

# PARTIAL DIFFERENTIAL EQUATIONS

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CHAPTER  
**ONE**

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# THEORY OF (UNBOUNDED) OPERATORS

## 1.1 Preliminaries on Operators

Let  $(X, \|\cdot\|)$  be a Banach space and  $\mathcal{L}(X)$  be the Banach space of bounded linear operators.

**Definition 1.1.1.** An operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow X$  is called an unbounded linear operator (UBLO) if  $D(\mathcal{A})$  is a subspace of  $X$  and  $\sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|} = +\infty$

### Exercise 1: 1

Let  $\mathcal{A} : H^1 \rightarrow L^2$ , such that  $f \mapsto f'$  and  $D(\mathcal{A}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f, f' \in L^2\}$ . Show  $\mathcal{A}$  is an unbounded linear operator.

**Notation:** If  $\mathcal{A}$  and  $\mathcal{B}$  are unbounded linear operators, then  $\mathcal{A} \supset \mathcal{B}$  if and only if  $D(\mathcal{A}) \supset D(\mathcal{B})$ . That is, for all  $x \in D(\mathcal{B})$ ,  $\mathcal{A}x = \mathcal{B}x$ .

### 1.1.1 Resolvent Operator

**Definition 1.1.2.** Let  $\mathcal{A} : D(\mathcal{A}) \rightarrow X$  be a UBLO.

$$\rho(\mathcal{A}) = \text{Resolvent of } \mathcal{A} = \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} (\lambda I - \mathcal{A}) : D(\mathcal{A}) \rightarrow X \text{ is bijective, and} \\ (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X) \end{array} \right\}$$

$$\sigma(\mathcal{A}) = \text{Spectrum of } \mathcal{A} = \mathbb{C} \setminus \rho(\mathcal{A}).$$

**Definition 1.1.3.**  $\mathcal{A}$  is closed if and only if the graph of  $\mathcal{A}$ , denoted as  $G(\mathcal{A})$  is closed. Also  $\mathcal{A}$  is closable if and only if there exists  $\tilde{\mathcal{A}} \supset \mathcal{A}$  such that  $G(\tilde{\mathcal{A}}) = \overline{G(\mathcal{A})}$ .

Note that  $G(\mathcal{A}) = \{(x, \mathcal{A}x) \mid x \in D(\mathcal{A})\}$ .

### Exercise 2: 2

1. Prove that if it exists,  $\tilde{\mathcal{A}}$  is unique, it then denoted by  $\overline{\mathcal{A}}$  called closure of  $\mathcal{A}$ .
2. Let  $\mathcal{A}_\ell = \frac{d}{dx}$  with  $(X = C^0([a, b], \mathbb{R}), \|\cdot\| = \sup |f(x)|)$  and  $D(\mathcal{A}_\ell) = C^\ell([a, b], \mathbb{R})$ .  
Prove  $\overline{\mathcal{A}_\ell} = \mathcal{A}_1$ .

**Lemma 1.1.1.** If  $\mathcal{A}$  an unbounded linear operator is closable, then  $\rho(\overline{\mathcal{A}}) = \rho(\mathcal{A})$ . If  $\mathcal{A}$  is closed then  $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$ .

*Hints (Exercise): If  $\rho(\mathcal{A}) \neq 0$  implies  $\mathcal{A}$  is closed. (Show first that if  $T$  is a UBLO with  $T^{-1} \in \mathcal{L}(X)$  implies  $T$  is closed).*

**Corollary 1.1.1.** Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a closed UBLO then  $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$  where

1.  $\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) \neq \{0\}\}$  (punctual spectrum and  $\lambda$ 's are the eigenvalue).
2.  $\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) = \{0\}, \overline{Rg(\lambda I - \mathcal{A})} \subset X \right\}$  (continuous spectrum).
3.  $\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is injective, } \overline{Rg(\lambda I - \mathcal{A})} = X, Rg(\lambda I - \mathcal{A}) \neq X \right\}$  (residual spectrum).

### Exercise 3: 1

Let

$$X = \ell^2(\mathbb{C}) = \left\{ (x_n)_{n \geq 0} : \sum_n |x_n|^2 < \infty \right\},$$

with  $(\mathcal{A}x_n)_{n \geq 1} = \left( \frac{x_n}{1+n} \right)_{n \geq 0}$ . Prove that  $\mathcal{A}$  is a BLO, injective,  $\overline{Rg(\mathcal{A})} = X$  and  $Rg(\mathcal{A}) \subset X$ .

**Theorem 1.1.1.** If  $\mathcal{A}$  is a closed UBLO then  $\rho(\mathcal{A})$  is open. If  $\mu \in \rho(\mathcal{A})$ , then for all  $\lambda \in \mathbb{C}$  with  $r := |\mu - \lambda|, \|(\mu I - \mathcal{A})^{-1}\| < 1$  then  $\lambda \in \rho(\mathcal{A})$  and

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathcal{A})^{-(n+1)}$$

*To do Question: do you need  $\mathcal{A}$  closed?*

**Theorem 1.1.2** (Resolvent Identity). Let  $\mathcal{A}$  be a UBLO. For  $\lambda \in \rho(\mathcal{A})$ , define the resolvent operator

$$R(\lambda) := (\lambda I - \mathcal{A})^{-1}.$$

Then for all  $\lambda, \mu \in \rho(\mathcal{A})$ ,

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

**Corollary 1.1.2.** The mapping  $\lambda \mapsto R(\lambda)$  from  $\rho(\mathcal{A})$  into  $\mathcal{L}(X)$  is analytic. Moreover,

$$\frac{d^n}{d\lambda^n} (\lambda I - \mathcal{A})^{-1} = (-1)^n n! [(\lambda I - \mathcal{A})^{-1}]^{(n+1)}.$$

## 1.1.2 Dual Operators

Let  $X \cong X^*$  and  $\mathcal{A}$  a closed UBLO with  $\overline{D(\mathcal{A})} = X$  a dense UBLO.

If  $X$  and  $Y$  are Banach spaces with duals  $X^*$  and  $Y^*$ , then for  $x \in X$  and  $x^* \in X^*$ , we define the duality product as  $\langle x^*, x \rangle$ .

**Definition 1.1.4** (Dual Operator of  $\mathcal{A}$ ). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow Y$  (UBLO), be such that  $\overline{D(\mathcal{A})} = X$ . The dual operator  $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y^* \rightarrow X^*$  is a UBLO defined as follows:*

$$D(\mathcal{A}^*) := \{y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x \rangle \forall x \in D(\mathcal{A})\}.$$

and  $y^* \in D(\mathcal{A}^*)$ , the element  $z^*$  is unique and we define  $A^*y^* := z^*$ .

**Lemma 1.1.2.** *Let  $X, Y$  be Banach spaces and let  $\mathcal{A} \in \mathcal{L}(X, Y)$ . Then  $\mathcal{A}^* \in \mathcal{L}(Y^*, X^*)$  and*

$$\|\mathcal{A}^*\|_{\mathcal{L}(Y^*, X^*)} = \|\mathcal{A}\|_{\mathcal{L}(X, Y)}.$$

**Lemma 1.1.3.** *Let  $X$  be a reflexive Banach space with  $X = X^*$  and let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a closedly dense UBLO. Then  $\overline{D(\mathcal{A}^*)} = X^*(\cong X)$ , and  $\mathcal{A}^*$  is closed.*

**Theorem 1.1.3.** *Let  $\mathcal{A}$  be a closedly dense UBLO. Then  $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$  and for all  $\lambda \in \rho(A)$ ,*

$$[(\lambda I - \mathcal{A})^{-1}]^* = (\bar{\lambda} I - \mathcal{A}^*)^{-1}.$$

### Exercise 4: 2

1. Let  $\mathcal{A} = \frac{d}{dx}$  on  $X = L^2(\mathbb{R})$  and  $D(\mathcal{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$ . Show the following:
  - a.  $\rho(\mathcal{A}) = \mathbb{C} \setminus i\mathbb{R}$  which implies  $\sigma(\mathcal{A}) = i\mathbb{R}$ .
  - b.  $\mathcal{A}$  is a closed unbounded linear operator.
  - c. If  $\lambda \in \rho(\mathcal{A})$  then  $(\lambda I - \mathcal{A})^{-1} : X \rightarrow D(\mathcal{A})$  is bounded.

For  $\Re(\lambda) \neq 0$ ; show for all  $g \in X$ , there exists uniquely  $f \in D(\mathcal{A})$  such that  $(\lambda I - \mathcal{A})f = g$ .

For  $\Re(\lambda) = 0$ ; show for all  $f_n \in X$  with  $\|f_n\|_{\ell^2} = 1$  then  $(i\omega I - \mathcal{A})f_n \rightarrow 0$ .

2. Do same for  $\mathcal{A} = -i\frac{d}{dx}$ .

## 1.2 Compact Operators

Let  $X$  and  $Y$  be Banach spaces on  $\mathbb{K}$ .

**Definition 1.2.1.** *Let  $K : X \rightarrow Y$  be a BLO (in  $\mathcal{L}(X, Y)$ ), then  $K$  compact iff  $K(B_1^X(0))$  is relatively compact in  $Y$  (i.e.  $\overline{K(B_1^X(0))}$  compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) \mid K \text{ is compact}\}.$$

### Exercise 5: 1

Let  $X = C([a, b], \mathbb{C})$  and  $k \in C^0([a, b] \times [c, d], \mathbb{C})$

Define  $K \in \mathcal{L}(X)$  by

$$(Kx)(t) = \int_a^b k(t, s)x(s) ds.$$

Show  $K \in \mathcal{K}(X)$ .

**Theorem 1.2.1.**  $\mathcal{K}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ .

*Proof.* We shall show this in two steps

1.  $\mathcal{K}(X, Y)$  is a vector space (Do it).
2. Closed:  $K_n \rightarrow K$  ([Prof. Yacine said he would send a different proof](#)).

□

### Exercise 6: 1

Let  $X = \ell^2(\mathbb{C})$ ,

$$\mathcal{A}((x_n)_{n \geq 0}) = \left( \frac{x_n}{n+1} \right)_{n \geq 0}.$$

Show that  $\mathcal{A}$  is compact.

**Theorem 1.2.2.** Let  $X, Y$  and  $Z$  be Banach spaces on  $\mathbb{K}$ .

$$X \xrightarrow{\mathcal{A}} Y \xrightarrow{\mathcal{B}} Z, \quad \mathcal{A} \in \mathcal{L}(X, Y), \mathcal{B} \in \mathcal{L}(Y, Z).$$

1. If  $\mathcal{A}$  is compact or  $\mathcal{B}$  is compact, then  $\mathcal{B}\mathcal{A}$  is compact.
2. If  $\mathcal{A}$  is compact then  $\mathcal{A}^* \in \mathcal{K}(Y^*, X^*)$ .
3. If  $\mathcal{A}$  is compact and  $\text{Rg}(\mathcal{A})$  is closed (in  $Y$ ), then it is finite dimensional.

To proceed with further results on compact operators, we need the following lemma

**Lemma 1.2.1** (Riesz Lemma). Let  $E$  be a normed vector space,  $F = \overline{F} \subset E$ . Then  $\forall r \in (0, 1)$ ,  $\exists x_r \in E$ , such that

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

*Proof.* Since  $F \neq E$  then this implies  $\exists z \in E \setminus F$ . Let  $d = d(z, F) > 0$ .

For  $0 < r < 1$ ,  $\exists y_r \in F$  s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all  $y \in F$ ,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|} \|z - (y_r + \|z - y_r\| y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

□

**Proposition 1.2.1.** Let  $\mathcal{A} \in \mathcal{K}(X)$ , such that  $X$  is a Banach space on  $\mathbb{C}$ . If  $\lambda \in \mathbb{C}^*$ , then  $\ker((\lambda I - \mathcal{A})^n)$  has finite dimension.

*Proof.* Only for  $n = 1$ . (do it for  $n \geq 2$ ). Now, let

$$\tilde{K} := \ker(\lambda I - \mathcal{A}) = \{x \in X : \mathcal{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda} \mathcal{A}x\right\} \subset \text{Rg}(\mathcal{A}).$$

So  $\tilde{K}$  is closed in  $\text{Rg}(\mathcal{A})$ . Suppose  $\dim \tilde{K} = +\infty$ . By Riesz lemma,  $\exists (x_n)$  in  $\tilde{K}$ , such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|} \|\mathcal{A}x_n - \mathcal{A}x_m\| \geq \frac{1}{2}, \quad \forall n \neq m$$

and so we have  $\|\mathcal{A}x_n\| \leq \|\mathcal{A}\|$ . So  $(\mathcal{A}x_n)$  is not Cauchy, hence a contradiction.

□

### Exercise 7: 1

Let  $X$  be a Banach space on  $\mathbb{K}$ . If  $\mathcal{A} \in \mathcal{L}(X)$ , assume  $\exists n_0$  s.t.  $\ker(\mathcal{A}^{n_0}) = \ker(\mathcal{A}^{n_0+1})$ .

Then  $\forall n \geq n_0$ ,

$$\ker(\mathcal{A}^n) = \ker(\mathcal{A}^{n_0}).$$

**Proposition 1.2.2.** Let  $\mathcal{A} \in \mathcal{K}(X)$  and  $X$  be a Banach space on  $\mathbb{C}$ ,  $\lambda \neq 0$ . Then  $\exists n_0$  such that

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathcal{A})^n) = \ker((\lambda I - \mathcal{A})^{n_0}).$$

*Proof.* Using the previous exercise and arguing by contradiction, that for all  $n \geq 1$   $\ker((\lambda I - \mathcal{A})^n) \subset \ker((\lambda I - \mathcal{A})^{n+1})$  and each of them is closed.

**RL:** with  $r = \frac{1}{2}$  with  $(x_n)_{n \geq 1} \in X$ , such that  $\|x_n\| = 1$ . Then,  $x_n \in \ker((\lambda I - \mathcal{A})^{n+1})$ . Thus,

$$d(x_n, \ker((\lambda I - \mathcal{A})^n)) \geq \frac{1}{2}.$$

For  $n = 1, x \in \ker(\lambda I - \mathcal{A}) \Rightarrow x = \frac{\mathcal{A}}{\lambda} x$ . For all  $1 \leq m < n$ ,

$$\begin{aligned} \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} &= x_n - x_m + \frac{\mathcal{A}x_n}{\lambda} - \left( x_m - x_m - \frac{\mathcal{A}x_m}{\lambda} \right) \\ &= x_n - \left[ \frac{(\lambda I - \mathcal{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathcal{A})x_m}{\lambda} \right]. \end{aligned}$$

So,

$$\left\| \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} \right\| \geq d(x_n, \ker(\lambda I - \mathcal{A})^n) \geq \frac{1}{2}.$$

which is a contradiction.  $\square$

Notice that if  $\ker(\lambda I - \mathcal{A}) \neq \{0\}$ , then  $\lambda \in \sigma_p(\mathcal{A})$ . Notice,

$$\dim \ker(\lambda I - \mathcal{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2  $\Rightarrow \exists n_0$  (smallest one) such that

$$\ker((\lambda I - \mathcal{A})^{n_0}) = \ker((\lambda I - \mathcal{A})^n), \quad \forall n \geq n_0.$$

Note that,

$$\ker((\lambda I - \mathcal{A})^{n_0}) := \text{generalized eigenspace}.$$

$$\dim \ker((\lambda I - \mathcal{A})^{n_0}) := \text{algebraic multiplicity of } \lambda.$$

**Proposition 1.2.3** (Fredholm alternative). *Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ .*

$$\text{Rg}(\lambda I - \mathcal{A}) = X \iff \ker(\lambda I - \mathcal{A}) = \{0\}.$$

**Proposition 1.2.4.** *Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ ,  $\dim X = \infty$ . If  $\lambda_n \rightarrow \lambda$ ,  $\lambda_n \in \sigma(\mathcal{A}) \setminus \{0\}$ , pairwise distinct, then  $\lambda = 0$ . Hence every  $\lambda \in \sigma(\mathcal{A}) \setminus \{0\}$  is isolated.*

*Proof.* Let  $\lambda_n \in \sigma_p(\mathcal{A})$ ,  $\exists \|x_n\| = 1$  such that  $\mathcal{A}x_n = \lambda_n x_n$ . Let

$$X_n = \text{span}(x_1, \dots, x_n), \quad X_n \subset X_{n+1}.$$

Let us prove that  $\dim X_n = n$ .

By induction:  $n = 1$  is OK.

$$\dim X_n = n \Rightarrow \dim X_{n+1} = n + 1.$$

By contradiction,  $x_{n+1} \in X_n$ .

$$x_{n+1} = \sum_{i=1}^n \alpha_i x_i, \text{ which implies } \lambda_{n+1} x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathcal{A}x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^n \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since  $(x_i)$  are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \quad 1 \leq i \leq n.$$

which implies  $\Rightarrow \alpha_i = 0, \Rightarrow x_{n+1} = 0$ , (Impossible).

Notice:

$$(\lambda_n I - \mathcal{A})X_n \subset X_{n-1}, \quad \forall n \geq 2.$$

**Recall:**

$$\|y_n\| = 1, \quad y_n \in X_n,$$

$$d(y_n, X_{n-1}) \geq \frac{1}{2}.$$

For  $2 \leq m < n$ ,

$$\begin{aligned} \left\| \frac{\mathcal{A}y_n}{\lambda_n} - \frac{\mathcal{A}y_m}{\lambda_m} \right\| &= \left\| y_n - \left[ \frac{\lambda_n I - \mathcal{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathcal{A}}{\lambda_m} y_m \right] \right\| \\ &\geq d(y_n, X_{n-1}) \geq \frac{1}{2}. \end{aligned}$$

Assume that

$$\lambda_n \rightarrow \lambda \quad (n \rightarrow \infty).$$

Suppose  $\lambda \neq 0$ , then

$$\left| \frac{1}{\lambda_n} \right| \leq C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left( \frac{\mathcal{A}y_n}{\lambda_n} \right)_{n \geq 1}$$

is a bounded sequence.

Then we have built a sequence in  $\mathcal{A}(B_M^X(0))$ ,  $M > 0$  which does not admit a convergent subsequence. Which is a Contradiction.  $\square$

**Theorem 1.2.3.** Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ . Then  $\sigma(\mathcal{A}) \setminus \{0\}$  is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in  $\{0\}$ .

## Main use of compact operators (in PDEs)

They appear as “inverse” of UBLO.

**Definition 1.2.2.** Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  UBLO, closed,  $\rho(\mathcal{A}) \neq \{0\}$ .  $\mathcal{A}$  is said to have compact resolvent if

$$(\lambda I - \mathcal{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathcal{A}).$$

**Main Example:**  $\mathcal{A} = -\Delta$  on  $\Omega$  with  $\mathcal{A}u = -u_{xx}$ .

## 1.3 Adjoints, Symmetric and Self-adjoint Operators

Let  $\mathcal{H}$  be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}.$$

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ UBLO}, \overline{D(\mathcal{A})} = \mathcal{H}.$$

**Definition 1.3.1** (Adjoint Operator  $\mathcal{A}^\circ$ ).

$$D(\mathcal{A}^\circ) = \{x \in \mathcal{H} : v \mapsto \langle \mathcal{A}v, x \rangle_{\mathcal{H}} : D(\mathcal{A}) \rightarrow \mathbb{C} \text{ bdd operator}\}.$$

If  $x \in D(\mathcal{A}^\circ)$ , then there exists uniquely  $z \in \mathcal{H}$  such that  $\langle v, z \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$  for all  $v \in D(\mathcal{A})$ .

Observe, with Riesz representation and the fact that  $\overline{D(\mathcal{A})} = \mathcal{H}$ , we have that  $z := \mathcal{A}^\circ x$  and  $\langle v, \mathcal{A}^\circ x \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$  for all  $v \in D(\mathcal{A})$ .

**Remark 1.3.1.** Let  $\mathcal{H}$  be a Hilbert space,  $E : \mathcal{H} \rightarrow \mathcal{H}^*$ ,  $x \mapsto \langle x, \cdot \rangle$ . Linear isometry between  $\mathcal{H}$  and  $\mathcal{H}^*$ . (One can identify  $\mathcal{H}$  and  $\mathcal{H}^*$ ). Now, we define the Dual operator as the following:

$$\mathcal{A}^* : D(\mathcal{A}^*) \subset \mathcal{H}^* \rightarrow \mathcal{H}, \quad \mathcal{A}^\circ = E^{-1} \mathcal{A}^* E.$$

**Definition 1.3.2** (Symmetric and Self-adjoint Operator). Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  be a UBLO, then

1. **Symmetric:** If  $\overline{D(\mathcal{A})} = \mathcal{H}$  and  $\mathcal{A}^\circ \supset \mathcal{A}$  with  $D(\mathcal{A}^*) \supset D(\mathcal{A})$  and for all  $x, y \in D(\mathcal{A})$ ,  $\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle$ .
2. **Self-adjoint:** If  $\overline{D(\mathcal{A})} = \mathcal{H}$  and  $\mathcal{A}^\circ = \mathcal{A}$ .

### Exercise 8: 2

1. Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ ,  $\overline{D(\mathcal{A})} = \mathcal{H}$ . If  $\mathcal{A}$  is closed, then  $\overline{D(\mathcal{A}^\circ)} = \mathcal{H}$ .
2. Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ ,  $\overline{D(\mathcal{A})} = \mathcal{H}$ . Suppose  $\mathcal{A}$  is symmetric and if  $0 \in \rho(\mathcal{A})$ , then prove that  $\lambda \in \mathbb{R}$  and

$$\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle.$$

**Proposition 1.3.1.** Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be an Hilbert space over  $\mathbb{C}$ , If  $\mathcal{A}$  is self-adjoint, injective and  $\overline{D(\mathcal{A})} = \mathcal{H}$ . Then  $\mathcal{A}^{-1} : \text{Rg}(\mathcal{A}) \rightarrow \mathcal{H}$  is self-adjoint.

*Proof.* Since  $\mathcal{A}$  is injective then  $\mathcal{A}^{-1}$  is well defined. Now, for all  $z \in D(\mathcal{A})$ ,  $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}^\circ z \rangle$  which implies  $\langle y, z \rangle = \langle x, \mathcal{A}^\circ z \rangle$  and so we have  $x \in D(\mathcal{A}^\circ) = D(\mathcal{A})$  and  $y = \mathcal{A}^\circ$ . Notice  $\overline{\text{Rg}(\mathcal{A})} = \ker(\mathcal{A})^\perp$  (because of self-adjointness).

Injectivity implies  $\overline{\text{Rg}(\mathcal{A})} = \mathcal{H}$  which implies  $\overline{D(\mathcal{A}^{-1})} = \mathcal{H}$ . So  $\mathcal{A}^{-1}$  is densely defined. Now, observe for all  $u, v \in D(\mathcal{A}^{-1})$ ,  $u = \mathcal{A}^\circ x$  and  $v = \mathcal{A} y$  with  $x, y \in D(\mathcal{A})$ . Hence,

$$\langle \mathcal{A}^{-1}u, v \rangle = \langle x, \mathcal{A}y \rangle = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}^{-1}y \rangle.$$

To this end,  $(\mathcal{A}^{-1})^\circ \subset \mathcal{A}^{-1}$ ,  $\forall z \in D((\mathcal{A}^{-1})^\circ) \exists w, \forall u \in D(\mathcal{A}^{-1}) = R(\mathcal{A})$  (i.e.  $u = \mathcal{A}x$ )

$$\langle \mathcal{A}^{-1}u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathcal{A}) \quad \langle x, z \rangle = \langle \mathcal{A}x, w \rangle$$

By definition  $w \in D(\mathcal{A}^\circ)$  and  $\mathcal{A}^\circ w = z$ .  $\mathcal{A}w = z \Rightarrow z \in \text{Rg}(\mathcal{A}) = D(\mathcal{A}^{-1})$ .  $\square$

**Theorem 1.3.1.** *Let  $\mathcal{H}$  be a Hilbert space, suppose  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is symmetric and surjective then  $\mathcal{A}$  is self-adjoint.*

*Proof.*  $\mathcal{A}$  and  $\mathcal{A}^\circ$  are injective. Do it only for  $\mathcal{A}$ , let  $x \in D(\mathcal{A})$  and  $\mathcal{A}x = 0$ .

$$\forall y \in D(\mathcal{A}), \quad 0 = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle \Rightarrow x \perp \text{Rg}(\mathcal{A}) = \mathcal{H}.$$

which implies  $x = 0$ .

Next, we show  $\mathcal{A}$  closed.

$$(x_n)_{n \geq 1} \subset D(\mathcal{A}), \quad x_n \rightarrow x \text{ in } \mathcal{H}, \quad \mathcal{A}x_n \rightarrow y \text{ in } \mathcal{H}$$

We shall show  $y = \mathcal{A}x$ . Now,  $\forall z \in D(\mathcal{A})$  then  $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}z \rangle$ , which implies  $\langle y, z \rangle = \langle x, \mathcal{A}z \rangle$  which implies  $x \in D(\mathcal{A}^\circ)$  and  $y = \mathcal{A}^\circ x$ . Since  $\mathcal{A}$  surjective  $\Rightarrow \exists w \in D(\mathcal{A})$  s.t.  $\mathcal{A}w = y$  and  $\mathcal{A}^\circ x = y$ .

Since  $\mathcal{A}$  is symmetric:  $\mathcal{A}^\circ w = \mathcal{A}w$ . Then  $\mathcal{A}^\circ w = \mathcal{A}^\circ x$ ,  $\mathcal{A}$  is injective  $\Rightarrow w = x$ . Hence  $\mathcal{A}x = \mathcal{A}w = y \Rightarrow y = \mathcal{A}x \Rightarrow \mathcal{A}$  is closed.

By closed graph theorem both  $\mathcal{A}$  and  $\mathcal{A}^{-1} \in \mathcal{L}(X)$ . We can conclude that  $\mathcal{A}$  is a self-adjoint operator.  $\square$

### Exercise 9: 2

Let  $\mathcal{H} = L^2(0, \pi)$  with  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  and

$$\mathcal{A}f = -f''$$

If  $D(\mathcal{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$  is  $\mathcal{A}$  a self-adjoint operator?

Similarly, if  $D(\mathcal{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$  is  $\mathcal{A}$  a self-adjoint operator?

**Theorem 1.3.2** (Fredrich's Extension). *Let  $\mathcal{H}$  be a Hilbert space on  $\mathbb{C}$  with  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ , symmetric then  $\mathcal{A}$  admits a unique self adjoint extension. If either*

- a.  $\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle > -\infty$
  - b.  $\sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle < +\infty$
- such that,  $\mathcal{A} \subset \mathcal{A}^\circ \subset (\mathcal{A}^\circ)^\circ \subset \dots$ . If (a) or (b) holds, then;  $\mathcal{A} \subset \mathcal{A}^\circ = (\mathcal{A}^\circ)^\circ$ .

## 1.4 Dissipative Operator and Numerical range

**Definition 1.4.1** (Duality Map). *Let  $X$  be a Banach space on  $\mathbb{K}$ . The duality map is defined as  $J : X \rightarrow 2^{X^*}$ ,  $x \mapsto J(x) = \{x^* \in X^* \mid \operatorname{Re} \langle x^*, x \rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$ . By the Hahn-Banach theorem,  $J(x) \neq \emptyset$ .*

**Question:** What can you say about  $J(X)$  when  $X$  is an Hilbert space or Reflexive?

**Definition 1.4.2.** *A map  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  (UBLO) is dissipative if for all  $x \in D(\mathcal{A})$ , there exists  $x^* \in J(X)$  such that  $\operatorname{Re} \langle x^*, \mathcal{A}x \rangle \leq 0$ .*

**Lemma 1.4.1.**  *$\mathcal{A}$  is dissipative if and only if for all  $\lambda > 0$ ,  $x \in D(\mathcal{A})$  we have that*

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

*Proof.* let  $x^* \in J(x)$ . Then

$$\begin{aligned} \|(\lambda I - \mathcal{A})x\| \|x^*\| &\geq |\langle x^*, (\lambda I - \mathcal{A})x \rangle| \geq \operatorname{Re} \langle x^*, (\lambda I - \mathcal{A})x \rangle, \\ &= \lambda \operatorname{Re} \langle x^*, x \rangle - \operatorname{Im} \langle x^*, \mathcal{A}x \rangle \geq \lambda \|x\|^2. \end{aligned}$$

Hence, if  $\|x\| \neq 0$ , then

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

( $\Leftarrow$ ) Let  $x \in D(\mathcal{A})$ ,  $x \neq 0$ , and  $\lambda > 0$ . Let  $y_\lambda^* \in J((\lambda I - \mathcal{A})x)$  and set  $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$ . Then

$$\|(\lambda I - \mathcal{A})x\|^2 = \|(\lambda I - \mathcal{A})x\| \|y_\lambda^*\| = \operatorname{Re} \langle y_\lambda^*, (\lambda I - \mathcal{A})x \rangle.$$

Since  $y_\lambda^* \neq 0$ , we have

$$\lambda \|x\| \leq \|(\lambda I - \mathcal{A})x\| = \operatorname{Re} \langle g_\lambda^*, (\lambda I - \mathcal{A})x \rangle = \lambda \langle g_\lambda^*, x \rangle - \operatorname{Im} \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Hence,

$$\operatorname{Im} \langle g_\lambda^*, \mathcal{A}x \rangle \leq \lambda \langle g_\lambda^*, x \rangle - \lambda \|x\| \leq \|g_\lambda^*\| \|x\| = \|x\|.$$

Therefore,

$$\operatorname{Im} \langle g_\lambda^*, \mathcal{A}x \rangle \leq 0. \quad (**)$$

**Idea:** Let  $\lambda \rightarrow +\infty$ .

Unit ball in  $X^*$  is compact for weak\* topology (Banach–Alaoglu).

(Up to subsequence)

$$g_\lambda^* \rightharpoonup g^* \in X^*, \quad \|g^*\| \leq 1.$$

Then from (\*\*),

$$\operatorname{Im} \langle g^*, \mathcal{A}x \rangle \leq 0.$$

$$(*) \quad \|x\| \leq \langle g_\lambda^*, x \rangle - \frac{1}{\lambda} \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Let  $\lambda \rightarrow +\infty$ . Then  $\|x\| \leq \langle g^*, x \rangle$ . Hence,  $\|g^*\| = 1$  and  $\langle g^*, x \rangle = \|x\|$ . Set  $x^* = \|x\|g^*$ . Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x \rangle = \|x\|^2,$$

that is,  $x^* \in J(x)$ . □

**Theorem 1.4.1** (Lumer-Phillips). *Let  $X$  be a Banach space and  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBL. Assume that  $\mathcal{A}$  is dissipative and that there exists  $\lambda_0 > 0$  such that  $\text{Rg}(\lambda_0 I - \mathcal{A}) = X$ .*

*Then  $\mathcal{A}$  is closed,  $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$ , and for all  $\lambda > 0$ ,*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}. \quad (1.4)$$

*Proof.* Let  $\lambda_0 > 0$ .

(1) To prove:  $(\lambda_0 I - A)$  is bijective.

- surj: Assumption,
- inj: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathcal{A})^{-1} : X \longrightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any  $y \in X$ , there exists a unique  $y \in X$  such that

$$x = (\lambda_0 I - \mathcal{A})^{-1}y, \quad (\lambda_0 I - \mathcal{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0} \|y\| \geq \|(\lambda_0 I - \mathcal{A})^{-1}y\|.$$

Hence,

$$\|(\lambda_0 I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_0}, \quad \lambda_0 \in \rho(\mathcal{A}).$$

(2)  $\mathcal{A}$  is closed.

Let  $x_n \rightarrow x$ ,  $x_n \in D(\mathcal{A})$ , and  $Ax_n \rightarrow y$ .

Then

$$(\lambda_0 I - \mathcal{A})x_n \rightarrow \lambda_0 x - y.$$

Since  $(\lambda_0 I - \mathcal{A})^{-1} \in \mathcal{L}(X)$ , we have

$$x_n \longrightarrow (\lambda_0 I - \mathcal{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathcal{A})x \iff y = \mathcal{A}x.$$

Therefore,  $\mathcal{A}$  is closed.

(3)  $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$  and (1.4).

Since  $\mathcal{A}$  is closed and  $\rho(\mathcal{A}) \neq \emptyset$ , we know that  $\rho(\mathcal{A})$  is open.

Let  $\Lambda = \rho(\mathcal{A}) \cap \mathbb{R}_+^*$ , which is open in  $\mathbb{R}_+^*$ . We show that it is closed.

Let  $(\lambda_n)_{n \in \mathbb{N}} \subset \Lambda$  such that  $\lambda_n \rightarrow \lambda \in \mathbb{R}_+^*$ . Note that since  $\lambda_n \in \Lambda$ , we have

$$\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$\begin{aligned} (\lambda I - \mathcal{A}) &= [I + u_n](\lambda_n I - \mathcal{A}) \implies \lambda I - \mathcal{A} = \lambda_n I - \mathcal{A} + u_n(\lambda_n I - \mathcal{A}), \\ &\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathcal{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathcal{A})^{-1} = u_n. \end{aligned}$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For  $n$  large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that  $\lambda \in \rho(\mathcal{A})$ . Hence,  $\Lambda$  is closed, and therefore  $\Lambda = \mathbb{R}_+^*$ .

□

**Corollary 1.4.1.** *Let  $X$  be a Banach space and  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO, closed, with  $\overline{D(\mathcal{A})} = X$ . Assume that  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative. Then*

$$\rho(\mathcal{A}) \supset \mathbb{R}_+^*, \quad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathcal{A})^{-1}\| \leq 1.$$

*Proof.* It is easy to show that  $\text{Rg}(I - A) = X$  (i.e.  $\lambda_0 = 1$  + Theorem 1.4.1).

$\mathcal{A}$  dissipative and closed implies

$\text{Rg}(I - \mathcal{A})$  is a closed subspace of  $X$ .

(give details!!!)

Let  $x^* \in X^*$  such that

$$\langle x^*, (I - \mathcal{A})x \rangle = 0, \quad \forall x \in D(\mathcal{A}). \tag{**}$$

Let us prove that  $x^* = 0$ .

Then  $x^* \in D(\mathcal{A}^*)$  and

$$(I - \mathcal{A}^*)x^* = 0.$$

Since  $\mathcal{A}^*$  is dissipative, by Lemma 1.4.1, we have  $x^* = 0$ . This implies that

$$\overline{\text{Rg}(I - \mathcal{A})} = X.$$

Since  $\text{Rg}(I - \mathcal{A})$  is closed, we obtain

$$\text{Rg}(I - \mathcal{A}) = X.$$

By contradiction and using Hahn–Banach.  $\square$

**Definition 1.4.3** (Numerical Range). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be UBLO. The numerical range of  $\mathcal{A}$ , denoted by  $W(\mathcal{A})$ ,*

$$W(\mathcal{A}) = \{\langle x^*, \mathcal{A}x \rangle \mid x^* \in J(x), x \in D(\mathcal{A}), \|x\| = \|x^*\| = 1, \langle x^*, x \rangle = 1\}.$$

In case of a Hilbert space, we have that  $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid x \in D(\mathcal{A}), \|x\| = 1\}$ .

Linear algebra in finite dimension  $\mathcal{A} \in \mathcal{M}_n(\mathbb{K})$ , we have that  $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid \|x\| = 1\}$ .

**Theorem 1.4.2** (Home-work). *Let  $\mathcal{A} : D(\mathcal{A}) \rightarrow X$  be closed, with  $\overline{D(\mathcal{A})} = X$ .*

1) If  $\lambda \notin \overline{W(\mathcal{A})}$ , then  $(\lambda I - \mathcal{A})$  is injective, has closed image, and for all  $x \in D(\mathcal{A})$ ,

$$\|(\lambda I - \mathcal{A})x\| \geq d(\lambda, W(\mathcal{A})) \|x\|.$$

Moreover, if  $\lambda \in \rho(\mathcal{A})$ , then

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{d(\lambda, W(\mathcal{A}))}. \quad (**)$$

2) If  $\Lambda$  is a connected open subset of  $\mathbb{C} \setminus W(\mathcal{A})$  such that  $\rho(\mathcal{A}) \cap \Lambda \neq \emptyset$ , then  $\rho(\mathcal{A}) \supset \Lambda$  and  $(**)$  holds true.

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CHAPTER  
**TWO**

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## INTRODUCTION TO THE THEORY OF SEMI-GROUPS

### 2.1 Intro to the Introduction

**Definition 2.1.1.** Let  $X$  be a Banach space over  $\mathbb{K}$ . A one-parameter family of bounded linear operators on  $X$ ,  $(T(t))_{t \geq 0}$ , is a semigroup (SG) of bounded linear operators on  $X$  if:

1.  $T(0) = Id_X$ ,
2.  $\forall (t, s) \in \mathbb{R}_+^2 : T(t + s) = T(t) \cdot T(s)$  (SG property).

**Remark 2.1.1.**  $T(t)$  and  $T(s)$  commute.

#### 2. Infinitesimal generator of SG-LO $(T(t))_{t \geq 0}$

Let  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  be an unbounded linear operator with

$$D(\mathcal{A}) = \left\{ x \in X \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \mathcal{A}x, \quad x \in D(\mathcal{A}).$$

where  $D(\mathcal{A}) = \text{domain of } \mathcal{A}$ .

### 2.2 Uniformly Continuous SG-BLO

**Definition 2.2.1.** A SG-BLO on  $X$ ,  $(T(t))_{t \geq 0}$  is uniformly continuous if

$$\|T(t) - Id\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as } t \rightarrow 0^+$$

**Lemma 2.2.1.** Let  $(T(t))_{t \geq 0}$  be a SG-BLO which is uniformly continuous. Then,  $\forall t > 0$ ,

$$\|T(s) - T(t)\| \xrightarrow{s \rightarrow t} 0$$

(continuity everywhere).

*Proof.* Let  $t$  be fixed.  $T(s) = T(s - t + t)$ ,  $s - t \geq 0$ .

$$s \geq t \Rightarrow T(s) = T(s - t)T(t) \Rightarrow T(s) - T(t) = T(t)[T(s - t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\| \|T(s - t) - I_d\| \xrightarrow[s \rightarrow t]{} 0.$$

For  $s \leq t$

$$T(t) = T(t - s)T(s) \Rightarrow T(t) - T(s) = T(s)[T(t - s) - I_d]$$

(Prove that  $\sup_{[0,t]} \|T(t)\| < +\infty$ )

Then

$$\begin{aligned} \|T(t) - T(s)\| &\leq \|T(s)\| \|T(t - s) - I_d\| \\ &\leq \sup \|T(s)\| \|T(t - s) - I_d\| \xrightarrow[s \rightarrow t]{} 0. \end{aligned}$$

□

**Theorem 2.2.1.** A linear operator  $\mathcal{A}$  is the infinitesimal generator of a uniformly continuous semigroup if and only if  $\mathcal{A}$  is a bounded linear operator.

*Proof.* Let  $\mathcal{A}$  be a bounded linear operator on  $X$  and set

$$T(t) = e^{t\mathcal{A}} = \sum_{n=0}^{\infty} \frac{(t\mathcal{A})^n}{n!}. \quad (1.5)$$

The right-hand side of (1.5) converges in norm for every  $t \geq 0$  and defines, for each such  $t$ , a bounded linear operator  $T(t)$ . It is clear that  $T(0) = I$  and a straightforward computation with the power series shows that  $T(t + s) = T(t)T(s)$ . Estimating the power series yields

$$\|T(t) - I\| \leq |t| \|\mathcal{A}\| e^{\|\mathcal{A}\| t}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathcal{A} \right\| \leq \|\mathcal{A}\| \cdot \max_{0 \leq s \leq t} \|T(s) - I\|$$

which imply that  $T(t)$  is a uniformly continuous semigroup of bounded linear operators on  $X$  and that  $\mathcal{A}$  is its infinitesimal generator.

Let  $T(t)$  be a uniformly continuous semigroup of bounded linear operators on  $X$ . Fix  $\rho > 0$ , small enough, such that

$$\left\| I - \rho \int_0^\rho T(s) ds \right\| < 1.$$

This implies that  $\rho^{-1} \int_0^\rho T(s) ds$  is invertible. Now,

$$h^{-1}(T(h) - I) \int_0^\rho T(s) ds = h^{-1} \left( \int_0^\rho T(s + h) ds - \int_0^\rho T(s) ds \right)$$

$$= h^{-1} \left( \int_{\rho}^{\rho+h} T(s) ds - \int_0^h T(s) ds \right)$$

and therefore

$$h^{-1}(T(h) - I) = \left( h^{-1} \int_{\rho}^{\rho+h} T(s) ds - h^{-1} \int_0^h T(s) ds \right) \left( \int_0^{\rho} T(s) ds \right)^{-1}. \quad (1.6)$$

Letting  $h \rightarrow 0$  in (1.6) shows that  $h^{-1}(T(h) - I)$  converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I) \left( \int_0^{\rho} T(s) ds \right)^{-1}$$

which is the infinitesimal generator of  $T(t)$ .  $\square$

**Remark 2.2.1.** *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

**Theorem 2.2.2.** *Let  $T(t)$  and  $S(t)$  be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t \rightarrow 0} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t \rightarrow 0} \frac{S(t) - I}{t}. \quad (1.7)$$

*then  $T(t) = S(t)$  for  $t \geq 0$ .*

*Proof.* We will show that given  $T > 0$ ,  $S(t) = T(t)$  for  $0 \leq t \leq T$ . Let  $T > 0$  be fixed, since  $t \mapsto \|T(t)\|$  and  $t \mapsto \|S(t)\|$  are continuous there is a constant  $C$  such that

$$\|T(t)\| \|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given  $\varepsilon > 0$  it follows from (1.7) that there is a  $\delta > 0$  such that

$$h^{-1} \|T(h) - S(h)\| < \varepsilon / TC \quad \text{for } 0 \leq h \leq \delta. \quad (1.8)$$

Let  $0 \leq t \leq T$  and choose  $n \geq 1$  such that  $t/n \leq \delta$ . From the semigroup property and (1.8) it then follows that

$$\begin{aligned} \|T(t) - S(t)\| &= \left\| T\left(n \frac{t}{n}\right) - S\left(n \frac{t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k) \frac{t}{n}\right) S\left(\frac{kt}{n}\right) - T\left((n-k-1) \frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k-1) \frac{t}{n}\right) \right\| \left\| T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{t}{n}\right) \right\| \leq Cn \frac{\varepsilon}{TC} \frac{t}{n} \leq \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary  $T(t) = S(t)$  for  $0 \leq t \leq T$  and the proof is complete.  $\square$

**Corollary 2.2.1.** Let  $T(t)$  be a uniformly continuous semigroup of bounded linear operators. Then

- (a) There exists a constant  $\omega \geq 0$  such that  $\|T(t)\| \leq e^{\omega t}$ .
- (b) There exists a unique bounded linear operator  $A$  such that  $T(t) = e^{tA}$ .
- (c) The operator  $A$  in part (b) is the infinitesimal generator of  $T(t)$ .
- (d)  $t \mapsto T(t)$  is differentiable in norm and satisfies

$$\frac{dT(t)}{dt} = AT(t) = T(t)A. \quad (1.9)$$

*Proof.* All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of  $T(t)$  is a bounded linear operator  $A$ .  $A$  is also the infinitesimal generator of  $e^{tA}$  defined by (1.5) and therefore, by Theorem 2.2.2,  $T(t) = e^{tA}$ .  $\square$

**Remark 2.2.2.** The proofs above are from the recommended text (*Semigroups of Linear Operators and Applications to Partial Differential Equations*) Page 3. [Theorem 1.3 and Corollary 1.4].

## 2.3 Strongly Continuous Semigroups ( $C_0$ -Semigroups)

**Definition 2.3.1.** The SG-BLO  $(T(t))_{t \geq 0}$  is strongly continuous (SC or  $C_0$ ) if  $\forall x \in X$

$$\|T(t)x - x\|_X \xrightarrow{t \rightarrow 0^+} 0 \quad (2.1)$$

**Theorem 2.3.1.** Let  $(T(t))_{t \geq 0}$ ,  $C_0$ -SG then  $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0 \|T(t)\| \leq M e^{\omega t}$

*Proof.* First we want to show that  $\exists \eta > 0, \sup_{t \in [0, \eta]} \|T(t)\| < +\infty$ .

By contradiction, assume that  $\sup \|T(t)\| = +\infty$ . Then  $\exists (t_n)_{n \geq 0} \searrow 0$  such that  $\|T(t_n)\| \geq n$  or  $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive)  $\exists x \in X$  such that  $\sup \|T(t_n)x\| = +\infty$ , but this contradicts the strong convergence 2.3.1.

Now take  $M := \sup \|T(t)\| \geq 1$  (This is because  $T(0) = Id$ ).

$\forall t \geq 0$  write  $t = k\eta + \eta_t$  where  $k = \lfloor \frac{t}{\eta} \rfloor$  and  $\eta_t \in [0, \eta]$ , then

$$\begin{aligned} \|T(t)\| &= \|T(k\eta + \eta_t)\| \\ &= \|[T(\eta)]^k T(\eta_t)\| && \text{by the SG property} \\ &= \|T(\eta)\|^k \|T(\eta_t)\| \\ &\leq M \cdot M^k && M \text{ is upperbound} \\ &\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\ &= M(e^{\ln M})^{t/\eta} \\ &= M e^{t \frac{\ln M}{\eta}} = M e^{\omega t} \end{aligned}$$

And since  $\omega = \frac{\ln M}{\eta}$  and  $M \geq 1$  then  $\omega \geq 0$ . □

**Corollary 2.3.1.** Let  $\{T(t)\}_{t \geq 0}$  be a  $C_0$ -semigroup then  $\forall x \in X, t \mapsto T(t)x$  is continuous

*Proof.* For  $h > 0$  and  $t > 0$ :

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \rightarrow 0$  as  $h \rightarrow 0$  by definition 2.3.1.
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$ . Since  $\|T(t-h)\|$  is bounded by theorem 2.3.1, so this tends to 0 as  $h \rightarrow 0$ .

□