PARTIAL DIFFERENTIAL EQUATIONS

International Mathematics Masters
National Higher School of Mathematics (NHSM)

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CONTENTS

CHAPTER

ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathscr{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. *An operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathscr{A})$ is a subspace of X and* $\sup_{x\neq 0} \frac{\|\mathscr{A}x\|}{\|x\|} = +\infty$

Exercise 1: 1

Let $\mathscr{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathscr{A}) = \{f : \mathbb{R} \to \mathbb{R} : f, f' \in L^2\}$. Show $\mathscr{A}$ is an unbounded linear operator.

Notation: If $\mathscr{A}$ and $\mathscr{B}$ are unbounded linear operators, then $\mathscr{A} \supset \mathscr{B}$ if and only if $D(\mathscr{A}) \supset D(\mathscr{B})$ That is, for all $x \in D(\mathscr{B})$, $\mathscr{A}x = \mathscr{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be a UBLO.*

$$\rho(\mathscr{A}) = \textit{Resolvent of } \mathscr{A} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} (\lambda I - \mathscr{A}) : D(\mathscr{A}) \to X \textit{ is bijective, and} \\ (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X) \end{array} \right\}$$

$$\sigma(\mathscr{A}) = \textit{Spectrum of } \mathscr{A} = \mathbb{C} \setminus \rho(\mathscr{A}).$$

Definition 1.1.3. *$\mathscr{A}$ is closed if and only if the graph of $\mathscr{A}$, denoted as $G(\mathscr{A})$ is closed. Also $\mathscr{A}$ is closable if and only if there exists $\tilde{\mathscr{A}} \supset \mathscr{A}$ such that $G(\tilde{\mathscr{A}}) = \overline{G(\mathscr{A})}$.*

Note that $G(\mathscr{A}) = \big\{(x, \mathscr{A}x) \big| x \in D(\mathscr{A})\big\}$.

Exercise 2: 2

1. Prove that if it exists, $\tilde{\mathscr{A}}$ is unique, it then denoted by $\overline{\mathscr{A}}$ called closure of $\mathscr{A}$.

2. Let $\mathscr{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a,b],\mathbb{R}), \|.\| = \sup|f(x)|)$ and $D(\mathscr{A}_\ell) = C^\ell([a,b],\mathbb{R})$. Prove $\overline{\mathscr{A}_\ell} = \mathscr{A}_1$.

Lemma 1.1.1. *If $\mathscr{A}$ an unbounded linear operator is closable, then $\rho(\overline{\mathscr{A}}) = \rho(\mathscr{A})$. If $\mathscr{A}$ is closed then $\rho(\mathscr{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X)\}$.*

Hints (Exercise): If $\rho(\mathscr{A}) \neq 0$ implies $\mathscr{A}$ is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathscr{L}(X)$ implies T is closed).

Corollary 1.1.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \cup \sigma_c(\mathscr{A})$ where*

1. *$\sigma_p(\mathscr{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) \neq \{0\}\}$ (punctual spectrum and λ's are the eigenvalue).*
2. *$\sigma_c(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) = \{0\}, \overline{Rg(\lambda I - \mathscr{A})} \subset X\right\}$ (continuous spectrum).*
3. *$\sigma_r(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \lambda I - \mathscr{A} \text{ is injective}, \overline{Rg(\lambda I - \mathscr{A})} = X,\ Rg(\lambda I - \mathscr{A}) \neq X\right\}$ (residual spectrum).*

Exercise 3: 1

Let

$$X = \ell^2(\mathbb{C}) = \left\{(x_n)_{n\geq 0} : \sum_n |x_n|^2 < \infty\right\},$$

with $(\mathscr{A}x_n)_{n\geq 1} = \left(\frac{x_n}{1+n}\right)_{n\geq 0}$. Prove that $\mathscr{A}$ is a BLO, injective, $\overline{\mathrm{Rg}(\mathscr{A})} = X$ and $\mathrm{Rg}(\mathscr{A}) \subset X$.

Theorem 1.1.1. *If $\mathscr{A}$ is a closed UBLO then $\rho(\mathscr{A})$ is open. If $\mu \in \rho(\mathscr{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathscr{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathscr{A})$ and*

$$(\lambda I - \mathscr{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathscr{A})^{-(n+1)}$$

To do Question: do you need $\mathscr{A}$ closed?

Theorem 1.1.2 (Resolvent Identity)**.** *Let $\mathscr{A}$ be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator*

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all $\lambda, \mu \in \rho(A)$,

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

Corollary 1.1.2. *The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathscr{A})$ into $\mathscr{L}(X)$ is analytic. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and $\mathscr{A}$ a closed UBLO with $\overline{D(\mathscr{A})} = X$ a dense UBLO.
If X and Y are Banach spaces with duals X^* and Y^*, then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x\rangle$.

Definition 1.1.4 (Dual Operator of $\mathscr{A}$)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to Y$ (UBLO), be such that $\overline{D(\mathscr{A})} = X$. The* dual operator *$\mathscr{A}^* : D(\mathscr{A}^*) \subset Y^* \to X^*$ is a UBLO defined as follows:*

$$D(\mathscr{A}^*) := \{y^* \in Y^* \,|\, \exists z^* \in X^* \text{ such that } \langle y^*, Ax\rangle = \langle z^*, x\rangle \ \forall x \in D(\mathscr{A})\}\,.$$

and $y^* \in D(\mathscr{A}^*)$, the element z^* is unique and we define $A^*y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathscr{A} \in \mathscr{L}(X, Y)$. Then $\mathscr{A}^* \in \mathscr{L}(Y^*, X^*)$ and*

$$\|\mathscr{A}^*\|_{\mathscr{L}(Y^*,X^*)} = \|\mathscr{A}\|_{\mathscr{L}(X,Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closedly dense UBLO. Then $\overline{D(\mathscr{A}^*)} = X^*(\cong X)$, and $\mathscr{A}^*$ is closed.*

Theorem 1.1.3. *Let $\mathscr{A}$ be a closedly dense UBLO. Then $\rho(\mathscr{A}) = \rho(\mathscr{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$\left[(\lambda I - \mathscr{A})^{-1}\right]^* = (\overline{\lambda} I - \mathscr{A}^*)^{-1}.$$

Exercise 4: 2

1. Let $\mathscr{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathscr{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:

   a. $\rho(\mathscr{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathscr{A}) = i\mathbb{R}$.

   b. $\mathscr{A}$ is a closed unbounded linear operator.

   c. If $\lambda \in \rho(\mathscr{A})$ then $(\lambda I - \mathscr{A})^{-1} : X \longrightarrow D(\mathscr{A})$ is bounded.

   For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathscr{A})$ such that $(\lambda I - \mathscr{A})f = g$.
   For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathscr{A})f_n \to 0$.

2. Do same for $\mathscr{A} = -i\frac{d}{dx}$.

1.2 Compact Operators

Let X and Y be Banach spaces on $\mathbb{K}$.

Definition 1.2.1. *Let $K : X \to Y$ be a BLO (in $\mathscr{L}(X, Y)$), then K compact iff $K(B_1^X(0))$ is relatively compact in Y (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathscr{L}(X, Y) \mid K \text{ is compact}\}.$$

Exercise 5: 1

Let $X = C([a,b],\mathbb{C})$ and $k \in C^0([a,b]\times[c,d],\mathbb{C})$
Define $K \in \mathscr{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t,s)x(s)\,ds.$$

Show $K \in \mathcal{K}(X)$.

Theorem 1.2.1. *$\mathcal{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$.*

Proof. We shall show this in two steps

1. $\mathcal{K}(X,Y)$ is a vector space (Do it).

2. Closed: $K_n \to K$ (Prof. Yacine said he would send a different proof).

□

Exercise 6: 1

Let $X = \ell^2(\mathbb{C})$,

$$\mathscr{A}((x_n)_{n\geq 0}) = \left(\frac{x_n}{n+1}\right)_{n\geq 0}.$$

Show that $\mathscr{A}$ is compact.

Theorem 1.2.2. *Let X, Y and Z be Banach spaces on $\mathbb{K}$.*

$$X \xrightarrow{\mathscr{A}} Y \xrightarrow{\mathscr{B}} Z, \quad \mathscr{A} \in \mathscr{L}(X,Y),\ \mathscr{B} \in \mathscr{L}(Y,Z).$$

1. If $\mathscr{A}$ is compact or $\mathscr{B}$ is compact, then $\mathscr{B}\mathscr{A}$ is compact.

2. If $\mathscr{A}$ is compact then $\mathscr{A}^* \in \mathcal{K}(Y^*, X^*)$.

3. If $\mathscr{A}$ is compact and $\mathrm{Rg}(\mathscr{A})$ is closed (in Y), then it is finite dimensional.

To proceed with further results on compact operators, we need the following lemma

Lemma 1.2.1 (Riesz Lemma)**.** *Let E be a normed vector space, $F = \overline{F} \subset E$. Then $\forall r \in (0,1)$, $\exists\, x_r \in E$, such that*

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

Proof. Since $F \neq E$ then this implies $\exists\, z \in E \setminus F$. Let $d = d(z,F) > 0$.

For $0 < r < 1$, $\exists\, y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|}\|z - (y_r + \|z - y_r\| y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

$\square$

Proposition 1.2.1. *Let $\mathscr{A} \in \mathcal{K}(X)$, such that X is a Banach space on $\mathbb{C}$. If $\lambda \in \mathbb{C}^*$, then $\ker((\lambda I - \mathscr{A})^n)$ has finite dimension.*

Proof. Only for $n = 1$. (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathscr{A}) = \{x \in X : \mathscr{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda}\mathscr{A}x\right\} \subset \mathrm{Rg}(\mathscr{A}).$$

So $\tilde{K}$ is closed in $\mathrm{Rg}(\mathscr{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists\, (x_n)$ in $\tilde{K}$, such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|}\|\mathscr{A}x_n - \mathscr{A}x_m\| \geq \frac{1}{2}, \qquad \forall\, n \neq m$$

and so we have $\|\mathscr{A}x_n\| \leq \|\mathscr{A}\|$. So $(\mathscr{A}x_n)$ is not Cauchy, hence a contradiction.

$\square$

Exercise 7: 1

Let X be a Banach space on $\mathbb{K}$. If $\mathscr{A} \in \mathscr{L}(X)$, assume $\exists\, n_0$ s.t. $\ker(\mathscr{A}^{n_0}) = \ker(\mathscr{A}^{n_0+1})$. Then $\forall n \geq n_0$,

$$\ker(\mathscr{A}^n) = \ker(\mathscr{A}^{n_0}).$$

Proposition 1.2.2. *Let $\mathscr{A} \in \mathcal{K}(X)$ and X be a Banach space on $\mathbb{C}$, $\lambda \neq 0$. Then $\exists\, n_0$ such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathscr{A})^n) = \ker((\lambda I - \mathscr{A})^{n_0}).$$

Proof. Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathscr{A})^n) \subset \ker((\lambda I - \mathscr{A})^{n+1})$ and each of them is closed.
RL: with $r = \frac{1}{2}$ with $(x_n)_{n\geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathscr{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathscr{A})^n))) \geq \frac{1}{2}.$$

For $n = 1$, $x \in \ker(\lambda I - \mathscr{A}) \implies x = \frac{\mathscr{A}}{\lambda}x$. For all $1 \leq m < n$,

$$\begin{aligned}\frac{\mathscr{A}x_n}{\lambda} - \frac{\mathscr{A}x_m}{\lambda} &= x_n - x_n + \frac{\mathscr{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathscr{A}x_m}{\lambda}\right) \\ &= x_n - \left[\frac{(\lambda I - \mathscr{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathscr{A})x_m}{\lambda}\right].\end{aligned}$$

So,

$$\left\| \frac{\mathscr{A} x_n}{\lambda} - \frac{\mathscr{A} x_m}{\lambda} \right\| \geq d\big(x_n, \ker(\lambda I - \mathscr{A})^n\big) \geq \frac{1}{2}.$$

which is a contradiction. □

Notice that if $\ker(\lambda I - \mathscr{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathscr{A})$. Notice,

$$\dim \ker(\lambda I - \mathscr{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2 $\Rightarrow \exists\, n_0$ (smallest one) such that

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) = \ker\big((\lambda I - \mathscr{A})^{n}\big), \qquad \forall n \geq n_0.$$

Note that,

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{generalized eigenspace}.$$

$$\dim \ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{algebraic multiplicity of } \lambda.$$

Proposition 1.2.3 (Fredholm aternative)**.** *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$.*

$$\operatorname{Rg}(\lambda I - \mathscr{A}) = X \iff \ker(\lambda I - \mathscr{A}) = \{0\}.$$

Proposition 1.2.4. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$, $\dim X = \infty$. If $\lambda_n \to \lambda$, $\lambda_n \in \sigma(\mathscr{A}) \setminus \{0\}$, pairwise distinct,then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathscr{A}) \setminus \{0\}$ is isolated.*

Proof. Let $\lambda_n \in \sigma_p(\mathscr{A})$, $\exists\, \|x_n\| = 1$ such that $\mathscr{A} x_n = \lambda_n x_n$. Let

$$X_n = \operatorname{span}(x_1, \ldots, x_n), \qquad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \;\Rightarrow\; \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i, \text{ which implies } \quad \lambda_{n+1} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathscr{A} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since (x_i) are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \qquad 1 \le i \le n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathscr{A})X_n \subset X_{n-1}, \qquad \forall n \ge 2.$$

Recall:

$$\|y_n\| = 1, \qquad y_n \in X_n,$$
$$d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

For $2 \le m < n$,

$$\left\| \frac{\mathscr{A} y_n}{\lambda_n} - \frac{\mathscr{A} y_m}{\lambda_m} \right\| = \left\| y_n - \left[\frac{\lambda_n I - \mathscr{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathscr{A}}{\lambda_m} y_m \right] \right\|$$
$$\ge d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

Assume that

$$\lambda_n \to \lambda \quad (n \to \infty).$$

Suppose $\lambda \neq 0$, then

$$\left| \frac{1}{\lambda_n} \right| \le C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathscr{A} y_n}{\lambda_n} \right)_{n \ge 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathscr{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. □

Theorem 1.2.3. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$. Then $\sigma(\mathscr{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.*

Main use of compact operators (in PDEs)

They appear as "inverse" of UBLO.

Definition 1.2.2. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ UBLO, closed, $\rho(\mathscr{A}) \neq \{0\}$. $\mathscr{A}$ is said to have compact resolvent if*

$$(\lambda I - \mathscr{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathscr{A}).$$

Main Example: $\mathscr{A} = -\Delta$ on Ω with $\mathscr{A} u = -u_{xx}$.

1.3 Adjoints, Symmetric and Self-adjoint Operators

Let $\mathscr{H}$ be a Hilbert space, with inner product

$$\langle\cdot,\cdot\rangle_{\mathscr{H}} : \mathscr{H} \times \mathscr{H} \to \mathbb{C}.$$

$\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ UBLO, $\overline{D(\mathscr{A})} = \mathscr{H}$.

Definition 1.3.1 (Adjoint Operator $\mathscr{A}^\circ$)**.**

$$D(\mathscr{A}^\circ) = \{x \in \mathscr{H} : v \mapsto \langle \mathscr{A} v, x\rangle_{\mathscr{H}} : D(\mathscr{A}) \to \mathbb{C} \text{ bdd operator}\}.$$

If $x \in D(\mathscr{A}^\circ)$, then there exists uniquely $z \in \mathscr{H}$ such that $\langle v, z\rangle_{\mathscr{H}} = \langle \mathscr{A} v, x\rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.
Observe, with Riesz representation and the fact that $\overline{D(\mathscr{A})} = \mathscr{H}$, we have that $z := \mathscr{A}^\circ x$ and $\langle v, \mathscr{A}^\circ\rangle_{\mathscr{H}} = \langle \mathscr{A} v, x\rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.

Remark 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, $E : \mathscr{H} \to \mathscr{H}^*$, $x \mapsto \langle x, \cdot\rangle$. Linear isometry between $\mathscr{H}$ and $\mathscr{H}^*$. (One can identify $\mathscr{H}$ and $\mathscr{H}^*$). Now, we define the Dual operator as the following:*

$$\mathscr{A}^* : D(\mathscr{A}^*) \subset \mathscr{H}^* \to \mathscr{H}, \quad \mathscr{A}^\circ = E^{-1}\mathscr{A}^* E.$$

Definition 1.3.2 (Symmetric and Self-adjoint Operator)**.** *Let $(\mathscr{H}, \langle,\rangle)$ be a Hilbert space and $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ be a UBLO, then*

1. ***Symmetric:*** *If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ \supset \mathscr{A}$ with $D(\mathscr{A}^*) \supset D(\mathscr{A})$ and for all $x, y \in D(\mathscr{A})$, $\langle \mathscr{A} x, y\rangle = \langle x, \mathscr{A}\rangle$.*
2. ***Self-adjoint:*** *If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ = \mathscr{A}$.*

Exercise 8: 2

1. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. If $\mathscr{A}$ is closed, then $\overline{D(\mathscr{A}^\circ)} = \mathscr{H}$.
2. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. Suppose $\mathscr{A}$ is symmetric and if $0 \in \rho(\mathscr{A})$, then prove that $\lambda \in \mathbb{R}$ and

$$\inf_{\|x\|=1, x\in D(\mathscr{A})} \langle \mathscr{A} x, x\rangle \le \lambda \le \sup_{\|x\|=1, x\in D(\mathscr{A})} \langle \mathscr{A} x, x\rangle.$$

Proposition 1.3.1. *Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be an Hilbert space over $\mathbb{C}$, If $\mathscr{A}$ is self-adjoint, injective and $\overline{D(\mathscr{A})} = \mathscr{H}$. Then $\mathscr{A}^{-1} : \mathrm{Rg}(\mathscr{A}) \to \mathscr{H}$ is self-adjoint.*

Proof. Since $\mathscr{A}$ is injective then $\mathscr{A}^{-1}$ is well defined. Now, for all $z \in D(\mathscr{A})$, $\langle \mathscr{A} x_n, z\rangle = \langle x_n, \mathscr{A}^\circ z\rangle$ which implies $\langle y, z\rangle = \langle x, \mathscr{A}^\circ z\rangle$ and so we have $x \in D(\mathscr{A}^\circ) = D(\mathscr{A})$ and $y = \mathscr{A}^\circ$. Notice $\overline{\mathrm{Rg}(\mathscr{A})} = \ker(\mathscr{A})^\perp$ (because of self-adjointness).

Injectivity implies $\overline{\mathrm{Rg}(\mathscr{A})} = \mathscr{H}$ which implies $\overline{D(\mathscr{A}^{-1})} = \mathscr{H}$. So $\mathscr{A}^{-1}$ is densely defined.
Now, observe for all $u, v \in D(\mathscr{A}^{-1})$, $u = \mathscr{A}^{\circ}x$ and $v = \mathscr{A}y$ with $x, y \in D(\mathscr{A})$. Hence,

$$\langle \mathscr{A}^{-1}u, v\rangle = \langle x, \mathscr{A}y\rangle = \langle \mathscr{A}x, y\rangle = \langle x, \mathscr{A}^{-1}y\rangle.$$

To this end, $(\mathscr{A}^{-1})^{\circ} \subset \mathscr{A}^{-1}$, $\forall\, z \in D\big((\mathscr{A}^{-1})^{\circ}\big)\ \exists\, w, \forall\, u \in D(\mathscr{A}^{-1}) = R(\mathscr{A})\ (i.e.\ u = \mathscr{A}x)$

$$\langle \mathscr{A}^{-1}u, z\rangle = \langle u, w\rangle \Rightarrow \forall x \in D(\mathscr{A}) \quad \langle x, z\rangle = \langle \mathscr{A}x, w\rangle$$

By definition $w \in D(\mathscr{A}^{\circ})$ and $\mathscr{A}^{\circ}w = z$. $\mathscr{A}w = z \Rightarrow z \in Rg(\mathscr{A}) = D(\mathscr{A}^{-1})$. $\square$

Theorem 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, suppose $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is symmetric and surjective then $\mathscr{A}$ is self-adjoint.*

Proof. $\mathscr{A}$ and $\mathscr{A}^{\circ}$ are injective. Do it only for $\mathscr{A}$, let $x \in D(\mathscr{A})$ and $\mathscr{A}x = 0$.

$$\forall y \in D(\mathscr{A}), \quad 0 = \langle \mathscr{A}x, y\rangle = \langle x, \mathscr{A}y\rangle \Rightarrow x \perp Rg(\mathscr{A}) = \mathscr{H}.$$

which implies $x = 0$.
Next, we show $\mathscr{A}$ closed.

$$(x_n)_{n\geq 1} \subset D(\mathscr{A}), \quad x_n \to x \text{ in } \mathscr{H}, \quad \mathscr{A}x_n \to y \text{ in } \mathscr{H}$$

We shall show $y = \mathscr{A}x$. Now, $\forall z \in D(\mathscr{A})$ then $\langle \mathscr{A}x_n, z\rangle = \langle x_n, \mathscr{A}z\rangle$, which implies $\langle y, z\rangle = \langle x, \mathscr{A}z\rangle$ which implies $x \in D(\mathscr{A}^{\circ})$ and $y = \mathscr{A}^{\circ}x$. Since $\mathscr{A}$ surjective $\Rightarrow \exists w \in D(\mathscr{A})$ s.t. $\mathscr{A}w = y$ and $\mathscr{A}^{\circ}x = y$.
Since $\mathscr{A}$ is symmetric: $\mathscr{A}^{\circ}w = \mathscr{A}w$. Then $\mathscr{A}^{\circ}w = \mathscr{A}^{\circ}x$, $\mathscr{A}$ is injective $\Rightarrow w = x$. Hence $\mathscr{A}x = \mathscr{A}w = y \Rightarrow y = \mathscr{A}x \Rightarrow \mathscr{A}$ is closed.
By closed graph theorem both $\mathscr{A}$ and $\mathscr{A}^{-1} \in \mathscr{L}(X)$. We can conclude that $\mathscr{A}$ is a self-adjoint operator. $\square$

Exercise 9: 2

Let $\mathscr{H} = L^2(0, \pi)$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ and

$$\mathscr{A}f = -f''$$

If $D(\mathscr{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?
Similarly, if $D(\mathscr{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?

Theorem 1.3.2 (Fredrich's Extension)**.** *Let $\mathscr{H}$ be a Hilbert space on $\mathbb{C}$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, symmetric then $\mathscr{A}$ admits a unique self adjoint extension. If either*

a. $\inf_{\|x\|=1, x\in D(\mathscr{A})} \langle \mathscr{A}x, x\rangle > -\infty$

b. $\sup_{\|x\|=1, x\in D(\mathscr{A})} \langle \mathscr{A}x, x\rangle < +\infty$

such that, $\mathscr{A} \subset \mathscr{A}^{\circ} \subset (\mathscr{A}^{\circ})^{\circ} \subset \cdots$. If (a) or (b) holds, then; $\mathscr{A} \subset \mathscr{A}^{\circ} = (\mathscr{A}^{\circ})^{\circ}$.