

PARTIAL DIFFERENTIAL EQUATIONS

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CONTENTS

1	Theory of (Unbounded) Operators	2
1.1	Preliminaries on Operators	2
1.1.1	Resolvent Operator	2
1.1.2	Dual Operators	4

CHAPTER
ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. An operator $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathcal{A})$ is a subspace of X and $\sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|} = +\infty$

Exercise 1: L

Let $\mathcal{A} : H^1 \rightarrow L^2$, such that $f \mapsto f'$ and $D(\mathcal{A}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f, f' \in L^2\}$. Show \mathcal{A} is an unbounded linear operator.

Notation: If \mathcal{A} and \mathcal{B} are unbounded linear operators, then $\mathcal{A} \supset \mathcal{B}$ if and only if $D(\mathcal{A}) \supset D(\mathcal{B})$. That is, for all $x \in D(\mathcal{B})$, $\mathcal{A}x = \mathcal{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. Let $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ be a UBLO.

$$\rho(\mathcal{A}) = \text{Resolvent of } \mathcal{A} = \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} (\lambda I - \mathcal{A}) : D(\mathcal{A}) \rightarrow X \text{ is bijective, and} \\ (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X) \end{array} \right\}$$

$$\sigma(\mathcal{A}) = \text{Spectrum of } \mathcal{A} = \mathbb{C} \setminus \rho(\mathcal{A}).$$

Definition 1.1.3. \mathcal{A} is closed if and only if the graph of \mathcal{A} , denoted as $G(\mathcal{A})$ is closed. Also \mathcal{A} is closable if and only if there exists $\tilde{\mathcal{A}} \supset \mathcal{A}$ such that $G(\tilde{\mathcal{A}}) = \overline{G(\mathcal{A})}$.

Note that $G(\mathcal{A}) = \{(x, \mathcal{A}x) \mid x \in D(\mathcal{A})\}$.

Exercise 2: 3

1. Prove that if it exists, $\tilde{\mathcal{A}}$ is unique, it then denoted by $\overline{\mathcal{A}}$ called closure of \mathcal{A} .
2. Let $\mathcal{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a, b], \mathbb{R}), \|\cdot\| = \sup |f(x)|)$ and $D(\mathcal{A}_\ell) = C^\ell([a, b], \mathbb{R})$.
Prove $\overline{\mathcal{A}_\ell} = \mathcal{A}_1$.

Lemma 1.1.1. If \mathcal{A} an unbounded linear operator is closable, then $\rho(\overline{\mathcal{A}}) = \rho(\mathcal{A})$. If \mathcal{A} is closed then $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$.

Hints (Exercise): If $\rho(\mathcal{A}) \neq 0$ implies \mathcal{A} is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathcal{L}(X)$ implies T is closed).

Corollary 1.1.1. Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closed UBLO then $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$ where

1. $\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) \neq \{0\}\}$ (punctual spectrum and λ 's are the eigenvalue).
2. $\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) = \{0\}, \overline{Rg(\lambda I - \mathcal{A})} \subset X \right\}$ (continuous spectrum).
3. $\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is injective}, \overline{Rg(\lambda I - \mathcal{A})} = X, Rg(\lambda I - \mathcal{A}) \neq X \right\}$ (residual spectrum).

Exercise 3: L

Let

$$X = \ell^2(\mathbb{C}) = \left\{ (x_n)_{n \geq 0} : \sum_n |x_n|^2 < \infty \right\},$$

with $(\mathcal{A}x_n)_{n \geq 1} = \left(\frac{x_n}{1+n} \right)_{n \geq 0}$. Prove that \mathcal{A} is a BLO, injective, $\overline{Rg(\mathcal{A})} = X$ and $Rg(\mathcal{A}) \subset X$.

Theorem 1.1.1. If \mathcal{A} is a closed UBLO then $\rho(\mathcal{A})$ is open. If $\mu \in \rho(\mathcal{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathcal{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathcal{A})$ and

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathcal{A})^{-(n+1)}$$

To do Question: do you need \mathcal{A} closed?

Theorem 1.1.2 (Resolvent Identity). Let \mathcal{A} be a UBLO. For $\lambda \in \rho(\mathcal{A})$, define the resolvent operator

$$R(\lambda) := (\lambda I - \mathcal{A})^{-1}.$$

Then for all $\lambda, \mu \in \rho(\mathcal{A})$,

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

Corollary 1.1.2. The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathcal{A})$ into $\mathcal{L}(X)$ is analytic. Moreover,

$$\frac{d^n}{d\lambda^n} (\lambda I - \mathcal{A})^{-1} = (-1)^n n! [(\lambda I - \mathcal{A})^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and \mathcal{A} a closed UBLO with $\overline{D(\mathcal{A})} = X$ a dense UBLO.

If X and Y are Banach spaces with duals X^* and Y^* , then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x \rangle$.

Definition 1.1.4 (Dual Operator of \mathcal{A}). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow Y$ (UBLO), be such that $\overline{D(\mathcal{A})} = X$. The dual operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y^* \rightarrow X^*$ is a UBLO defined as follows:*

$$D(\mathcal{A}^*) := \{y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x^* \rangle \forall x \in D(\mathcal{A})\}.$$

and $y^* \in D(\mathcal{A}^*)$, the element z^* is unique and we define $A^*y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathcal{A} \in \mathcal{L}(X, Y)$. Then $\mathcal{A}^* \in \mathcal{L}(Y^*, X^*)$ and*

$$\|\mathcal{A}^*\|_{\mathcal{L}(Y^*, X^*)} = \|\mathcal{A}\|_{\mathcal{L}(X, Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closedly dense UBLO. Then $\overline{D(\mathcal{A}^*)} = X^*(\cong X)$, and \mathcal{A}^* is closed.*

Theorem 1.1.3. *Let \mathcal{A} be a closedly dense UBLO. Then $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$ and for all $\lambda \in \rho(\mathcal{A})$,*

$$[(\lambda I - \mathcal{A})^{-1}]^* = (\lambda I - \mathcal{A}^*).$$

Exercise 4: 1

1. Let $\mathcal{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathcal{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:

- a. $\rho(\mathcal{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathcal{A}) = i\mathbb{R}$.
- b. \mathcal{A} is a closed unbounded linear operator.
- c. If $\lambda \in \rho(\mathcal{A})$ then $(\lambda I - \mathcal{A})^{-1} : X \rightarrow D(\mathcal{A})$ is bounded.

For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})f = g.$$

For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathcal{A})f_n \rightarrow 0$.

2. Do same for $\mathcal{A} = -i\frac{d}{dx}$.