PARTIAL DIFFERENTIAL EQUATIONS

International Mathematics Masters
National Higher School of Mathematics (NHSM)

2026

CONTENTS

CHAPTER

ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathscr{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. *An operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathscr{A})$ is a subspace of X and* $\sup_{x\neq 0} \frac{\|\mathscr{A}x\|}{\|x\|} = +\infty$

Exercise 1

Let $\mathscr{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathscr{A}) = \{f : \mathbb{R} \to \mathbb{R} : f, f' \in L^2\}$. Show $\mathscr{A}$ is an unbounded linear operator.

Notation: If $\mathscr{A}$ and $\mathscr{B}$ are unbounded linear operators, then $\mathscr{A} \supset \mathscr{B}$ if and only if $D(\mathscr{A}) \supset D(\mathscr{B})$ and for all $x \in D(\mathscr{B})$, $\mathscr{A}x = \mathscr{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be a UBLO.*

$$\rho(\mathscr{A}) = \textit{ Resolvent of } \mathscr{A} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} (\lambda I - \mathscr{A}) : D(\mathscr{A}) \to X \textit{ is bijective, and} \\ (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X) \end{array} \right\}$$

$$\sigma(\mathscr{A}) = \textit{ Spectrum of } \mathscr{A} = \mathbb{C} \setminus \rho(\mathscr{A}).$$

Definition 1.1.3. *$\mathscr{A}$ is closed if and only if the graph of $\mathscr{A}$, denoted as $G(\mathscr{A})$ is closed. Also $\mathscr{A}$ is closable if and only if there exists $\tilde{\mathscr{A}} \supset \mathscr{A}$ such that $G(\tilde{\mathscr{A}}) = \overline{G(\mathscr{A})}$.*

Note that $G(\mathscr{A}) = \big\{(x, \mathscr{A}x) \big| x \in D(\mathscr{A})\big\}$.

Exercise 2

1. Prove that if it exists, $\tilde{\mathscr{A}}$ is unique, it then denoted by $\overline{\mathscr{A}}$ called closure of $\mathscr{A}$.
2. Let $\mathscr{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a,b],\mathbb{R}), \|\cdot\|_\infty = \sup|f(x)|$ and $D(\mathscr{A}_\ell) = C^\ell([a,b],\mathbb{R})$. Prove $\overline{\mathscr{A}_\ell} = \mathscr{A}_1$.

Lemma 1.1.1. *If $\mathscr{A}$ an unbounded linear operator is closable, then $\rho(\overline{\mathscr{A}}) = \rho(\mathscr{A})$. If $\mathscr{A}$ is closed then $\rho(\mathscr{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X)\}$.*
Hints (Exercise): If $\rho(\mathscr{A}) \neq \emptyset$ implies $\mathscr{A}$ is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathscr{L}(X)$ implies T is closed).

Corollary 1.1.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \cup \sigma_c(\mathscr{A})$ where*

1. $\sigma_p(\mathscr{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) \neq \{0\}\}$ *(punctual spectrum and λ's are the eigenvalue).*
2. $\sigma_c(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) = \{0\}, \overline{Rg(\lambda I - \mathscr{A})} \subsetneq X\right\}$ *(continuous spectrum).*
3. $\sigma_r(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \lambda I - \mathscr{A} \text{ is injective}, \overline{Rg(\lambda I - \mathscr{A})} = X,\ Rg(\lambda I - \mathscr{A}) \neq X\right\}$ *(residual spectrum).*

Exercise 3

Let

$$X = \ell^2(\mathbb{C}) = \left\{(x_n)_{n\geq 0} : \sum_n |x_n|^2 < \infty\right\},$$

with $(\mathscr{A}x_n)_{n\geq 0} = \left(\frac{x_n}{1+n}\right)_{n\geq 0}$. Prove that $\mathscr{A}$ is a BLO, injective, $\overline{\mathrm{Rg}(\mathscr{A})} = X$ and $\mathrm{Rg}(\mathscr{A}) \subsetneq X$.

Theorem 1.1.1. *If $\mathscr{A}$ is a closed UBLO then $\rho(\mathscr{A})$ is open. If $\mu \in \rho(\mathscr{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathscr{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathscr{A})$ and*

$$(\lambda I - \mathscr{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathscr{A})^{-(n+1)}$$

To do Question: do you need $\mathscr{A}$ closed?

Theorem 1.1.2 (Resolvent Identity)**.** *Let $\mathscr{A}$ be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator*

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all $\lambda, \mu \in \rho(A)$,

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

Corollary 1.1.2. *The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathscr{A})$ into $\mathscr{L}(X)$ is analytic. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and $\mathscr{A}$ a closed UBLO with $\overline{D(\mathscr{A})} = X$ a dense UBLO.
If X and Y are Banach spaces with duals X^* and Y^*, then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x\rangle$.

Definition 1.1.4 (Dual Operator of $\mathscr{A}$)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to Y$ (UBLO), be such that $\overline{D(\mathscr{A})} = X$. The* dual operator *$\mathscr{A}^* : D(\mathscr{A}^*) \subset Y^* \to X^*$ is a UBLO defined as follows:*

$$D(\mathscr{A}^*) := \{y^* \in Y^* \,|\, \exists z^* \in X^* \text{ such that } \langle y^*, Ax\rangle = \langle z^*, x\rangle \; \forall x \in D(\mathscr{A})\}\,.$$

and $y^* \in D(\mathscr{A}^*)$, the element z^* is unique and we define $A^*y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathscr{A} \in \mathscr{L}(X, Y)$. Then $\mathscr{A}^* \in \mathscr{L}(Y^*, X^*)$ and*

$$\|\mathscr{A}^*\|_{\mathscr{L}(Y^*,X^*)} = \|\mathscr{A}\|_{\mathscr{L}(X,Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closedly dense UBLO. Then $\overline{D(\mathscr{A}^*)} = X^*(\cong X)$, and $\mathscr{A}^*$ is closed.*

Theorem 1.1.3. *Let $\mathscr{A}$ be a closedly dense UBLO. Then $\rho(\mathscr{A}) = \rho(\mathscr{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$\left[(\lambda I - \mathscr{A})^{-1}\right]^* = (\lambda I - \mathscr{A}^*)^{-1}.$$

Exercise 4

1. Let $\mathscr{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathscr{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:

   a. $\rho(\mathscr{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathscr{A}) = i\mathbb{R}$.

   b. $\mathscr{A}$ is a closed unbounded linear operator.

   c. If $\lambda \in \rho(\mathscr{A})$ then $(\lambda I - \mathscr{A})^{-1} : X \longrightarrow D(\mathscr{A})$ is bounded.

   For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathscr{A})$ such that $(\lambda I - \mathscr{A})f = g$.
   For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathscr{A})f_n \to 0$.

2. Do same for $\mathscr{A} = -i\frac{d}{dx}$.

1.2 Compact Operators

Let X and Y be Banach spaces on $\mathbb{K}$.

Definition 1.2.1. *Let $K : X \to Y$ be a BLO (in $\mathscr{L}(X, Y)$), then K compact iff $K(B_1^X(0))$ is relatively compact in Y (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathscr{L}(X, Y) \mid K \text{ is compact}\}.$$

Exercise 5

Let $X = C([a,b],\mathbb{C})$ and $k \in C^0([a,b]\times[c,d],\mathbb{C})$
Define $K \in \mathscr{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t,s)x(s)\,ds.$$

Show $K \in \mathcal{K}(X)$.

Theorem 1.2.1. *$\mathcal{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$.*

Proof. We shall show this in two steps

1. $\mathcal{K}(X,Y)$ is a vector space (Do it).

2. Closed: $K_n \to K$ (Prof. Yacine said he would send a different proof).

□

Exercise 6

Let $X = \ell^2(\mathbb{C})$,

$$\mathscr{A}((x_n)_{n\geq 0}) = \left(\frac{x_n}{n+1}\right)_{n\geq 0}.$$

Show that $\mathscr{A}$ is compact.

Theorem 1.2.2. *Let X, Y and Z be Banach spaces on $\mathbb{K}$.*

$$X \xrightarrow{\mathscr{A}} Y \xrightarrow{\mathscr{B}} Z, \quad \mathscr{A} \in \mathscr{L}(X,Y),\ \mathscr{B} \in \mathscr{L}(Y,Z).$$

1. *If $\mathscr{A}$ is compact or $\mathscr{B}$ is compact, then $\mathscr{B}\mathscr{A}$ is compact.*

2. *If $\mathscr{A}$ is compact then $\mathscr{A}^* \in \mathcal{K}(Y^*, X^*)$.*

3. *If $\mathscr{A}$ is compact and $\mathrm{Rg}(\mathscr{A})$ is closed (in Y), then it is finite dimensional.*

To proceed with further results on compact operators, we need the following lemma

Lemma 1.2.1 (Riesz Lemma)**.** *Let E be a normed vector space, $F = \overline{F} \subset E$. Then $\forall r \in (0,1)$, $\exists\, x_r \in E$, such that*

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

Proof. Since $F \neq E$ then this implies $\exists\, z \in E \setminus F$. Let $d = d(z,F) > 0$.
For $0 < r < 1$, $\exists\, y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|}\|z - (y_r + \|z - y_r\|y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

$\square$

Proposition 1.2.1. *Let $\mathscr{A} \in \mathcal{K}(X)$, such that X is a Banach space on $\mathbb{C}$. If $\lambda \in \mathbb{C}^*$, then $\ker((\lambda I - \mathscr{A})^n)$ has finite dimension.*

Proof. Only for $n = 1$. (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathscr{A}) = \{x \in X : \mathscr{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda}\mathscr{A}x\right\} \subset \mathrm{Rg}(\mathscr{A}).$$

So $\tilde{K}$ is closed in $\mathrm{Rg}(\mathscr{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists\, (x_n)$ in $\tilde{K}$, such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|}\|\mathscr{A}x_n - \mathscr{A}x_m\| \geq \frac{1}{2}, \qquad \forall\, n \neq m$$

and so we have $\|\mathscr{A}x_n\| \leq \|\mathscr{A}\|$. So $(\mathscr{A}x_n)$ is not Cauchy, hence a contradiction.

$\square$

Exercise 7

Let X be a Banach space on $\mathbb{K}$. If $\mathscr{A} \in \mathscr{L}(X)$, assume $\exists\, n_0$ s.t. $\ker(\mathscr{A}^{n_0}) = \ker(\mathscr{A}^{n_0+1})$. Then $\forall n \geq n_0$,

$$\ker(\mathscr{A}^n) = \ker(\mathscr{A}^{n_0}).$$

Proposition 1.2.2. *Let $\mathscr{A} \in \mathcal{K}(X)$ and X be a Banach space on $\mathbb{C}$, $\lambda \neq 0$. Then $\exists\, n_0$ such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathscr{A})^n) = \ker((\lambda I - \mathscr{A})^{n_0}).$$

Proof. Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathscr{A})^n) \subset \ker((\lambda I - \mathscr{A})^{n+1})$ and each of them is closed.
RL: with $r = \frac{1}{2}$ with $(x_n)_{n\geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathscr{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathscr{A})^n)) \geq \frac{1}{2}.$$

For $n = 1, x \in \ker(\lambda I - \mathscr{A}) \;\Rightarrow\; x = \frac{\mathscr{A}}{\lambda}x$. For all $1 \leq m < n$,

$$\begin{aligned}\frac{\mathscr{A}x_n}{\lambda} - \frac{\mathscr{A}x_m}{\lambda} &= x_n - x_n + \frac{\mathscr{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathscr{A}x_m}{\lambda}\right) \\ &= x_n - \left[\frac{(\lambda I - \mathscr{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathscr{A})x_m}{\lambda}\right].\end{aligned}$$

So,

$$\left\| \frac{\mathscr{A} x_n}{\lambda} - \frac{\mathscr{A} x_m}{\lambda} \right\| \geq d\big(x_n, \ker(\lambda I - \mathscr{A})^n\big) \geq \frac{1}{2}.$$

which is a contradiction. □

Notice that if $\ker(\lambda I - \mathscr{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathscr{A})$. Notice,

$$\dim \ker(\lambda I - \mathscr{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2 $\Rightarrow \exists\, n_0$ (smallest one) such that

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) = \ker\big((\lambda I - \mathscr{A})^{n}\big), \qquad \forall n \geq n_0.$$

Note that,

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{generalized eigenspace}.$$

$$\dim \ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{algebraic multiplicity of } \lambda.$$

Proposition 1.2.3 (Fredholm aternative)**.** *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$.*

$$\operatorname{Rg}(\lambda I - \mathscr{A}) = X \iff \ker(\lambda I - \mathscr{A}) = \{0\}.$$

Proposition 1.2.4. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$, $\dim X = \infty$. If $\lambda_n \to \lambda$, $\lambda_n \in \sigma(\mathscr{A}) \setminus \{0\}$, pairwise distinct,then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathscr{A}) \setminus \{0\}$ is isolated.*

Proof. Let $\lambda_n \in \sigma_p(\mathscr{A})$, $\exists\, \|x_n\| = 1$ such that $\mathscr{A} x_n = \lambda_n x_n$. Let

$$X_n = \operatorname{span}(x_1, \dots, x_n), \qquad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \;\Rightarrow\; \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i, \text{ which implies } \quad \lambda_{n+1} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathscr{A} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since (x_i) are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \qquad 1 \leq i \leq n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathscr{A})X_n \subset X_{n-1}, \qquad \forall n \geq 2.$$

Recall:

$$\|y_n\| = 1, \qquad y_n \in X_n,$$
$$d(y_n, X_{n-1}) \geq \frac{1}{2}.$$

For $2 \leq m < n$,

$$\begin{aligned}\left\| \frac{\mathscr{A} y_n}{\lambda_n} - \frac{\mathscr{A} y_m}{\lambda_m} \right\| &= \left\| y_n - \left[\frac{\lambda_n I - \mathscr{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathscr{A}}{\lambda_m} y_m \right] \right\| \\ &\geq d(y_n, X_{n-1}) \geq \frac{1}{2}.\end{aligned}$$

Assume that

$$\lambda_n \to \lambda \quad (n \to \infty).$$

Suppose $\lambda \neq 0$, then

$$\left| \frac{1}{\lambda_n} \right| \leq C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathscr{A} y_n}{\lambda_n} \right)_{n \geq 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathscr{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. □

Theorem 1.2.3. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$. Then $\sigma(\mathscr{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.*

Main use of compact operators (in PDEs)

They appear as "inverse" of UBLO.

Definition 1.2.2. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ UBLO, closed, $\rho(\mathscr{A}) \neq \{0\}$. $\mathscr{A}$ is said to have compact resolvent if*

$$(\lambda I - \mathscr{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathscr{A}).$$

Main Example: $\mathscr{A} = -\Delta$ on Ω with $\mathscr{A} u = -u_{xx}$.

1.3 Adjoints, Symmetric and Self-adjoint Operators

Let $\mathscr{H}$ be a Hilbert space, with inner product

$$\langle\cdot,\cdot\rangle_{\mathscr{H}} : \mathscr{H}\times\mathscr{H}\to\mathbb{C}.$$

$\mathscr{A} : D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$ UBLO, $\overline{D(\mathscr{A})}=\mathscr{H}$.

Definition 1.3.1 (Adjoint Operator $\mathscr{A}^\circ$)**.**

$$D(\mathscr{A}^\circ)=\{x\in\mathscr{H} : v\mapsto\langle\mathscr{A}v,x\rangle_{\mathscr{H}} : D(\mathscr{A})\to\mathbb{C}\ \textit{bdd operator}\}.$$

If $x\in D(\mathscr{A}^\circ)$, then there exists uniquely $z\in\mathscr{H}$ such that $\langle v,z\rangle_{\mathscr{H}}=\langle\mathscr{A}v,x\rangle_{\mathscr{H}}$ for all $v\in D(\mathscr{A})$.
Observe, with Riesz representation and the fact that $\overline{D(\mathscr{A})}=\mathscr{H}$, we have that $z:=\mathscr{A}^\circ x$ and $\langle v,\mathscr{A}^\circ x\rangle_{\mathscr{H}}=\langle\mathscr{A}v,x\rangle_{\mathscr{H}}$ for all $v\in D(\mathscr{A})$.

Remark 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, $E : \mathscr{H}\to\mathscr{H}^*$, $x\mapsto\langle x,\cdot\rangle$. Linear isometry between $\mathscr{H}$ and $\mathscr{H}^*$. (One can identify $\mathscr{H}$ and $\mathscr{H}^*$). Now, we define the Dual operator as the following:*

$$\mathscr{A}^* : D(\mathscr{A}^*)\subset\mathscr{H}^*\to\mathscr{H},\quad \mathscr{A}^\circ=E^{-1}\mathscr{A}^*E.$$

Definition 1.3.2 (Symmetric and Self-adjoint Operator)**.** *Let $(\mathscr{H},\langle,\rangle)$ be a Hilbert space and $\mathscr{A} : D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$ be a UBLO, then*

1. ***Symmetric:*** *If $\overline{D(\mathscr{A})}=\mathscr{H}$ and $\mathscr{A}^\circ\supset\mathscr{A}$ with $D(\mathscr{A}^\circ)\supset D(\mathscr{A})$ and for all $x,y\in D(\mathscr{A})$, $\langle\mathscr{A}x,y\rangle=\langle x,\mathscr{A}y\rangle$.*

2. ***Self-adjoint:*** *If $\overline{D(\mathscr{A})}=\mathscr{H}$ and $\mathscr{A}^\circ=\mathscr{A}$.*

Exercise 8

1. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$, $\overline{D(\mathscr{A})}=\mathscr{H}$. If $\mathscr{A}$ is closed, then $\overline{D(\mathscr{A}^\circ)}=\mathscr{H}$.

2. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$, $\overline{D(\mathscr{A})}=\mathscr{H}$. Suppose $\mathscr{A}$ is symmetric and if $\lambda\in\sigma_p(\mathscr{A})$, then prove that $\lambda\in\mathbb{R}$ and

$$\inf_{\|x\|=1,x\in D(\mathscr{A})}\langle\mathscr{A}x,x\rangle\le\lambda\le\sup_{\|x\|=1,x\in D(\mathscr{A})}\langle\mathscr{A}x,x\rangle.$$

Proposition 1.3.1. *Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be an Hilbert space over $\mathbb{C}$, If $\mathscr{A}$ is self-adjoint, injective and $\overline{D(\mathscr{A})}=\mathscr{H}$. Then $\mathscr{A}^{-1} : \mathrm{Rg}(\mathscr{A})\to\mathscr{H}$ is self-adjoint.*

Proof. Since $\mathscr{A}$ is injective then $\mathscr{A}^{-1}$ is well defined, and since $\mathscr{A}=\mathscr{A}^\circ$ then $\mathscr{A}$ is closed.

Now assume $(x_n)\subset D(\mathscr{A})$ such that $x_n\to x\in D(A)$ (because $\mathscr{A}$ is closed) and $\mathscr{A}x_n-\to y$ then for all $z\in D(\mathscr{A})$, $\langle\mathscr{A}x_n,z\rangle=\langle x_n,\mathscr{A}^\circ z\rangle$ which implies $\langle y,z\rangle=\langle x,\mathscr{A}^\circ z\rangle$ and so we have

$x \in D(\mathscr{A}^\circ) = D(\mathscr{A})$ and $y = \mathscr{A}^\circ$. Notice $\overline{\mathrm{Rg}(\mathscr{A})} = \ker(\mathscr{A})^\perp$ (because of self-adjointness). Injectivity implies $\overline{\mathrm{Rg}(\mathscr{A})} = \mathscr{H}$ which implies $\overline{D(\mathscr{A}^{-1})} = \mathscr{H}$. So $\mathscr{A}^{-1}$ is densely defined.
Now, observe for all $u, v \in D(\mathscr{A}^{-1})$, $u = \mathscr{A}^\circ x$ and $v = \mathscr{A} y$ with $x, y \in D(\mathscr{A})$. Hence,

$$\langle \mathscr{A}^{-1} u, v \rangle = \langle x, \mathscr{A} y \rangle = \langle \mathscr{A} x, y \rangle = \langle x, \mathscr{A}^{-1} y \rangle.$$

To this end, $(\mathscr{A}^{-1})^\circ \subset \mathscr{A}^{-1}$, $\forall\, z \in D\big((\mathscr{A}^{-1})^\circ\big)\ \exists\, w, \forall\, u \in D(\mathscr{A}^{-1}) = R(\mathscr{A})\ (i.e.\ u = \mathscr{A} x)$

$$\langle \mathscr{A}^{-1} u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathscr{A}) \quad \langle x, z \rangle = \langle \mathscr{A} x, w \rangle$$

By definition $w \in D(\mathscr{A}^\circ)$ and $\mathscr{A}^\circ w = z$. $\mathscr{A} w = z \Rightarrow z \in Rg(\mathscr{A}) = D(\mathscr{A}^{-1})$. □

Theorem 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, suppose $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is symmetric and surjective then $\mathscr{A}$ is self-adjoint.*

Proof. $\mathscr{A}$ and $\mathscr{A}^\circ$ are injective. Do it only for $\mathscr{A}$, let $x \in D(\mathscr{A})$ and $\mathscr{A} x = 0$.

$$\forall y \in D(\mathscr{A}), \quad 0 = \langle \mathscr{A} x, y \rangle = \langle x, \mathscr{A} y \rangle \Rightarrow x \perp Rg(\mathscr{A}) = \mathscr{H}.$$

which implies $x = 0$.
Next, we show $\mathscr{A}$ closed.

$$(x_n)_{n \geq 1} \subset D(\mathscr{A}), \quad x_n \to x \text{ in } \mathscr{H}, \quad \mathscr{A} x_n \to y \text{ in } \mathscr{H}$$

We shall show $y = \mathscr{A} x$. Now, $\forall z \in D(\mathscr{A})$ then $\langle \mathscr{A} x_n, z \rangle = \langle x_n, \mathscr{A} z \rangle$, which implies $\langle y, z \rangle = \langle x, \mathscr{A} z \rangle$ which implies $x \in D(\mathscr{A}^\circ)$ and $y = \mathscr{A}^\circ x$. Since $\mathscr{A}$ surjective $\Rightarrow \exists w \in D(\mathscr{A})$ s.t. $\mathscr{A} w = y$ and $\mathscr{A}^\circ x = y$.
Since $\mathscr{A}$ is symmetric: $\mathscr{A}^\circ w = \mathscr{A} w$. Then $\mathscr{A}^\circ w = \mathscr{A}^\circ x$, $\mathscr{A}$ is injective $\Rightarrow w = x$. Hence $\mathscr{A} x = \mathscr{A} w = y \Rightarrow y = \mathscr{A} x \Rightarrow \mathscr{A}$ is closed.
By closed graph theorem both $\mathscr{A}$ and $\mathscr{A}^{-1} \in \mathscr{L}(X)$. We can conclude that $\mathscr{A}$ is a self-adjoint operator. □

Exercise 9

Let $\mathscr{H} = L^2(0, \pi)$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ and

$$\mathscr{A} f = -f''$$

If $D(\mathscr{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?
Similarly, if $D(\mathscr{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?

Theorem 1.3.2 (Fredrich's Extension)**.** *Let $\mathscr{H}$ be a Hilbert space on $\mathbb{C}$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, symmetric then $\mathscr{A}$ admits a unique self adjoint extension. If either*

a. $\inf_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A} x, x \rangle > -\infty$

b. $\sup_{\|x\|=1, x\in D(\mathscr{A})}\langle \mathscr{A}x, x\rangle < +\infty$

such that, $\mathscr{A} \subset \mathscr{A}^{\circ} \subset (\mathscr{A}^{\circ})^{\circ} \subset \cdots$. *If (a) or (b) holds, then;* $\mathscr{A} \subset \mathscr{A}^{\circ} = (\mathscr{A}^{\circ})^{\circ}$.

1.4 Dissipative Operator and Numerical range

Definition 1.4.1 (Duality Map)**.** *Let X be a Banach space on $\mathbb{K}$. The duality map is defined as* $J : X \longrightarrow 2^{X^*}$, $x \mapsto J(x) = \{x^* \in X^* \mid Re\langle x^*, x\rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$.
By the Hahn-Banach theorem, $J(x) \neq \emptyset$.

Question: What can you say about $J(X)$ when X is an Hilbert space or Reflexive?

Definition 1.4.2. *A map $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ (UBLO) is dissipative if for all $x \in D(\mathscr{A})$, there exists $x^* \in J(X)$ such that $Re\langle x^*, \mathscr{A}x\rangle \leq 0$.*

Lemma 1.4.1. *$\mathscr{A}$ is dissipative if and only if for all $\lambda > 0$, $x \in D(\mathscr{A})$ we have that* $\|(\lambda I - \mathscr{A})x\| \geq \lambda\|x\|$.

Proof. let $x^* \in J(x)$. Then

$$\|(\lambda I - \mathscr{A})x\|\,\|x^*\| \geq |\langle x^*, (\lambda I - \mathscr{A})x\rangle| \geq \Re\langle x^*, (\lambda I - \mathscr{A})x\rangle,$$

$$= \lambda\Re\langle x^*, x\rangle - \Re\langle x^*, \mathscr{A}x\rangle \geq \lambda\|x\|^2.$$

Hence, if $\|x\| \neq 0$, then

$$\|(\lambda I - \mathscr{A})x\| \geq \lambda\|x\|.$$

($\Leftarrow$) Let $x \in D(\mathscr{A})$, $x \neq 0$, and $\lambda > 0$. Let $y^*_\lambda \in J((\lambda I - \mathscr{A})x)$ and set $g^*_\lambda = \frac{y^*_\lambda}{\|y^*_\lambda\|}$. Then

$$\|(\lambda I - \mathscr{A})x\|^2 = \|(\lambda I - \mathscr{A})x\|\,\|y^*_\lambda\| = \Re\langle y^*_\lambda, (\lambda I - \mathscr{A})x\rangle.$$

Since $y^*_\lambda \neq 0$, we have

$$\lambda\|x\| \leq \|(\lambda I - \mathscr{A})x\| = \Re\langle g^*_\lambda, (\lambda I - \mathscr{A})x\rangle = \lambda\langle g^*_\lambda, x\rangle - \Re\langle g^*_\lambda, \mathscr{A}x\rangle.$$

Hence,

$$\Re\langle g^*_\lambda, \mathscr{A}x\rangle \leq \lambda\langle g^*_\lambda, x\rangle - \lambda\|x\| \leq \|g^*_\lambda\|\,\|x\| = \|x\|.$$

Therefore,

$$\Re\langle g^*_\lambda, \mathscr{A}x\rangle \leq 0. \qquad (**)$$

Idea: Let $\lambda \to +\infty$.

Unit ball in X^* is compact for weak* topology (Banach–Alaoglu).

(Up to subsequence)

$$g^*_\lambda \longrightarrow g^* \in X^*, \qquad \|g^*\| \leq 1.$$

Then from $(**)$,

$$\Re\langle g^*, \mathscr{A}x\rangle \leq 0.$$

$(*)$ $\qquad \|x\| \leq \langle g_\lambda^*, x\rangle - \frac{1}{\lambda}\Re\langle g_\lambda^*, \mathscr{A}x\rangle.$

Let $\lambda \to +\infty$. Then $\|x\| \leq \langle g^*, x\rangle$. Hence, $\|g^*\| = 1$ and $\langle g^*, x\rangle = \|x\|$. Set $x^* = \|x\| g^*$. Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x\rangle = \|x\|^2,$$

that is, $x^* \in J(x)$. □

Theorem 1.4.1 (Lumer-Phillips)**.** *Let X be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO. Assume that $\mathscr{A}$ is dissipative and that there exists $\lambda_0 > 0$ such that* $\mathrm{Rg}(\lambda_0 I - \mathscr{A}) = X$.

Then $\mathscr{A}$ is closed, $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$, and for all $\lambda > 0$,

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}. \tag{1.4}$$

Proof. Let $\lambda_0 > 0$.

(1) To prove: $(\lambda_0 I - A)$ is bijective.

- Surjective: Assumption,
- Injective: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathscr{A})^{-1} : X \longrightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any $y \in X$, there exists a unique $y \in X$ such that

$$x = (\lambda_0 I - \mathscr{A})^{-1} y, \qquad (\lambda_0 I - \mathscr{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0}\|y\| \geq \|(\lambda_0 I - \mathscr{A})^{-1} y\|.$$

Hence,

$$\|(\lambda_0 I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_0}, \qquad \lambda_0 \in \rho(\mathscr{A}).$$

(2) $\mathscr{A}$ is closed.

Let $x_n \to x$, $x_n \in D(\mathscr{A})$, and $Ax_n \to y$.
Then

$$(\lambda_0 I - \mathscr{A})x_n \to \lambda_0 x - y.$$

Since $(\lambda_0 I - \mathscr{A})^{-1} \in \mathcal{L}(X)$, we have

$$x_n \longrightarrow (\lambda_0 I - \mathscr{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathscr{A})x \iff y = \mathscr{A}x.$$

Therefore, $\mathscr{A}$ is closed.

(3) $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$ and (1.4).

Since $\mathscr{A}$ is closed and $\rho(\mathscr{A}) \neq \emptyset$, we know that $\rho(\mathscr{A})$ is open.

Let $\Lambda = \rho(\mathscr{A}) \cap \mathbb{R}_+^*$, which is open in $\mathbb{R}_+^*$. We show that it is closed.

Let $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda$ such that $\lambda_n \longrightarrow \lambda \in \mathbb{R}_+^*$. Note that since $\lambda_n \in \Lambda$, we have

$$\|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$(\lambda I - \mathscr{A}) = \big[I + u_n\big](\lambda_n I - \mathscr{A}) \Longrightarrow \lambda I - \mathscr{A} = \lambda_n I - \mathscr{A} + u_n(\lambda_n I - \mathscr{A}),$$

$$\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathscr{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathscr{A})^{-1} = u_n.$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For n large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that $\lambda \in \rho(\mathscr{A})$. Hence, Λ is closed, and therefore $\Lambda = \mathbb{R}_+^*$.

□

Corollary 1.4.1. *Let X be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO, closed, with $\overline{D(\mathscr{A})} = X$. Assume that $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative. Then*

$$\rho(\mathscr{A}) \supset \mathbb{R}_+^*, \qquad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathscr{A})^{-1}\| \leq 1.$$

Proof. It is easy to show that $\mathrm{Rg}(I - A) = X$ (i.e. $\lambda_0 = 1$ + Theorem 1.4.1).
$\mathscr{A}$ dissipative and closed implies

$$\mathrm{Rg}(I - \mathscr{A}) \text{ is a closed subspace of } X.$$

(give details!!!)

Let $x^* \in X^*$ such that

$$\langle x^*, (I - \mathscr{A})x \rangle = 0, \qquad \forall x \in D(\mathscr{A}). \tag{$**$}$$

Let us prove that $x^* = 0$.

Then $x^* \in D(\mathscr{A}^*)$ and

$$(I - \mathscr{A}^*)x^* = 0.$$

Since $\mathscr{A}^*$ is dissipative, by Lemma 1.4.1, we have $x^* = 0$. This implies that

$$\overline{\mathrm{Rg}(I - \mathscr{A})} = X.$$

Since $\mathrm{Rg}(I - \mathscr{A})$ is closed, we obtain

$$\mathrm{Rg}(I - \mathscr{A}) = X.$$

By contradiction and using Hahn–Banach. □

Definition 1.4.3 (Numerical Range)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be UBLO. The numerical range of $\mathscr{A}$, denoted by $W(\mathscr{A})$,*

$$W(\mathscr{A}) = \{\langle x^*, \mathscr{A}x\rangle \mid x^* \in J(x),\ x \in D(\mathscr{A}),\ \|x\| = \|x^*\| = 1,\ \langle x^*, x\rangle = 1\}.$$

In case of a Hilbert space, we have that $W(\mathscr{A}) = \{\langle x, \mathscr{A}x\rangle \mid x \in D(\mathscr{A}),\ \|x\| = 1\}$.

Linear algebra in finite dimension $\mathscr{A} \in \mathcal{M}_n(\mathbb{K})$, we have that $W(\mathscr{A}) = \{\langle x, \mathscr{A}x\rangle \mid \|x\| = 1\}$.

Theorem 1.4.2 (Home-work)**.** *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be closed, with $\overline{D(\mathscr{A})} = X$.*

1) If $\lambda \notin \overline{W(\mathscr{A})}$, then $(\lambda I - \mathscr{A})$ is injective, has closed image, and for all $x \in D(\mathscr{A})$,

$$\|(\lambda I - \mathscr{A})x\| \geq d(\lambda, W(\mathscr{A}))\,\|x\|.$$

Moreover, if $\lambda \in \rho(\mathscr{A})$, then

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, W(\mathscr{A}))}. \tag{$**$}$$

2) If Λ is a connected open subset of $\mathbb{C} \setminus W(\mathscr{A})$ such that $\rho(\mathscr{A}) \cap \Lambda \neq \emptyset$, then $\rho(\mathscr{A}) \supset \Lambda$ and $(**)$ holds true.

CHAPTER

TWO

INTRODUCTION TO THE THEORY OF SEMI-GROUPS

2.1 Intro to the Introduction

Definition 2.1.1. *Let X be a Banach space over $\mathbb{K}$. A one-parameter family of bounded linear operators on X, $(T(t))_{t\geq 0}$, is a semigroup (SG) of bounded linear operators on X if:*

1. $T(0) = Id_X$,
2. $\forall (t,s) \in \mathbb{R}_+^2 : \; T(t+s) = T(t)\cdot T(s)$ *(SG property).*

Remark 2.1.1. *$T(t)$ and $T(s)$ commute.*

2. **Infinitesimal generator of SG–LO** $(T(t))_{t\geq 0}$
Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be an unbounded linear operator with

$$D(\mathscr{A}) = \left\{ x \in X \;\middle|\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = \mathscr{A}x, \qquad x \in D(\mathscr{A}).$$

where $D(\mathscr{A})$ = domain of $\mathscr{A}$.

2.2 Uniformly Continuous SG–BLO

Definition 2.2.1. *A SG–BLO on X, $(T(t))_{t\geq 0}$ is uniformly continuous if*

$$\|T(t) - Id\|_{\mathscr{L}(X)} \to 0 \quad \text{as } t \to 0^+$$

Lemma 2.2.1. *Let $(T(t))_{t\geq 0}$ be a SG–BLO which is uniformly continuous. Then, $\forall t > 0$,*

$$\|T(s) - T(t)\| \underset{s\to t}{\longrightarrow} 0$$

(continuity everywhere).

Proof. Let t be fixed. $T(s) = T(s-t+t), \qquad s-t \geq 0.$

$$s \geq t \;\Rightarrow\; T(s) = T(s-t)T(t) \Rightarrow\; T(s) - T(t) = T(t)\,[T(s-t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\|\,\|T(s-t) - I_d\| \underset{s\to t}{\longrightarrow} 0.$$

For $s \leq t$

$$T(t) = T(t-s)T(s) \Rightarrow T(t) - T(s) = T(s)\,[T(t-s) - I_d]$$

(Prove that $\sup_{[0,t]} \|T(t)\| < +\infty$)

Then

$$\|T(t) - T(s)\| \leq \|T(s)\|\,\|T(t-s) - I_d\|$$

$$\leq \sup \|T(s)\|\,\|T(t-s) - I_d\| \underset{s\to t}{\longrightarrow} 0.$$

$\square$

Theorem 2.2.1. *A linear operator $\mathscr{A}$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $\mathscr{A}$ is a bounded linear operator.*

Proof. Let $\mathscr{A}$ be a bounded linear operator on X and set

$$T(t) = e^{t\mathscr{A}} = \sum_{n=0}^{\infty} \frac{(t\mathscr{A})^n}{n!}. \tag{1.5}$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such t, a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t+s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq |t|\|\mathscr{A}\|e^{|t|\|\mathscr{A}\|}$$

and

$$\left\|\frac{T(t) - I}{t} - \mathscr{A}\right\| \leq \|\mathscr{A}\| \cdot \max_{0\leq s\leq t} \|T(s) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on X and that $\mathscr{A}$ is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on X. Fix $\rho > 0$, small enough, such that

$$\left\|I - \rho\int_0^\rho T(s)\,ds\right\| < 1.$$

This implies that $\rho^{-1}\int_0^\rho T(s)\,ds$ is invertible. Now,

$$h^{-1}(T(h) - I)\int_0^\rho T(s)\,ds = h^{-1}\left(\int_0^\rho T(s+h)\,ds - \int_0^\rho T(s)\,ds\right)$$

$$= h^{-1}\left(\int_{\rho}^{\rho+h} T(s)\,ds - \int_0^h T(s)\,ds\right)$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1}\int_{\rho}^{\rho+h} T(s)\,ds - h^{-1}\int_0^h T(s)\,ds\right)\left(\int_0^{\rho} T(s)\,ds\right)^{-1}. \tag{1.6}$$

Letting $h \to 0$ in (1.6) shows that $h^{-1}(T(h) - I)$ converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I)\left(\int_0^{\rho} T(s)\,ds\right)^{-1}$$

which is the infinitesimal generator of $T(t)$. □

Remark 2.2.1. *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

Theorem 2.2.2. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\to 0}\frac{T(t) - I}{t} = \mathscr{A} = \lim_{t\to 0}\frac{S(t) - I}{t}. \tag{1.7}$$

then $T(t) = S(t)$ for $t \geq 0$.

Proof. We will show that given $T > 0$, $S(t) = T(t)$ for $0 \leq t \leq T$. Let $T > 0$ be fixed, since $t \mapsto \|T(t)\|$ and $t \mapsto \|S(t)\|$ are continuous there is a constant C such that

$$\|T(t)\|\|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given $\varepsilon > 0$ it follows from (1.7) that there is a $\delta > 0$ such that

$$h^{-1}\|T(h) - S(h)\| < \varepsilon/TC \qquad \text{for } 0 \leq h \leq \delta. \tag{1.8}$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n \leq \delta$. From the semigroup property and (1.8) it then follows that

$$\begin{aligned}
\|T(t) - S(t)\| &= \left\|T\left(n\frac{t}{n}\right) - S\left(n\frac{t}{n}\right)\right\| \\
&\leq \sum_{k=0}^{n-1}\left\|T\left((n-k)\frac{t}{n}\right)S\left(\frac{kt}{n}\right) - T\left((n-k-1)\frac{t}{n}\right)S\left(\frac{(k+1)t}{n}\right)\right\| \\
&\leq \sum_{k=0}^{n-1}\left\|T\left((n-k-1)\frac{t}{n}\right)\right\|\left\|T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right)\right\|\left\|S\left(\frac{t}{n}\right)\right\| \leq Cn\frac{\varepsilon}{TC}\frac{t}{n} \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $T(t) = S(t)$ for $0 \leq t \leq T$ and the proof is complete. □

Corollary 2.2.1. *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

(a) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.

(b) There exists a unique bounded linear operator $\mathscr{A}$ such that $T(t) = e^{t\mathscr{A}}$.

(c) The operator $\mathscr{A}$ in part (b) is the infinitesimal generator of $T(t)$.

(d) $t \mapsto T(t)$ is differentiable in norm and satisfies

$$\frac{dT(t)}{dt} = \mathscr{A}T(t) = T(t)\mathscr{A}. \tag{1.9}$$

Proof. All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator $\mathscr{A}$. $\mathscr{A}$ is also the infinitesimal generator of $e^{t\mathscr{A}}$ defined by (1.5) and therefore, by Theorem 2.2.2, $T(t) = e^{t\mathscr{A}}$. □

Remark 2.2.2. *The proofs above are from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 3. [Theorem 1.3 and Corollary 1.4].*

2.3 Strongly Continuous Semigroups (C_0-Semigroups)

Definition 2.3.1. *The SG-BLO $(T(t))_{t\geq 0}$ is strongly continous (SC or C_0) if $\forall x \in X$*

$$\|T(t)x - x\|_X \xrightarrow{t\to 0^+} 0$$

Theorem 2.3.1. *Let $(T(t))_{t\geq 0}$, C_0-SG then $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0$ $\|T(t)\| \leq Me^{\omega t}$*

Proof. First we want to show that $\exists \eta > 0$, $\sup_{t\in[0,\eta]} \|T(t)\| < +\infty$.

By contradiction, assume that $\sup \|T(t)\| = +\infty$. Then $\exists (t_n)_{n\geq 0} \searrow 0$ such that $\|T(t_n)\| \geq n$ or $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive) $\exists x \in X$ such that $\sup \|T(t_n)x\| = +\infty$, but this contradicts the strong convergance 2.3.1.

Now take $M := \sup \|T(t)\| \geq 1$ (This is because $T(0) = Id$).

$\forall t \geq 0$ write $t = k\eta + \eta_t$ where $k = \left\lfloor \frac{t}{\eta} \right\rfloor$ and $\eta_t \in [0, \eta]$, then

$$\begin{aligned}
\|T(t)\| &= \|T(k\eta + \eta_t)\| \\
&= \left\| [T(\eta)]^k \, T(\eta_t) \right\| && \text{by the SG property} \\
&= \|T(\eta)\|^k \, \|T(\eta_t)\| \\
&\leq M \cdot M^k && \text{M is upperbound} \\
&\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\
&= M(e^{\ln M})^{t/\eta} \\
&= Me^{t\frac{\ln M}{\eta}} = Me^{\omega t}
\end{aligned}$$

And since $\omega = \dfrac{\ln M}{\eta}$ and $M \geq 1$ then $\omega \geq 0$. $\square$

Corollary 2.3.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup then $\forall x \in X$, $t \mapsto T(t)x$ is continuous*

Proof. For $h > 0$ and $t > 0$:

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \to 0$ as $h \to 0$ by definition 2.3.1.
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$. Since $\|T(t-h)\|$ is bounded by theorem 2.3.1, so this tends to 0 as $h \to 0$.

$\square$

Theorem. *Let $\mathscr{A}$ be the IG of C_0-SG $T(t)$, then:*

2.3.2 $\forall x \in X, \forall t \geq 0,\ \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$ *for all* $x \in X$.

2.3.3 $\forall x \in X, \forall t \geq 0,\ \int_0^t T(s)x\,ds \in D(\mathscr{A})$ *and* $\mathscr{A}\int_0^t T(s)x\,ds = T(t)x - x$.

2.3.4 $\forall x \in D(\mathscr{A})$, $T(t)x \in D(\mathscr{A})$ *and* $\frac{d}{dt}T(t)x = \mathscr{A}T(t)x = T(t)\mathscr{A}x$.

2.3.5 $\forall x \in D(\mathscr{A}), \forall t \geq 0, \forall s \geq 0,\ T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x\,du$ *for* $x \in D(\mathscr{A})$.

*Proof of **2.3.2**.* Consider the small interval $[t, t+h]$ relative to its value at t:

$$\frac{1}{h}\int_t^{t+h} T(s)x\,ds - T(t)x = \frac{1}{h}\int_t^{t+h} [T(s)x - T(t)x]\,ds$$

By the continuity (corollary 2.3.1) of the map $s \mapsto T(s)x$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all s satisfying $|s - t| < \delta$, we have $\|T(s)x - T(t)x\| < \epsilon$.

Taking $0 < h < \delta$, we can estimate the norm of the integral:

$$\left\| \frac{1}{h}\int_t^{t+h} [T(s)x - T(t)x]\,ds \right\| \leq \frac{1}{h}\int_t^{t+h} \|T(s)x - T(t)x\|\,ds < \frac{1}{h}\cdot h\epsilon = \epsilon$$

Then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$$

$\square$

*Proof of **2.3.3**.* Let $h > 0$. Consider the difference quotient for the integral $y = \int_0^t T(s)x\,ds$:

$$\begin{aligned}
\frac{T(h) - Id}{h}\int_0^t T(s)x\,ds &= \frac{1}{h}\left[\int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds\right] \\
&= \frac{1}{h}\left[\int_h^{t+h} T(u)x\,du - \int_0^t T(u)x\,du\right] \\
&= \frac{1}{h}\int_t^{t+h} T(u)x\,du - \frac{1}{h}\int_0^h T(u)x\,du
\end{aligned}$$

As $h \to 0^+$, the first term converges to $T(t)x$ and the second to $T(0)x = x$ by 2.3.2. Thus the limit exists, $y \in D(\mathscr{A})$, and $\mathscr{A}y = T(t)x - x$. $\square$

Proof of 2.3.4. If $x \in D(\mathscr{A})$, then $T(t)\mathscr{A}x = T(t)\lim_{h\to 0}\frac{T(h)x-x}{h} = \lim_{h\to 0}\frac{T(h)T(t)x-T(t)x}{h}$. This limit exists and equals $\mathscr{A}(T(t)x)$, proving $T(t)x \in D(\mathscr{A})$ and $T(t)\mathscr{A}x = \mathscr{A}T(t)x$. This also shows the right-derivative of $T(t)x$ is $\mathscr{A}T(t)x$. A similar argument for the left-derivative completes the differentiability. $\square$

Proof of 2.3.5. By Property 2.3.4, the function $f(u) = T(u)x$ is differentiable with $f'(u) = T(u)\mathscr{A}x$. Since f' is continuous, we integrate f' over $[s,t]$ to obtain $f(t) - f(s) = \int_s^t f'(u)\,du$, which is $T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x\,du$. $\square$

Theorem 2.3.6. *The IG $\mathscr{A}$ of C_0-SG is a closed linear operator and $\overline{D(\mathscr{A})} = X$.*

Proof. For any $x \in X$, let $x_t = \frac{1}{t}\int_0^t T(s)x\,ds$. By 2.3.3, $x_t \in D(\mathscr{A})$. By 2.3.2, $x_t \to T(0)x = x$ as $t \to 0^+$, which shows $\overline{D(\mathscr{A})} = X$.

Let $x_n \in D(\mathscr{A})$ such that $x_n \to x$ and $\mathscr{A}x_n \to y$. From 2.3.3 take $s = 0$, we have $T(t)x_n - x_n = \int_0^t T(s)\mathscr{A}x_n\,ds$. Passing to the limit $n \to \infty$, we get $T(t)x - x = \int_0^t T(s)y\,ds$. Dividing by t and letting $t \to 0^+$, the Right-Hand Side (RHS) converges to y. Thus $x \in D(\mathscr{A})$ and $\mathscr{A}x = y$. $\square$

Theorem 2.3.7. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be two C_0-SG with infinitesimal generators $\mathscr{A}$ and B, respectively. If $\mathscr{A} = \mathscr{B}$, then $T(t) = S(t)$ for all $t \geq 0$.*

Proof. Assume $\mathscr{A} = \mathscr{B}$, then $D(\mathscr{A}) = D(\mathscr{B})$. Let $x \in D(\mathscr{A})$ be fixed, and for a fixed $t > 0$, define

$$\varphi : [0,t] \to X, \quad \varphi(s) = T(t-s)S(s)x$$

Since $x \in D(\mathscr{A})$, the map φ is of class C^1 on $[0,t]$. We differentiate φ with respect to s using the product rule and 2.3.4, we get:

$$\begin{aligned}\frac{d}{ds}\varphi(s) &= \frac{d}{ds}[T(t-s)]S(s)x + T(t-s)\frac{d}{ds}[S(s)x] \\ &= -\mathscr{A}T(t-s)S(s)x + T(t-s)BS(s)x\end{aligned}$$

Because $T(t-s)$ commutes with its generator $\mathscr{A}$, and given $\mathscr{A} = B$, we have:

$$\frac{d}{ds}\varphi(s) = -T(t-s)\mathscr{A}S(s)x + T(t-s)\mathscr{A}S(s)x = 0$$

Since the derivative is zero for all $s \in [0,t]$, the function φ must be constant. Evaluating φ at the endpoints $s = 0$ and $s = t$ yields

$$\varphi(0) = T(t)S(0)x = T(t)x \quad \text{and} \quad \varphi(t) = T(0)S(t)x = S(t)x$$

Thus, $T(t)x = S(t)x$ for all $x \in D(\mathscr{A})$. Since $D(\mathscr{A})$ is dense in X and $T(t), S(t)$ are bounded linear operators, this identity extends to all $x \in X$ by continuity. Therefore, $T(t) = S(t)$ for all $t \geq 0$. □

Theorem 2.3.8. *Let $\mathscr{A}$ be the IG of a C_0-SG $\{T(t)\}_{t\geq 0}$ on a Banach space X. Then the subspace*

$$X = \overline{\bigcap_{n\geq 1} D(\mathscr{A}^n)}$$

Proof. Let $\mathscr{D} = \{\varphi : \mathbb{R} \to \mathbb{C} |\, \varphi \text{ has compact support in } \mathbb{R}_+^* \text{ and is smooth } C^\infty\}$. Let $x \in X$ and consider a test function $\varphi \in \mathscr{D}$. Define

$$x_\varphi = \int_0^\infty \varphi(s)T(s)x\, ds$$

First, we show that $x_\varphi \in D(\mathscr{A})$. Consider

$$\begin{aligned} \frac{T(h) - Id}{h} x_\varphi &= \frac{1}{h}\int_0^\infty \varphi(s)[T(s+h)x - T(s)x]\, ds \\ &= \frac{1}{h}\left[\int_h^\infty \varphi(u-h)T(u)x\, du - \int_0^\infty \varphi(u)T(u)x\, du\right] \\ &= \int_0^\infty \frac{\varphi(u-h) - \varphi(u)}{h} T(u)x\, du \end{aligned}$$

As $h \to 0$, the quotient $\frac{\varphi(u-h)-\varphi(u)}{h}$ converges uniformly to $-\varphi'(u)$ because φ is C^∞ and has compact support. Thus:

$$\mathscr{A}x_\varphi = -\int_0^\infty \varphi'(s)T(s)x\, ds$$

Since $\varphi' \in C_c^\infty(0, \infty)$, we can repeat this process inductively. For any $n \geq 1$, we find:

$$\mathscr{A}^n x_\varphi = (-1)^n \int_0^\infty \varphi^{(n)}(s)T(s)x\, ds$$

This proves that $x_\varphi \in D(\mathscr{A}^n)$ for all n.

To prove density, suppose $\overline{\bigcap_{n\geq 1} D(\mathscr{A}^n)} \neq X$. By the Hahn-Banach Theorem, there exists a non-zero functional $x^* \in X^*$ such that $\langle x^*, y\rangle = 0$ for all $y \in \bigcap_{n\geq 1} D(\mathscr{A}^n)$. Specifically, for any $x \in X$ and $\varphi \in C_c^\infty(0, +\infty)$:

$$\langle x^*, x_\varphi\rangle = \int_0^{+\infty} \varphi(s)\langle x^*, T(s)x\rangle\, ds = 0$$

This identity holds for all C^∞ functions φ with compact support. Then $\langle x^*, T(s)x\rangle$ must be zero for all $s > 0$.

By the strong continuity of the semigroup at $s = 0$:

$$\langle x^*, x\rangle = \lim_{s\to 0^+} \langle x^*, T(s)x\rangle = 0$$

Since this holds for all $x \in X$, it implies $x^* = 0$, which contradicts our assumption that x^* was non-zero. Thus, $\bigcap_{n\geq 1} D(\mathscr{A}^n)$ must be dense in X. $\square$

Exercise 10

Let $X = \{f : \mathbb{R} \to \mathbb{C} \mid f$ is continuous and uniformly bounded$\}$ equipped with the supremum norm $\|f\|_\infty = \sup_{s\in\mathbb{R}} |f(s)|$. Define the family of operators $(T(t))_{t\geq 0}$ by:

$$(T(t)f)(s) = f(s+t), \quad s \in \mathbb{R}, t \geq 0$$

Prove that this family is C_0-SG, its IG is $\mathscr{A} f = f'$, and $\|T(t)\| = 1$.

2.4 Hille-Yosida Theorem

Definition 2.4.1. *A C_0-SG $\{T(t)\}_{t\geq 0}$ is called uniformly bounded semigroup if $\exists M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.*

Definition 2.4.2. *A C_0-SG $\{T(t)\}_{t\geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.*

Theorem 2.4.1 (Hille-Yosida Theorem (Contraction Case))**.** *A linear unbounded operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ is the IG of a C_0-SG of contractions if and only if:*

(i) $\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = X$.

(ii) $\mathbb{R}_+^* \subset \rho(\mathscr{A})$ and $\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

($\Rightarrow$). (i) follows directly from 2.3.6.

If $\mathscr{A}$ generates a contraction semigroup $\{T(t)\}_{t\geq 0}$, we define the resolvent for $\lambda > 0$ as follows

$$R(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t)\, dt$$

Taking the norm, we obtain:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)x\| \; dt \leq \int_0^{+\infty} e^{-\lambda t} \|x\| \; dt = \frac{1}{\lambda} \|x\|$$

$\square$

Remark 2.4.1. *Note for all real numbers λ, a with $\lambda > a$, we have the following:*

$$\frac{1}{\lambda - a} = \int\limits_0^{+\infty} e^{-(\lambda - a)t}\, dt$$

Extending this to the vector space we get the way of writing the resolvent operator from above.

2.4.1 The Yosida Approximation

To prove if ($\Leftarrow$), we introduce a family of bounded operators that approximate the unbounded generator A.

Definition 2.4.3. *For $\lambda > 0$, the Yosida Approximation of $\mathscr{A}$ is defined as:*

$$\mathscr{A}_\lambda := \lambda \mathscr{A} R(\lambda, \mathscr{A}) = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$$

Note that $\mathscr{A}_\lambda$ is a bounded linear operator for each $\lambda \in \rho(\mathscr{A})$.

Claim 2.4.1. *For $\lambda \in \rho(\mathscr{A})$ and $x \in D(\mathscr{A})$, the following identity holds:*

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A} x$$

Proof. By the definition of the resolvent as the inverse of the operator $(\lambda I - \mathscr{A})$, we have:

$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A}) = Id_X$$

Applying this to any $x \in X$:

$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A})x = x$$

Distributing the operators on the left-hand side gives:

$$\lambda R(\lambda, \mathscr{A})x - \mathscr{A} R(\lambda, \mathscr{A})x = x$$

Rearranging the terms to isolate the $\mathscr{A} R(\lambda, \mathscr{A})x$ term yields:

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x \tag{*}$$

This identity holds for all $x \in X$ because $R(\lambda, \mathscr{A})$ maps X into $D(\mathscr{A})$.

For the other side, let $x \in D(\mathscr{A})$. We use the fact that the resolvent also satisfies:

$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A}) = Id_{D(\mathscr{A})}$$

Applying this to $x \in D(\mathscr{A})$:

$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A})x = x$$

Distributing $R(\lambda, \mathscr{A})$ gives:

$$\lambda R(\lambda, \mathscr{A})x - R(\lambda, \mathscr{A})\mathscr{A} x = x$$

Rearranging the terms:

$$\lambda R(\lambda, \mathscr{A})x - x = R(\lambda, \mathscr{A})\mathscr{A} x \tag{**}$$

From (*) and (**) we get:

$$\mathscr{A}R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A}x$$

$\square$

Theorem 2.4.2. *For all $x \in X$, $\lim\limits_{\lambda \to +\infty} \mathscr{A}_\lambda x = \mathscr{A}x$.*

Proof. Using the identity $\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A}R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A}x$ for $x \in D(\mathscr{A})$, we observe:

$$\|\lambda R(\lambda, \mathscr{A})x - x\| = \|R(\lambda, \mathscr{A})\mathscr{A}x\| \leq \frac{\|\mathscr{A}x\|}{\lambda} \xrightarrow{\lambda \to +\infty} 0$$

Since $\mathscr{A}_\lambda x = \lambda R(\lambda, \mathscr{A})\mathscr{A}x$, and we just showed $\lambda R(\lambda, \mathscr{A})x \to x, \ \forall x \in D(\mathscr{A})$, it follows that $\mathscr{A}_\lambda x \to \mathscr{A}x$. By density and the uniform bound $\|\lambda R(\lambda, \mathscr{A})\| \leq 1$, this convergence holds for all $x \in X$. $\square$

Lemma 2.4.1. *For each $\lambda > 0$, $\mathscr{A}_\lambda$ generates a uniformly continuous semigroup of contractions $\{e^{t\mathscr{A}_\lambda}\}_{t \geq 0}$.*

Proof. Since $\mathscr{A}_\lambda = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$, we have:

$$\left\|e^{t\mathscr{A}_\lambda}\right\| = \left\|e^{-t\lambda} e^{t\lambda^2 R(\lambda, \mathscr{A})}\right\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda, \mathscr{A})\|} \leq e^{-t\lambda} e^{t\lambda^2 \frac{1}{\lambda}} = e^{-t\lambda} e^{t\lambda} = 1$$

This confirms the contraction property for the approximating semigroups. $\square$

Lemma 2.4.2. *For any $x \in X$ and $\lambda, \mu > 0$, the following estimate holds:*

$$\left\|e^{t\mathscr{A}_\lambda} x - e^{t\mathscr{A}_\mu} x\right\| \leq t \left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\|$$

Proof. Fix $x \in X$, consider the function $\phi : [0, 1] \to X$ defined by:

$$\phi(s) = e^{st\mathscr{A}_\lambda} e^{(1-s)t\mathscr{A}_\mu} x$$

Since $\mathscr{A}_\lambda$ and $\mathscr{A}_\mu$ commute (as resolvents commute), the semigroups $e^{t\mathscr{A}_\lambda}$ and $e^{t\mathscr{A}_\mu}$ also commute. The function ϕ is differentiable with respect to s:

$$\begin{aligned}\frac{d}{ds}\phi(s) &= t\mathscr{A}_\lambda e^{st\mathscr{A}_\lambda} e^{(1-s)t\mathscr{A}_\mu} x - e^{st\mathscr{A}_\lambda} t\mathscr{A}_\mu e^{(1-s)t\mathscr{A}_\mu} x \\ &= t e^{st\mathscr{A}_\lambda} e^{(1-s)t\mathscr{A}_\mu} (\mathscr{A}_\lambda - \mathscr{A}_\mu) x\end{aligned}$$

Integrating from 0 to 1:

$$\phi(1) - \phi(0) = e^{t\mathscr{A}_\lambda} x - e^{t\mathscr{A}_\mu} x = \int_0^1 t e^{st\mathscr{A}_\lambda} e^{(1-s)t\mathscr{A}_\mu} (\mathscr{A}_\lambda - \mathscr{A}_\mu) x \, ds$$

Taking the norm and using the contraction property $\|e^{t\mathscr{A}_\lambda}\| \leq e^{t\|\mathscr{A}_\lambda\|} \leq 1$ (since $\mathscr{A}_\lambda$ is dissipative):

$$\begin{aligned}\left\|e^{t\mathscr{A}_\lambda}x - e^{t\mathscr{A}_\mu}x\right\| &\leq \int_0^1 t\left\|e^{st\mathscr{A}_\lambda}\right\|\left\|e^{(1-s)t\mathscr{A}_\mu}\right\|\left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\|\, ds \\ &\leq t\left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\|\end{aligned}$$

□

Now let's start proving the only if direction of Hille-Yosida ($\Longleftarrow$).

Proof. We need the sequence $(e^{t\mathscr{A}_\lambda}x)_{\lambda>0}$ to be converging. But doing so is hard, what we can do is show that it is a Cauchy sequence in X, since X is complete.

Taking $x \in D(\mathscr{A})$ we have the following:

$$\left\|e^{t\mathscr{A}_\lambda}x - e^{t\mathscr{A}_\mu}x\right\| \leq t\left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\| \leq t(\|\mathscr{A}_\lambda x - \mathscr{A}x\| + \left\|\mathscr{A}x - \mathscr{A}_\mu x\right\|) \xrightarrow[]{\substack{\lambda\to+\infty\\ \mu\to+\infty}} 0$$

This is possible because $\lim_{\lambda\to+\infty}\mathscr{A}_\lambda x = \mathscr{A}x$.

Define:

$$T(t)x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}x$$

This limit is well defined by the argument above for all $x \in D(\mathscr{A})$. Moreover, since $D(A)$ is dense in X and $\|e^{tA_\lambda}\| \leq 1$, we can extend $T(t)$ to a bounded linear operator on all of X by density.

Now we show that the family $(T(t))_{t\geq 0}$ defined above is a C_0-semigroup of contractions.

We verify the semigroup properties:

1. Identity: $T(0)x = \lim_{\lambda\to+\infty} e^{0\cdot\mathscr{A}_\lambda}x = x$.

2. Semigroup Property: For $x \in X$,

$$T(t+s)x = \lim_{\lambda\to+\infty} e^{(t+s)\mathscr{A}_\lambda}x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}e^{s\mathscr{A}_\lambda}x = T(t)T(s)x.$$

3. Strong Continuity: For $x \in D(\mathscr{A})$, convergence is uniform on compact intervals of t, because of lemma 2.4.2. Thus $t \mapsto T(t)x$ is continuous. By the density of $D(A)$ and uniform boundedness $\|T(t)\| \leq 1$, continuity extends to all $x \in X$.

We must show that the generator of the constructed semigroup $T(t)$ is indeed $\mathscr{A}$. Let $\mathscr{B}$ be the generator of $T(t)$, we show that $\mathscr{A} = \mathscr{B}$.

For any $x \in D(\mathscr{A})$, we have the identity:

$$e^{t\mathscr{A}_\lambda}x - x = \int_0^t e^{s\mathscr{A}_\lambda}\mathscr{A}_\lambda x\, ds$$

As $\lambda \to +\infty$, $e^{s\mathscr{A}_\lambda} \to T(s)$ strongly and uniformly on compact sets, and $\mathscr{A}_\lambda x \to \mathscr{A}x$. Passing to the limit:

$$T(t)x - x = \int_0^t T(s)\mathscr{A}x\, ds$$

Dividing by t and taking $t \to 0^+$:

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t T(s)\mathscr{A}x\, ds = T(0)\mathscr{A}x = \mathscr{A}x$$

Thus, $x \in D(\mathscr{B})$ and $\mathscr{B}x = \mathscr{A}x$, implying $\mathscr{A} \subset \mathscr{B}$. Since $\mathscr{B}$ is the generator of a C_0-semigroup of contractions, $1 \in \rho(\mathscr{B})$. By hypothesis, $1 \in \rho(\mathscr{A})$. Since $\mathscr{A} \subset \mathscr{B}$ and both $(I - \mathscr{A})$ and $(I - \mathscr{B})$ are surjective (mapping onto X), it follows that $\mathscr{A} = \mathscr{B}$. □

Corollary 2.4.1. *Let $\mathscr{A}$ be the IG of a C_0-SG of contractions $(T(t))_{t\geq 0}$. Then for every $x \in X$, the semigroup is given by the limit:*

$$T(t)x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}x$$

Proof. In the previous proof, we constructed a SG, let us call it $S(t)$, defined by $S(t)x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}x$. We proved that the generator of $S(t)$ is exactly the operator $\mathscr{A}$.

Since $\mathscr{A}$ is the generator of the original semigroup $T(t)$ by hypothesis, and we know that a C_0-SG is uniquely determined by its generator (Uniqueness Theorem), it follows that:

$$T(t) = S(t), \quad \forall t \geq 0$$

□

Corollary 2.4.2. *Let $\mathscr{A}$ be IG of a C_0-SG of contractions $(T(t))_{t\geq 0}$. Then the resolvent set $\rho(\mathscr{A})$ contains the open right half-plane:*

$$\rho(\mathscr{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$$

Furthermore, for all λ with $\Re(\lambda) > 0$, the following estimate holds:

$$\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\Re(\lambda)}$$

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. We define the operator $R(\lambda)$ on X by the Laplace transform of the semigroup:

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x\, dt, \quad \forall x \in X$$

Since $T(t)$ is a contraction semigroup ($\|T(t)\| \leq 1$) and $\Re(\lambda) > 0$, the integrand is exponentially bounded:

$$\left\| e^{-\lambda t} T(t)x \right\| = e^{-\Re(\lambda)t} \|T(t)x\| \leq e^{-\Re(\lambda)t} \|x\|$$

Thus, the integral converges absolutely, defining a bounded linear operator. We calculate its norm:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\Re(\lambda)t} \|x\| \, dt = \|x\| \left[\frac{-e^{-\Re(\lambda)t}}{\Re(\lambda)} \right]_0^{+\infty} = \frac{1}{\Re(\lambda)} \|x\|$$

This proves the bound $\|R(\lambda)\| \leq \dfrac{1}{\Re(\lambda)}$.

It remains to show that this integral operator $R(\lambda)$ is indeed the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$. For any $x \in X$ and $h > 0$:

$$\begin{aligned}
\frac{T(h) - I}{h} R(\lambda)x &= \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t+h)x \, dt - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \\
&= \frac{e^{\lambda h}}{h} \int_h^{+\infty} e^{-\lambda s} T(s)x \, ds - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt - \frac{e^{\lambda h}}{h} \int_0^{h} e^{-\lambda t} T(t)x \, dt
\end{aligned}$$

Taking the limit as $h \to 0^+$, the first term converges to $\lambda R(\lambda)x$ and the second term converges to $-x$. Thus, for any $x \in X$, $R(\lambda)x \in D(\mathscr{A})$ and $\mathscr{A} R(\lambda)x = \lambda R(\lambda)x - x$, which implies $(\lambda I - \mathscr{A})R(\lambda)x = x$. Similarly, one can show $R(\lambda)(\lambda I - \mathscr{A})x = x$ for $x \in D(\mathscr{A})$.

Therefore, $R(\lambda) = (\lambda I - \mathscr{A})^{-1}$. □

Exercise 11

Let $X = BVC(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{C} \mid f \text{ is bounded and uniformly continuous}\}$. Equipped with the supremum norm $\|f\|_\infty = \sup_{t \geq 0} |f(t)|$, $(X, \|\cdot\|_\infty)$ is a Banach space.
For $t \geq 0$, define the operator $T(t)$ by the left shift:

$$(T(t)f)(s) = f(s+t), \quad \forall s \geq 0$$

Check the following:

1. $(T(t))_{t \geq 0}$ is a C_0-semigroup of contractions (i.e., $\|T(t)\| \leq 1$).
2. Show that $\|T(t)\| = 1$.
3. The infinitesimal generator is the differentiation operator $\mathscr{A} f = f'$, with an appropriate domain $D(\mathscr{A})$.
4. Verify that $\rho(\mathscr{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$.

2.4.2 The General Hille-Yosida Theorem

We now consider the general case where the semigroup is not necessarily a contraction.

Theorem 2.4.3. *A linear operator $\mathscr{A}$ generates a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq M e^{\omega t}$ if and only if:*

(i) A is closed and densely defined.

(ii) $(\omega, +\infty) \subset \rho(\mathscr{A})$ *and for all* $\lambda > \omega$ *and* $n \geq 1$*:*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}.$$

2.4.3 Reduction to the Case $\omega = 0$

If $\|T(t)\| \leq Me^{\omega t}$, consider the rescaled semigroup $S(t) = e^{-\omega t}T(t)$. The generator of $S(t)$ is $\mathscr{A} - \omega I$ where $\mathscr{A}$ is the IG of $(T(t))_{t\geq 0}$, and $\|S(t)\| \leq M$. Conversely, if we prove the theorem for $\omega = 0$, the general case follows by applying the result to $\mathscr{A} - \omega I$.

Now we have the following corollary:

Corollary 2.4.3 (Hille-Yosida for $(1, \omega)$)**.** *A linear operator* $\mathscr{A}$ *is the IG of a* C_0*-SG satisfying* $\|T(t)\| \leq e^{\omega t}$ *if and only if:*

(i) $\mathscr{A}$ *is closed and* $\overline{D(\mathscr{A})} = X$*.*

(ii) $\rho(\mathscr{A}) \supset (\omega, +\infty)$ *and for all* $\lambda > \omega$*, the following estimate holds:*

$$\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda - \omega}.$$

Proof. ($\Rightarrow$): Suppose $\mathscr{A}$ generates a semigroup $T(t)$ such that $\|T(t)\| \leq e^{\omega t}$. Consider the rescaled family of operators $S(t) = e^{-\omega t}T(t)$. It is easy to verify that $S(t)$ is a C_0-SG. Furthermore, it is a contraction:

$$\|S(t)\| = e^{-\omega t}\|T(t)\| \leq e^{-\omega t}e^{\omega t} = 1$$

Let $\mathscr{B}$ be the generator of $S(t)$. By the definition of the generator:

$$\mathscr{B}x = \lim_{t\to 0^+} \frac{e^{-\omega t}T(t)x - x}{t} = \lim_{t\to 0^+} \left(e^{-\omega t}\frac{T(t)x - x}{t} + \frac{e^{-\omega t} - 1}{t}x \right) = \mathscr{A}x - \omega x$$

Thus, $\mathscr{B} = \mathscr{A} - \omega I$. Since $\mathscr{B}$ generates a contraction semigroup, by the Hille-Yosida Theorem for contractions (Case $M = 1, \omega = 0$), we know that for any $\mu > 0$, $\mu \in \rho(\mathscr{B})$ and $\|R(\mu, \mathscr{B})\| \leq \dfrac{1}{\mu}$.

Let $\lambda = \mu + \omega$. Then $\lambda > \omega$. Since $R(\mu, \mathscr{B}) = (\mu I - \mathscr{B})^{-1} = (\mu I - (\mathscr{A} - \omega I))^{-1} = ((\mu + \omega)I - \mathscr{A})^{-1}$, we have:

$$R(\lambda, \mathscr{A}) = R(\lambda - \omega, \mathscr{B})$$

Substituting the norm bound:

$$\|R(\lambda, \mathscr{A})\| = \|R(\lambda - \omega, \mathscr{B})\| \leq \frac{1}{\lambda - \omega}$$

($\Leftarrow$): Conversely, suppose $\mathscr{A}$ satisfies conditions (1)-(3). Define $\mathscr{B} = \mathscr{A} - \omega I$. Clearly, $\mathscr{B}$ is closed and densely defined. For any $\mu > 0$, let $\lambda = \mu + \omega > \omega$. Then $\lambda \in \rho(\mathscr{A})$, which implies

$\mu \in \rho(\mathscr{B})$. The resolvent satisfies:

$$\|R(\mu, \mathscr{B})\| = \|R(\mu + \omega, \mathscr{A})\| \leq \frac{1}{(\mu + \omega) - \omega} = \frac{1}{\mu}$$

Thus, $\mathscr{B}$ satisfies the Hille-Yosida conditions for the contraction case ($M = 1, \omega = 0$). Therefore, $\mathscr{B}$ generates a contraction semigroup $S(t)$ with $\|S(t)\| \leq 1$. Defining $T(t) = e^{\omega t} S(t)$, we see that $T(t)$ is a C_0-semigroup generated by $\mathscr{A} = \mathscr{B} + \omega I$, and it satisfies:

$$\|T(t)\| = e^{\omega t} \|S(t)\| \leq e^{\omega t}$$

□

This corollary provides us with a method to rescale the bound. Thus, we focus on the case $(M, 0)$, i.e., $\|T(t)\| \leq M$. The condition on the resolvent becomes $\|R(\lambda, \mathscr{A})^n\| \leq \dfrac{M}{\lambda^n}$.

Lemma 2.4.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup satisfying $\|T(t)\| \leq M$ for all $t \geq 0$. Let $\mathscr{A}$ be its infinitesimal generator. Then for all $\lambda > 0$ and all integers $n \geq 0$:*

$$\|R(\lambda, \mathscr{A})^n\| \leq \frac{M}{\lambda^n}$$

Equivalently, $\|\lambda^n R(\lambda, \mathscr{A})^n\| \leq M$.

Proof. For $\lambda > 0$, the resolvent is given by the Laplace transform of the semigroup:

$$R(\lambda, \mathscr{A})x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since the integral converges absolutely (due to the exponential decay $e^{-\lambda t}$ and bounded $T(t)$), we can differentiate this expression with respect to λ inside the integral sign. Differentiating $n - 1$ times:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathscr{A})x = \int_0^{+\infty} (-t)^{n-1} e^{-\lambda t} T(t)x \, dt$$

On the other hand, from the general theory of resolvents, we have the identity:

$$\frac{d^k}{d\lambda^k} R(\lambda, \mathscr{A}) = (-1)^k k! R(\lambda, \mathscr{A})^{k+1}$$

Setting $k = n - 1$, we get:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathscr{A}) = (-1)^{n-1} (n-1)! R(\lambda, \mathscr{A})^n$$

Equating the two expressions for the derivative:

$$(-1)^{n-1} (n-1)! R(\lambda, \mathscr{A})^n x = \int_0^{+\infty} (-1)^{n-1} t^{n-1} e^{-\lambda t} T(t)x \, dt$$

Simplifying and solving for $R(\lambda, \mathscr{A})^n x$:

$$R(\lambda, \mathscr{A})^n x = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t) x \, dt$$

Now, we take the norm and use the bound $\|T(t)\| \le M$:

$$\begin{aligned} \|R(\lambda, \mathscr{A})^n x\| &\le \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \|T(t)x\| \, dt \\ &\le \frac{M \|x\|}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt \end{aligned}$$

The integral on the right is the Gamma function definition. Substituting $u = \lambda t$ ($dt = du/\lambda$):

$$\int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt = \frac{1}{\lambda^n} \int_0^{+\infty} u^{n-1} e^{-u} \, du = \frac{\Gamma(n)}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

Substituting this back into the inequality:

$$\|R(\lambda, \mathscr{A})^n x\| \le \frac{M \|x\|}{(n-1)!} \cdot \frac{(n-1)!}{\lambda^n} = \frac{M}{\lambda^n} \|x\|$$

Thus, $\|R(\lambda, \mathscr{A})^n\| \le \dfrac{M}{\lambda^n}$. □

2.4.4 Renorming Lemma

The idea is to construct an equivalent norm on X under which $\mathscr{A}$ becomes dissipative (generating a contraction semigroup), allowing us to apply the contraction case result.

Lemma 2.4.4. *Let $A : D(A) \subset X \to X$ be a linear operator with $\rho(A) \supset \mathbb{R}_+^*$ such that for all $\lambda > 0$ and $n \ge 0$, $\|\lambda^n R(\lambda, A)^n\| \le M$. Then, there exists a norm $\|\cdot\|_\mu$ on X such that:*

1. *The norms are equivalent: $\|x\| \le \|x\|_\mu \le M\|x\|$ for all $x \in X$.*
2. *For all $\lambda > 0$, the operator $\lambda R(\lambda, A)$ is a contraction in the new norm: $\|\lambda R(\lambda, A)x\|_\mu \le \|x\|_\mu$.*

Proof. Fix $\mu > 0$. We define the new norm $\|\cdot\|_\mu$ by:

$$\|x\|_\mu = \sup_{n \ge 0} \|\mu^n R(\mu, A)^n x\|$$

To show that the norms are equivalent, let $n = 0$, the term is $\|Ix\| = \|x\|$ so $\|x\| \le \|x\|_\mu$. Using the hypothesis $\|\mu^n R(\mu, A)^n\| \le M$, we have:

$$\|x\|_\mu = \sup_{n \ge 0} \|\mu^n R(\mu, A)^n x\| \le \sup_{n \ge 0} M \|x\| = M \|x\|$$

Thus, $\|x\| \le \|x\|_\mu \le M \|x\|$.

Now to show that $\mu R(\mu, A)$ is a contraction, we check the contraction property for the specific value μ:

$$\begin{aligned}
\|\mu R(\mu, A)x\|_\mu &= \sup_{n\geq 0} \|\mu^n R(\mu, A)^n (\mu R(\mu, A)x)\| \\
&= \sup_{n\geq 0} \left\|\mu^{n+1} R(\mu, A)^{n+1} x\right\| \\
&= \sup_{k\geq 1} \left\|\mu^k R(\mu, A)^k x\right\| \\
&\leq \sup_{k\geq 0} \left\|\mu^k R(\mu, A)^k x\right\| = \|x\|_\mu .
\end{aligned}$$

So, $\|\mu R(\mu, A)\|_\mu \leq 1$.

Finally, we show the contraction for $0 < \lambda \leq \mu$:

Let $x \in X$ and define $y = R(\lambda, \mathscr{A})x$. We want to show $\lambda \|y\|_\mu \leq \|x\|_\mu$. Recall the Resolvent Identity:

$$R(\lambda, \mathscr{A}) - R(\mu, \mathscr{A}) = (\mu - \lambda) R(\mu, \mathscr{A}) R(\lambda, \mathscr{A})$$

Applying this to x:

$$y = R(\mu, \mathscr{A})x + (\mu - \lambda) R(\mu, \mathscr{A}) y = R(\mu, \mathscr{A})[x + (\mu - \lambda) y].$$

Taking the $\|\cdot\|_\mu$ norm and using the fact that $\|\mu R(\mu, A)z\|_\mu \leq \|z\|_\mu \implies \|R(\mu, A)z\|_\mu \leq \frac{1}{\mu} \|z\|_\mu$:

$$\begin{aligned}
\|y\|_\mu &= \|R(\mu, A)[x + (\mu - \lambda) y]\|_\mu \\
&\leq \frac{1}{\mu} \|x + (\mu - \lambda) y\|_\mu \\
&\leq \frac{1}{\mu} \left(\|x\|_\mu + (\mu - \lambda) \|y\|_\mu\right)
\end{aligned}$$

Multiplying by μ:

$$\mu \|y\|_\mu \leq \|x\|_\mu + (\mu - \lambda) \|y\|_\mu$$

Subtracting $(\mu - \lambda) \|y\|_\mu$ from both sides (since $\mu - \lambda \geq 0$):

$$\lambda \|y\|_\mu \leq \|x\|_\mu .$$

Thus, $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$ for $0 < \lambda \leq \mu$. Since this holds for any sufficiently large μ, and the definition of the norm can be adjusted, this property extends to all $\lambda > 0$. □

Remark 2.4.2. *In equivalent norm, the following are presarved:*

1. *The closeness of an operator.*

2. *The density of the image of an operator.*

3. *The strong continuity of the C_0-SG.*

Theorem 2.4.4 (Hille-Yosida for $(M, 0)$)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a linear operator on a Banach space X. Then $\mathscr{A}$ is the IG of a C_0-SG $T(t)$ satisfying $\|T(t)\| \leq M$ for all $t \geq 0$ if and only if:*

(i) *$\mathscr{A}$ is closed and ($\overline{D(\mathscr{A})} = X$).*

(ii) *$\rho(\mathscr{A}) \supset (0, \infty)$ and for every $\lambda > 0$:*

$$\|(\lambda^n R(\lambda, \mathscr{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

Proof. $\Rightarrow$
Let $\mathscr{A}$ be the generator of a C_0-semigroup $T(t)$ with $\|T(t)\| \leq M$. By theorem 2.3.6 part (i) is done.

For part (ii) is direct from lemma 2.4.3

$\Leftarrow$
By using lemma 2.4.4 and apply it on the space the assumptions change to the following:

(i) $\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = X$.

(ii) $\mathbb{R}_+^* \subset \rho(\mathscr{A})$ and $\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

Which are just the conditions on Hille-Yosida for contractions and we get the desired result. □

Theorem 2.4.5 (Hille-Yosida for the Generale Case (M, ω))**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a linear operator on a Banach space X. Then $\mathscr{A}$ is the IG of a C_0-SG $T(t)$ satisfying $\|T(t)\| \leq M \cdot e^{\omega t}$ for all $t \geq 0$ if and only if:*

(i) *$\mathscr{A}$ is closed and ($\overline{D(\mathscr{A})} = X$).*

(ii) *$\rho(\mathscr{A}) \supset (0, \infty)$ and for every $\lambda > 0$:*

$$\|(\lambda^n R(\lambda, \mathscr{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

Theorem 2.4.6. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a UBLO and the IG of a C_0-SG $T(t)_{t \geq 0}$ then for all $x \in X$:*

$$T(t)x = \lim_{\lambda \to +\infty} e^{t\mathscr{A}_\lambda} x$$

Where $\mathscr{A}_\lambda$ is the Yosida Approximation of $\mathscr{A}$.

Proof. This can have 2 cases depending on ω.

First case, let $\omega = 0$. Then by applying lemma 2.4.4 we get a contraction and it is direct by corollary 2.4.1

Second case, let $\omega > 0$. We want to get a bound on the approximation.

$$\begin{aligned}
\left\|e^{t\mathscr{A}_\lambda}\right\| &= \left\|e^{t(\lambda^2 R(\lambda,\mathscr{A})-\lambda I)}\right\| \\
&= e^{-\lambda t}\left\|e^{t\lambda^2 R(\lambda,\mathscr{A})}\right\| \\
&\leq e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(t\lambda^2)^k}{k!}\left\|R(\lambda,\mathscr{A})^k\right\| \\
&\leq e^{-\lambda t} M\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t\lambda^2}{\lambda-\omega}\right)^k \qquad\qquad \text{since } \left\|R(\lambda,\mathscr{A})^k\right\| \leq \frac{M}{(\lambda-\omega)^k} \\
&= Me^{-\lambda t}\exp\left(\frac{t\lambda^2}{\lambda-\omega}\right) \\
&= M\exp\left(-\lambda t+\frac{t\lambda^2}{\lambda-\omega}\right)
\end{aligned}$$

We can do the following calculation in the exponent:

$$\begin{aligned}
-\lambda+\frac{\lambda^2}{\lambda-\omega} &= \left(\frac{-\lambda(\lambda-\omega)+\lambda^2}{\lambda-\omega}\right) \\
&= \frac{\lambda\omega}{\lambda-\omega} \\
&= \omega\left(\frac{\lambda-\omega+\omega}{\lambda-\omega}\right) \\
&= \omega\left(1+\frac{\omega}{\lambda-\omega}\right) \\
&= \omega+\frac{\omega^2}{\lambda-\omega}
\end{aligned}$$

Now for all $\lambda \geq 2\omega$ we have $\lambda-\omega \geq \omega > 0$ and $\frac{1}{\lambda-\omega} \leq \frac{1}{\omega}$, hence:

$$\omega+\frac{\omega^2}{\lambda-\omega} \leq \omega+\frac{\omega^2}{\omega} = 2\omega$$

And we get the bound:

$$\left\|e^{t\mathscr{A}_\lambda}\right\| \leq M\cdot e^{2t\omega} \tag{*}$$

Let $S(t) = e^{-\omega t}T(t)$, this C_0-SG is generated by $\mathscr{A}-\omega I$ (rescaling of $T(t)$).
Since $S(t)$ is $(M,0)$ then by the first case we have:

$$\forall t \geq 0,\ \forall x \in X \quad S(t)x = \lim_{\lambda\to+\infty} e^{t(\mathscr{A}-\lambda I)_\lambda}$$

Equivalently:

$$e^{-\omega t}T(t)x = \lim_{\lambda\to+\infty} e^{t(\mathscr{A}-\lambda I)_\lambda}$$

Now we have to show that $\forall t \geq 0, \forall x \in X$ the following:

$$e^{t[(\mathscr{A}-\omega I)_\lambda+\omega I]}x - e^{t\mathscr{A}_\lambda}x = 0$$

We have the following:

$$\begin{aligned}
e^{t[(\mathscr{A}-\omega I)_\lambda+\omega I]}x - e^{t\mathscr{A}_\lambda}x &= e^{t[(\mathscr{A}-\omega I)_\lambda+\omega I-\mathscr{A}_\lambda+\mathscr{A}_\lambda]}x - e^{t\mathscr{A}_\lambda}x \\
&= e^{t(A_\lambda+H(\lambda))}x - e^{tA_\lambda}x && \text{by setting } H(\lambda) := (\mathscr{A}-\omega I)_\lambda + \omega I - \mathscr{A}_\lambda \\
&= e^{tA_\lambda}e^{tH(\lambda)}x - e^{tA_\lambda}x \\
&= e^{tA_\lambda}\left(e^{tH(\lambda)} - I\right)x && \mathscr{A}_\lambda \text{ and } H(\lambda) \text{ commutes (why?)}
\end{aligned}$$

Now, we aim to show that for any fixed $x \in D(A)$, $H(\lambda)x \to 0$ as $\lambda \to +\infty$.

By theorem 2.4.2 we have:

$$\begin{aligned}
\lim_{\lambda\to\infty} H(\lambda)x &= \lim_{\lambda\to\infty}((A-\omega I)_\lambda x + \omega x - A_\lambda x) \\
&= (Ax - \omega x) + \omega x - Ax \\
&= 0
\end{aligned}$$

Thus, $H(\lambda)x \to 0$ for all $x \in D(A)$.

Now we have:

$$e^{t[(\mathscr{A}-\omega I)_\lambda+\omega I]}x - e^{t\mathscr{A}_\lambda}x = e^{tA_\lambda}\left(e^{tH(\lambda)} - I\right)x$$

By taking the norm:

$$\left\| e^{t[(\mathscr{A}-\omega I)_\lambda+\omega I]}x - e^{t\mathscr{A}_\lambda}x \right\| \le \left\| e^{tA_\lambda} \right\| \left\| \left(e^{tH(\lambda)} - I\right)x \right\|$$

From (*), we have that $\left\| e^{tA_\lambda} \right\|$ is uniformaly bounded hence we need to work on $\left\| \left(e^{tH(\lambda)} - I\right)x \right\|$.

Note that $e^{tH(\lambda)} = e^{t(B_\lambda+\omega I)}e^{-tA_\lambda}$. From the bound estimates shown earlier, both $\|e^{s(B_\lambda+\omega I)}\|$ and $\|e^{-sA_\lambda}\|$ are uniformly bounded for s in a compact interval $[0, T]$ and sufficiently large λ. Let C be this bound. Then:

$$\|(e^{tH(\lambda)} - I)x\| \le t \cdot C \cdot \|H(\lambda)x\|$$

Since we proved that $\lim_{\lambda\to\infty} \|H(\lambda)x\| = 0$, it follows immediately that:

$$\lim_{\lambda\to\infty} \|(e^{tH(\lambda)} - I)x\| = 0$$

Finally, combining this with the uniform bound on $\|e^{tA_\lambda}\|$, we obtain:

$$\lim_{\lambda\to\infty} \|e^{t(B_\lambda+\omega I)}x - e^{tA_\lambda}x\| = 0$$

This proves that the limit is the same, regardless of the shift. This completed our proof. □

2.5 Inverse Laplace Transform and Resolvent Formula

We have established that for a C_0-SG $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq Me^{t\omega}$, the resolvent is given using the following formula:

$$R(\lambda, \mathscr{A})x = \int_0^\infty e^{-\lambda t} T(t)x\, dt \quad \text{for } \mathrm{Re}(\lambda) > \omega.$$

which is just the Laplace Tranfrom of the semigroup.

A major problem in semigroup theory is recovering $T(t)$ from $R(\lambda, \mathscr{A})x$ (Inverting the Laplace Transform).

Lemma 2.5.1. *Let X be BS and $\mathscr{B} \in \mathscr{L}(X)$ then $\forall \gamma > \|\mathscr{B}\|$ we have:*

$$\begin{aligned} e^{t\mathscr{B}} &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda, \mathscr{B})d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{(\gamma+is)t} R(\gamma+is, \mathscr{B})i\,ds \end{aligned}$$

Proof. To make sense and motivate this proof we start by showing that this is correct for the scalar case. Let $b \in \mathbb{C}$ and the exponential function $f(t) = e^{tb}$. The Laplace transform of $f(t)$ is:

$$\mathcal{L}\{e^{tb}\}(\lambda) = \int_0^\infty e^{-\lambda t} e^{tb}\, dt = \frac{1}{\lambda - b}, \quad \text{for } \Re(\lambda) > \Re(b)$$

Now if we choose $|\lambda| > |b|$. We can expand $(\lambda - b)^{-1}$ as a convergent geometric series: Now if we choose γ such that $\Re(\gamma) > |b|$. Let show that:

$$e^{tb} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{1}{\lambda - b}\, d\lambda$$

For a circle $C(0, r)$ with radius $r > |b|$, we can expand the term $(\lambda - b)^{-1}$ as a geometric series for $|\lambda| > |b|$:

$$\frac{1}{\lambda - b} = \frac{1}{\lambda(1 - \frac{b}{\lambda})} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{b^k}{\lambda^k} = \sum_{k=0}^{\infty}\frac{b^k}{\lambda^{k+1}}$$

Plugging this into the formula above we get:

$$\begin{aligned} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{1}{\lambda - b}\, d\lambda &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\sum_{k=0}^{\infty}\frac{b^k}{\lambda^{k+1}}\, d\lambda \quad \text{(the choice of } \gamma \text{ allow for this)} \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty} b^k \int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda^{k+1}}\, d\lambda \quad \text{(The sum is uniformly convergent)} \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty} 2\pi i\frac{b^k t^k}{k!} \quad \text{(Cauchy Integral Formula)} \\ &= e^{bt} \end{aligned}$$

This shows that the formula works for the scalar case.

The case of the operator follows the same steps and same resoning.

Again let $C(0,r)$ with radius r such that $r > \|\mathscr{B}\|$ For any $\lambda \in C(0,r)$, we have $|\lambda| > \|\mathscr{B}\|$, which implies $\|\frac{\mathscr{B}}{\lambda}\| < 1$.

In this region, the resolvent $R(\lambda, \mathscr{B}) = (\lambda I - \mathscr{B})^{-1}$ admits a uniformly convergent Neumann series expansion:

$$R(\lambda, \mathscr{B}) = \frac{1}{\lambda}\left(I - \frac{\mathscr{B}}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{\mathscr{B}^k}{\lambda^{k+1}}.$$

We substitute this series into the contour integral over $C(0,r)$:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C(0,r)} e^{\lambda t} R(\lambda, \mathscr{B})\, d\lambda &= \frac{1}{2\pi i}\int_{C0,r} e^{\lambda t}\left(\sum_{k=0}^{\infty} \frac{\mathscr{B}^k}{\lambda^{k+1}}\right) d\lambda \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty} \mathscr{B}^k \left[\int_{C_r} \frac{e^{\lambda t}}{\lambda^{k+1}}\, d\lambda\right] \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty} \mathscr{B}^k \left[2\pi i \frac{t^k}{k!}\right] \\
&= \sum_{k=0}^{\infty} \frac{t^k \mathscr{B}^k}{k!} \\
&= e^{t\mathscr{B}}
\end{aligned}$$

□

2.6 Spectral Mapping Theorem

Let $(T(t))_{t\geq 0}$ be C_0-SG with $\mathscr{A}$ as its IG, then what is the relation betweem $\sigma(T(t))$ and $\sigma(\mathscr{A})$?

Exercise 12

Let $(T(t))_{t\geq 0}$ be UC-SG with $\mathscr{A}$ as its IG then $e^{t\sigma(\mathscr{A})} = \sigma(T(t))$.

Exercise 13

Take $X = \{f : [0,1] \to \mathbb{C} \mid f \text{ is continuous and } f(1) = 0\}$ then $(X, \|\cdot\|_\infty)$ is BS. Define:

$$T(t)f(x) = \begin{cases} \text{f(x + t)} & x + t \leq 1 \\ 0 & x + t > 1 \end{cases}$$

Then:

1. $(T(t))_{t \geq 0}$ is a C_0-SG.
2. $\mathscr{A}$ the IG and $\mathscr{A}f = f'$. Determine $D(\mathscr{A})$.
3. Check $\forall \lambda \in \mathbb{C}, \exists! f \in X$ such that $(\lambda I - \mathscr{A})f = g$ for all $g \in X$.
4. $\sigma(\mathscr{A}) = \emptyset$.
5. $\sigma(T(t)) \neq \emptyset$.

Lemma 2.6.1. *[Preparatory Lemma] Let $(T(t))_{t \geq 0}$ be a C_0-SG and let $\mathscr{A}$ be its IG. For all $t \geq 0$ and for all $x \in X$ define:*

$$\mathscr{B}_\lambda(t)x = \int_0^t e^{\lambda(t-s)} T(s)\, x\, ds$$

then $(\lambda I - \mathscr{A})\mathscr{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$.

Proof. It is enough to show that:

$$\mathscr{A}\mathscr{B}_\lambda(t)x = T(t)x + \lambda \mathscr{B}_\lambda(t)x - e^{\lambda t}x$$

Recall that:

$$\mathscr{A} = \lim_{h \to 0^+} \frac{T(h) - Id}{h}$$

Then we have:

$$\begin{aligned}
\frac{T(h) - Id}{h} B_\lambda(t)x &= \frac{1}{h} \int_0^t e^{\lambda(t-s)} \left(T(h)T(s) - T(s)\right)\, x\, ds \\
&= \frac{1}{h} \left[\int_0^t e^{\lambda(t-s)} T(s+h)\, x\, ds - \int_0^t e^{\lambda(t-s)} T(s)\, x\, ds \right] \\
&= \frac{1}{h} \left[\int_h^{t+h} e^{\lambda(t-s+h)} T(s)\, x\, ds - \int_0^t e^{\lambda(t-s)} T(s)\, x\, ds \right] \\
&= \frac{1}{h} \left[e^{\lambda h} \int_h^{t+h} e^{\lambda(t-s)} T(s)\, x\, ds - \int_0^t e^{\lambda(t-s)} T(s)\, x\, ds \right]
\end{aligned}$$

$$= \frac{1}{h}\left[\left(e^{\lambda h}-1\right)\int_{h}^{t+h} e^{\lambda(t-s)}T(s)\,x\,ds + \int_{h}^{t+h} e^{\lambda(t-s)}T(s)\,x\,ds - \int_{0}^{t} e^{\lambda(t-s)}T(s)\,x\,ds\right]$$

$$= \frac{1}{h}\left[\left(e^{\lambda h}-1\right)\int_{h}^{t+h} e^{\lambda(t-s)}T(s)\,x\,ds + \int_{t}^{t+h} e^{\lambda(t-s)}T(s)\,x\,ds - \int_{0}^{h} e^{\lambda(t-s)}T(s)\,x\,ds\right]$$

Now taking $\lim\limits_{h\to 0^+}$ we get:

$$\mathscr{A}\mathscr{B}_\lambda(t)x = \lambda\mathscr{B}_\lambda(t)x + T(t)x - e^{\lambda t}x$$

$\square$

Remark 2.6.1. *In the lemma above $\mathscr{A}$ commutes if $x \in D(\mathscr{A})$, and we get:*

$$\mathscr{B}_\lambda(t)(\lambda I - \mathscr{A})x = (\lambda I - \mathscr{A})\mathscr{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$$

This is true because of the fact that $\mathscr{A}$ is the limit.

Theorem 2.6.1. *Let $(T(t))_{t\geq 0}$ be a C_0-SG generated by $\mathscr{A}$. Then for all $t \geq 0$:*

$$e^{t\sigma(A)} \subset \sigma(T(t))$$

Proof. We prove the equivalent statement: if $e^{\lambda t} \in \rho(T(t))$, then $\lambda \in \rho(\mathscr{A})$.

Assume $e^{\lambda t} \in \rho(T(t))$. Then the operator $(e^{\lambda t}I - T(t))$ is invertible with a bounded inverse $Q = (e^{\lambda t}I - T(t))^{-1} \in \mathcal{L}(X)$.

From Lemma 2.6.1, we have:

$$(\lambda I - \mathscr{A})\mathscr{B}_\lambda(t) = e^{\lambda t}I - T(t).$$

Claim 2.6.1. *Q and $\mathscr{B}_\lambda(t)$ commute. (Prove it)*

Multiplying by Q on the right:

$$(\lambda I - \mathscr{A})\mathscr{B}_\lambda(t)Q = I.$$

Similarly, for $x \in D(\mathscr{A})$:

$$Q\mathscr{B}_\lambda(t)(\lambda I - \mathscr{A})x = x.$$

Defining $R(\lambda) = \mathscr{B}_\lambda(t)Q$, we see that $R(\lambda)$ acts as a two-sided inverse for $(\lambda I - A)$. Since $B_\lambda(t)$ and Q are bounded, $R(\lambda)$ is bounded. Thus $\lambda \in \rho(\mathscr{A})$.

By contraposition, $\lambda \in \sigma(\mathscr{A}) \implies e^{\lambda t} \in \sigma(T(t))$. $\square$

While the full spectral mapping theorem fails, a precise relationship holds for the point spectrum σ_p and the residual spectrum σ_r.

Recall the decomposition of the spectrum:

$$\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_c(\mathscr{A}) \cup \sigma_r(\mathscr{A})$$

where σ_p denotes point spectrum (eigenvalues), σ_c continuous spectrum, and σ_r residual spectrum.

Theorem 2.6.2. *Let $(T(t))_{t\geq 0}$ be a C_0-SG generated by $\mathscr{A}$. Then:*

$$e^{t\sigma_p(\mathscr{A})} \subseteq \sigma_p(T(t)) \subseteq e^{t\sigma_p(\mathscr{A})} \cup \{0\}$$

Precisely, if $e^{\lambda t} \in \sigma_p\left(T(t)\right)$, then there exists $k \in \mathbb{Z}$ such that:

$$\lambda_k = \lambda + \frac{2\pi i k}{t} \in \sigma_p(\mathscr{A})$$

Proof. First inclusion, $\lambda \in \sigma_p(\mathscr{A}) \implies e^{\lambda t} \in \sigma_p(T(t))$.

Let $\lambda \in \sigma_p(\mathscr{A})$. Then there exists $x_0 \neq 0$ such that $\mathscr{A}x_0 = \lambda x_0$ and by lemma 2.6.1 we have:

$$\mathscr{B}_\lambda(t)(\lambda I - \mathscr{A})x = \left(e^{\lambda t} I - T(t)\right) x \qquad \forall x \in D(\mathscr{A})$$

Now take $x = x_0$ and note the fact that $(\lambda I - \mathscr{A})x_0 = 0$:

$$\left(e^{\lambda t} I - T(t)\right) x_0 = 0$$

Hence $\ker\left(e^{\lambda t} I - T(t)\right) \neq \{0\}$ then $e^{\lambda t} \in \sigma_p(T(t))$.

For the second inclusion, let $\mu \in \sigma_p(T(t))$ with $\mu \neq 0$. Write $\mu = e^{\lambda t}$ for some $\lambda \in \mathbb{C}$.

There exists $x_0 \neq 0$ such that $T(t)x_0 = e^{\lambda t}x_0$.

Consider the function $\phi_{x_0}(s)$ defined by:

$$\phi_{x_0}(s) = e^{-\lambda s} T(s) x_0$$

Clearly:

$$\phi_{x_0}(0) = x_0 = \phi_{x_0}(t)$$

We check if ϕ_{x_0} is periodic with period t.

$$\phi_{x_0}(s+t) = e^{-\lambda(s+t)} T(s+t)x_0 = e^{-\lambda s} e^{-\lambda t} T(s)T(t)x_0$$

Using the eigen-property $T(t)x_0 = e^{\lambda t}x_0$:

$$\phi_{x_0}(s+t) = e^{-\lambda s} e^{-\lambda t} T(s)(e^{\lambda t}x_0) = e^{-\lambda s} T(s)x_0 = \phi_{x_0}(s)$$

Since $\phi_{x_0}(s)$ is a t-periodic continuous function (and assuming $x_0 \in X$), we can expand it into a

Fourier series. The k-th Fourier coefficient is:

$$\hat{x}_k = \frac{1}{t}\int_0^t e^{-2\pi iks/t}\phi_{x_0}(s)\,ds$$

Substituting $\phi_{x_0}(s) = e^{-\lambda s}T(s)x_0$:

$$\hat{x}_k = \frac{1}{t}\int_0^t e^{-(\lambda+\frac{2\pi ik}{t})s}T(s)x_0\,ds$$

Let $\lambda_k = \lambda + \frac{2\pi ik}{t}$. Since ϕ_{x_0} is not identically zero (as $\phi_{x_0}(0) = x_0 \neq 0$), at least one coefficient $\hat{x}_k$ must be non-zero.

We will show that if $\hat{x}_k \neq 0$, then $\lambda_k \in \sigma_p(\mathscr{A})$.

We use the resolvent formula. For $\Re(\gamma)$ sufficiently large, $R(\gamma,\mathscr{A}) = \int_0^\infty e^{-\gamma s}T(s)ds$. Applying this to x_0:

$$R(\gamma,\mathscr{A})x_0 = \int_0^\infty e^{-\gamma s}T(s)x_0\,ds$$

Decompose the integral over intervals $[nt,(n+1)t]$:

$$\begin{aligned}
R(\gamma,\mathscr{A})x_0 &= \sum_{n=0}^{\infty}\int_{nt}^{(n+1)t} e^{-\gamma s}T(s)x_0\,ds \\
&= \sum_{n=0}^{\infty}\int_0^t e^{-\gamma(nt+\tau)}T(nt+\tau)x_0\,d\tau \quad (\text{let } s = nt+\tau)
\end{aligned}$$

Using the semigroup property $T(nt+\tau)x_0 = T(\tau)T(t)^n x_0 = T(\tau)e^{n\lambda t}x_0$:

$$\begin{aligned}
R(\gamma,\mathscr{A})x_0 &= \sum_{n=0}^{\infty} e^{-n\gamma t}e^{n\lambda t}\int_0^t e^{-\gamma\tau}T(\tau)x_0\,d\tau \\
&= \left(\sum_{n=0}^{\infty} e^{-n(\gamma-\lambda)t}\right)\int_0^t e^{-\gamma\tau}T(\tau)x_0\,d\tau
\end{aligned}$$

The geometric series converges if $\Re(\gamma) > \Re(\lambda)$:

$$\sum_{n=0}^{\infty}(e^{-(\gamma-\lambda)t})^n = \frac{1}{1-e^{-(\gamma-\lambda)t}}$$

Thus:

$$R(\gamma,\mathscr{A})x_0 = \frac{1}{1-e^{(\lambda-\gamma)t}}\int_0^t e^{-\gamma\tau}T(\tau)x_0\,d\tau$$

The function $\gamma \mapsto R(\gamma,\mathscr{A})x_0$ is meromorphic. The poles of the resolvent indicate the spectrum. The denominator vanishes when:

$$e^{(\lambda-\gamma)t} = 1 \iff (\lambda-\gamma)t = 2\pi ik \iff \gamma = \lambda - \frac{2\pi ik}{t}$$

Let $\mu_k = \lambda + \frac{2\pi i k}{t}$. The resolvent has a pole at μ_k. Specifically, the residue near the pole relates to the existence of an eigenvector. If we analyze the limit:

$$\lim_{\gamma \to \lambda_k} (\gamma - \lambda_k) R(\gamma, \mathscr{A}) x_0 \neq 0 \implies \lambda_k \in \sigma_p(\mathscr{A}).$$

This confirms that the eigenvalues of $\mathscr{A}$ are exactly the logarithms of the eigenvalues of $T(t)$ (modulo $2\pi i/t$). $\square$

CHAPTER

THREE

APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Let X be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. We define the homogeneous and inhomogeneous abstract Cauchy problems as follows:

- Homogeneous Cauchy Problem $h - (CP)_{0,x_0}$:

$$\frac{du}{dt} = Au, \quad t \geq 0$$

$$u(0) = x_0$$

- Inhomogeneous Cauchy Problem $inh - (CP)_{0,x_0}$:

$$\frac{du}{dt} = Au + f(t), \quad t \geq 0$$

$$u(0) = x_0$$

where $f : \mathbb{R}_+ \to X$.

3.1 The Cauchy Problem

Definition 3.1.1. *We distinguish three types of solutions for the Cauchy problem:*

1. ***Strong Solution:*** *A function $x(\cdot) \in C^1(\mathbb{R}_+, D(\mathscr{A}))$ such that for all $t \geq 0$, $x(t) \in D(\mathscr{A})$ and $\frac{dx}{dt} = \mathscr{A}x(t)$.*
2. ***Mild Solution:*** *A function $x(\cdot) \in C^0(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+^*, D(\mathscr{A}))$ with $x(0) = x_0$, and for all $t > 0$, $x(t) \in D(\mathscr{A})$ such that $\frac{dx}{dt} = \mathscr{A}x(t)$.*
3. ***Weak Solution:*** *A function $x(\cdot) \in C^0(\mathbb{R}_+, X)$ such that $x(0) = x_0$, and for all $x^* \in D(A^*)$, the map $t \mapsto \langle x^*, x(t)\rangle$ is C^1 and satisfies:*

$$\frac{d}{dt}\langle x^*, x(t)\rangle = \langle A^* x^*, x(t)\rangle$$

Remark 3.1.1. *Some properties of the solutions and constraints on the initial value:*

1. *If $x_0 \notin D(A)$, there is no valid strong solution for $(CP)_{0,x_0}$.*

2. *For all $x_0 \in X$, $t \mapsto T(t)x_0$ is the unique solution of $(CP)_{0,x_0}$.*

Example 3.1.1. *Consider the transport equation:*

$$u_t = u_x \qquad x \in [0,1],\ t \geq 0$$

subject to boundary conditions at $t = 0$, $u(0,\cdot) = u_0(\cdot) \in X$.

We associate this problem with the operator $\mathscr{A}$ defined by $\mathscr{A}u = \frac{du}{dx}$. To define the weak solution, we need to determine the action of the adjoint operator $\mathscr{A}^*$.

Let φ be a test function (of $D(A^*)$). We compute the pairing $\langle \mathscr{A}u, \varphi\rangle$ using the inner product in $L^2([0,1])$:

$$\langle \mathscr{A}u, \varphi\rangle = \int_0^1 (\mathscr{A}u)(x)\varphi(x)\,dx = \int_0^1 \frac{du}{dx}(x)\varphi(x)\,dx$$

By integration by parts:

$$\int_0^1 u'(x)\varphi(x)\,dx = [u(x)\varphi(x)]_0^1 - \int_0^1 u(x)\varphi'(x)\,dx$$

Using the Dirichlet boundary conditions ($u(t,1) = u(t,0) = 0$), the boundary term $[u(x)\varphi(x)]_0^1$ vanishes. Thus, we obtain:

$$\langle \mathscr{A}u, \varphi\rangle = -\int_0^1 u(x)\varphi'(x)\,dx = \int_0^1 u(x)\left(-\frac{d\varphi}{dx}\right)\,dx$$

This can be rewritten in terms of the inner product as:

$$\langle \mathscr{A}u, \varphi\rangle = \langle u, -\varphi'\rangle$$

From the definition of the adjoint operator $\langle \mathscr{A}u, \varphi\rangle = \langle u, \mathscr{A}^*\varphi\rangle$, we identify:

$$\mathscr{A}^*\varphi = -\frac{d\varphi}{dx}$$

Therefore, the condition for $u(t)$ to be a weak solution:

$$\frac{d}{dt}\langle \varphi, u(t)\rangle = \langle A^*\varphi, u(t)\rangle$$

becomes:

$$\frac{d}{dt}\int_0^1 u(t,x)\varphi(x)\,dx = -\int_0^1 u(t,x)\varphi'(x)\,dx$$

3.2 Homogeneous Cauchy Problem

Theorem 3.2.1. *Let $\mathscr{A}$ be the IG of a C_0-SG $(T(t))_{t\geq 0}$, then:*

1. *For all $x_0 \in D(\mathscr{A})$, there exists a unique strong solution of $(CP)_{0,x_0}$ given by $u(t) = T(t)x_0$, which is also a mild and weak solution.*

2. *For all $x_0 \in X$, $t \mapsto T(t)x_0$ is a weak solution of $(CP)_{0,x_0}$.*

Proof. The proof of part 1 is direct.

For part 2: Let $x_0 \in X$ and choose a sequence $x_n \in D(\mathscr{A})$ such that $x_n \to x_0$ (why is this possible?). For all $x^* \in D(\mathscr{A}^*)$ and $t \geq 0$, the function $t \mapsto \langle x^*, T(t)x_n\rangle$ is C^1, and we have:

$$\frac{d}{dt}\langle x^*, T(t)x_n\rangle = \langle x^*, \mathscr{A}T(t)x_n\rangle = \langle \mathscr{A}^*x^*, T(t)x_n\rangle$$

Integrating yields:

$$\langle x^*, T(t)x_n\rangle = \langle x^*, x_n\rangle + \int_0^t \langle A^*x^*, T(s)x_n\rangle ds$$

By passing to the limit as $n \to \infty$ (show how), we extend this formula for all $x_0 \in X$, showing $t \mapsto T(t)x_0$ is a weak solution.

Now to show the uniqueness of the solution, consider two weak solutions $x_1(\cdot)$ and $x_2(\cdot)$ of $(CP)_{0,x_0}$. Set $u = x_1 - x_2$. Then u is a weak solution of the problem with initial condition 0. For all $t \geq 0$ and $x^* \in D(A^*)$, we have:

$$\langle x^*, u(t)\rangle = \int_0^t \langle A^*x^*, u(s)\rangle ds$$

Let $U(t) = \int_0^t u(s)ds$, with $U(0) = 0$. Then, $\langle x^*, u(t)\rangle = \langle A^*x^*, U(t)\rangle$, which implies:

$$\frac{d}{dt}\langle x^*, U(t)\rangle = \langle A^*x^*, U(t)\rangle \tag{***}$$

Claim 3.2.1. *If $(T(t))_{t\geq 0}$ is a C_0-SG with $\mathscr{A}$ as its IG then $T(t)^*D(\mathscr{A}^*) \subset D(\mathscr{A}^*)$ and $T(t)^*$ and $\mathscr{A}^*$ commute, that is:*

$$T(t)^*\mathscr{A}^* = \mathscr{A}^*T(t)^*$$

Proof of claim. We have to show that $\forall x^* \in D(\mathscr{A}^*)$, $\forall x \in D(\mathscr{A})$, $\forall t \geq 0$, the follwing:

$$\exists \xi^* \in X^*, \langle T(t)^*x^*, \mathscr{A}x\rangle = \langle \xi^*, x\rangle$$

We have the following:

$$\begin{aligned}
\langle T(t)^* x^*, \mathscr{A} x \rangle &= \langle x^*, T(t)\mathscr{A} x \rangle \\
&= \langle x^*, \mathscr{A} T(t) x \rangle \\
&= \langle \mathscr{A}^* x^*, T(t) x \rangle \\
&= \langle T(t)^* \mathscr{A}^* x^*, x \rangle
\end{aligned}$$

Then $T(t)^*\mathscr{A}^*x^* \in D(\mathscr{A}^*)$ and $T(t)^*\mathscr{A}^* = \mathscr{A}^*T(t)^*$. □

Apply (***) replacing x^* with $T^*(t^*-t)x^*$, where $0 \le t \le t^*$:

$$\frac{d}{dt}\langle T^*(t^*-t)x^*, U(t)\rangle = \langle -\mathscr{A}^*T^*(t^*-t)x^*, U(t)\rangle + \langle T^*(t^*-t)x^*, u(t)\rangle$$

Since $\langle T^*(t^*-t)x^*, u(t)\rangle = \langle \mathscr{A}^*T^*(t^*-t)x^*, U(t)\rangle$, the derivative is 0.

Integrating from $t = 0$ to $t = t^*$ yields:

$$\langle x^*, U(t^*)\rangle - \langle T^*(t^*)x^*, U(0)\rangle = 0$$

Since $U(0) = 0$, we get $\langle x^*, U(t^*)\rangle = 0$ for all $t^* \ge 0$. Since this holds for all $x^* \in D(\mathscr{A}^*)$ and $D(\mathscr{A}^*)$ is weak*-dense, $U(t) \equiv 0$.

Differentiating gives $u(t) = 0$, proving uniqueness. □

Theorem 3.2.2. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closed, densely defined operator. If there exists ω such that for $\lambda > \omega$, $\lambda \in \rho(\mathscr{A})$ and $\|R(\lambda, \mathscr{A})\| = o(e^{\sigma\lambda})$, then $(CP)_{0,x_0}$ admits a unique mild solution.*

We first prove a necessary lemma for uniqueness:

Lemma 3.2.1. *Let $u : [0, T] \to X$ be continuous with $u(0) = 0$. Assume there exists $M \ge 0$ such that for all $n \ge 0$,*

$$\left\| \int_0^T e^{ns} u(s) ds \right\| \le M$$

Then $u(t) = 0$ for all $t \in [0, T]$.

Proof. For any $x^* \in X^*$, define the continuous function $\varphi(s) := \langle x^*, u(s)\rangle$. By assumption, $\left\| \int_0^T e^{ns}\varphi(s)ds \right\| \le M$ for all n. Define the sequence of functions:

$$\psi_n(t) = \sum_{k\ge1} \frac{(-1)^{k+1}}{k!} e^{knt} = 1 - e^{-e^{nt}}$$

We consider the integral:

$$\int_0^T \psi_n(t - T + s)\varphi(s)ds$$

By substituting the sum definition of ψ_n and using the bound on φ, we have:

$$\begin{aligned}\left\| \int_0^T \sum_{k\geq 1} \frac{(-1)^{k+1}}{k!} e^{kn(t-T+s)}\varphi(s)ds \right\| &\leq \sum_{k\geq 1} \frac{1}{k!} e^{kn(t-T)} \left\| \int_0^T e^{kns}\varphi(s)ds \right\| \\ &\leq M \sum_{k\geq 1} \frac{(e^{n(t-T)})^k}{k!} \\ &= M(e^{e^{n(t-T)}} - 1)\end{aligned}$$

For $t < T$, as $n \to \infty$, $n(t-T) \to -\infty$, thus $e^{n(t-T)} \to 0$, meaning the integral evaluates to 0.

On the other hand, analyzing the behavior of $\psi_n(t-T+s)$ directly as $n \to \infty$:

- If $s > T - t$, then $t - T + s > 0$, so $\psi_n \to 1$.
- If $s < T - t$, then $t - T + s < 0$, so $\psi_n \to 0$.

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^T \psi_n(t-T+s)\varphi(s)ds = \int_{T-t}^T \varphi(s)ds$$

Equating the two limits, we deduce $\int_{T-t}^T \varphi(s)ds = 0$ for all t. Differentiating with respect to t gives $\varphi(T-t) = 0$, implying $\varphi \equiv 0$, and thus $u \equiv 0$. □

Proof of theorem 3.2.2. Without loss of generality we can assume $\omega = 0$ since if it is not then the transform $\mathscr{A} \mapsto \mathscr{A} - zI$, shifts the coefficient by e^{zt}, and:

$$\frac{du_z}{dt} = (\mathscr{A} + zI)u_z$$

$$\frac{du}{dt} = \mathscr{A}u$$

Assume u is a mild solution of $(CP)_{0,0}$, meaning $u(0) = 0$ and $\frac{du}{dt} = \mathscr{A}u$. For $\lambda > 0$, define $v(t) = R(\lambda, \mathscr{A})u(t)$. Since

$$\frac{d}{dt}R(\lambda,\mathscr{A})u(t) = R(\lambda,\mathscr{A})\mathscr{A}u(t) = R(\lambda,\mathscr{A})(\mathscr{A} - \lambda I + \lambda I)u(t) = -u(t) + \lambda R(\lambda,\mathscr{A})u(t)$$

we get $\frac{dv}{dt} - \lambda v(t) = -u(t)$. Solving this ODE yields:

$$R(\lambda,\mathscr{A})u(t) = \int_0^t e^{\lambda(t-s)}u(s)ds$$

Multiplying by $e^{-\sigma\lambda}$, we get:

$$e^{-\sigma\lambda}R(\lambda,\mathscr{A})u(t) = \int_0^t e^{\lambda(t-\sigma-s)}u(s)ds$$

Claim 3.2.2. $\lim_{\lambda \to +\infty} \int_0^t e^{\lambda(t-\sigma-s)} u(s) ds = 0.$

Proof. Do it. (Hint: use the bond from the assumption and lemma 3.2.1) ☐

By the claim we get $u \equiv 0$. ☐

Exercise 14: Homework

Let X be a Hilbert Space and $\mathscr{A}$ be defined by

$$\mathscr{A} y = \sum \lambda_n \langle y, e_n \rangle e_n$$

over an orthonormal basis $\{e_n\}$, with real numbers $\lambda_n \nearrow +\infty$. Its domain is $D(\mathscr{A}) = \{y \in X \mid \sum \lambda_n^2 \langle y, e_n \rangle^2 < +\infty\}$.

1. Prove $\mathscr{A}$ is self-adjoint.
2. Prove it generates a C_0-SG.

(Hint: $\lambda I - \mathscr{A}$ is formally bounded if and only if $\lim_{n \geq 1} |\lambda - \lambda_n| > 0$)

What can we say without $\mathscr{A}$ being the IG of some C_0-SG?

Theorem 3.2.3 (Existence of Solution)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a UBLO with $\overline{D(\mathscr{A})} = X$ such that $\rho(\mathscr{A}) \supset (\lambda_0, \infty)$, $\lambda_0 \in \mathbb{R}$ and*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \ln \|R(\lambda, \mathscr{A})\| = 0$$

then $\forall x \in X$ $(CP)_{0,x}$ admits a mild solution.

Note the difference between the assumption here and Hille-Yosida assumption, so one can not use Hille-Yosida directly.

Lemma 3.2.2. *Let $\mathscr{A} : D(\mathscr{A}) \supset X \to X$ be a UBLO then:*

1. *$\forall \lambda \in \rho(\mathscr{A})$ $D(\mathscr{A}^2) = (\lambda I - \mathscr{A})^{-1} D(\mathscr{A})$.*
2. *If $\rho(\mathscr{A}) \neq \emptyset$ and $\overline{D(\mathscr{A})} = X$ then $\overline{D(\mathscr{A}^2)} = X$.*

Proof. Left as an exercise. ☐

Theorem 3.2.4 (Uniqueness of Solution)**.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a UBLO with $\overline{D(\mathscr{A})} = X$ and $\rho(\mathscr{A}) \neq \emptyset$. $\forall x \in D(\mathscr{A})$, $(CP)_{0,x}$ admits a mild solution if and only if $\mathscr{A}$ is the IG of some C_0-SG.*

Proof. Let's start with ($\Longleftarrow$). Assume $\mathscr{A}$ is the IG of some C_0-SG then by theorem 3.2.1 part 1 we have a unique strong solution which is also mild.

For ($\Longrightarrow$). We need to construct a C_0-SG with $\mathscr{A}$ as it IG. For this we use the flow of the solution. For all $x \in D(\mathscr{A})$ we have:

$$\frac{du}{dt} = \mathscr{A}u$$
$$u(0) = x$$

Since u is the unique mild solutions of $(CP)_{0,x}$ (actually it is a strong solution as well, why?). Then the mapping:

$$u : \mathbb{R}_+ \times D(\mathscr{A}) \to D(\mathscr{A})$$
$$(t, x) \mapsto u(t, x)$$

is well degined. We can now consider the flow associated with $(CP)_{0,x}$. For each $t \geq 0$, define the operator $T(t) : D(\mathscr{A}) \to D(\mathscr{A})$ by:

$$T(t)x = u(t, x)$$

We must prove that the family $(T(t))_{t\geq 0}$ forms a C_0-SG.

1. Identity: $T(0)x = u(0, x) = x$. Thus, $T(0) = Id_{D(\mathscr{A})}$.
2. Linearity: The operator $T(t)$ is linear due to the linearity of the derivative and the operator $\mathscr{A}$.
3. Semigroup Property: We need to show $T(t + s) = T(t)T(s)$.

By definition, $T(t)T(s)x = u(t, T(s)x)$, which is the solution evaluated at time t for the initial value problem starting at $T(s)x$. Similarly, $T(t + s)x = u(t + s, x)$, which is the solution evaluated at time $t + s$ for the initial value problem starting at x. Due to the time-invariance of the equation $\frac{du}{dt} = Au$ and the uniqueness of the mild solution, advancing the flow by s and then by t is equivalent to advancing it by $t + s$. Hence, $T(t + s) = T(t)T(s)$.

Now we need to find the bound on $T(t)$ to show it is C_0-SG. To do this we need to equip the domain $D(\mathscr{A})$ with a suitable Banach space structure (put a complete norm on it).

We put the graph norm on $D(\mathscr{A})$:

$$|x|_G = \|x\| + \|\mathscr{A}x\|$$

With this and the fact that $\mathscr{A}$ is closed (why?), $(D(\mathscr{A}), |\cdot|_G)$ is a Banach space. This will be denoted by $[D(\mathscr{A})]$.

Let $t_0 > 0$, define X_{t_0} as follows:

$$X_{t_0} = C^0([0, t_0], [D(\mathscr{A})])$$

Clearly, $(X_{t_0}, \|\cdot\|_\infty)$ is a Banach space.

Define the operator S_{t_0} by:

$$\begin{aligned} S_{t_0} : [D(A)] &\to X_{t_0} \\ x &\mapsto (t \mapsto u(t,x) = T(t)x) \end{aligned}$$

S_{t_0} is well-defined and linear because of the uniqueness of solutions of $(CP)_{0,x}$.

Claim 3.2.3. *The operator S_{t_0} is closed.*

Proof. Suppose $x_n \to x$ in $[D(\mathscr{A})]$ and $S_{t_0}(x_n) \to v$ in X_{t_0}. We aim to show that $S_{t_0}(x) = v$.

By definition, $S_{t_0}(x_n) = u(\cdot, x_n)$. The convergence in X_{t_0} implies:

$$||S_{t_0}(x_n) - v||_\infty = \sup_{t\in[0,t_0]} |u(t,x_n) - v(t)|_G \to 0 \quad \text{as } n \to \infty$$

Because the map is continuous, and u is the unique solution, we must have $v(t) = u(t,x)$. Therefore, $S_{t_0}(x) = v$. □

Because S_{t_0} is a closed operator acting between two Banach spaces, the Closed Graph Theorem implies that S_{t_0} is bounded. Thus, there exists a constant $C_{t_0} > 0$ such that for all $x \in [D(\mathscr{A})]$:

$$\|S_{t_0}(x)\|_\infty \le C_{t_0}|x|_G$$

Consequently, for all $t \in [0,t_0]$, we have $|T(t)x|_G \le C_{t_0}|x|_G$. This demonstrates that $T(t) \in \mathcal{L}([D(\mathscr{A})])$, meaning $(T(t))_{t\ge 0}$ is a C_0-SG on $[D(\mathscr{A})]$.

Now, we prove that the semigroup commutes with its generator on appropriate domains.

Claim 3.2.4. *For all $y \in D(\mathscr{A}^2)$, $T(t)\mathscr{A}y = \mathscr{A}T(t)y$.*

Proof. Let $y \in D(\mathscr{A}^2)$. We define the function $w(t)$:

$$w(t) = y + \int_0^t u(s, \mathscr{A}y)ds$$

Since $u(s,\mathscr{A}y)$ is continuous, $w(t)$ is continuously differentiable (C^1) with respect to t, and:

$$w'(t) = u(t, \mathscr{A}y)$$

We can also pull the closed operator $\mathscr{A}$ outside the integral (since $u(s,\mathscr{A}y) \in D(\mathscr{A})$):

$$w'(t) = \mathscr{A}y + \int_0^t \frac{d}{ds}u(s,\mathscr{A}y)ds = \mathscr{A}\left[y + \int_0^t u(s,\mathscr{A}y)ds\right] = Aw(t)$$

At $t = 0$, $w(0) = y$.

This shows that $w(t)$ is a mild solution to $(CP)_{0,y}$. Because solutions are unique, it must coincide with our defined flow:

$$\forall t \geq 0, \quad w(t) = u(t, y) = T(t)y$$

Differentiating both sides with respect to t yields:

$$w'(t) = \frac{d}{dt}[u(t, y)] = \mathscr{A}u(t, y) = \mathscr{A}T(t)y$$

However, from our earlier calculation, we also know $w'(t) = u(t, \mathscr{A}y) = T(t)\mathscr{A}y$. Equating the two expressions for $w'(t)$ gives the desired result:

$$T(t)\mathscr{A}y = \mathscr{A}T(t)y$$

$\square$

Now since $\rho(\mathscr{A}) \neq \emptyset$, then $\exists \lambda_0 \in \rho(\mathscr{A}), \forall y \in D(\mathscr{A}^2)$ set $x = (\lambda_0 I - \mathscr{A})y$ then:

$$T(t)x = T(t)(\lambda_0 I - \mathscr{A})y = (\lambda_0 I - \mathscr{A})T(t)y$$

then

$$\|T(t)x\| \leq |\lambda_0| \|T(t)y\| + \|\mathscr{A}T(t)y\| \leq C_{\lambda_0} \|T(t)y\| \leq C_{\lambda_0} M e^{\omega t} |y|_G$$

Where

$$\begin{aligned} |y|_G = \|y\| + \|\mathscr{A}y\| &= \|y\| + \|(\mathscr{A} - \lambda_0 I)y + \lambda_0 y\| \\ &\leq C_1 \|x\| + \|x\| + |\lambda_0| C_2 \|x\| \\ &\leq C_3 \|x\| \end{aligned}$$

Finally we get:

$$\|T(t)x\| \leq M' e^{\omega t} \|x\|$$

Now by lemma 3.2.2 and the fact that $\rho(\mathscr{A}) \neq \emptyset$ this bound can be extended to all X. And the operator commutes as well.

Finally, we need to show that $\mathscr{A}$ is the IG of $(T(t))_{t \geq 0}$. Let $\mathscr{A}_1$ be the IG of $(T(t))_{t \geq 0}$ then by definition of $T(t)$ we have $T(t)x = u(t, x)$ $\forall x \in D(\mathscr{A})$ then:

$$\frac{d}{dt} T(t)x = \mathscr{A}T(t)x \qquad \text{for } t \geq 0$$

Which means in particular that $\frac{d}{dt} T(t)x \mid_{t=0} = \mathscr{A}x$ then $\mathscr{A}_1 \supset \mathscr{A}$.

Let $\Re(\lambda) > \omega$ and let $y \in D(\mathscr{A}^2)$. Then:

$$e^{-\lambda t} \mathscr{A}T(t)y = e^{-\lambda t} T(t)\mathscr{A}y = e^{-\lambda t} T(t)\mathscr{A}_1 y$$

By integrating both sides from 0 to ∞ we get:

$$\mathscr{A} R(\lambda, \mathscr{A}_1) y = R(\lambda, \mathscr{A}_1) \mathscr{A} y = \mathscr{A}_1 R(\lambda, \mathscr{A}_1) y$$

Since $R(\lambda, \mathscr{A}_1)\mathscr{A}_1 y = \mathscr{A}_1 R(\lambda, \mathscr{A}_1) y$ then $R(\lambda, \mathscr{A}_1)\mathscr{A} y = R(\lambda, \mathscr{A}_1)\mathscr{A}_1 y\ \forall y \in D(\mathscr{A}^2)$. Since $R(\lambda, \mathscr{A}_1)\mathscr{A}_1$ is uniformly bounded, $\mathscr{A}$ is closed and $\overline{D(\mathscr{A}^2)} = X$ we get $R(\lambda, \mathscr{A}_1)\mathscr{A} y = R(\lambda, \mathscr{A}_1)\mathscr{A}_1 y\ \forall y \in X$. This implies that $D(\mathscr{A}) \supset Rg(R(\lambda, \mathscr{A}_1)) = D(\mathscr{A}_1)$ hence $\mathscr{A}_1 \subset \mathscr{A}$ then $\mathscr{A}_1 = \mathscr{A}$ and the proof is complete. $\square$