# PARTIAL DIFFERENTIAL EQUATIONS

International Mathematics Masters
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# CONTENTS

CHAPTER

# ONE

# THEORY OF (UNBOUNDED) OPERATORS

## 1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathscr{L}(X)$ be the Banach space of bounded linear operators.

**Definition 1.1.1.** *An operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathscr{A})$ is a subspace of $X$ and* $\sup_{x\neq 0} \frac{\|\mathscr{A}x\|}{\|x\|} = +\infty$

**Exercise 1**

Let $\mathscr{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathscr{A}) = \{f : \mathbb{R} \to \mathbb{R} : f, f' \in L^2\}$. Show $\mathscr{A}$ is an unbounded linear operator.

**Notation:** If $\mathscr{A}$ and $\mathscr{B}$ are unbounded linear operators, then $\mathscr{A} \supset \mathscr{B}$ if and only if $D(\mathscr{A}) \supset D(\mathscr{B})$ and for all $x \in D(\mathscr{B})$, $\mathscr{A}x = \mathscr{B}x$.

### 1.1.1 Resolvent Operator

**Definition 1.1.2.** *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be a UBLO.*

$$\rho(\mathscr{A}) = \textit{ Resolvent of } \mathscr{A} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} (\lambda I - \mathscr{A}) : D(\mathscr{A}) \to X \textit{ is bijective, and} \\ (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X) \end{array} \right\}$$

$$\sigma(\mathscr{A}) = \textit{ Spectrum of } \mathscr{A} = \mathbb{C} \setminus \rho(\mathscr{A}).$$

**Definition 1.1.3.** *$\mathscr{A}$ is closed if and only if the graph of $\mathscr{A}$, denoted as $G(\mathscr{A})$ is closed. Also $\mathscr{A}$ is closable if and only if there exists $\tilde{\mathscr{A}} \supset \mathscr{A}$ such that $G(\tilde{\mathscr{A}}) = \overline{G(\mathscr{A})}$.*

Note that $G(\mathscr{A}) = \big\{(x, \mathscr{A}x) \big| x \in D(\mathscr{A})\big\}$.

**Exercise 2**

1. Prove that if it exists, $\tilde{\mathscr{A}}$ is unique, it then denoted by $\overline{\mathscr{A}}$ called closure of $\mathscr{A}$.
2. Let $\mathscr{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a,b],\mathbb{R}), \|.\| = \sup|f(x)|)$ and $D(\mathscr{A}_\ell) = C^\ell([a,b],\mathbb{R})$. Prove $\overline{\mathscr{A}_\ell} = \mathscr{A}_1$.

**Lemma 1.1.1.** *If $\mathscr{A}$ an unbounded linear operator is closable, then $\rho(\overline{\mathscr{A}}) = \rho(\mathscr{A})$. If $\mathscr{A}$ is closed then $\rho(\mathscr{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X)\}$.*
*Hints (Exercise): If $\rho(\mathscr{A}) \neq \emptyset$ implies $\mathscr{A}$ is closed. (Show first that if $T$ is a UBLO with $T^{-1} \in \mathscr{L}(X)$ implies $T$ is closed).*

**Corollary 1.1.1.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \cup \sigma_c(\mathscr{A})$ where*

1. *$\sigma_p(\mathscr{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) \neq \{0\}\}$ (punctual spectrum and $\lambda$'s are the eigenvalue).*
2. *$\sigma_c(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) = \{0\}, \overline{Rg(\lambda I - \mathscr{A})} \subsetneq X\right\}$ (continuous spectrum).*
3. *$\sigma_r(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \lambda I - \mathscr{A} \text{ is injective}, \overline{Rg(\lambda I - \mathscr{A})} = X,\ Rg(\lambda I - \mathscr{A}) \neq X\right\}$ (residual spectrum).*

**Exercise 3**

Let

$$X = \ell^2(\mathbb{C}) = \left\{(x_n)_{n\geq 0} : \sum_n |x_n|^2 < \infty\right\},$$

with $(\mathscr{A}x_n)_{n\geq 1} = \left(\frac{x_n}{1+n}\right)_{n\geq 0}$. Prove that $\mathscr{A}$ is a BLO, injective, $\overline{\mathrm{Rg}(\mathscr{A})} = X$ and $\mathrm{Rg}(\mathscr{A}) \subsetneq X$.

**Theorem 1.1.1.** *If $\mathscr{A}$ is a closed UBLO then $\rho(\mathscr{A})$ is open. If $\mu \in \rho(\mathscr{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathscr{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathscr{A})$ and*

$$(\lambda I - \mathscr{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathscr{A})^{-(n+1)}$$

*To do Question: do you need $\mathscr{A}$ closed?*

**Theorem 1.1.2** (Resolvent Identity)**.** *Let $\mathscr{A}$ be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator*

$$R(\lambda) := (\lambda I - A)^{-1}.$$

*Then for all $\lambda, \mu \in \rho(A)$,*

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

**Corollary 1.1.2.** *The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathscr{A})$ into $\mathscr{L}(X)$ is analytic. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

### 1.1.2 Dual Operators

Let $X \cong X^*$ and $\mathscr{A}$ a closed UBLO with $\overline{D(\mathscr{A})} = X$ a dense UBLO.
If $X$ and $Y$ are Banach spaces with duals $X^*$ and $Y^*$, then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x\rangle$.

**Definition 1.1.4** (Dual Operator of $\mathscr{A}$). *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to Y$ (UBLO), be such that $\overline{D(\mathscr{A})} = X$. The* dual operator *$\mathscr{A}^* : D(\mathscr{A}^*) \subset Y^* \to X^*$ is a UBLO defined as follows:*

$$D(\mathscr{A}^*) := \{y^* \in Y^* \,|\, \exists z^* \in X^* \text{ such that } \langle y^*, Ax\rangle = \langle z^*, x\rangle \;\forall x \in D(\mathscr{A})\}.$$

*and $y^* \in D(\mathscr{A}^*)$, the element $z^*$ is unique and we define $A^*y^* := z^*$.*

**Lemma 1.1.2.** *Let $X, Y$ be Banach spaces and let $\mathscr{A} \in \mathscr{L}(X,Y)$. Then $\mathscr{A}^* \in \mathscr{L}(Y^*, X^*)$ and*

$$\|\mathscr{A}^*\|_{\mathscr{L}(Y^*,X^*)} = \|\mathscr{A}\|_{\mathscr{L}(X,Y)}.$$

**Lemma 1.1.3.** *Let $X$ be a reflexive Banach space with $X = X^*$ and let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closedly dense UBLO. Then $\overline{D(\mathscr{A}^*)} = X^*(\cong X)$, and $\mathscr{A}^*$ is closed.*

**Theorem 1.1.3.** *Let $\mathscr{A}$ be a closedly dense UBLO. Then $\rho(\mathscr{A}) = \rho(\mathscr{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$\left[(\lambda I - \mathscr{A})^{-1}\right]^* = (\lambda I - \mathscr{A}^*)^{-1}.$$

**Exercise 4**

1. Let $\mathscr{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathscr{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:
   a. $\rho(\mathscr{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathscr{A}) = i\mathbb{R}$.
   b. $\mathscr{A}$ is a closed unbounded linear operator.
   c. If $\lambda \in \rho(\mathscr{A})$ then $(\lambda I - \mathscr{A})^{-1} : X \longrightarrow D(\mathscr{A})$ is bounded.

   For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathscr{A})$ such that $(\lambda I - \mathscr{A})f = g$.
   For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathscr{A})f_n \to 0$.
2. Do same for $\mathscr{A} = -i\frac{d}{dx}$.

## 1.2 Compact Operators

Let $X$ and $Y$ be Banach spaces on $\mathbb{K}$.

**Definition 1.2.1.** *Let $K : X \to Y$ be a BLO (in $\mathscr{L}(X,Y)$), then $K$ compact iff $K(B_1^X(0))$ is relatively compact in $Y$ (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X,Y) = \{K \in \mathscr{L}(X,Y) \mid K \text{ is compact}\}.$$

**Exercise 5**

Let $X = C([a,b],\mathbb{C})$ and $k \in C^0([a,b]\times[c,d],\mathbb{C})$
Define $K \in \mathscr{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t,s)x(s)\,ds.$$

Show $K \in \mathcal{K}(X)$.

**Theorem 1.2.1.** *$\mathcal{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$.*

*Proof.* We shall show this in two steps

1. $\mathcal{K}(X,Y)$ is a vector space (Do it).
2. Closed: $K_n \to K$ (Prof. Yacine said he would send a different proof).

□

**Exercise 6**

Let $X = \ell^2(\mathbb{C})$,

$$\mathscr{A}((x_n)_{n\geq 0}) = \left(\frac{x_n}{n+1}\right)_{n\geq 0}.$$

Show that $\mathscr{A}$ is compact.

**Theorem 1.2.2.** *Let $X, Y$ and $Z$ be Banach spaces on $\mathbb{K}$.*

$$X \xrightarrow{\mathscr{A}} Y \xrightarrow{\mathscr{B}} Z, \quad \mathscr{A} \in \mathscr{L}(X,Y),\ \mathscr{B} \in \mathscr{L}(Y,Z).$$

1. *If $\mathscr{A}$ is compact or $\mathscr{B}$ is compact, then $\mathscr{B}\mathscr{A}$ is compact.*
2. *If $\mathscr{A}$ is compact then $\mathscr{A}^* \in \mathcal{K}(Y^*, X^*)$.*
3. *If $\mathscr{A}$ is compact and $\mathrm{Rg}(\mathscr{A})$ is closed (in $Y$), then it is finite dimensional.*

To proceed with further results on compact operators, we need the following lemma

**Lemma 1.2.1** (Riesz Lemma)**.** *Let $E$ be a normed vector space, $F = \overline{F} \subset E$. Then $\forall r \in (0,1)$, $\exists\, x_r \in E$, such that*

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

*Proof.* Since $F \neq E$ then this implies $\exists\, z \in E \setminus F$. Let $d = d(z,F) > 0$.
For $0 < r < 1$, $\exists\, y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|}\|z - (y_r + \|z - y_r\|y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

□

**Proposition 1.2.1.** *Let $\mathscr{A} \in \mathcal{K}(X)$, such that $X$ is a Banach space on $\mathbb{C}$. If $\lambda \in \mathbb{C}^*$, then* $\ker((\lambda I - \mathscr{A})^n)$ *has finite dimension.*

*Proof. Only for $n = 1$.* (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathscr{A}) = \{x \in X : \mathscr{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda}\mathscr{A}x\right\} \subset \mathrm{Rg}(\mathscr{A}).$$

So $\tilde{K}$ is closed in $\mathrm{Rg}(\mathscr{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists\ (x_n)$ in $\tilde{K}$, such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|}\|\mathscr{A}x_n - \mathscr{A}x_m\| \geq \frac{1}{2}, \qquad \forall\ n \neq m$$

and so we have $\|\mathscr{A}x_n\| \leq \|\mathscr{A}\|$. So $(\mathscr{A}x_n)$ is not Cauchy, hence a contradiction.

□

**Exercise 7**

Let $X$ be a Banach space on $\mathbb{K}$. If $\mathscr{A} \in \mathscr{L}(X)$, assume $\exists\ n_0$ s.t. $\ker(\mathscr{A}^{n_0}) = \ker(\mathscr{A}^{n_0+1})$. Then $\forall n \geq n_0$,

$$\ker(\mathscr{A}^n) = \ker(\mathscr{A}^{n_0}).$$

**Proposition 1.2.2.** *Let $\mathscr{A} \in \mathcal{K}(X)$ and $X$ be a Banach space on $\mathbb{C}$, $\lambda \neq 0$. Then $\exists\ n_0$ such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathscr{A})^n) = \ker((\lambda I - \mathscr{A})^{n_0}).$$

*Proof.* Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathscr{A})^n) \subset \ker((\lambda I - \mathscr{A})^{n+1})$ and each of them is closed.
**RL:** with $r = \frac{1}{2}$ with $(x_n)_{n\geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathscr{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathscr{A})^n)) \geq \frac{1}{2}.$$

For $n = 1, x \in \ker(\lambda I - \mathscr{A}) \ \Rightarrow \ x = \frac{\mathscr{A}}{\lambda}x$. For all $1 \leq m < n$,

$$\begin{aligned}\frac{\mathscr{A}x_n}{\lambda} - \frac{\mathscr{A}x_m}{\lambda} &= x_n - x_n + \frac{\mathscr{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathscr{A}x_m}{\lambda}\right) \\ &= x_n - \left[\frac{(\lambda I - \mathscr{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathscr{A})x_m}{\lambda}\right].\end{aligned}$$

So,

$$\left\| \frac{\mathscr{A} x_n}{\lambda} - \frac{\mathscr{A} x_m}{\lambda} \right\| \geq d\big(x_n, \ker(\lambda I - \mathscr{A})^n\big) \geq \frac{1}{2}.$$

which is a contradiction. $\square$

Notice that if $\ker(\lambda I - \mathscr{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathscr{A})$. Notice,

$$\dim \ker(\lambda I - \mathscr{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2 $\Rightarrow \exists\, n_0$ (smallest one) such that

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) = \ker\big((\lambda I - \mathscr{A})^n\big), \qquad \forall n \geq n_0.$$

Note that,

$$\ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{generalized eigenspace}.$$

$$\dim \ker\big((\lambda I - \mathscr{A})^{n_0}\big) := \text{algebraic multiplicity of } \lambda.$$

**Proposition 1.2.3** (Fredholm aternative)**.** *Let $\mathscr{A} \in \mathcal{K}(X)$, and let $X$ be a Banach space on $\mathbb{C}$.*

$$\operatorname{Rg}(\lambda I - \mathscr{A}) = X \iff \ker(\lambda I - \mathscr{A}) = \{0\}.$$

**Proposition 1.2.4.** *Let $\mathscr{A} \in \mathcal{K}(X)$, and let $X$ be a Banach space on $\mathbb{C}$,* $\dim X = \infty$*. If $\lambda_n \to \lambda$, $\lambda_n \in \sigma(\mathscr{A}) \setminus \{0\}$, pairwise distinct,then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathscr{A}) \setminus \{0\}$ is isolated.*

*Proof.* Let $\lambda_n \in \sigma_p(\mathscr{A})$, $\exists\, \|x_n\| = 1$ such that $\mathscr{A} x_n = \lambda_n x_n$. Let

$$X_n = \operatorname{span}(x_1, \ldots, x_n), \qquad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \;\Rightarrow\; \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i, \text{ which implies } \quad \lambda_{n+1} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathscr{A} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since $(x_i)$ are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \qquad 1 \le i \le n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathscr{A})X_n \subset X_{n-1}, \qquad \forall n \ge 2.$$

**Recall:**

$$\|y_n\| = 1, \qquad y_n \in X_n,$$
$$d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

For $2 \le m < n$,

$$\left\|\frac{\mathscr{A} y_n}{\lambda_n} - \frac{\mathscr{A} y_m}{\lambda_m}\right\| = \left\|y_n - \left[\frac{\lambda_n I - \mathscr{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathscr{A}}{\lambda_m} y_m\right]\right\|$$
$$\ge d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

Assume that

$$\lambda_n \to \lambda \quad (n \to \infty).$$

Suppose $\lambda \neq 0$, then

$$\left|\frac{1}{\lambda_n}\right| \le C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathscr{A} y_n}{\lambda_n}\right)_{n \ge 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathscr{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. $\square$

**Theorem 1.2.3.** *Let $\mathscr{A} \in \mathcal{K}(X)$, and let $X$ be a Banach space on $\mathbb{C}$. Then $\sigma(\mathscr{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.*

## Main use of compact operators (in PDEs)

They appear as "inverse" of UBLO.

**Definition 1.2.2.** *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ UBLO, closed, $\rho(\mathscr{A}) \neq \{0\}$. $\mathscr{A}$ is said to have compact resolvent if*

$$(\lambda I - \mathscr{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathscr{A}).$$

**Main Example:** $\mathscr{A} = -\Delta$ on $\Omega$ with $\mathscr{A} u = -u_{xx}$.

## 1.3 Adjoints, Symmetric and Self-adjoint Operators

Let $\mathscr{H}$ be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathscr{H}} : \mathscr{H} \times \mathscr{H} \to \mathbb{C}.$$

$\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ UBLO, $\overline{D(\mathscr{A})} = \mathscr{H}$.

**Definition 1.3.1** (Adjoint Operator $\mathscr{A}^\circ$)**.**

$$D(\mathscr{A}^\circ) = \{x \in \mathscr{H} : v \mapsto \langle \mathscr{A} v, x \rangle_{\mathscr{H}} : D(\mathscr{A}) \to \mathbb{C} \textit{ bdd operator}\}.$$

*If $x \in D(\mathscr{A}^\circ)$, then there exists uniquely $z \in \mathscr{H}$ such that $\langle v, z \rangle_{\mathscr{H}} = \langle \mathscr{A} v, x \rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.*
*Observe, with Riesz representation and the fact that $\overline{D(\mathscr{A})} = \mathscr{H}$, we have that $z := \mathscr{A}^\circ x$ and $\langle v, \mathscr{A}^\circ \rangle_{\mathscr{H}} = \langle \mathscr{A} v, x \rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.*

**Remark 1.3.1.** *Let $\mathscr{H}$ be a Hilbert space, $E : \mathscr{H} \to \mathscr{H}^*$, $x \mapsto \langle x, \cdot \rangle$. Linear isometry between $\mathscr{H}$ and $\mathscr{H}^*$. (One can identify $\mathscr{H}$ and $\mathscr{H}^*$). Now, we define the Dual operator as the following:*

$$\mathscr{A}^* : D(\mathscr{A}^*) \subset \mathscr{H}^* \to \mathscr{H}, \quad \mathscr{A}^\circ = E^{-1} \mathscr{A}^* E.$$

**Definition 1.3.2** (Symmetric and Self-adjoint Operator)**.** *Let $(\mathscr{H}, \langle , \rangle)$ be a Hilbert space and $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ be a UBLO, then*

1. ***Symmetric:*** *If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ \supset \mathscr{A}$ with $D(\mathscr{A}^\circ) \supset D(\mathscr{A})$ and for all $x, y \in D(\mathscr{A})$, $\langle \mathscr{A} x, y \rangle = \langle x, \mathscr{A} y \rangle$.*

2. ***Self-adjoint:*** *If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ = \mathscr{A}$.*

**Exercise 8**

1. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. If $\mathscr{A}$ is closed, then $\overline{D(\mathscr{A}^\circ)} = \mathscr{H}$.

2. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. Suppose $\mathscr{A}$ is symmetric and if $0 \in \sigma_p(\mathscr{A})$, then prove that $\lambda \in \mathbb{R}$ and

$$\inf_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A} x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A} x, x \rangle.$$

**Proposition 1.3.1.** *Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be an Hilbert space over $\mathbb{C}$, If $\mathscr{A}$ is self-adjoint, injective and $\overline{D(\mathscr{A})} = \mathscr{H}$. Then $\mathscr{A}^{-1} : \mathrm{Rg}(\mathscr{A}) \to \mathscr{H}$ is self-adjoint.*

*Proof.* Since $\mathscr{A}$ is injective then $\mathscr{A}^{-1}$ is well defined, and since $\mathscr{A} = \mathscr{A}^\circ$ then $\mathscr{A}$ is closed.

Now assume $(x_n) \subset D(\mathscr{A})$ such that $x_n \to x \in D(A)$ (because $\mathscr{A}$ is closed) and $\mathscr{A} x_n - \to y$ then for all $z \in D(\mathscr{A})$, $\langle \mathscr{A} x_n, z \rangle = \langle x_n, \mathscr{A}^\circ z \rangle$ which implies $\langle y, z \rangle = \langle x, \mathscr{A}^\circ z \rangle$ and so we have

$x \in D(\mathscr{A}^\circ) = D(\mathscr{A})$ and $y = \mathscr{A}^\circ$. Notice $\overline{\text{Rg}(\mathscr{A})} = \ker(\mathscr{A})^\perp$ (because of self-adjointness). Injectivity implies $\overline{\text{Rg}(\mathscr{A})} = \mathscr{H}$ which implies $\overline{D(\mathscr{A}^{-1})} = \mathscr{H}$. So $\mathscr{A}^{-1}$ is densely defined. Now, observe for all $u, v \in D(\mathscr{A}^{-1})$, $u = \mathscr{A}^\circ x$ and $v = \mathscr{A} y$ with $x, y \in D(\mathscr{A})$. Hence,

$$\langle \mathscr{A}^{-1} u, v\rangle = \langle x, \mathscr{A} y\rangle = \langle \mathscr{A} x, y\rangle = \langle x, \mathscr{A}^{-1} y\rangle.$$

To this end, $(\mathscr{A}^{-1})^\circ \subset \mathscr{A}^{-1}$, $\forall\, z \in D\big((\mathscr{A}^{-1})^\circ\big)\ \exists\, w, \forall\, u \in D(\mathscr{A}^{-1}) = R(\mathscr{A})\ (i.e.\ u = \mathscr{A} x)$

$$\langle \mathscr{A}^{-1} u, z\rangle = \langle u, w\rangle \Rightarrow \forall x \in D(\mathscr{A}) \quad \langle x, z\rangle = \langle \mathscr{A} x, w\rangle$$

By definition $w \in D(\mathscr{A}^\circ)$ and $\mathscr{A}^\circ w = z$. $\mathscr{A} w = z \Rightarrow z \in Rg(\mathscr{A}) = D(\mathscr{A}^{-1})$. □

**Theorem 1.3.1.** *Let $\mathscr{H}$ be a Hilbert space, suppose $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is symmetric and surjective then $\mathscr{A}$ is self-adjoint.*

*Proof.* $\mathscr{A}$ and $\mathscr{A}^\circ$ are injective. Do it only for $\mathscr{A}$, let $x \in D(\mathscr{A})$ and $\mathscr{A} x = 0$.

$$\forall y \in D(\mathscr{A}), \quad 0 = \langle \mathscr{A} x, y\rangle = \langle x, \mathscr{A} y\rangle \Rightarrow x \perp Rg(\mathscr{A}) = \mathscr{H}.$$

which implies $x = 0$.
Next, we show $\mathscr{A}$ closed.

$$(x_n)_{n\geq 1} \subset D(\mathscr{A}), \quad x_n \to x \text{ in } \mathscr{H}, \quad \mathscr{A} x_n \to y \text{ in } \mathscr{H}$$

We shall show $y = \mathscr{A} x$. Now, $\forall z \in D(\mathscr{A})$ then $\langle \mathscr{A} x_n, z\rangle = \langle x_n, \mathscr{A} z\rangle$, which implies $\langle y, z\rangle = \langle x, \mathscr{A} z\rangle$ which implies $x \in D(\mathscr{A}^\circ)$ and $y = \mathscr{A}^\circ x$. Since $\mathscr{A}$ surjective $\Rightarrow \exists w \in D(\mathscr{A})$ s.t. $\mathscr{A} w = y$ and $\mathscr{A}^\circ x = y$.
Since $\mathscr{A}$ is symmetric: $\mathscr{A}^\circ w = \mathscr{A} w$. Then $\mathscr{A}^\circ w = \mathscr{A}^\circ x$, $\mathscr{A}$ is injective $\Rightarrow w = x$. Hence $\mathscr{A} x = \mathscr{A} w = y \Rightarrow y = \mathscr{A} x \Rightarrow \mathscr{A}$ is closed.
By closed graph theorem both $\mathscr{A}$ and $\mathscr{A}^{-1} \in \mathscr{L}(X)$. We can conclude that $\mathscr{A}$ is a self-adjoint operator. □

**Exercise 9**

Let $\mathscr{H} = L^2(0, \pi)$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ and

$$\mathscr{A} f = -f''$$

If $D(\mathscr{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?
Similarly, if $D(\mathscr{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?

**Theorem 1.3.2** (Fredrich's Extension)**.** *Let $\mathscr{H}$ be a Hilbert space on $\mathbb{C}$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, symmetric then $\mathscr{A}$ admits a unique self adjoint extension. If either*

*a.* $\inf_{\|x\|=1, x\in D(\mathscr{A})} \langle \mathscr{A} x, x\rangle > -\infty$

*b.* $\sup_{\|x\|=1, x\in D(\mathscr{A})}\langle \mathscr{A}x, x\rangle < +\infty$

*such that,* $\mathscr{A} \subset \mathscr{A}^\circ \subset (\mathscr{A}^\circ)^\circ \subset \cdots$. *If (a) or (b) holds, then;* $\mathscr{A} \subset \mathscr{A}^\circ = (\mathscr{A}^\circ)^\circ$.

## 1.4 Dissipative Operator and Numerical range

**Definition 1.4.1** (Duality Map)**.** *Let $X$ be a Banach space on $\mathbb{K}$. The duality map is defined as* $J : X \longrightarrow 2^{X^*}$, $x \mapsto J(x) = \{x^* \in X^* \mid Re\langle x^*, x\rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$.
*By the Hahn-Banach theorem,* $J(x) \neq \emptyset$.

**Question:** What can you say about $J(X)$ when $X$ is an Hilbert space or Reflexive?

**Definition 1.4.2.** *A map $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ (UBLO) is dissipative if for all $x \in D(\mathscr{A})$, there exists $x^* \in J(X)$ such that $Re\langle x^*, \mathscr{A}x\rangle \leq 0$.*

**Lemma 1.4.1.** *$\mathscr{A}$ is dissipative if and only if for all $\lambda > 0$, $x \in D(\mathscr{A})$ we have that* $\|(\lambda I - \mathscr{A})x\| \geq \lambda\|x\|$.

*Proof.* let $x^* \in J(x)$. Then

$$\|(\lambda I - \mathscr{A})x\|\,\|x^*\| \geq |\langle x^*, (\lambda I - \mathscr{A})x\rangle| \geq \Re\langle x^*, (\lambda I - \mathscr{A})x\rangle,$$

$$= \lambda\Re\langle x^*, x\rangle - \Re\langle x^*, \mathscr{A}x\rangle \geq \lambda\|x\|^2.$$

Hence, if $\|x\| \neq 0$, then

$$\|(\lambda I - \mathscr{A})x\| \geq \lambda\|x\|.$$

($\Leftarrow$) Let $x \in D(\mathscr{A})$, $x \neq 0$, and $\lambda > 0$. Let $y_\lambda^* \in J((\lambda I - \mathscr{A})x)$ and set $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$. Then

$$\|(\lambda I - \mathscr{A})x\|^2 = \|(\lambda I - \mathscr{A})x\|\,\|y_\lambda^*\| = \Re\langle y_\lambda^*, (\lambda I - \mathscr{A})x\rangle.$$

Since $y_\lambda^* \neq 0$, we have

$$\lambda\|x\| \leq \|(\lambda I - \mathscr{A})x\| = \Re\langle g_\lambda^*, (\lambda I - \mathscr{A})x\rangle = \lambda\langle g_\lambda^*, x\rangle - \Re\langle g_\lambda^*, \mathscr{A}x\rangle.$$

Hence,

$$\Re\langle g_\lambda^*, \mathscr{A}x\rangle \leq \lambda\langle g_\lambda^*, x\rangle - \lambda\|x\| \leq \|g_\lambda^*\|\,\|x\| = \|x\|.$$

Therefore,

$$\Re\langle g_\lambda^*, \mathscr{A}x\rangle \leq 0. \tag{**}$$

**Idea:** Let $\lambda \to +\infty$.

Unit ball in $X^*$ is compact for weak* topology (Banach–Alaoglu).

(Up to subsequence)

$$g_\lambda^* \longrightarrow g^* \in X^*, \qquad \|g^*\| \leq 1.$$

Then from $(**)$,

$$\Re\langle g^*, \mathscr{A}x\rangle \le 0.$$

$(*)$ $\qquad \|x\| \le \langle g_\lambda^*, x\rangle - \frac{1}{\lambda}\Re\langle g_\lambda^*, \mathscr{A}x\rangle.$

Let $\lambda \to +\infty$. Then $\|x\| \le \langle g^*, x\rangle$. Hence, $\|g^*\| = 1 \quad$ and $\quad \langle g^*, x\rangle = \|x\|$. Set $x^* = \|x\| g^*$. Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x\rangle = \|x\|^2,$$

that is, $x^* \in J(x)$. $\square$

**Theorem 1.4.1** (Lumer-Phillips)**.** *Let $X$ be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO. Assume that $\mathscr{A}$ is dissipative and that there exists $\lambda_0 > 0$ such that* $\mathrm{Rg}(\lambda_0 I - \mathscr{A}) = X$.

*Then $\mathscr{A}$ is closed, $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$, and for all $\lambda > 0$,*

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \le \frac{1}{\lambda}. \tag{1.4}$$

*Proof.* Let $\lambda_0 > 0$.

(1) To prove: $(\lambda_0 I - A)$ is bijective.

- surj: Assumption,
- inj: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathscr{A})^{-1} : X \longrightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any $y \in X$, there exists a unique $y \in X$ such that

$$x = (\lambda_0 I - \mathscr{A})^{-1} y, \qquad (\lambda_0 I - \mathscr{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0}\|y\| \ge \|(\lambda_0 I - \mathscr{A})^{-1} y\|.$$

Hence,

$$\|(\lambda_0 I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \le \frac{1}{\lambda_0}, \qquad \lambda_0 \in \rho(\mathscr{A}).$$

(2) $\mathscr{A}$ is closed.

Let $x_n \to x$, $x_n \in D(\mathscr{A})$, and $Ax_n \to y$.
Then

$$(\lambda_0 I - \mathscr{A})x_n \to \lambda_0 x - y.$$

Since $(\lambda_0 I - \mathscr{A})^{-1} \in \mathcal{L}(X)$, we have

$$x_n \longrightarrow (\lambda_0 I - \mathscr{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,
$$\lambda_0 x - y = (\lambda_0 I - \mathscr{A})x \iff y = \mathscr{A}x.$$

Therefore, $\mathscr{A}$ is closed.

(3) $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$ and (1.4).

Since $\mathscr{A}$ is closed and $\rho(\mathscr{A}) \neq \emptyset$, we know that $\rho(\mathscr{A})$ is open.

Let $\Lambda = \rho(\mathscr{A}) \cap \mathbb{R}_+^*$, which is open in $\mathbb{R}_+^*$. We show that it is closed.

Let $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda$ such that $\lambda_n \longrightarrow \lambda \in \mathbb{R}_+^*$. Note that since $\lambda_n \in \Lambda$, we have
$$\|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write
$$(\lambda I - \mathscr{A}) = \big[I + u_n\big](\lambda_n I - \mathscr{A}) \Longrightarrow \lambda I - \mathscr{A} = \lambda_n I - \mathscr{A} + u_n(\lambda_n I - \mathscr{A}),$$
$$\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathscr{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathscr{A})^{-1} = u_n.$$

Hence,
$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For $n$ large enough,
$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that $\lambda \in \rho(\mathscr{A})$. Hence, $\Lambda$ is closed, and therefore $\Lambda = \mathbb{R}_+^*$.

□

**Corollary 1.4.1.** *Let $X$ be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO, closed, with $\overline{D(\mathscr{A})} = X$. Assume that $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative. Then*
$$\rho(\mathscr{A}) \supset \mathbb{R}_+^*, \qquad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathscr{A})^{-1}\| \leq 1.$$

*Proof.* It is easy to show that $\mathrm{Rg}(I - A) = X$ (i.e. $\lambda_0 = 1$ + Theorem 1.4.1).
$\mathscr{A}$ dissipative and closed implies
$$\mathrm{Rg}(I - \mathscr{A}) \text{ is a closed subspace of } X.$$

(give details!!!)

Let $x^* \in X^*$ such that
$$\langle x^*, (I - \mathscr{A})x \rangle = 0, \qquad \forall x \in D(\mathscr{A}). \tag{$**$}$$

Let us prove that $x^* = 0$.

Then $x^* \in D(\mathscr{A}^*)$ and

$$(I - \mathscr{A}^*)x^* = 0.$$

Since $\mathscr{A}^*$ is dissipative, by Lemma 1.4.1, we have $x^* = 0$. This implies that

$$\overline{\mathrm{Rg}(I - \mathscr{A})} = X.$$

Since $\mathrm{Rg}(I - \mathscr{A})$ is closed, we obtain

$$\mathrm{Rg}(I - \mathscr{A}) = X.$$

By contradiction and using Hahn–Banach. □

**Definition 1.4.3** (Numerical Range). *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be UBLO. The numerical range of $\mathscr{A}$, denoted by $W(\mathscr{A})$,*

$$W(\mathscr{A}) = \{\langle x^*, \mathscr{A}x\rangle \mid x^* \in J(x),\ x \in D(\mathscr{A}),\ \|x\| = \|x^*\| = 1,\ \langle x^*, x\rangle = 1\}.$$

*In case of a Hilbert space, we have that $W(\mathscr{A}) = \{\langle x, \mathscr{A}x\rangle \mid x \in D(\mathscr{A}),\ \|x\| = 1\}$.*

*Linear algebra in finite dimension $\mathscr{A} \in \mathcal{M}_n(\mathbb{K})$, we have that $W(\mathscr{A}) = \{\langle x, \mathscr{A}x\rangle \mid \|x\| = 1\}$.*

**Theorem 1.4.2** (Home-work). *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be closed, with $\overline{D(\mathscr{A})} = X$.*

*1) If $\lambda \notin \overline{W(\mathscr{A})}$, then $(\lambda I - \mathscr{A})$ is injective, has closed image, and for all $x \in D(\mathscr{A})$,*

$$\|(\lambda I - \mathscr{A})x\| \geq d(\lambda, W(\mathscr{A}))\,\|x\|.$$

*Moreover, if $\lambda \in \rho(\mathscr{A})$, then*

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, W(\mathscr{A}))}. \tag{$**$}$$

*2) If $\Lambda$ is a connected open subset of $\mathbb{C} \setminus W(\mathscr{A})$ such that $\rho(\mathscr{A}) \cap \Lambda \neq \emptyset$, then $\rho(\mathscr{A}) \supset \Lambda$ and $(**)$ holds true.*

CHAPTER

# TWO

# INTRODUCTION TO THE THEORY OF SEMI-GROUPS

## 2.1 Intro to the Introduction

**Definition 2.1.1.** *Let $X$ be a Banach space over $\mathbb{K}$. A one-parameter family of bounded linear operators on $X$, $(T(t))_{t\geq 0}$, is a semigroup (SG) of bounded linear operators on $X$ if:*

1. $T(0) = Id_X$,
2. $\forall (t,s) \in \mathbb{R}_+^2 :\ T(t+s) = T(t)\cdot T(s)$ *(SG property).*

**Remark 2.1.1.** *$T(t)$ and $T(s)$ commute.*

2. **Infinitesimal generator of SG–LO** $(T(t))_{t\geq 0}$
Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be an unbounded linear operator with

$$D(\mathscr{A}) = \left\{ x \in X \;\middle|\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = \mathscr{A}x, \qquad x \in D(\mathscr{A}).$$

where $D(\mathscr{A})$ = domain of $\mathscr{A}$.

## 2.2 Uniformly Continuous SG–BLO

**Definition 2.2.1.** *A SG–BLO on $X$, $(T(t))_{t\geq 0}$ is uniformly continuous if*

$$\|T(t) - Id\|_{\mathscr{L}(X)} \to 0 \quad \textit{as } t \to 0^+$$

**Lemma 2.2.1.** *Let $(T(t))_{t\geq 0}$ be a SG–BLO which is uniformly continuous. Then, $\forall t > 0$,*

$$\|T(s) - T(t)\| \underset{s\to t}{\longrightarrow} 0$$

*(continuity everywhere).*

*Proof.* Let $t$ be fixed. $T(s) = T(s - t + t), \qquad s - t \geq 0.$

$$s \geq t \;\Rightarrow\; T(s) = T(s-t)T(t) \Rightarrow\; T(s) - T(t) = T(t)\,[T(s-t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\|\,\|T(s-t) - I_d\| \xrightarrow[s\to t]{} 0.$$

For $s \leq t$

$$T(t) = T(t-s)T(s) \Rightarrow T(t) - T(s) = T(s)\,[T(t-s) - I_d]$$

(Prove that $\sup_{[0,t]} \|T(t)\| < +\infty$)

Then

$$\|T(t) - T(s)\| \leq \|T(s)\|\,\|T(t-s) - I_d\|$$

$$\leq \sup \|T(s)\|\,\|T(t-s) - I_d\| \xrightarrow[s\to t]{} 0.$$

□

**Theorem 2.2.1.** *A linear operator $\mathscr{A}$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $\mathscr{A}$ is a bounded linear operator.*

*Proof.* Let $\mathscr{A}$ be a bounded linear operator on $X$ and set

$$T(t) = e^{t\mathscr{A}} = \sum_{n=0}^{\infty} \frac{(t\mathscr{A})^n}{n!}. \tag{1.5}$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such $t$, a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t+s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq |t|\|\mathscr{A}\|e^{|t|\|\mathscr{A}\|}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathscr{A} \right\| \leq \|\mathscr{A}\| \cdot \max_{0\leq s\leq t} \|T(s) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on $X$ and that $\mathscr{A}$ is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on $X$. Fix $\rho > 0$, small enough, such that

$$\left\| I - \rho \int_0^{\rho} T(s)\,ds \right\| < 1.$$

This implies that $\rho^{-1} \int_0^{\rho} T(s)\,ds$ is invertible. Now,

$$h^{-1}(T(h) - I) \int_0^{\rho} T(s)\,ds = h^{-1} \left( \int_0^{\rho} T(s+h)\,ds - \int_0^{\rho} T(s)\,ds \right)$$

$$= h^{-1}\left(\int_{\rho}^{\rho+h} T(s)\,ds - \int_{0}^{h} T(s)\,ds\right)$$

and therefore

$$h^{-1}(T(h)-I) = \left(h^{-1}\int_{\rho}^{\rho+h} T(s)\,ds - h^{-1}\int_{0}^{h} T(s)\,ds\right)\left(\int_{0}^{\rho} T(s)\,ds\right)^{-1}. \tag{1.6}$$

Letting $h \to 0$ in (1.6) shows that $h^{-1}(T(h)-I)$ converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho)-I)\left(\int_{0}^{\rho} T(s)\,ds\right)^{-1}$$

which is the infinitesimal generator of $T(t)$. □

**Remark 2.2.1.** *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

**Theorem 2.2.2.** *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\to 0}\frac{T(t)-I}{t} = \mathscr{A} = \lim_{t\to 0}\frac{S(t)-I}{t}. \tag{1.7}$$

*then $T(t) = S(t)$ for $t \geq 0$.*

*Proof.* We will show that given $T > 0$, $S(t) = T(t)$ for $0 \leq t \leq T$. Let $T > 0$ be fixed, since $t \mapsto \|T(t)\|$ and $t \mapsto \|S(t)\|$ are continuous there is a constant $C$ such that

$$\|T(t)\|\|S(s)\| \leq C \quad \text{for } 0 \leq s,t \leq T.$$

Given $\varepsilon > 0$ it follows from (1.7) that there is a $\delta > 0$ such that

$$h^{-1}\|T(h)-S(h)\| < \varepsilon/TC \qquad \text{for } 0 \leq h \leq \delta. \tag{1.8}$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n \leq \delta$. From the semigroup property and (1.8) it then follows that

$$\begin{aligned}
\|T(t)-S(t)\| &= \left\|T\left(n\frac{t}{n}\right) - S\left(n\frac{t}{n}\right)\right\| \\
&\leq \sum_{k=0}^{n-1}\left\|T\left((n-k)\frac{t}{n}\right)S\left(\frac{kt}{n}\right) - T\left((n-k-1)\frac{t}{n}\right)S\left(\frac{(k+1)t}{n}\right)\right\| \\
&\leq \sum_{k=0}^{n-1}\left\|T\left((n-k-1)\frac{t}{n}\right)\right\|\left\|T\left(\frac{t}{n}\right)-S\left(\frac{t}{n}\right)\right\|\left\|S\left(\frac{t}{n}\right)\right\| \leq Cn\frac{\varepsilon}{TC}\frac{t}{n} \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $T(t) = S(t)$ for $0 \leq t \leq T$ and the proof is complete. □

**Corollary 2.2.1.** *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

*(a) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*

*(b) There exists a unique bounded linear operator $A$ such that $T(t) = e^{t\mathscr{A}}$.*

*(c) The operator $\mathscr{A}$ in part (b) is the infinitesimal generator of $T(t)$.*

*(d) $t \mapsto T(t)$ is differentiable in norm and satisfies*

$$\frac{dT(t)}{dt} = \mathscr{A}T(t) = T(t)\mathscr{A}. \tag{1.9}$$

*Proof.* All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator $\mathscr{A}$. $\mathscr{A}$ is also the infinitesimal generator of $e^{t\mathscr{A}}$ defined by (1.5) and therefore, by Theorem 2.2.2, $T(t) = e^{t\mathscr{A}}$. □

**Remark 2.2.2.** *The proofs above are from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 3. [Theorem 1.3 and Corollary 1.4].*

## 2.3 Strongly Continuos Semigroups ($C_0$-Semigroups)

**Definition 2.3.1.** *The SG-BLO $(T(t))_{t\geq 0}$ is strongly continous (SC or $C_0$) if $\forall x \in X$*

$$\|T(t)x - x\|_X \xrightarrow{t\to 0^+} 0 \tag{2.1}$$

**Theorem 2.3.1.** *Let $(T(t))_{t\geq 0}$, $C_0$-SG then $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0$ $\|T(t)\| \leq Me^{\omega t}$*

*Proof.* First we want to show that $\exists \eta > 0$, $\sup_{t\in[0,\eta]} \|T(t)\| < +\infty$.

By contradiction, assume that $\sup \|T(t)\| = +\infty$. Then $\exists (t_n)_{n\geq 0} \searrow 0$ such that $\|T(t_n)\| \geq n$ or $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive) $\exists x \in X$ such that $\sup \|T(t_n)x\| = +\infty$, but this contradicts the strong convergance 2.3.1.

Now take $M := \sup \|T(t)\| \geq 1$ (This is because $T(0) = Id$).

$\forall t \geq 0$ write $t = k\eta + \eta_t$ where $k = \lfloor \frac{t}{\eta} \rfloor$ and $\eta_t \in [0, \eta]$, then

$$\begin{aligned}
\|T(t)\| &= \|T(k\eta + \eta_t)\| \\
&= \left\| [T(\eta)]^k \, T(\eta_t) \right\| && \text{by the SG property} \\
&= \|T(\eta)\|^k \, \|T(\eta_t)\| \\
&\leq M \cdot M^k && \text{M is upperbound} \\
&\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\
&= M(e^{\ln M})^{t/\eta} \\
&= Me^{t\frac{\ln M}{\eta}} = Me^{\omega t}
\end{aligned}$$

And since $\omega = \dfrac{\ln M}{\eta}$ and $M \geq 1$ then $\omega \geq 0$. □

**Corollary 2.3.1.** *Let $\{T(t)\}_{t\geq 0}$ be a $C_0$-semigroup then $\forall x \in X$, $t \mapsto T(t)x$ is continuous*

*Proof.* For $h > 0$ and $t > 0$:

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \to 0$ as $h \to 0$ by definition 2.3.1.
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$. Since $\|T(t-h)\|$ is bounded by theorem 2.3.1, so this tends to 0 as $h \to 0$.

□

**Theorem.** *Let $\mathscr{A}$ be the IG of $C_0$-SG $T(t)$, then:*

***2.3.2*** $\forall x \in X, \forall t \geq 0,\ \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$ *for all* $x \in X$.

***2.3.3*** $\forall x \in X, \forall t \geq 0,\ \int_0^t T(s)x\,ds \in D(\mathscr{A})$ *and* $\mathscr{A}\int_0^t T(s)x\,ds = T(t)x - x$.

***2.3.4*** $\forall x \in D(\mathscr{A})$, $T(t)x \in D(\mathscr{A})$ *and* $\frac{d}{dt}T(t)x = \mathscr{A}T(t)x = T(t)\mathscr{A}x$.

***2.3.5*** $\forall x \in D(\mathscr{A}), \forall t \geq 0, \forall s \geq 0,\ T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x\,du$ *for* $x \in D(\mathscr{A})$.

*Proof of* ***2.3.2***. Consider the small interval $[t, t+h]$ relative to its value at $t$:

$$\frac{1}{h}\int_t^{t+h} T(s)x\,ds - T(t)x = \frac{1}{h}\int_t^{t+h} [T(s)x - T(t)x]\,ds$$

By the continuity (corollary 2.3.1) of the map $s \mapsto T(s)x$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $s$ satisfying $|s - t| < \delta$, we have $\|T(s)x - T(t)x\| < \epsilon$.

Taking $0 < h < \delta$, we can estimate the norm of the integral:

$$\left\|\frac{1}{h}\int_t^{t+h} [T(s)x - T(t)x]\,ds\right\| \leq \frac{1}{h}\int_t^{t+h} \|T(s)x - T(t)x\|\,ds < \frac{1}{h}\cdot h\epsilon = \epsilon$$

Then

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$$

□

*Proof of* ***2.3.3***. Let $h > 0$. Consider the difference quotient for the integral $y = \int_0^t T(s)x\,ds$:

$$\begin{aligned}
\frac{T(h) - Id}{h}\int_0^t T(s)x\,ds &= \frac{1}{h}\left[\int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds\right] \\
&= \frac{1}{h}\left[\int_h^{t+h} T(u)x\,du - \int_0^t T(u)x\,du\right] \\
&= \frac{1}{h}\int_t^{t+h} T(u)x\,du - \frac{1}{h}\int_0^h T(u)x\,du
\end{aligned}$$

As $h \to 0^+$, the first term converges to $T(t)x$ and the second to $T(0)x = x$ by 2.3.2. Thus the limit exists, $y \in D(\mathscr{A})$, and $\mathscr{A}y = T(t)x - x$. □

*Proof of 2.3.4.* If $x \in D(\mathscr{A})$, then $T(t)\mathscr{A}x = T(t)\lim_{h\to 0}\frac{T(h)x-x}{h} = \lim_{h\to 0}\frac{T(h)T(t)x-T(t)x}{h}$. This limit exists and equals $\mathscr{A}(T(t)x)$, proving $T(t)x \in D(\mathscr{A})$ and $T(t)\mathscr{A}x = \mathscr{A}T(t)x$. This also shows the right-derivative of $T(t)x$ is $\mathscr{A}T(t)x$. A similar argument for the left-derivative completes the differentiability. □

*Proof of 2.3.5.* By Property 2.3.4, the function $f(u) = T(u)x$ is differentiable with $f'(u) = T(u)\mathscr{A}x$. Since $f'$ is continuous, we integrate $f'$ over $[s,t]$ to obtain $f(t) - f(s) = \int_s^t f'(u)\,du$, which is $T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x\,du$. □

**Theorem 2.3.6.** *The IG $\mathscr{A}$ of $C_0$-SG is a closed linear operator and $\overline{D(\mathscr{A})} = X$.*

*Proof.* For any $x \in X$, let $x_t = \frac{1}{t}\int_0^t T(s)x\,ds$. By 2.3.3, $x_t \in D(\mathscr{A})$. By 2.3.2, $x_t \to T(0)x = x$ as $t \to 0^+$, which shows $\overline{D(\mathscr{A})} = X$.

Let $x_n \in D(\mathscr{A})$ such that $x_n \to x$ and $\mathscr{A}x_n \to y$. From 2.3.3 take $s = 0$, we have $T(t)x_n - x_n = \int_0^t T(s)\mathscr{A}x_n\,ds$. Passing to the limit $n \to \infty$, we get $T(t)x - x = \int_0^t T(s)y\,ds$. Dividing by $t$ and letting $t \to 0^+$, the Right-Hand Side (RHS) converges to $y$. Thus $x \in D(\mathscr{A})$ and $\mathscr{A}x = y$. □

**Theorem 2.3.7.** *Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be two $C_0$-SG with infinitesimal generators $\mathscr{A}$ and $B$, respectively. If $\mathscr{A} = \mathscr{B}$, then $T(t) = S(t)$ for all $t \geq 0$.*

*Proof.* Assume $\mathscr{A} = \mathscr{B}$, then $D(\mathscr{A}) = D(\mathscr{B})$. Let $x \in D(\mathscr{A})$ be fixed, and for a fixed $t > 0$, define

$$\varphi : [0,t] \to X, \quad \varphi(s) = T(t-s)S(s)x$$

Since $x \in D(\mathscr{A})$, the map $\varphi$ is of class $C^1$ on $[0,t]$. We differentiate $\varphi$ with respect to $s$ using the product rule and 2.3.4, we get:

$$\begin{aligned}\frac{d}{ds}\varphi(s) &= \frac{d}{ds}[T(t-s)]S(s)x + T(t-s)\frac{d}{ds}[S(s)x] \\ &= -\mathscr{A}T(t-s)S(s)x + T(t-s)BS(s)x\end{aligned}$$

Because $T(t-s)$ commutes with its generator $\mathscr{A}$, and given $\mathscr{A} = B$, we have:

$$\frac{d}{ds}\varphi(s) = -T(t-s)\mathscr{A}S(s)x + T(t-s)\mathscr{A}S(s)x = 0$$

Since the derivative is zero for all $s \in [0,t]$, the function $\varphi$ must be constant. Evaluating $\varphi$ at the endpoints $s = 0$ and $s = t$ yields

$$\varphi(0) = T(t)S(0)x = T(t)x \quad \text{and} \quad \varphi(t) = T(0)S(t)x = S(t)x$$

Thus, $T(t)x = S(t)x$ for all $x \in D(\mathscr{A})$. Since $D(\mathscr{A})$ is dense in $X$ and $T(t), S(t)$ are bounded linear operators, this identity extends to all $x \in X$ by continuity. Therefore, $T(t) = S(t)$ for all $t \geq 0$. $\square$

**Theorem 2.3.8.** *Let $\mathscr{A}$ be the IG of a $C_0$-SG $\{T(t)\}_{t\geq 0}$ on a Banach space $X$. Then the subspace*

$$X = \overline{\bigcap_{n\geq 1} D(\mathscr{A}^n)}$$

*Proof.* Let $\mathscr{D} = \{\varphi : \mathbb{R} \to \mathbb{C} | \, \varphi \text{ has compact support in } \mathbb{R}_+^* \text{ and is smooth } C^\infty\}$. Let $x \in X$ and consider a test function $\varphi \in \mathscr{D}$. Define

$$x_\varphi = \int_0^\infty \varphi(s)T(s)x\, ds$$

First, we show that $x_\varphi \in D(\mathscr{A})$. Consider

$$\begin{aligned}
\frac{T(h) - Id}{h} x_\varphi &= \frac{1}{h}\int_0^\infty \varphi(s)[T(s+h)x - T(s)x]\, ds \\
&= \frac{1}{h}\left[\int_h^\infty \varphi(u-h)T(u)x\, du - \int_0^\infty \varphi(u)T(u)x\, du\right] \\
&= \int_0^\infty \frac{\varphi(u-h) - \varphi(u)}{h} T(u)x\, du
\end{aligned}$$

As $h \to 0$, the quotient $\frac{\varphi(u-h)-\varphi(u)}{h}$ converges uniformly to $-\varphi'(u)$ because $\varphi$ is $C^\infty$ and has compact support. Thus:

$$\mathscr{A} x_\varphi = -\int_0^\infty \varphi'(s)T(s)x\, ds$$

Since $\varphi' \in C_c^\infty(0, \infty)$, we can repeat this process inductively. For any $n \geq 1$, we find:

$$\mathscr{A}^n x_\varphi = (-1)^n \int_0^\infty \varphi^{(n)}(s)T(s)x\, ds$$

This proves that $x_\varphi \in D(\mathscr{A}^n)$ for all $n$.

To prove density, suppose $\overline{\bigcap_{n\geq 1} D(\mathscr{A}^n)} \neq X$. By the Hahn-Banach Theorem, there exists a non-zero functional $x^* \in X^*$ such that $\langle x^*, y\rangle = 0$ for all $y \in \bigcap_{n\geq 1} D(\mathscr{A}^n)$. Specifically, for any $x \in X$ and $\varphi \in C_c^\infty(0, \infty)$:

$$\langle x^*, x_\varphi\rangle = \int_0^\infty \varphi(s)\langle x^*, T(s)x\rangle\, ds = 0$$

This identity holds for all $C^\infty$ functions $\varphi$ with compact support. Then $\langle x^*, T(s)x\rangle$ must be zero for all $s > 0$.

By the strong continuity of the semigroup at $s = 0$:

$$\langle x^*, x\rangle = \lim_{s\to 0^+} \langle x^*, T(s)x\rangle = 0$$

Since this holds for all $x \in X$, it implies $x^* = 0$, which contradicts our assumption that $x^*$ was non-zero. Thus, $\bigcap_{n \geq 1} D(\mathscr{A}^n)$ must be dense in $X$. $\square$

**Exercise 10**

Let $X = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and uniformly bounded}\}$ equipped with the supremum norm $\|f\|_\infty = \sup_{s \in \mathbb{R}} |f(s)|$. Define the family of operators $(T(t))_{t \geq 0}$ by:

$$(T(t)f)(s) = f(s+t), \quad s \in \mathbb{R}, t \geq 0$$

Prove that this family is $C_0$-SG and its IG is $\mathscr{A} f = f'$.

## 2.4 Hille-Yosida Theorem

**Definition 2.4.1.** *A $C_0$-SG $\{T(t)\}_{t \geq 0}$ is called uniformly bounded semigroup if $\exists M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.*

**Definition 2.4.2.** *A $C_0$-SG $\{T(t)\}_{t \geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.*

**Theorem 2.4.1** (Hille-Yosida Theorem)**.** *A linear unbounded operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ is the IG of a $C_0$-SG of contractions if and only if:*

*(i) $\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = X$.*

*(ii) $\mathbb{R}_+^* \subset \rho(\mathscr{A})$ and $\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.*

*($\Rightarrow$).* (i) follows directly from 2.3.6.

If $\mathscr{A}$ generates a contraction semigroup $\{T(t)\}_{t \geq 0}$, we define the resolvent for $\lambda > 0$ as follows

$$R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\, dt$$

Taking the norm, we obtain:

$$\|R(\lambda)x\| \leq \int_0^\infty e^{-\lambda t} \|T(t)x\|\, dt \leq \int_0^\infty e^{-\lambda t} \|x\|\, dt = \frac{1}{\lambda} \|x\|$$

$\square$

**Remark 2.4.1.** *Note for all real numbers $\lambda, a$ with $\lambda > a$, we have the following:*

$$\frac{1}{\lambda - a} = \int_0^\infty e^{-(\lambda - a)t}\, dt$$

*Extending this to the vector space we get the way of writing the resolvent operator from above.*

### 2.4.1 The Yosida Approximation

To prove if ($\Leftarrow$), we introduce a family of bounded operators that approximate the unbounded generator $A$.

**Definition 2.4.3.** *For $\lambda > 0$, the Yosida Approximation of $\mathscr{A}$ is defined as:*

$$\mathscr{A}_\lambda := \lambda \mathscr{A} R(\lambda, \mathscr{A}) = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$$

Note that $\mathscr{A}_\lambda$ is a bounded linear operator for each $\lambda \in \rho(\mathscr{A})$.

**Theorem 2.4.2.** *For $\lambda \in \rho(\mathscr{A})$ and $x \in D(\mathscr{A})$, the following identity holds:*

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A}x$$

*Proof.* By the definition of the resolvent as the inverse of the operator $(\lambda I - \mathscr{A})$, we have:

$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A}) = Id_X$$

Applying this to any $x \in X$:

$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A})x = x$$

Distributing the operators on the left-hand side gives:

$$\lambda R(\lambda, \mathscr{A})x - \mathscr{A} R(\lambda, \mathscr{A})x = x$$

Rearranging the terms to isolate the $\mathscr{A} R(\lambda, \mathscr{A})x$ term yields:

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x \qquad (*)$$

This identity holds for all $x \in X$ because $R(\lambda, \mathscr{A})$ maps $X$ into $D(\mathscr{A})$.

For the other side, let $x \in D(\mathscr{A})$. We use the fact that the resolvent also satisfies:

$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A}) = Id_{D(\mathscr{A})}$$

Applying this to $x \in D(\mathscr{A})$:

$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A})x = x$$

Distributing $R(\lambda, \mathscr{A})$ gives:

$$\lambda R(\lambda, \mathscr{A})x - R(\lambda, \mathscr{A})\mathscr{A}x = x$$

Rearranging the terms:

$$\lambda R(\lambda, \mathscr{A})x - x = R(\lambda, \mathscr{A})\mathscr{A}x \qquad (**)$$

From (*) and (**) we get:

$$\mathscr{A} R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A} x$$

$\square$

**Theorem 2.4.3.** *For all $x \in D(\mathscr{A})$, $\lim_{\lambda\to\infty} \mathscr{A}_\lambda x = \mathscr{A} x$.*

*Proof.* Using the identity $\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A} x$, we observe:

$$\|\lambda R(\lambda, \mathscr{A})x - x\| = \|R(\lambda, \mathscr{A})\mathscr{A} x\| \leq \frac{\|\mathscr{A} x\|}{\lambda} \xrightarrow{\lambda\to\infty} 0$$

Since $\mathscr{A}_\lambda x = \lambda R(\lambda, \mathscr{A})\mathscr{A} x$, and we just showed $\lambda R(\lambda, \mathscr{A})x \to x,\ \forall x \in D(\mathscr{A})$, it follows that $\mathscr{A}_\lambda x \to \mathscr{A} x$. By density and the uniform bound $\|\lambda R(\lambda, \mathscr{A})\| \leq 1$, this convergence holds for all $x \in X$. $\square$

**Lemma 2.4.1.** *For each $\lambda > 0$, $\mathscr{A}_\lambda$ generates a uniformly continuous semigroup of contractions $\{e^{t\mathscr{A}_\lambda}\}_{t\geq 0}$.*

*Proof.* Since $\mathscr{A}_\lambda = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$, we have:

$$\left\|e^{t\mathscr{A}_\lambda}\right\| = \left\|e^{-t\lambda} e^{t\lambda^2 R(\lambda,\mathscr{A})}\right\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda,\mathscr{A})\|} \leq e^{-t\lambda} e^{t\lambda^2 \frac{1}{\lambda}} = e^{-t\lambda} e^{t\lambda} = 1$$

This confirms the contraction property for the approximating semigroups. $\square$