|T(s - t) - I_d\| \xrightarrow[s \rightarrow t]{} 0.$$

For $s \leq t$

$$T(t) = T(t - s)T(s) \Rightarrow T(t) - T(s) = T(s)[T(t - s) - I_d]$$

(Prove that $\sup_{[0,t]} \|T(t)\| < +\infty$)

Then

$$\begin{aligned} \|T(t) - T(s)\| &\leq \|T(s)\| \|T(t - s) - I_d\| \\ &\leq \sup \|T(s)\| \|T(t - s) - I_d\| \xrightarrow[s \rightarrow t]{} 0. \end{aligned}$$

□

Theorem 2.2.1. A linear operator \mathcal{A} is the infinitesimal generator of a uniformly continuous semigroup if and only if \mathcal{A} is a bounded linear operator.

Proof. Let \mathcal{A} be a bounded linear operator on X and set

$$T(t) = e^{t\mathcal{A}} = \sum_{n=0}^{\infty} \frac{(t\mathcal{A})^n}{n!}. \quad (1.5)$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such t , a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t + s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq |t| \|\mathcal{A}\| e^{\|\mathcal{A}\| t}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathcal{A} \right\| \leq \|\mathcal{A}\| \cdot \max_{0 \leq s \leq t} \|T(s) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on X and that \mathcal{A} is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on X . Fix $\rho > 0$, small enough, such that

$$\left\| I - \rho \int_0^\rho T(s) ds \right\| < 1.$$

This implies that $\rho^{-1} \int_0^\rho T(s) ds$ is invertible. Now,

$$h^{-1}(T(h) - I) \int_0^\rho T(s) ds = h^{-1} \left(\int_0^\rho T(s + h) ds - \int_0^\rho T(s) ds \right)$$

$$= h^{-1} \left(\int_{\rho}^{\rho+h} T(s) ds - \int_0^h T(s) ds \right)$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1} \int_{\rho}^{\rho+h} T(s) ds - h^{-1} \int_0^h T(s) ds \right) \left(\int_0^{\rho} T(s) ds \right)^{-1}. \quad (1.6)$$

Letting $h \rightarrow 0$ in (1.6) shows that $h^{-1}(T(h) - I)$ converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I) \left(\int_0^{\rho} T(s) ds \right)^{-1}$$

which is the infinitesimal generator of $T(t)$. \square

Remark 2.2.1. *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

Theorem 2.2.2. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t \rightarrow 0} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t \rightarrow 0} \frac{S(t) - I}{t}. \quad (1.7)$$

then $T(t) = S(t)$ for $t \geq 0$.

Proof. We will show that given $T > 0$, $S(t) = T(t)$ for $0 \leq t \leq T$. Let $T > 0$ be fixed, since $t \mapsto \|T(t)\|$ and $t \mapsto \|S(t)\|$ are continuous there is a constant C such that

$$\|T(t)\| \|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given $\varepsilon > 0$ it follows from (1.7) that there is a $\delta > 0$ such that

$$h^{-1} \|T(h) - S(h)\| < \varepsilon / TC \quad \text{for } 0 \leq h \leq \delta. \quad (1.8)$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n \leq \delta$. From the semigroup property and (1.8) it then follows that

$$\begin{aligned} \|T(t) - S(t)\| &= \left\| T\left(n \frac{t}{n}\right) - S\left(n \frac{t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k) \frac{t}{n}\right) S\left(\frac{kt}{n}\right) - T\left((n-k-1) \frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k-1) \frac{t}{n}\right) \right\| \left\| T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{t}{n}\right) \right\| \leq Cn \frac{\varepsilon}{TC} \frac{t}{n} \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $T(t) = S(t)$ for $0 \leq t \leq T$ and the proof is complete. \square

Corollary 2.2.1. Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then

- (a) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.
- (b) There exists a unique bounded linear operator \mathcal{A} such that $T(t) = e^{t\mathcal{A}}$.
- (c) The operator \mathcal{A} in part (b) is the infinitesimal generator of $T(t)$.
- (d) $t \mapsto T(t)$ is differentiable in norm and satisfies

$$\frac{dT(t)}{dt} = \mathcal{A}T(t) = T(t)\mathcal{A}. \quad (1.9)$$

Proof. All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator \mathcal{A} . \mathcal{A} is also the infinitesimal generator of $e^{t\mathcal{A}}$ defined by (1.5) and therefore, by Theorem 2.2.2, $T(t) = e^{t\mathcal{A}}$. \square

Remark 2.2.2. The proofs above are from the recommended text (*Semigroups of Linear Operators and Applications to Partial Differential Equations*) Page 3. [Theorem 1.3 and Corollary 1.4].

2.3 Strongly Continuous Semigroups (C_0 -Semigroups)

Definition 2.3.1. The SG-BLO $(T(t))_{t \geq 0}$ is strongly continuous (SC or C_0) if $\forall x \in X$

$$\|T(t)x - x\|_X \xrightarrow{t \rightarrow 0^+} 0$$

Theorem 2.3.1. Let $(T(t))_{t \geq 0}$, C_0 -SG then $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0 \|T(t)\| \leq M e^{\omega t}$

Proof. First we want to show that $\forall \eta > 0, \sup_{t \in [0, \eta]} \|T(t)\| < +\infty$.

By contradiction, assume that $\sup \|T(t)\| = +\infty$. Then $\exists (t_n)_{n \geq 0} \searrow 0$ such that $\|T(t_n)\| \geq n$ or $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive) $\exists x \in X$ such that $\sup \|T(t_n)x\| = +\infty$, but this contradicts the strong convergance 2.3.1.

Now take $M := \sup \|T(t)\| \geq 1$ (This is because $T(0) = Id$).

$\forall t \geq 0$ write $t = k\eta + \eta_t$ where $k = \left\lfloor \frac{t}{\eta} \right\rfloor$ and $\eta_t \in [0, \eta]$, then

$$\begin{aligned} \|T(t)\| &= \|T(k\eta + \eta_t)\| \\ &= \|[T(\eta)]^k T(\eta_t)\| && \text{by the SG property} \\ &= \|T(\eta)\|^k \|T(\eta_t)\| \\ &\leq M \cdot M^k && M \text{ is upperbound} \\ &\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\ &= M(e^{\ln M})^{t/\eta} \\ &= M e^{t \frac{\ln M}{\eta}} = M e^{\omega t} \end{aligned}$$

And since $\omega = \frac{\ln M}{\eta}$ and $M \geq 1$ then $\omega \geq 0$. □

Corollary 2.3.1. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup then $\forall x \in X, t \mapsto T(t)x$ is continuous

Proof. For $h > 0$ and $t > 0$:

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \rightarrow 0$ as $h \rightarrow 0$ by definition 2.3.1.
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$. Since $\|T(t-h)\|$ is bounded by theorem 2.3.1, so this tends to 0 as $h \rightarrow 0$.

□

Theorem. Let \mathcal{A} be the IG of C_0 -SG $T(t)$, then:

2.3.2 $\forall x \in X, \forall t \geq 0, \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$ for all $x \in X$.

2.3.3 $\forall x \in X, \forall t \geq 0, \int_0^t T(s)x ds \in D(\mathcal{A})$ and $\mathcal{A} \int_0^t T(s)x ds = T(t)x - x$.

2.3.4 $\forall x \in D(\mathcal{A}), T(t)x \in D(\mathcal{A})$ and $\frac{d}{dt} T(t)x = \mathcal{A} T(t)x = T(t)\mathcal{A}x$.

2.3.5 $\forall x \in D(\mathcal{A}), \forall t \geq 0, \forall s \geq 0, T(t)x - T(s)x = \int_s^t T(u)\mathcal{A}x du$ for $x \in D(\mathcal{A})$.

Proof of 2.3.2. Consider the small interval $[t, t+h]$ relative to its value at t :

$$\frac{1}{h} \int_t^{t+h} T(s)x ds - T(t)x = \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] ds$$

By the continuity (corollary 2.3.1) of the map $s \mapsto T(s)x$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all s satisfying $|s-t| < \delta$, we have $\|T(s)x - T(t)x\| < \epsilon$.

Taking $0 < h < \delta$, we can estimate the norm of the integral:

$$\left\| \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] ds \right\| \leq \frac{1}{h} \int_t^{t+h} \|T(s)x - T(t)x\| ds < \frac{1}{h} \cdot h\epsilon = \epsilon$$

Then

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$$

□

Proof of 2.3.3. Let $h > 0$. Consider the difference quotient for the integral $y = \int_0^t T(s)x ds$:

$$\begin{aligned} \frac{T(h) - Id}{h} \int_0^t T(s)x ds &= \frac{1}{h} \left[\int_0^t T(s+h)x ds - \int_0^t T(s)x ds \right] \\ &= \frac{1}{h} \left[\int_h^{t+h} T(u)x du - \int_0^t T(u)x du \right] \\ &= \frac{1}{h} \int_t^{t+h} T(u)x du - \frac{1}{h} \int_0^h T(u)x du \end{aligned}$$

As $h \rightarrow 0^+$, the first term converges to $T(t)x$ and the second to $T(0)x = x$ by 2.3.2. Thus the limit exists, $y \in D(\mathcal{A})$, and $\mathcal{A}y = T(t)x - x$. \square

Proof of 2.3.4. If $x \in D(\mathcal{A})$, then $T(t)\mathcal{A}x = T(t) \lim_{h \rightarrow 0} \frac{T(h)x - x}{h} = \lim_{h \rightarrow 0} \frac{T(h)T(t)x - T(t)x}{h}$. This limit exists and equals $\mathcal{A}(T(t)x)$, proving $T(t)x \in D(\mathcal{A})$ and $T(t)\mathcal{A}x = \mathcal{A}T(t)x$. This also shows the right-derivative of $T(t)x$ is $\mathcal{A}T(t)x$. A similar argument for the left-derivative completes the differentiability. \square

Proof of 2.3.5. By Property 2.3.4, the function $f(u) = T(u)x$ is differentiable with $f'(u) = T(u)\mathcal{A}x$. Since f' is continuous, we integrate f' over $[s, t]$ to obtain $f(t) - f(s) = \int_s^t f'(u) du$, which is $T(t)x - T(s)x = \int_s^t T(u)\mathcal{A}x du$. \square

Theorem 2.3.6. *The IG \mathcal{A} of C_0 -SG is a closed linear operator and $\overline{D(\mathcal{A})} = X$.*

Proof. For any $x \in X$, let $x_t = \frac{1}{t} \int_0^t T(s)x ds$. By 2.3.3, $x_t \in D(\mathcal{A})$. By 2.3.2, $x_t \rightarrow T(0)x = x$ as $t \rightarrow 0^+$, which shows $\overline{D(\mathcal{A})} = X$.

Let $x_n \in D(\mathcal{A})$ such that $x_n \rightarrow x$ and $\mathcal{A}x_n \rightarrow y$. From 2.3.3 take $s = 0$, we have $T(t)x_n - x_n = \int_0^t T(s)\mathcal{A}x_n ds$. Passing to the limit $n \rightarrow \infty$, we get $T(t)x - x = \int_0^t T(s)y ds$. Dividing by t and letting $t \rightarrow 0^+$, the Right-Hand Side (RHS) converges to y . Thus $x \in D(\mathcal{A})$ and $\mathcal{A}x = y$. \square

Theorem 2.3.7. *Let $\{T(t)\}_{t \geq 0}$ and $\{S(t)\}_{t \geq 0}$ be two C_0 -SG with infinitesimal generators \mathcal{A} and B , respectively. If $\mathcal{A} = B$, then $T(t) = S(t)$ for all $t \geq 0$.*

Proof. Assume $\mathcal{A} = B$, then $D(\mathcal{A}) = D(B)$. Let $x \in D(\mathcal{A})$ be fixed, and for a fixed $t > 0$, define

$$\varphi : [0, t] \rightarrow X, \quad \varphi(s) = T(t-s)S(s)x$$

Since $x \in D(\mathcal{A})$, the map φ is of class C^1 on $[0, t]$. We differentiate φ with respect to s using the product rule and 2.3.4, we get:

$$\begin{aligned} \frac{d}{ds}\varphi(s) &= \frac{d}{ds}[T(t-s)]S(s)x + T(t-s)\frac{d}{ds}[S(s)x] \\ &= -\mathcal{A}T(t-s)S(s)x + T(t-s)BS(s)x \end{aligned}$$

Because $T(t-s)$ commutes with its generator \mathcal{A} , and given $\mathcal{A} = B$, we have:

$$\frac{d}{ds}\varphi(s) = -T(t-s)\mathcal{A}S(s)x + T(t-s)\mathcal{A}S(s)x = 0$$

Since the derivative is zero for all $s \in [0, t]$, the function φ must be constant. Evaluating φ at the endpoints $s = 0$ and $s = t$ yields

$$\varphi(0) = T(t)S(0)x = T(t)x \quad \text{and} \quad \varphi(t) = T(0)S(t)x = S(t)x$$

Thus, $T(t)x = S(t)x$ for all $x \in D(\mathcal{A})$. Since $D(\mathcal{A})$ is dense in X and $T(t), S(t)$ are bounded linear operators, this identity extends to all $x \in X$ by continuity. Therefore, $T(t) = S(t)$ for all $t \geq 0$. \square

Theorem 2.3.8. *Let \mathcal{A} be the IG of a C_0 -SG $\{T(t)\}_{t \geq 0}$ on a Banach space X . Then the subspace*

$$X = \overline{\bigcap_{n \geq 1} D(\mathcal{A}^n)}$$

Proof. Let $\mathcal{D} = \{\varphi : \mathbb{R} \rightarrow \mathbb{C} \mid \varphi \text{ has compact support in } \mathbb{R}_+^* \text{ and is smooth } C^\infty\}$. Let $x \in X$ and consider a test function $\varphi \in \mathcal{D}$. Define

$$x_\varphi = \int_0^\infty \varphi(s) T(s)x \, ds$$

First, we show that $x_\varphi \in D(\mathcal{A})$. Consider

$$\begin{aligned} \frac{T(h) - Id}{h} x_\varphi &= \frac{1}{h} \int_0^\infty \varphi(s)[T(s+h)x - T(s)x] \, ds \\ &= \frac{1}{h} \left[\int_h^\infty \varphi(u-h)T(u)x \, du - \int_0^\infty \varphi(u)T(u)x \, du \right] \\ &= \int_0^\infty \frac{\varphi(u-h) - \varphi(u)}{h} T(u)x \, du \end{aligned}$$

As $h \rightarrow 0$, the quotient $\frac{\varphi(u-h) - \varphi(u)}{h}$ converges uniformly to $-\varphi'(u)$ because φ is C^∞ and has compact support. Thus:

$$\mathcal{A}x_\varphi = - \int_0^\infty \varphi'(s) T(s)x \, ds$$

Since $\varphi' \in C_c^\infty(0, \infty)$, we can repeat this process inductively. For any $n \geq 1$, we find:

$$\mathcal{A}^n x_\varphi = (-1)^n \int_0^\infty \varphi^{(n)}(s) T(s)x \, ds$$

This proves that $x_\varphi \in D(\mathcal{A}^n)$ for all n .

To prove density, suppose $\overline{\bigcap_{n \geq 1} D(\mathcal{A}^n)} \neq X$. By the Hahn-Banach Theorem, there exists a non-zero functional $x^* \in X^*$ such that $\langle x^*, y \rangle = 0$ for all $y \in \bigcap_{n \geq 1} D(\mathcal{A}^n)$. Specifically, for any $x \in X$ and $\varphi \in C_c^\infty(0, +\infty)$:

$$\langle x^*, x_\varphi \rangle = \int_0^{+\infty} \varphi(s) \langle x^*, T(s)x \rangle \, ds = 0$$

This identity holds for all C^∞ functions φ with compact support. Then $\langle x^*, T(s)x \rangle$ must be zero for all $s > 0$.

By the strong continuity of the semigroup at $s = 0$:

$$\langle x^*, x \rangle = \lim_{s \rightarrow 0^+} \langle x^*, T(s)x \rangle = 0$$

Since this holds for all $x \in X$, it implies $x^* = 0$, which contradicts our assumption that x^* was non-zero. Thus, $\bigcap_{n \geq 1} D(\mathcal{A}^n)$ must be dense in X . \square

Exercise 10

Let $X = \{f : \mathbb{R} \rightarrow \mathbb{C} \mid f \text{ is continuous and uniformly bounded}\}$ equipped with the supremum norm $\|f\|_\infty = \sup_{s \in \mathbb{R}} |f(s)|$. Define the family of operators $(T(t))_{t \geq 0}$ by:

$$(T(t)f)(s) = f(s + t), \quad s \in \mathbb{R}, t \geq 0$$

Prove that this family is C_0 -SG, its IG is $\mathcal{A}f = f'$, and $\|T(t)\| = 1$.

2.4 Hille-Yosida Theorem

Definition 2.4.1. A C_0 -SG $\{T(t)\}_{t \geq 0}$ is called uniformly bounded semigroup if $\exists M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

Definition 2.4.2. A C_0 -SG $\{T(t)\}_{t \geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.

Theorem 2.4.1 (Hille-Yosida Theorem (Contraction Case)). A linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ is the IG of a C_0 -SG of contractions if and only if:

- (i) \mathcal{A} is closed and $\overline{D(\mathcal{A})} = X$.
- (ii) $\mathbb{R}_+^* \subset \rho(\mathcal{A})$ and $\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

(\Rightarrow). (i) follows directly from 2.3.6.

If \mathcal{A} generates a contraction semigroup $\{T(t)\}_{t \geq 0}$, we define the resolvent for $\lambda > 0$ as follows

$$R(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t) dt$$

Taking the norm, we obtain:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)x\| dt \leq \int_0^{+\infty} e^{-\lambda t} \|x\| dt = \frac{1}{\lambda} \|x\|$$

\square

Remark 2.4.1. Note for all real numbers λ, a with $\lambda > a$, we have the following:

$$\frac{1}{\lambda - a} = \int_0^{+\infty} e^{-(\lambda-a)t} dt$$

Extending this to the vector space we get the way of writing the resolvent operator from above.

2.4.1 The Yosida Approximation

To prove if (\Leftarrow), we introduce a family of bounded operators that approximate the unbounded generator A .

Definition 2.4.3. For $\lambda > 0$, the Yosida Approximation of \mathcal{A} is defined as:

$$\mathcal{A}_\lambda := \lambda \mathcal{A} R(\lambda, \mathcal{A}) = \lambda^2 R(\lambda, \mathcal{A}) - \lambda I$$

Note that \mathcal{A}_λ is a bounded linear operator for each $\lambda \in \rho(\mathcal{A})$.

Claim 2.4.1. For $\lambda \in \rho(\mathcal{A})$ and $x \in D(\mathcal{A})$, the following identity holds:

$$\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A} R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$$

Proof. By the definition of the resolvent as the inverse of the operator $(\lambda I - \mathcal{A})$, we have:

$$(\lambda I - \mathcal{A})R(\lambda, \mathcal{A}) = Id_X$$

Applying this to any $x \in X$:

$$(\lambda I - \mathcal{A})R(\lambda, \mathcal{A})x = x$$

Distributing the operators on the left-hand side gives:

$$\lambda R(\lambda, \mathcal{A})x - \mathcal{A} R(\lambda, \mathcal{A})x = x$$

Rearranging the terms to isolate the $\mathcal{A} R(\lambda, \mathcal{A})x$ term yields:

$$\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A} R(\lambda, \mathcal{A})x \quad (*)$$

This identity holds for all $x \in X$ because $R(\lambda, \mathcal{A})$ maps X into $D(\mathcal{A})$.

For the other side, let $x \in D(\mathcal{A})$. We use the fact that the resolvent also satisfies:

$$R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}) = Id_{D(\mathcal{A})}$$

Applying this to $x \in D(\mathcal{A})$:

$$R(\lambda, \mathcal{A})(\lambda I - \mathcal{A})x = x$$

Distributing $R(\lambda, \mathcal{A})$ gives:

$$\lambda R(\lambda, \mathcal{A})x - R(\lambda, \mathcal{A})\mathcal{A}x = x$$

Rearranging the terms:

$$\lambda R(\lambda, \mathcal{A})x - x = R(\lambda, \mathcal{A})\mathcal{A}x \quad (**)$$

From (*) and (**) we get:

$$\mathcal{A}R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$$

□

Theorem 2.4.2. For all $x \in X$, $\lim_{\lambda \rightarrow +\infty} \mathcal{A}_\lambda x = \mathcal{A}x$.

Proof. Using the identity $\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A}R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$ for $x \in D(\mathcal{A})$, we observe:

$$\|\lambda R(\lambda, \mathcal{A})x - x\| = \|R(\lambda, \mathcal{A})\mathcal{A}x\| \leq \frac{\|\mathcal{A}x\|}{\lambda} \xrightarrow{\lambda \rightarrow +\infty} 0$$

Since $\mathcal{A}_\lambda x = \lambda R(\lambda, \mathcal{A})\mathcal{A}x$, and we just showed $\lambda R(\lambda, \mathcal{A})x \rightarrow x$, $\forall x \in D(\mathcal{A})$, it follows that $\mathcal{A}_\lambda x \rightarrow \mathcal{A}x$. By density and the uniform bound $\|\lambda R(\lambda, \mathcal{A})\| \leq 1$, this convergence holds for all $x \in X$. □

Lemma 2.4.1. For each $\lambda > 0$, \mathcal{A}_λ generates a uniformly continuous semigroup of contractions $\{e^{t\mathcal{A}_\lambda}\}_{t \geq 0}$.

Proof. Since $\mathcal{A}_\lambda = \lambda^2 R(\lambda, \mathcal{A}) - \lambda I$, we have:

$$\|e^{t\mathcal{A}_\lambda}\| = \left\| e^{-t\lambda} e^{t\lambda^2 R(\lambda, \mathcal{A})} \right\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda, \mathcal{A})\|} \leq e^{-t\lambda} e^{t\lambda^2 \frac{1}{\lambda}} = e^{-t\lambda} e^{t\lambda} = 1$$

This confirms the contraction property for the approximating semigroups. □

Lemma 2.4.2. For any $x \in X$ and $\lambda, \mu > 0$, the following estimate holds:

$$\|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| \leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\|$$

Proof. Fix $x \in X$, consider the function $\phi : [0, 1] \rightarrow X$ defined by:

$$\phi(s) = e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} x$$

Since \mathcal{A}_λ and \mathcal{A}_μ commute (as resolvents commute), the semigroups $e^{t\mathcal{A}_\lambda}$ and $e^{t\mathcal{A}_\mu}$ also commute. The function ϕ is differentiable with respect to s :

$$\begin{aligned} \frac{d}{ds} \phi(s) &= t \mathcal{A}_\lambda e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} x - e^{st\mathcal{A}_\lambda} t \mathcal{A}_\mu e^{(1-s)t\mathcal{A}_\mu} x \\ &= t e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} (\mathcal{A}_\lambda - \mathcal{A}_\mu) x \end{aligned}$$

Integrating from 0 to 1:

$$\phi(1) - \phi(0) = e^{t\mathcal{A}_\lambda} x - e^{t\mathcal{A}_\mu} x = \int_0^1 t e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} (\mathcal{A}_\lambda - \mathcal{A}_\mu) x \, ds$$

Taking the norm and using the contraction property $\|e^{t\mathcal{A}_\lambda}\| \leq e^{t\|\mathcal{A}_\lambda\|} \leq 1$ (since \mathcal{A}_λ is dissipative):

$$\begin{aligned}\|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| &\leq \int_0^1 t \|e^{st\mathcal{A}_\lambda}\| \|e^{(1-s)t\mathcal{A}_\mu}\| \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\| ds \\ &\leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\|\end{aligned}$$

□

Now let's start proving the only if direction of Hille-Yosida (\Leftarrow).

Proof. We need the sequence $(e^{t\mathcal{A}_\lambda}x)_{\lambda>0}$ to be converging. But doing so is hard, what we can do is show that it is a Cauchy sequence in X , since X is complete.

Taking $x \in D(\mathcal{A})$ we have the following:

$$\|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| \leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\| \leq t(\|\mathcal{A}_\lambda x - \mathcal{A}x\| + \|\mathcal{A}x - \mathcal{A}_\mu x\|) \xrightarrow[\mu \rightarrow +\infty]{} 0$$

This is possible because $\lim_{\lambda \rightarrow +\infty} \mathcal{A}_\lambda x = \mathcal{A}x$.

Define:

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda}x$$

This limit is well defined by the argument above for all $x \in D(\mathcal{A})$. Moreover, since $D(A)$ is dense in X and $\|e^{t\mathcal{A}_\lambda}\| \leq 1$, we can extend $T(t)$ to a bounded linear operator on all of X by density.

Now we show that the family $(T(t))_{t \geq 0}$ defined above is a C_0 -semigroup of contractions.

We verify the semigroup properties:

1. Identity: $T(0)x = \lim_{\lambda \rightarrow +\infty} e^{0\cdot\mathcal{A}_\lambda}x = x$.

2. Semigroup Property: For $x \in X$,

$$T(t+s)x = \lim_{\lambda \rightarrow +\infty} e^{(t+s)\mathcal{A}_\lambda}x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda}e^{s\mathcal{A}_\lambda}x = T(t)T(s)x.$$

3. Strong Continuity: For $x \in D(\mathcal{A})$, convergence is uniform on compact intervals of t , because of lemma 2.4.2. Thus $t \mapsto T(t)x$ is continuous. By the density of $D(A)$ and uniform boundedness $\|T(t)\| \leq 1$, continuity extends to all $x \in X$.

We must show that the generator of the constructed semigroup $T(t)$ is indeed \mathcal{A} . Let \mathcal{B} be the generator of $T(t)$, we show that $\mathcal{A} = \mathcal{B}$.

For any $x \in D(\mathcal{A})$, we have the identity:

$$e^{t\mathcal{A}_\lambda}x - x = \int_0^t e^{s\mathcal{A}_\lambda} \mathcal{A}_\lambda x ds$$

As $\lambda \rightarrow +\infty$, $e^{s\mathcal{A}\lambda} \rightarrow T(s)$ strongly and uniformly on compact sets, and $\mathcal{A}\lambda x \rightarrow \mathcal{A}x$. Passing to the limit:

$$T(t)x - x = \int_0^t T(s)\mathcal{A}x \, ds$$

Dividing by t and taking $t \rightarrow 0^+$:

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t T(s)\mathcal{A}x \, ds = T(0)\mathcal{A}x = \mathcal{A}x$$

Thus, $x \in D(\mathcal{B})$ and $\mathcal{B}x = \mathcal{A}x$, implying $\mathcal{A} \subset \mathcal{B}$. Since \mathcal{B} is the generator of a C_0 -semigroup of contractions, $1 \in \rho(\mathcal{B})$. By hypothesis, $1 \in \rho(\mathcal{A})$. Since $\mathcal{A} \subset \mathcal{B}$ and both $(I - \mathcal{A})$ and $(I - \mathcal{B})$ are surjective (mapping onto X), it follows that $\mathcal{A} = \mathcal{B}$. \square

Corollary 2.4.1. *Let \mathcal{A} be the IG of a C_0 -SG of contractions $(T(t))_{t \geq 0}$. Then for every $x \in X$, the semigroup is given by the limit:*

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}\lambda}x$$

Proof. In the previous proof, we constructed a SG, let us call it $S(t)$, defined by $S(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}\lambda}x$. We proved that the generator of $S(t)$ is exactly the operator \mathcal{A} .

Since \mathcal{A} is the generator of the original semigroup $T(t)$ by hypothesis, and we know that a C_0 -SG is uniquely determined by its generator (Uniqueness Theorem), it follows that:

$$T(t) = S(t), \quad \forall t \geq 0$$

\square

Corollary 2.4.2. *Let \mathcal{A} be IG of a C_0 -SG of contractions $(T(t))_{t \geq 0}$. Then the resolvent set $\rho(\mathcal{A})$ contains the open right half-plane:*

$$\rho(\mathcal{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$$

Furthermore, for all λ with $\Re(\lambda) > 0$, the following estimate holds:

$$\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\Re(\lambda)}$$

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. We define the operator $R(\lambda)$ on X by the Laplace transform of the semigroup:

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since $T(t)$ is a contraction semigroup ($\|T(t)\| \leq 1$) and $\Re(\lambda) > 0$, the integrand is exponentially bounded:

$$\left\| e^{-\lambda t} T(t)x \right\| = e^{-\Re(\lambda)t} \|T(t)x\| \leq e^{-\Re(\lambda)t} \|x\|$$

Thus, the integral converges absolutely, defining a bounded linear operator. We calculate its norm:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\Re(\lambda)t} \|x\| dt = \|x\| \left[\frac{-e^{-\Re(\lambda)t}}{\Re(\lambda)} \right]_0^{+\infty} = \frac{1}{\Re(\lambda)} \|x\|$$

This proves the bound $\|R(\lambda)\| \leq \frac{1}{\Re(\lambda)}$.

It remains to show that this integral operator $R(\lambda)$ is indeed the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$. For any $x \in X$ and $h > 0$:

$$\begin{aligned} \frac{T(h) - I}{h} R(\lambda)x &= \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t+h)x dt - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt \\ &= \frac{e^{\lambda h}}{h} \int_h^{+\infty} e^{-\lambda s} T(s)x ds - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T(t)x dt \end{aligned}$$

Taking the limit as $h \rightarrow 0^+$, the first term converges to $\lambda R(\lambda)x$ and the second term converges to $-x$. Thus, for any $x \in X$, $R(\lambda)x \in D(\mathcal{A})$ and $\mathcal{A}R(\lambda)x = \lambda R(\lambda)x - x$, which implies $(\lambda I - \mathcal{A})R(\lambda)x = x$. Similarly, one can show $R(\lambda)(\lambda I - \mathcal{A})x = x$ for $x \in D(\mathcal{A})$.

Therefore, $R(\lambda) = (\lambda I - \mathcal{A})^{-1}$. □

Exercise 11

Let $X = BVC(\mathbb{R}_+) = \{f : \mathbb{R}_+ \rightarrow \mathbb{C} \mid f \text{ is bounded and uniformly continuous}\}$. Equipped with the supremum norm $\|f\|_\infty = \sup_{t \geq 0} |f(t)|$, $(X, \|\cdot\|_\infty)$ is a Banach space.

For $t \geq 0$, define the operator $T(t)$ by the left shift:

$$(T(t)f)(s) = f(s + t), \quad \forall s \geq 0$$

Check the following:

1. $(T(t))_{t \geq 0}$ is a C_0 -semigroup of contractions (i.e., $\|T(t)\| \leq 1$).
2. Show that $\|T(t)\| = 1$.
3. The infinitesimal generator is the differentiation operator $\mathcal{A}f = f'$, with an appropriate domain $D(\mathcal{A})$.
4. Verify that $\rho(\mathcal{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$.

2.4.2 The General Hille-Yosida Theorem

We now consider the general case where the semigroup is not necessarily a contraction.

Theorem 2.4.3. *A linear operator \mathcal{A} generates a C_0 -semigroup $T(t)$ satisfying $\|T(t)\| \leq M e^{\omega t}$ if and only if:*

(i) A is closed and densely defined.

(ii) $(\omega, +\infty) \subset \rho(\mathcal{A})$ and for all $\lambda > \omega$ and $n \geq 1$:

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}.$$

2.4.3 Reduction to the Case $\omega = 0$

If $\|T(t)\| \leq M e^{\omega t}$, consider the rescaled semigroup $S(t) = e^{-\omega t} T(t)$. The generator of $S(t)$ is $\mathcal{A} - \omega I$ where \mathcal{A} is the IG of $(T(t))_{t \geq 0}$, and $\|S(t)\| \leq M$. Conversely, if we prove the theorem for $\omega = 0$, the general case follows by applying the result to $\mathcal{A} - \omega I$.

Now we have the following corollary:

Corollary 2.4.3 (Hille-Yosida for $(1, \omega)$). *A linear operator \mathcal{A} is the IG of a C_0 -SG satisfying $\|T(t)\| \leq e^{\omega t}$ if and only if:*

(i) \mathcal{A} is closed and $\overline{D(\mathcal{A})} = X$.

(ii) $\rho(\mathcal{A}) \supset (\omega, +\infty)$ and for all $\lambda > \omega$, the following estimate holds:

$$\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda - \omega}.$$

Proof. (\Rightarrow): Suppose \mathcal{A} generates a semigroup $T(t)$ such that $\|T(t)\| \leq e^{\omega t}$. Consider the rescaled family of operators $S(t) = e^{-\omega t} T(t)$. It is easy to verify that $S(t)$ is a C_0 -SG. Furthermore, it is a contraction:

$$\|S(t)\| = e^{-\omega t} \|T(t)\| \leq e^{-\omega t} e^{\omega t} = 1$$

Let \mathcal{B} be the generator of $S(t)$. By the definition of the generator:

$$\mathcal{B}x = \lim_{t \rightarrow 0^+} \frac{e^{-\omega t} T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \left(e^{-\omega t} \frac{T(t)x - x}{t} + \frac{e^{-\omega t} - 1}{t} x \right) = \mathcal{A}x - \omega x$$

Thus, $\mathcal{B} = \mathcal{A} - \omega I$. Since \mathcal{B} generates a contraction semigroup, by the Hille-Yosida Theorem for contractions (Case $M = 1, \omega = 0$), we know that for any $\mu > 0$, $\mu \in \rho(\mathcal{B})$ and $\|R(\mu, \mathcal{B})\| \leq \frac{1}{\mu}$.

Let $\lambda = \mu + \omega$. Then $\lambda > \omega$. Since $R(\mu, \mathcal{B}) = (\mu I - \mathcal{B})^{-1} = (\mu I - (\mathcal{A} - \omega I))^{-1} = ((\mu + \omega)I - \mathcal{A})^{-1}$, we have:

$$R(\lambda, \mathcal{A}) = R(\lambda - \omega, \mathcal{B})$$

Substituting the norm bound:

$$\|R(\lambda, \mathcal{A})\| = \|R(\lambda - \omega, \mathcal{B})\| \leq \frac{1}{\lambda - \omega}$$

(\Leftarrow): Conversely, suppose \mathcal{A} satisfies conditions (1)-(3). Define $\mathcal{B} = \mathcal{A} - \omega I$. Clearly, \mathcal{B} is closed and densely defined. For any $\mu > 0$, let $\lambda = \mu + \omega > \omega$. Then $\lambda \in \rho(\mathcal{A})$, which implies

$\mu \in \rho(\mathcal{B})$. The resolvent satisfies:

$$\|R(\mu, \mathcal{B})\| = \|R(\mu + \omega, \mathcal{A})\| \leq \frac{1}{(\mu + \omega) - \omega} = \frac{1}{\mu}$$

Thus, \mathcal{B} satisfies the Hille-Yosida conditions for the contraction case ($M = 1, \omega = 0$). Therefore, \mathcal{B} generates a contraction semigroup $S(t)$ with $\|S(t)\| \leq 1$. Defining $T(t) = e^{\omega t} S(t)$, we see that $T(t)$ is a C_0 -semigroup generated by $\mathcal{A} = \mathcal{B} + \omega I$, and it satisfies:

$$\|T(t)\| = e^{\omega t} \|S(t)\| \leq e^{\omega t}$$

□

This corollary provides us with a method to rescale the bound. Thus, we focus on the case $(M, 0)$, i.e., $\|T(t)\| \leq M$. The condition on the resolvent becomes $\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$.

Lemma 2.4.3. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup satisfying $\|T(t)\| \leq M$ for all $t \geq 0$. Let \mathcal{A} be its infinitesimal generator. Then for all $\lambda > 0$ and all integers $n \geq 0$:*

$$\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$$

Equivalently, $\|\lambda^n R(\lambda, \mathcal{A})^n\| \leq M$.

Proof. For $\lambda > 0$, the resolvent is given by the Laplace transform of the semigroup:

$$R(\lambda, \mathcal{A})x = \int_0^{+\infty} e^{-\lambda t} T(t)x dt, \quad \forall x \in X$$

Since the integral converges absolutely (due to the exponential decay $e^{-\lambda t}$ and bounded $T(t)$), we can differentiate this expression with respect to λ inside the integral sign. Differentiating $n - 1$ times:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathcal{A})x = \int_0^{+\infty} (-t)^{n-1} e^{-\lambda t} T(t)x dt$$

On the other hand, from the general theory of resolvents, we have the identity:

$$\frac{d^k}{d\lambda^k} R(\lambda, \mathcal{A}) = (-1)^k k! R(\lambda, \mathcal{A})^{k+1}$$

Setting $k = n - 1$, we get:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathcal{A}) = (-1)^{n-1} (n-1)! R(\lambda, \mathcal{A})^n$$

Equating the two expressions for the derivative:

$$(-1)^{n-1} (n-1)! R(\lambda, \mathcal{A})^n x = \int_0^{+\infty} (-1)^{n-1} t^{n-1} e^{-\lambda t} T(t)x dt$$

Simplifying and solving for $R(\lambda, \mathcal{A})^n x$:

$$R(\lambda, \mathcal{A})^n x = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t)x dt$$

Now, we take the norm and use the bound $\|T(t)\| \leq M$:

$$\begin{aligned} \|R(\lambda, \mathcal{A})^n x\| &\leq \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \|T(t)x\| dt \\ &\leq \frac{M \|x\|}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} dt \end{aligned}$$

The integral on the right is the Gamma function definition. Substituting $u = \lambda t$ ($dt = du/\lambda$):

$$\int_0^{+\infty} t^{n-1} e^{-\lambda t} dt = \frac{1}{\lambda^n} \int_0^{+\infty} u^{n-1} e^{-u} du = \frac{\Gamma(n)}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

Substituting this back into the inequality:

$$\|R(\lambda, \mathcal{A})^n x\| \leq \frac{M \|x\|}{(n-1)!} \cdot \frac{(n-1)!}{\lambda^n} = \frac{M}{\lambda^n} \|x\|$$

Thus, $\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$. □

2.4.4 Renorming Lemma

The idea is to construct an equivalent norm on X under which \mathcal{A} becomes dissipative (generating a contraction semigroup), allowing us to apply the contraction case result.

Lemma 2.4.4. *Let $A : D(A) \subset X \rightarrow X$ be a linear operator with $\rho(A) \supset \mathbb{R}_+^*$ such that for all $\lambda > 0$ and $n \geq 0$, $\|\lambda^n R(\lambda, A)^n\| \leq M$. Then, there exists a norm $\|\cdot\|_\mu$ on X such that:*

1. *The norms are equivalent: $\|x\| \leq \|x\|_\mu \leq M \|x\|$ for all $x \in X$.*
2. *For all $\lambda > 0$, the operator $\lambda R(\lambda, A)$ is a contraction in the new norm: $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$.*

Proof. Fix $\mu > 0$. We define the new norm $\|\cdot\|_\mu$ by:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\|$$

To show that the norms are equivalent, let $n = 0$, the term is $\|Ix\| = \|x\|$ so $\|x\| \leq \|x\|_\mu$. Using the hypothesis $\|\mu^n R(\mu, A)^n\| \leq M$, we have:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\| \leq \sup_{n \geq 0} M \|x\| = M \|x\|$$

Thus, $\|x\| \leq \|x\|_\mu \leq M \|x\|$.

Now to show that $\mu R(\mu, A)$ is a contraction, we check the contraction property for the specific value μ :

$$\begin{aligned}\|\mu R(\mu, A)x\|_\mu &= \sup_{n \geq 0} \|\mu^n R(\mu, A)^n (\mu R(\mu, A)x)\| \\ &= \sup_{n \geq 0} \|\mu^{n+1} R(\mu, A)^{n+1} x\| \\ &= \sup_{k \geq 1} \|\mu^k R(\mu, A)^k x\| \\ &\leq \sup_{k \geq 0} \|\mu^k R(\mu, A)^k x\| = \|x\|_\mu.\end{aligned}$$

So, $\|\mu R(\mu, A)\|_\mu \leq 1$.

Finally, we show the contraction for $0 < \lambda \leq \mu$:

Let $x \in X$ and define $y = R(\lambda, \mathcal{A})x$. We want to show $\lambda \|y\|_\mu \leq \|x\|_\mu$. Recall the Resolvent Identity:

$$R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}) = (\mu - \lambda)R(\mu, \mathcal{A})R(\lambda, \mathcal{A})$$

Applying this to x :

$$y = R(\mu, \mathcal{A})x + (\mu - \lambda)R(\mu, \mathcal{A})y = R(\mu, \mathcal{A})[x + (\mu - \lambda)y].$$

Taking the $\|\cdot\|_\mu$ norm and using the fact that $\|\mu R(\mu, A)z\|_\mu \leq \|z\|_\mu \implies \|R(\mu, A)z\|_\mu \leq \frac{1}{\mu} \|z\|_\mu$:

$$\begin{aligned}\|y\|_\mu &= \|R(\mu, A)[x + (\mu - \lambda)y]\|_\mu \\ &\leq \frac{1}{\mu} \|x + (\mu - \lambda)y\|_\mu \\ &\leq \frac{1}{\mu} (\|x\|_\mu + (\mu - \lambda) \|y\|_\mu)\end{aligned}$$

Multiplying by μ :

$$\mu \|y\|_\mu \leq \|x\|_\mu + (\mu - \lambda) \|y\|_\mu$$

Subtracting $(\mu - \lambda) \|y\|_\mu$ from both sides (since $\mu - \lambda \geq 0$):

$$\lambda \|y\|_\mu \leq \|x\|_\mu.$$

Thus, $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$ for $0 < \lambda \leq \mu$. Since this holds for any sufficiently large μ , and the definition of the norm can be adjusted, this property extends to all $\lambda > 0$. \square

Remark 2.4.2. In equivalent norm, the following are preserved:

1. The closeness of an operator.
2. The density of the image of an operator.

3. The strong continuity of the C_0 -SG.

Theorem 2.4.4 (Hille-Yosida for $(M, 0)$). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a linear operator on a Banach space X . Then \mathcal{A} is the IG of a C_0 -SG $T(t)$ satisfying $\|T(t)\| \leq M$ for all $t \geq 0$ if and only if:*

- (i) \mathcal{A} is closed and $(\overline{D(\mathcal{A})}) = X$.
- (ii) $\rho(\mathcal{A}) \supset (0, \infty)$ and for every $\lambda > 0$:

$$\|(\lambda^n R(\lambda, \mathcal{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

Proof. \Rightarrow

Let \mathcal{A} be the generator of a C_0 -semigroup $T(t)$ with $\|T(t)\| \leq M$. By theorem 2.3.6 part (i) is done.

For part (ii) is direct from lemma 2.4.3

\Leftarrow

By using lemma 2.4.4 and apply it on the space the assumptions change to the following:

- (i) \mathcal{A} is closed and $(\overline{D(\mathcal{A})}) = X$.
- (ii) $\mathbb{R}_+^* \subset \rho(\mathcal{A})$ and $\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

Which are just the conditions on Hille-Yosida for contractions and we get the desired result. \square

Theorem 2.4.5 (Hille-Yosida for the Generale Case (M, ω)). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a linear operator on a Banach space X . Then \mathcal{A} is the IG of a C_0 -SG $T(t)$ satisfying $\|T(t)\| \leq M \cdot e^{\omega t}$ for all $t \geq 0$ if and only if:*

- (i) \mathcal{A} is closed and $(\overline{D(\mathcal{A})}) = X$.
- (ii) $\rho(\mathcal{A}) \supset (0, \infty)$ and for every $\lambda > 0$:

$$\|(\lambda^n R(\lambda, \mathcal{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

Theorem 2.4.6. *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO and the IG of a C_0 -SG $T(t)_{t \geq 0}$ then for all $x \in X$:*

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda}x$$

Where \mathcal{A}_λ is the Yosida Approximation of \mathcal{A} .

Proof. This can have 2 cases depending on ω .

First case, let $\omega = 0$. Then by applying lemma 2.4.4 we get a contraction and it is direct by corollary 2.4.1

Second case, let $\omega > 0$. We want to get a bound on the approximation.

$$\begin{aligned}
\|e^{t\mathcal{A}_\lambda}\| &= \left\| e^{t(\lambda^2 R(\lambda, \mathcal{A}) - \lambda I)} \right\| \\
&= e^{-\lambda t} \left\| e^{t\lambda^2 R(\lambda, \mathcal{A})} \right\| \\
&\leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(t\lambda^2)^k}{k!} \|R(\lambda, \mathcal{A})^k\| \\
&\leq e^{-\lambda t} M \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t\lambda^2}{\lambda - \omega} \right)^k \quad \text{since } \|R(\lambda, \mathcal{A})^k\| \leq \frac{M}{(\lambda - \omega)^k} \\
&= M e^{-\lambda t} \exp\left(\frac{t\lambda^2}{\lambda - \omega}\right) \\
&= M \exp\left(-\lambda t + \frac{t\lambda^2}{\lambda - \omega}\right)
\end{aligned}$$

We can do the following calculation in the exponent:

$$\begin{aligned}
-\lambda + \frac{\lambda^2}{\lambda - \omega} &= \left(\frac{-\lambda(\lambda - \omega) + \lambda^2}{\lambda - \omega} \right) \\
&= \frac{\lambda\omega}{\lambda - \omega} \\
&= \omega \left(\frac{\lambda - \omega + \omega}{\lambda - \omega} \right) \\
&= \omega \left(1 + \frac{\omega}{\lambda - \omega} \right) \\
&= \omega + \frac{\omega^2}{\lambda - \omega}
\end{aligned}$$

Now for all $\lambda \geq 2\omega$ we have $\lambda - \omega \geq \omega > 0$ and $\frac{1}{\lambda - \omega} \leq \frac{1}{\omega}$, hence:

$$\omega + \frac{\omega^2}{\lambda - \omega} \leq \omega + \frac{\omega^2}{\omega} = 2\omega$$

And we get the bound:

$$\|e^{t\mathcal{A}_\lambda}\| \leq M \cdot e^{2t\omega} \tag{*}$$

Let $S(t) = e^{-\omega t} T(t)$, this C_0 -SG is generated by $\mathcal{A} - \omega I$ (rescaling of $T(t)$).

Since $S(t)$ is $(M, 0)$ then by the first case we have:

$$\forall t \geq 0, \forall x \in X \quad S(t)x = \lim_{\lambda \rightarrow +\infty} e^{t(\mathcal{A} - \lambda I)\lambda}$$

Equivalently:

$$e^{-\omega t} T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t(\mathcal{A} - \lambda I)\lambda}$$

Now we have to show that $\forall t \geq 0, \forall x \in X$ the following:

$$e^{t[(\mathcal{A}-\omega I)\lambda + \omega I]}x - e^{t\mathcal{A}\lambda}x = 0$$

We have the following:

$$\begin{aligned} e^{t[(\mathcal{A}-\omega I)\lambda + \omega I]}x - e^{t\mathcal{A}\lambda}x &= e^{t[(\mathcal{A}-\omega I)\lambda + \omega I - \mathcal{A}\lambda + \mathcal{A}\lambda]}x - e^{t\mathcal{A}\lambda}x \\ &= e^{t(A_\lambda + H(\lambda))}x - e^{tA_\lambda}x \quad \text{by setting } H(\lambda) := (\mathcal{A} - \omega I)_\lambda + \omega I - \mathcal{A}_\lambda \\ &= e^{tA_\lambda}e^{tH(\lambda)}x - e^{tA_\lambda}x \\ &= e^{tA_\lambda}(e^{tH(\lambda)} - I)x \quad \mathcal{A}_\lambda \text{ and } H(\lambda) \text{ commutes (why?)} \end{aligned}$$

Now, we aim to show that for any fixed $x \in D(A)$, $H(\lambda)x \rightarrow 0$ as $\lambda \rightarrow +\infty$.

By theorem 2.4.2 we have:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} H(\lambda)x &= \lim_{\lambda \rightarrow \infty} ((A - \omega I)_\lambda x + \omega x - A_\lambda x) \\ &= (Ax - \omega x) + \omega x - Ax \\ &= 0 \end{aligned}$$

Thus, $H(\lambda)x \rightarrow 0$ for all $x \in D(A)$.

Now we have:

$$e^{t[(\mathcal{A}-\omega I)\lambda + \omega I]}x - e^{t\mathcal{A}\lambda}x = e^{tA_\lambda}(e^{tH(\lambda)} - I)x$$

By taking the norm:

$$\|e^{t[(\mathcal{A}-\omega I)\lambda + \omega I]}x - e^{t\mathcal{A}\lambda}x\| \leq \|e^{tA_\lambda}\| \|(e^{tH(\lambda)} - I)x\|$$

From (*), we have that $\|e^{tA_\lambda}\|$ is uniformly bounded hence we need to work on $\|(e^{tH(\lambda)} - I)x\|$.

Note that $e^{tH(\lambda)} = e^{t(B_\lambda + \omega I)}e^{-tA_\lambda}$. From the bound estimates shown earlier, both $\|e^{s(B_\lambda + \omega I)}\|$ and $\|e^{-sA_\lambda}\|$ are uniformly bounded for s in a compact interval $[0, T]$ and sufficiently large λ . Let C be this bound. Then:

$$\|(e^{tH(\lambda)} - I)x\| \leq t \cdot C \cdot \|H(\lambda)x\|$$

Since we proved that $\lim_{\lambda \rightarrow \infty} \|H(\lambda)x\| = 0$, it follows immediately that:

$$\lim_{\lambda \rightarrow \infty} \|(e^{tH(\lambda)} - I)x\| = 0$$

Finally, combining this with the uniform bound on $\|e^{tA_\lambda}\|$, we obtain:

$$\lim_{\lambda \rightarrow \infty} \|e^{t(B_\lambda + \omega I)}x - e^{tA_\lambda}x\| = 0$$

This proves that the limit is the same, regardless of the shift. This completed our proof. \square

2.5 Inverse Laplace Transform and Resolvent Formula

We have established that for a C_0 -SG $(T(t))_{t \geq 0}$ with $\|T(t)\| \leq M e^{t\omega}$, the resolvent is given using the following formula:

$$R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x dt \quad \text{for } \operatorname{Re}(\lambda) > \omega.$$

which is just the Laplace Tranfrom of the semigroup.

A major problem in semigroup theory is recovering $T(t)$ from $R(\lambda, \mathcal{A})x$ (Inverting the Laplace Transform).

Lemma 2.5.1. *Let X be BS and $\mathcal{B} \in \mathcal{L}(X)$ then $\forall \gamma > \|\mathcal{B}\|$ we have:*

$$\begin{aligned} e^{t\mathcal{B}} &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda, \mathcal{B}) d\lambda \\ &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{(\gamma+is)t} R(\gamma + is, \mathcal{B}) i ds \end{aligned}$$

Proof. To make sense and motivate this proof we start by showing that this is correct for the scalar case. Let $b \in \mathbb{C}$ and the exponential function $f(t) = e^{tb}$. The Laplace transform of $f(t)$ is:

$$\mathcal{L}\{e^{tb}\}(\lambda) = \int_0^\infty e^{-\lambda t} e^{tb} dt = \frac{1}{\lambda - b}, \quad \text{for } \operatorname{Re}(\lambda) > \operatorname{Re}(b)$$

Now if we choose $|\lambda| > |b|$. We can expand $(\lambda - b)^{-1}$ as a convergent geometric series: Now if we choose γ such that $\operatorname{Re}(\gamma) > |b|$. Let show that:

$$e^{tb} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{1}{\lambda - b} d\lambda$$

For a circle $C(0, r)$ with radius $r > |b|$, we can expand the term $(\lambda - b)^{-1}$ as a geometric series for $|\lambda| > |b|$:

$$\frac{1}{\lambda - b} = \frac{1}{\lambda(1 - \frac{b}{\lambda})} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{b^k}{\lambda^k} = \sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}$$

Plugging this into the formula above we get:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{1}{\lambda - b} d\lambda &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}} d\lambda \quad (\text{the choice of } \gamma \text{ allow for this}) \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} b^k \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^{k+1}} d\lambda \quad (\text{The sum is uniformly convergent}) \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} 2\pi i \frac{b^k t^k}{k!} \quad (\text{Cauchy Integral Formula}) \\ &= e^{bt} \end{aligned}$$

This shows that the formula works for the scalar case.

The case of the operator follows the same steps and same reasoning.

Again let $C(0, r)$ with radius r such that $r > \|\mathcal{B}\|$. For any $\lambda \in C(0, r)$, we have $|\lambda| > \|\mathcal{B}\|$, which implies $\left\| \frac{\mathcal{B}}{\lambda} \right\| < 1$.

In this region, the resolvent $R(\lambda, \mathcal{B}) = (\lambda I - \mathcal{B})^{-1}$ admits a uniformly convergent Neumann series expansion:

$$R(\lambda, \mathcal{B}) = \frac{1}{\lambda} \left(I - \frac{\mathcal{B}}{\lambda} \right)^{-1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}^k}{\lambda^{k+1}}.$$

We substitute this series into the contour integral over $C(0, r)$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{C(0,r)} e^{\lambda t} R(\lambda, \mathcal{B}) d\lambda &= \frac{1}{2\pi i} \int_{C(0,r)} e^{\lambda t} \left(\sum_{k=0}^{\infty} \frac{\mathcal{B}^k}{\lambda^{k+1}} \right) d\lambda \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \mathcal{B}^k \left[\int_{C_r} \frac{e^{\lambda t}}{\lambda^{k+1}} d\lambda \right] \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \mathcal{B}^k \left[2\pi i \frac{t^k}{k!} \right] \\ &= \sum_{k=0}^{\infty} \frac{t^k \mathcal{B}^k}{k!} \\ &= e^{t\mathcal{B}} \end{aligned}$$

□

2.6 Spectral Mapping Theorem

Let $(T(t))_{t \geq 0}$ be C_0 -SG with \mathcal{A} as its IG, then what is the relation between $\sigma(T(t))$ and $\sigma(\mathcal{A})$?

Exercise 12

Let $(T(t))_{t \geq 0}$ be UC-SG with \mathcal{A} as its IG then $e^{t\sigma(\mathcal{A})} = \sigma(T(t))$.

Exercise 13

Take $X = \{f : [0, 1] \rightarrow \mathbb{C} \mid f \text{ is continuous and } f(1) = 0\}$ then $(X, \|\cdot\|_\infty)$ is BS. Define:

$$T(t)f(x) = \begin{cases} f(x+t) & x+t \leq 1 \\ 0 & x+t > 1 \end{cases}$$

Then:

1. $(T(t))_{t \geq 0}$ is a C_0 -SG.
2. \mathcal{A} the IG and $\mathcal{A}f = f'$. Determine $D(\mathcal{A})$.
3. Check $\forall \lambda \in \mathbb{C}, \exists! f \in X$ such that $(\lambda I - \mathcal{A})f = g$ for all $g \in X$.
4. $\sigma(\mathcal{A}) = \emptyset$.
5. $\sigma(T(t)) \neq \emptyset$.

Lemma 2.6.1. [Preparatory Lemma] Let $(T(t))_{t \geq 0}$ be a C_0 -SG and let \mathcal{A} be its IG. For all $t \geq 0$ and for all $x \in X$ define:

$$\mathcal{B}_\lambda(t)x = \int_0^t e^{\lambda(t-s)} T(s)x ds$$

then $(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$.

Proof. It is enough to show that:

$$\mathcal{A}\mathcal{B}_\lambda(t)x = T(t)x + \lambda\mathcal{B}_\lambda(t)x - e^{\lambda t}x$$

Recall that:

$$\mathcal{A} = \lim_{h \rightarrow 0^+} \frac{T(h) - Id}{h}$$

Then we have:

$$\begin{aligned} \frac{T(h) - Id}{h} \mathcal{B}_\lambda(t)x &= \frac{1}{h} \int_0^t e^{\lambda(t-s)} (T(h)T(s) - T(s)) x ds \\ &= \frac{1}{h} \left[\int_0^t e^{\lambda(t-s)} T(s+h)x ds - \int_0^t e^{\lambda(t-s)} T(s)x ds \right] \\ &= \frac{1}{h} \left[\int_h^{t+h} e^{\lambda(t-s+h)} T(s)x ds - \int_0^t e^{\lambda(t-s)} T(s)x ds \right] \\ &= \frac{1}{h} \left[e^{\lambda h} \int_h^{t+h} e^{\lambda(t-s)} T(s)x ds - \int_0^t e^{\lambda(t-s)} T(s)x ds \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h} \left[\left(e^{\lambda h} - 1 \right) \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds + \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds - \int_0^t e^{\lambda(t-s)} T(s) x \, ds \right] \\
&= \frac{1}{h} \left[\left(e^{\lambda h} - 1 \right) \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds + \int_t^{t+h} e^{\lambda(t-s)} T(s) x \, ds - \int_0^h e^{\lambda(t-s)} T(s) x \, ds \right]
\end{aligned}$$

Now taking $\lim_{h \rightarrow 0^+}$ we get:

$$\mathcal{A}\mathcal{B}_\lambda(t)x = \lambda\mathcal{B}_\lambda(t)x + T(t)x - e^{\lambda t}x$$

□

Remark 2.6.1. In the lemma above \mathcal{A} commutes if $x \in D(\mathcal{A})$, and we get:

$$\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = (\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$$

This is true because of the fact that \mathcal{A} is the limit.

Theorem 2.6.1. Let $(T(t))_{t \geq 0}$ be a C_0 -SG generated by \mathcal{A} . Then for all $t \geq 0$:

$$e^{t\sigma(\mathcal{A})} \subset \sigma(T(t))$$

Proof. We prove the equivalent statement: if $e^{\lambda t} \in \rho(T(t))$, then $\lambda \in \rho(\mathcal{A})$.

Assume $e^{\lambda t} \in \rho(T(t))$. Then the operator $(e^{\lambda t}I - T(t))$ is invertible with a bounded inverse $Q = (e^{\lambda t}I - T(t))^{-1} \in \mathcal{L}(X)$.

From Lemma 2.6.1, we have:

$$(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t) = e^{\lambda t}I - T(t).$$

Claim 2.6.1. Q and $\mathcal{B}_\lambda(t)$ commute. (Prove it)

Multiplying by Q on the right:

$$(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)Q = I.$$

Similarly, for $x \in D(\mathcal{A})$:

$$Q\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = x.$$

Defining $R(\lambda) = \mathcal{B}_\lambda(t)Q$, we see that $R(\lambda)$ acts as a two-sided inverse for $(\lambda I - \mathcal{A})$. Since $\mathcal{B}_\lambda(t)$ and Q are bounded, $R(\lambda)$ is bounded. Thus $\lambda \in \rho(\mathcal{A})$.

By contraposition, $\lambda \in \sigma(\mathcal{A}) \implies e^{\lambda t} \in \sigma(T(t))$. □

While the full spectral mapping theorem fails, a precise relationship holds for the point spectrum σ_p and the residual spectrum σ_r .

Recall the decomposition of the spectrum:

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where σ_p denotes point spectrum (eigenvalues), σ_c continuous spectrum, and σ_r residual spectrum.

Theorem 2.6.2. Let $(T(t))_{t \geq 0}$ be a C_0 -SG generated by \mathcal{A} . Then:

$$e^{t\sigma_p(\mathcal{A})} \subseteq \sigma_p(T(t)) \subseteq e^{t\sigma_p(\mathcal{A})} \cup \{0\}$$

Precisely, if $e^{\lambda t} \in \sigma_p(T(t))$, then there exists $k \in \mathbb{Z}$ such that:

$$\lambda_k = \lambda + \frac{2\pi i k}{t} \in \sigma_p(\mathcal{A})$$

Proof. First inclusion, $\lambda \in \sigma_p(\mathcal{A}) \implies e^{\lambda t} \in \sigma_p(T(t))$.

Let $\lambda \in \sigma_p(\mathcal{A})$. Then there exists $x_0 \neq 0$ such that $\mathcal{A}x_0 = \lambda x_0$ and by lemma 2.6.1 we have:

$$\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = (e^{\lambda t}I - T(t))x \quad \forall x \in D(\mathcal{A})$$

Now take $x = x_0$ and note the fact that $(\lambda I - \mathcal{A})x_0 = 0$:

$$(e^{\lambda t}I - T(t))x_0 = 0$$

Hence $\ker(e^{\lambda t}I - T(t)) \neq \{0\}$ then $e^{\lambda t} \in \sigma_p(T(t))$.

For the second inclusion, let $\mu \in \sigma_p(T(t))$ with $\mu \neq 0$. Write $\mu = e^{\lambda t}$ for some $\lambda \in \mathbb{C}$.

There exists $x_0 \neq 0$ such that $T(t)x_0 = e^{\lambda t}x_0$.

Consider the function $\phi_{x_0}(s)$ defined by:

$$\phi_{x_0}(s) = e^{-\lambda s}T(s)x_0$$

Clearly:

$$\phi_{x_0}(0) = x_0 = \phi_{x_0}(t)$$

We check if ϕ_{x_0} is periodic with period t .

$$\phi_{x_0}(s+t) = e^{-\lambda(s+t)}T(s+t)x_0 = e^{-\lambda s}e^{-\lambda t}T(s)T(t)x_0$$

Using the eigen-property $T(t)x_0 = e^{\lambda t}x_0$:

$$\phi_{x_0}(s+t) = e^{-\lambda s}e^{-\lambda t}T(s)(e^{\lambda t}x_0) = e^{-\lambda s}T(s)x_0 = \phi_{x_0}(s)$$

Since $\phi_{x_0}(s)$ is a t -periodic continuous function (and assuming $x_0 \in X$), we can expand it into a

Fourier series. The k -th Fourier coefficient is:

$$\hat{x}_k = \frac{1}{t} \int_0^t e^{-2\pi i ks/t} \phi_{x_0}(s) ds$$

Substituting $\phi_{x_0}(s) = e^{-\lambda s} T(s)x_0$:

$$\hat{x}_k = \frac{1}{t} \int_0^t e^{-(\lambda + \frac{2\pi i k}{t})s} T(s)x_0 ds$$

Let $\lambda_k = \lambda + \frac{2\pi i k}{t}$. Since ϕ_{x_0} is not identically zero (as $\phi_{x_0}(0) = x_0 \neq 0$), at least one coefficient \hat{x}_k must be non-zero.

We will show that if $\hat{x}_k \neq 0$, then $\lambda_k \in \sigma_p(\mathcal{A})$.

We use the resolvent formula. For $\Re(\gamma)$ sufficiently large, $R(\gamma, \mathcal{A}) = \int_0^\infty e^{-\gamma s} T(s) ds$. Applying this to x_0 :

$$R(\gamma, \mathcal{A})x_0 = \int_0^\infty e^{-\gamma s} T(s)x_0 ds$$

Decompose the integral over intervals $[nt, (n+1)t]$:

$$\begin{aligned} R(\gamma, \mathcal{A})x_0 &= \sum_{n=0}^{\infty} \int_{nt}^{(n+1)t} e^{-\gamma s} T(s)x_0 ds \\ &= \sum_{n=0}^{\infty} \int_0^t e^{-\gamma(nt+\tau)} T(nt+\tau)x_0 d\tau \quad (\text{let } s = nt + \tau) \end{aligned}$$

Using the semigroup property $T(nt + \tau)x_0 = T(\tau)T(t)^n x_0 = T(\tau)e^{n\lambda t}x_0$:

$$\begin{aligned} R(\gamma, \mathcal{A})x_0 &= \sum_{n=0}^{\infty} e^{-n\gamma t} e^{n\lambda t} \int_0^t e^{-\gamma\tau} T(\tau)x_0 d\tau \\ &= \left(\sum_{n=0}^{\infty} e^{-n(\gamma-\lambda)t} \right) \int_0^t e^{-\gamma\tau} T(\tau)x_0 d\tau \end{aligned}$$

The geometric series converges if $\Re(\gamma) > \Re(\lambda)$:

$$\sum_{n=0}^{\infty} (e^{-(\gamma-\lambda)t})^n = \frac{1}{1 - e^{-(\gamma-\lambda)t}}$$

Thus:

$$R(\gamma, \mathcal{A})x_0 = \frac{1}{1 - e^{(\lambda-\gamma)t}} \int_0^t e^{-\gamma\tau} T(\tau)x_0 d\tau$$

The function $\gamma \mapsto R(\gamma, \mathcal{A})x_0$ is meromorphic. The poles of the resolvent indicate the spectrum. The denominator vanishes when:

$$e^{(\lambda-\gamma)t} = 1 \iff (\lambda - \gamma)t = 2\pi i k \iff \gamma = \lambda - \frac{2\pi i k}{t}$$

Let $\mu_k = \lambda + \frac{2\pi i k}{t}$. The resolvent has a pole at μ_k . Specifically, the residue near the pole relates to the existence of an eigenvector. If we analyze the limit:

$$\lim_{\gamma \rightarrow \lambda_k} (\gamma - \lambda_k) R(\gamma, \mathcal{A}) x_0 \neq 0 \implies \lambda_k \in \sigma_p(\mathcal{A}).$$

This confirms that the eigenvalues of \mathcal{A} are exactly the logarithms of the eigenvalues of $T(t)$ (modulo $2\pi i/t$). \square

CHAPTER
THREE

APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Let X be a Banach space and $A : D(A) \subset X \rightarrow X$ be a linear operator. We define the homogeneous and inhomogeneous abstract Cauchy problems as follows:

- Homogeneous Cauchy Problem $h - (CP)_{0,x_0}$:

$$\frac{du}{dt} = Au, \quad t \geq 0$$

$$u(0) = x_0$$

- Inhomogeneous Cauchy Problem $inh - (CP)_{0,x_0}$:

$$\frac{du}{dt} = Au + f(t), \quad t \geq 0$$

$$u(0) = x_0$$

where $f : \mathbb{R}_+ \rightarrow X$.

3.1 The Cauchy Problem

Definition 3.1.1. *We distinguish three types of solutions for the Cauchy problem:*

1. **Strong Solution:** A function $x(\cdot) \in C^1(\mathbb{R}_+, D(\mathcal{A}))$ such that for all $t \geq 0$, $x(t) \in D(\mathcal{A})$ and $\frac{dx}{dt} = \mathcal{A}x(t)$.
2. **Mild Solution:** A function $x(\cdot) \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, D(\mathcal{A}))$ with $x(0) = x_0$, and for all $t > 0$, $x(t) \in D(\mathcal{A})$ such that $\frac{dx}{dt} = \mathcal{A}x(t)$.
3. **Weak Solution:** A function $x(\cdot) \in C^0(\mathbb{R}_+, X)$ such that $x(0) = x_0$, and for all $x^* \in D(A^*)$, the map $t \mapsto \langle x^*, x(t) \rangle$ is C^1 and satisfies:

$$\frac{d}{dt} \langle x^*, x(t) \rangle = \langle A^* x^*, x(t) \rangle$$

Remark 3.1.1. Some properties of the solutions and constraints on the initial value:

1. If $x_0 \notin D(A)$, there is no valid strong solution for $(CP)_{0,x_0}$.
2. For all $x_0 \in X$, $t \mapsto T(t)x_0$ is the unique solution of $(CP)_{0,x_0}$.

Example 3.1.1. Consider the transport equation:

$$u_t = u_x \quad x \in [0, 1], t \geq 0$$

subject to boundary conditions at $t = 0$, $u(0, \cdot) = u_0(\cdot) \in X$.

We associate this problem with the operator \mathcal{A} defined by $\mathcal{A}u = \frac{du}{dx}$. To define the weak solution, we need to determine the action of the adjoint operator \mathcal{A}^* .

Let φ be a test function (of $D(A^*)$). We compute the pairing $\langle \mathcal{A}u, \varphi \rangle$ using the inner product in $L^2([0, 1])$:

$$\langle \mathcal{A}u, \varphi \rangle = \int_0^1 (\mathcal{A}u)(x)\varphi(x) dx = \int_0^1 \frac{du}{dx}(x)\varphi(x) dx$$

By integration by parts:

$$\int_0^1 u'(x)\varphi(x) dx = [u(x)\varphi(x)]_0^1 - \int_0^1 u(x)\varphi'(x) dx$$

Using the Dirichlet boundary conditions ($u(t, 1) = u(t, 0) = 0$), the boundary term $[u(x)\varphi(x)]_0^1$ vanishes. Thus, we obtain:

$$\langle \mathcal{A}u, \varphi \rangle = - \int_0^1 u(x)\varphi'(x) dx = \int_0^1 u(x) \left(-\frac{d\varphi}{dx} \right) dx$$

This can be rewritten in terms of the inner product as:

$$\langle \mathcal{A}u, \varphi \rangle = \langle u, -\varphi' \rangle$$

From the definition of the adjoint operator $\langle \mathcal{A}u, \varphi \rangle = \langle u, \mathcal{A}^*\varphi \rangle$, we identify:

$$\mathcal{A}^*\varphi = -\frac{d\varphi}{dx}$$

Therefore, the condition for $u(t)$ to be a weak solution:

$$\frac{d}{dt} \langle \varphi, u(t) \rangle = \langle \mathcal{A}^*\varphi, u(t) \rangle$$

becomes:

$$\frac{d}{dt} \int_0^1 u(t, x)\varphi(x) dx = - \int_0^1 u(t, x)\varphi'(x) dx$$

3.2 Homogeneous Cauchy Problem

Theorem 3.2.1. Let \mathcal{A} be the IG of a C_0 -SG $(T(t))_{t \geq 0}$, then:

1. For all $x_0 \in D(\mathcal{A})$, there exists a unique strong solution of $(CP)_{0,x_0}$ given by $u(t) = T(t)x_0$, which is also a mild and weak solution.
2. For all $x_0 \in X$, $t \mapsto T(t)x_0$ is a weak solution of $(CP)_{0,x_0}$.

Proof. The proof of part 1 is direct.

For part 2: Let $x_0 \in X$ and choose a sequence $x_n \in D(\mathcal{A})$ such that $x_n \rightarrow x_0$ (why is this possible?). For all $x^* \in D(\mathcal{A}^*)$ and $t \geq 0$, the function $t \mapsto \langle x^*, T(t)x_n \rangle$ is C^1 , and we have:

$$\frac{d}{dt} \langle x^*, T(t)x_n \rangle = \langle x^*, \mathcal{A}T(t)x_n \rangle = \langle \mathcal{A}^*x^*, T(t)x_n \rangle$$

Integrating yields:

$$\langle x^*, T(t)x_n \rangle = \langle x^*, x_n \rangle + \int_0^t \langle A^*x^*, T(s)x_n \rangle ds$$

By passing to the limit as $n \rightarrow \infty$ (show how), we extend this formula for all $x_0 \in X$, showing $t \mapsto T(t)x_0$ is a weak solution.

Now to show the uniqueness of the solution, consider two weak solutions $x_1(\cdot)$ and $x_2(\cdot)$ of $(CP)_{0,x_0}$. Set $u = x_1 - x_2$. Then u is a weak solution of the problem with initial condition 0. For all $t \geq 0$ and $x^* \in D(A^*)$, we have:

$$\langle x^*, u(t) \rangle = \int_0^t \langle A^*x^*, u(s) \rangle ds$$

Let $U(t) = \int_0^t u(s) ds$, with $U(0) = 0$. Then, $\langle x^*, u(t) \rangle = \langle A^*x^*, U(t) \rangle$, which implies:

$$\frac{d}{dt} \langle x^*, U(t) \rangle = \langle A^*x^*, U(t) \rangle \tag{***}$$

Claim 3.2.1. If $(T(t))_{t \geq 0}$ is a C_0 -SG with \mathcal{A} as its IG then $T(t)^*D(\mathcal{A}^*) \subset D(\mathcal{A}^*)$ and $T(t)^*$ and \mathcal{A}^* commute, that is:

$$T(t)^*\mathcal{A}^* = \mathcal{A}^*T(t)^*$$

Proof of claim. We have to show that $\forall x^* \in D(\mathcal{A}^*)$, $\forall x \in D(\mathcal{A})$, $\forall t \geq 0$, the following:

$$\exists \xi^* \in X^*, \langle T(t)^*x^*, \mathcal{A}x \rangle = \langle \xi^*, x \rangle$$

We have the following:

$$\begin{aligned}
\langle T(t)^*x^*, \mathcal{A}x \rangle &= \langle x^*, T(t)\mathcal{A}x \rangle \\
&= \langle x^*, \mathcal{A}T(t)x \rangle \\
&= \langle \mathcal{A}^*x^*, T(t)x \rangle \\
&= \langle T(t)^*\mathcal{A}^*x^*, x \rangle
\end{aligned}$$

Then $T(t)^*\mathcal{A}^*x^* \in D(\mathcal{A}^*)$ and $T(t)^*\mathcal{A}^* = \mathcal{A}^*T(t)^*$. \square

Apply (***) replacing x^* with $T^*(t^* - t)x^*$, where $0 \leq t \leq t^*$:

$$\frac{d}{dt} \langle T^*(t^* - t)x^*, U(t) \rangle = \langle -\mathcal{A}^*T^*(t^* - t)x^*, U(t) \rangle + \langle T^*(t^* - t)x^*, u(t) \rangle$$

Since $\langle T^*(t^* - t)x^*, u(t) \rangle = \langle \mathcal{A}^*T^*(t^* - t)x^*, U(t) \rangle$, the derivative is 0.

Integrating from $t = 0$ to $t = t^*$ yields:

$$\langle x^*, U(t^*) \rangle - \langle T^*(t^*)x^*, U(0) \rangle = 0$$

Since $U(0) = 0$, we get $\langle x^*, U(t^*) \rangle = 0$ for all $t^* \geq 0$. Since this holds for all $x^* \in D(\mathcal{A}^*)$ and $D(\mathcal{A}^*)$ is weak*-dense, $U(t) \equiv 0$.

Differentiating gives $u(t) = 0$, proving uniqueness. \square

Theorem 3.2.2. *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closed, densely defined operator. If there exists ω such that for $\lambda > \omega$, $\lambda \in \rho(\mathcal{A})$ and $\|R(\lambda, \mathcal{A})\| = o(e^{\sigma\lambda})$, then $(CP)_{0,x_0}$ admits a unique mild solution.*

We first prove a necessary lemma for uniqueness:

Lemma 3.2.1. *Let $u : [0, T] \rightarrow X$ be continuous with $u(0) = 0$. Assume there exists $M \geq 0$ such that for all $n \geq 0$,*

$$\left\| \int_0^T e^{ns} u(s) ds \right\| \leq M$$

Then $u(t) = 0$ for all $t \in [0, T]$.

Proof. For any $x^* \in X^*$, define the continuous function $\varphi(s) := \langle x^*, u(s) \rangle$. By assumption, $\left\| \int_0^T e^{ns} \varphi(s) ds \right\| \leq M$ for all n . Define the sequence of functions:

$$\psi_n(t) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k!} e^{knt} = 1 - e^{-e^{nt}}$$

We consider the integral:

$$\int_0^T \psi_n(t - T + s) \varphi(s) ds$$

By substituting the sum definition of ψ_n and using the bound on φ , we have:

$$\begin{aligned} \left\| \int_0^T \sum_{k \geq 1} \frac{(-1)^{k+1}}{k!} e^{kn(t-T+s)} \varphi(s) ds \right\| &\leq \sum_{k \geq 1} \frac{1}{k!} e^{kn(t-T)} \left\| \int_0^T e^{kns} \varphi(s) ds \right\| \\ &\leq M \sum_{k \geq 1} \frac{(e^{n(t-T)})^k}{k!} \\ &= M(e^{e^{n(t-T)}} - 1) \end{aligned}$$

For $t < T$, as $n \rightarrow \infty$, $n(t - T) \rightarrow -\infty$, thus $e^{n(t-T)} \rightarrow 0$, meaning the integral evaluates to 0.

On the other hand, analyzing the behavior of $\psi_n(t - T + s)$ directly as $n \rightarrow \infty$:

- If $s > T - t$, then $t - T + s > 0$, so $\psi_n \rightarrow 1$.
- If $s < T - t$, then $t - T + s < 0$, so $\psi_n \rightarrow 0$.

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^T \psi_n(t - T + s) \varphi(s) ds = \int_{T-t}^T \varphi(s) ds$$

Equating the two limits, we deduce $\int_{T-t}^T \varphi(s) ds = 0$ for all t . Differentiating with respect to t gives $\varphi(T - t) = 0$, implying $\varphi \equiv 0$, and thus $u \equiv 0$. \square

Proof of theorem 3.2.2. Without loss of generality we can assume $\omega = 0$ since if it is not then the transform $\mathcal{A} \mapsto \mathcal{A} - zI$, shifts the coefficient by e^{zt} , and:

$$\frac{du_z}{dt} = (\mathcal{A} + zI)u_z$$

$$\frac{du}{dt} = \mathcal{A}u$$

Assume u is a mild solution of $(CP)_{0,0}$, meaning $u(0) = 0$ and $\frac{du}{dt} = \mathcal{A}u$. For $\lambda > 0$, define $v(t) = R(\lambda, \mathcal{A})u(t)$. Since

$$\frac{d}{dt} R(\lambda, \mathcal{A})u(t) = R(\lambda, \mathcal{A})\mathcal{A}u(t) = R(\lambda, \mathcal{A})(\mathcal{A} - \lambda I + \lambda I)u(t) = -u(t) + \lambda R(\lambda, \mathcal{A})u(t)$$

we get $\frac{dv}{dt} - \lambda v(t) = -u(t)$. Solving this ODE yields:

$$R(\lambda, \mathcal{A})u(t) = \int_0^t e^{\lambda(t-s)} u(s) ds$$

Multiplying by $e^{-\sigma\lambda}$, we get:

$$e^{-\sigma\lambda} R(\lambda, \mathcal{A})u(t) = \int_0^t e^{\lambda(t-\sigma-s)} u(s) ds$$

Claim 3.2.2. $\lim_{\lambda \rightarrow +\infty} \int_0^t e^{\lambda(t-\sigma-s)} u(s) ds = 0$.

Proof. Do it. (Hint: use the bond from the assumption and lemma 3.2.1) □

By the claim we get $u \equiv 0$. □

Exercise 14: Homework

Let X be a Hilbert Space and \mathcal{A} be defined by

$$\mathcal{A}y = \sum \lambda_n \langle y, e_n \rangle e_n$$

over an orthonormal basis $\{e_n\}$, with real numbers $\lambda_n \nearrow +\infty$. Its domain is $D(\mathcal{A}) = \{y \in X \mid \sum \lambda_n^2 \langle y, e_n \rangle^2 < +\infty\}$.

1. Prove \mathcal{A} is self-adjoint.
2. Prove it generates a C_0 -SG.

(Hint: $\lambda I - \mathcal{A}$ is formally bounded if and only if $\lim_{n \geq 1} |\lambda - \lambda_n| > 0$)

What can we say without \mathcal{A} being the IG of some C_0 -SG?

Theorem 3.2.3 (Existence of Solution). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO with $\overline{D(\mathcal{A})} = X$ such that $\rho(\mathcal{A}) \supset (\lambda_0, \infty)$, $\lambda_0 \in \mathbb{R}$ and*

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \ln \|R(\lambda, \mathcal{A})\| = 0$$

then $\forall x \in X$ $(CP)_{0,x}$ admits a mild solution.

Note the difference between the assumption here and Hille-Yosida assumption, so one can not use Hille-Yosida directly.

Lemma 3.2.2. *Let $\mathcal{A} : D(\mathcal{A}) \supset X \rightarrow X$ be a UBLO then:*

1. $\forall \lambda \in \rho(\mathcal{A}) D(\mathcal{A}^2) = (\lambda I - \mathcal{A})^{-1} D(\mathcal{A})$.
2. If $\rho(\mathcal{A}) \neq \emptyset$ and $\overline{D(\mathcal{A})} = X$ then $\overline{D(\mathcal{A}^2)} = X$.

Proof. Left as an exercise. □

Theorem 3.2.4 (Uniqueness of Solution). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO with $\overline{D(\mathcal{A})} = X$ and $\rho(\mathcal{A}) \neq \emptyset$. $\forall x \in D(\mathcal{A})$, $(CP)_{0,x}$ admits a mild solution if and only if \mathcal{A} is the IG of some C_0 -SG.*

Proof. Let's start with (\Leftarrow). Assume \mathcal{A} is the IG of some C_0 -SG then by theorem 3.2.1 part 1 we have a unique strong solution which is also mild.

For (\implies). We need to construct a C_0 -SG with \mathcal{A} as its IG. For this we use the flow of the solution. For all $x \in D(\mathcal{A})$ we have:

$$\begin{aligned}\frac{du}{dt} &= \mathcal{A}u \\ u(0) &= x\end{aligned}$$

Since u is the unique mild solution of $(CP)_{0,x}$ (actually it is a strong solution as well, why?). Then the mapping:

$$\begin{aligned}u : \mathbb{R}_+ \times D(\mathcal{A}) &\rightarrow D(\mathcal{A}) \\ (t, x) &\mapsto u(t, x)\end{aligned}$$

is well defined. We can now consider the flow associated with $(CP)_{0,x}$. For each $t \geq 0$, define the operator $T(t) : D(\mathcal{A}) \rightarrow D(\mathcal{A})$ by:

$$T(t)x = u(t, x)$$

We must prove that the family $(T(t))_{t \geq 0}$ forms a C_0 -SG.

1. Identity: $T(0)x = u(0, x) = x$. Thus, $T(0) = Id_{D(\mathcal{A})}$.
2. Linearity: The operator $T(t)$ is linear due to the linearity of the derivative and the operator \mathcal{A} .
3. Semigroup Property: We need to show $T(t+s) = T(t)T(s)$.

By definition, $T(t)T(s)x = u(t, T(s)x)$, which is the solution evaluated at time t for the initial value problem starting at $T(s)x$. Similarly, $T(t+s)x = u(t+s, x)$, which is the solution evaluated at time $t+s$ for the initial value problem starting at x . Due to the time-invariance of the equation $\frac{du}{dt} = \mathcal{A}u$ and the uniqueness of the mild solution, advancing the flow by s and then by t is equivalent to advancing it by $t+s$. Hence, $T(t+s) = T(t)T(s)$.

Now we need to find the bound on $T(t)$ to show it is C_0 -SG. To do this we need to equip the domain $D(\mathcal{A})$ with a suitable Banach space structure (put a complete norm on it).

We put the graph norm on $D(\mathcal{A})$:

$$|x|_G = \|x\| + \|\mathcal{A}x\|$$

With this and the fact that \mathcal{A} is closed (why?), $(D(\mathcal{A}), |\cdot|_G)$ is a Banach space. This will be denoted by $[D(\mathcal{A})]$.

Let $t_0 > 0$, define X_{t_0} as follows:

$$X_{t_0} = C^0([0, t_0], [D(\mathcal{A})])$$

Clearly, $(X_{t_0}, \|\cdot\|_\infty)$ is a Banach space.

Define the operator S_{t_0} by:

$$\begin{aligned} S_{t_0} : [D(A)] &\rightarrow X_{t_0} \\ x &\mapsto (t \mapsto u(t, x) = T(t)x) \end{aligned}$$

S_{t_0} is well-defined and linear because of the uniqueness of solutions of $(CP)_{0,x}$.

Claim 3.2.3. *The operator S_{t_0} is closed.*

Proof. Suppose $x_n \rightarrow x$ in $[D(\mathcal{A})]$ and $S_{t_0}(x_n) \rightarrow v$ in X_{t_0} . We aim to show that $S_{t_0}(x) = v$.

By definition, $S_{t_0}(x_n) = u(\cdot, x_n)$. The convergence in X_{t_0} implies:

$$\|S_{t_0}(x_n) - v\|_\infty = \sup_{t \in [0, t_0]} |u(t, x_n) - v(t)|_G \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Because the map is continuous, and u is the unique solution, we must have $v(t) = u(t, x)$. Therefore, $S_{t_0}(x) = v$. \square

Because S_{t_0} is a closed operator acting between two Banach spaces, the Closed Graph Theorem implies that S_{t_0} is bounded. Thus, there exists a constant $C_{t_0} > 0$ such that for all $x \in [D(\mathcal{A})]$:

$$\|S_{t_0}(x)\|_\infty \leq C_{t_0}|x|_G$$

Consequently, for all $t \in [0, t_0]$, we have $|T(t)x|_G \leq C_{t_0}|x|_G$. This demonstrates that $T(t) \in \mathcal{L}([D(\mathcal{A})])$, meaning $(T(t))_{t \geq 0}$ is a C_0 -SG on $[D(\mathcal{A})]$.

Now, we prove that the semigroup commutes with its generator on appropriate domains.

Claim 3.2.4. *For all $y \in D(\mathcal{A}^2)$, $T(t)\mathcal{A}y = \mathcal{A}T(t)y$.*

Proof. Let $y \in D(\mathcal{A}^2)$. We define the function $w(t)$:

$$w(t) = y + \int_0^t u(s, \mathcal{A}y)ds$$

Since $u(s, \mathcal{A}y)$ is continuous, $w(t)$ is continuously differentiable (C^1) with respect to t , and:

$$w'(t) = u(t, \mathcal{A}y)$$

We can also pull the closed operator \mathcal{A} outside the integral (since $u(s, \mathcal{A}y) \in D(\mathcal{A})$):

$$w'(t) = \mathcal{A}y + \int_0^t \frac{d}{ds}u(s, \mathcal{A}y)ds = \mathcal{A}\left[y + \int_0^t u(s, \mathcal{A}y)ds\right] = Aw(t)$$

At $t = 0$, $w(0) = y$.

This shows that $w(t)$ is a mild solution to $(CP)_{0,y}$. Because solutions are unique, it must coincide with our defined flow:

$$\forall t \geq 0, \quad w(t) = u(t, y) = T(t)y$$

Differentiating both sides with respect to t yields:

$$w'(t) = \frac{d}{dt}[u(t, y)] = \mathcal{A}u(t, y) = \mathcal{A}T(t)y$$

However, from our earlier calculation, we also know $w'(t) = u(t, \mathcal{A}y) = T(t)\mathcal{A}y$. Equating the two expressions for $w'(t)$ gives the desired result:

$$T(t)\mathcal{A}y = \mathcal{A}T(t)y$$

□

Now since $\rho(\mathcal{A}) \neq \emptyset$, then $\exists \lambda_0 \in \rho(\mathcal{A}), \forall y \in D(\mathcal{A}^2)$ set $x = (\lambda_0 I - \mathcal{A})y$ then:

$$T(t)x = T(t)(\lambda_0 I - \mathcal{A})y = (\lambda_0 I - \mathcal{A})T(t)y$$

then

$$\|T(t)x\| \leq |\lambda_0| \|T(t)y\| + \|\mathcal{A}T(t)y\| \leq C_{\lambda_0} \|T(t)y\| \leq C_{\lambda_0} M e^{\omega t} |y|_G$$

Where

$$\begin{aligned} |y|_G &= \|y\| + \|\mathcal{A}y\| = \|y\| + \|(\mathcal{A} - \lambda_0 I)y + \lambda_0 y\| \\ &\leq C_1 \|x\| + \|x\| + |\lambda_0| C_2 \|x\| \\ &\leq C_3 \|x\| \end{aligned}$$

Finally we get:

$$\|T(t)x\| \leq M' e^{\omega t} \|x\|$$

Now by lemma 3.2.2 and the fact that $\rho(\mathcal{A}) \neq \emptyset$ this bound can be extended to all X . And the operator commutes as well.

Finally, we need to show that \mathcal{A} is the IG of $(T(t))_{t \geq 0}$. Let \mathcal{A}_1 be the IG of $(T(t))_{t \geq 0}$ then by definition of $T(t)$ we have $T(t)x = u(t, x) \forall x \in D(\mathcal{A})$ then:

$$\frac{d}{dt} T(t)x = \mathcal{A}T(t)x \quad \text{for } t \geq 0$$

Which means in particular that $\frac{d}{dt} T(t)x |_{t=0} = \mathcal{A}x$ then $\mathcal{A}_1 \supset \mathcal{A}$.

Let $\Re(\lambda) > \omega$ and let $y \in D(\mathcal{A}^2)$. Then:

$$e^{-\lambda t} \mathcal{A}T(t)y = e^{-\lambda t} T(t)\mathcal{A}y = e^{-\lambda t} T(t)\mathcal{A}_1 y$$

By integrating both sides from 0 to ∞ we get:

$$\mathcal{A}R(\lambda, \mathcal{A}_1)y = R(\lambda, \mathcal{A}_1)\mathcal{A}y = \mathcal{A}_1R(\lambda, \mathcal{A}_1)y$$

Since $R(\lambda, \mathcal{A}_1)\mathcal{A}_1y = \mathcal{A}_1R(\lambda, \mathcal{A}_1)y$ then $R(\lambda, \mathcal{A}_1)\mathcal{A}y = R(\lambda, \mathcal{A}_1)\mathcal{A}_1y \quad \forall y \in D(\mathcal{A}^2)$. Since $R(\lambda, \mathcal{A}_1)\mathcal{A}_1$ is uniformly bounded, \mathcal{A} is closed and $\overline{D(\mathcal{A}^2)} = X$ we get $R(\lambda, \mathcal{A}_1)\mathcal{A}y = R(\lambda, \mathcal{A}_1)\mathcal{A}_1y \quad \forall y \in X$. This implies that $D(\mathcal{A}) \supset Rg(R(\lambda, \mathcal{A}_1)) = D(\mathcal{A}_1)$ hence $\mathcal{A}_1 \subset \mathcal{A}$ then $\mathcal{A}_1 = \mathcal{A}$ and the proof is complete. \square