

PARTIAL DIFFERENTIAL EQUATIONS

International Mathematics Masters
National Higher School of Mathematics (NHSM)



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THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. An operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathcal{A})$ is a subspace of X and $\sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|} = +\infty$

Exercise 1: 1

Let $\mathcal{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathcal{A}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f, f' \in L^2\}$. Show \mathcal{A} is an unbounded linear operator.

Notation: If \mathcal{A} and \mathcal{B} are unbounded linear operators, then $\mathcal{A} \supset \mathcal{B}$ if and only if $D(\mathcal{A}) \supset D(\mathcal{B})$. That is, for all $x \in D(\mathcal{B})$, $\mathcal{A}x = \mathcal{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. Let $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$ be a UBLO.

$$\rho(\mathcal{A}) = \text{Resolvent of } \mathcal{A} = \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} (\lambda I - \mathcal{A}) : D(\mathcal{A}) \rightarrow X \text{ is bijective, and} \\ (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X) \end{array} \right\}$$

$$\sigma(\mathcal{A}) = \text{Spectrum of } \mathcal{A} = \mathbb{C} \setminus \rho(\mathcal{A}).$$

Definition 1.1.3. \mathcal{A} is closed if and only if the graph of \mathcal{A} , denoted as $G(\mathcal{A})$ is closed. Also \mathcal{A} is closable if and only if there exists $\tilde{\mathcal{A}} \supset \mathcal{A}$ such that $G(\tilde{\mathcal{A}}) = \overline{G(\mathcal{A})}$.

Note that $G(\mathcal{A}) = \{(x, \mathcal{A}x) \mid x \in D(\mathcal{A})\}$.

Exercise 2: 2

1. Prove that if it exists, $\tilde{\mathcal{A}}$ is unique, it then denoted by $\overline{\mathcal{A}}$ called closure of \mathcal{A} .
2. Let $\mathcal{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a, b], \mathbb{R}), \|\cdot\| = \sup |f(x)|)$ and $D(\mathcal{A}_\ell) = C^\ell([a, b], \mathbb{R})$. Prove $\overline{\mathcal{A}_\ell} = \mathcal{A}_1$.

Lemma 1.1.1. If \mathcal{A} an unbounded linear operator is closable, then $\rho(\overline{\mathcal{A}}) = \rho(\mathcal{A})$. If \mathcal{A} is closed then $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$.

Hints (Exercise): If $\rho(\mathcal{A}) \neq 0$ implies \mathcal{A} is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathcal{L}(X)$ implies T is closed).

Corollary 1.1.1. Let $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$ where

1. $\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) \neq \{0\}\}$ (punctual spectrum and λ 's are the eigenvalue).
2. $\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) = \{0\}, \overline{\text{Rg}(\lambda I - \mathcal{A})} \subset X \right\}$ (continuous spectrum).
3. $\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is injective, } \overline{\text{Rg}(\lambda I - \mathcal{A})} = X, \text{Rg}(\lambda I - \mathcal{A}) \neq X \right\}$ (residual spectrum).

Exercise 3: 1

Let

$$X = \ell^2(\mathbb{C}) = \left\{ (x_n)_{n \geq 0} : \sum_n |x_n|^2 < \infty \right\},$$

with $(\mathcal{A} x_n)_{n \geq 1} = \left(\frac{x_n}{1+n} \right)_{n \geq 0}$. Prove that \mathcal{A} is a BLO, injective, $\overline{\text{Rg}(\mathcal{A})} = X$ and $\text{Rg}(\mathcal{A}) \subset X$.

Theorem 1.1.1. If \mathcal{A} is a closed UBLO then $\rho(\mathcal{A})$ is open. If $\mu \in \rho(\mathcal{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|$, $\|(\mu I - \mathcal{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathcal{A})$ and

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathcal{A})^{-(n+1)}$$

To do Question: do you need \mathcal{A} closed?

Theorem 1.1.2 (Resolvent Identity). Let \mathcal{A} be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all $\lambda, \mu \in \rho(A)$,

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

Corollary 1.1.2. The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathcal{A})$ into $\mathcal{L}(X)$ is analytic. Moreover,

$$\frac{d^n}{d\lambda^n} (\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and \mathcal{A} a closed UBLO with $\overline{D(\mathcal{A})} = X$ a dense UBLO.

If X and Y are Banach spaces with duals X^* and Y^* , then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x \rangle$.

Definition 1.1.4 (Dual Operator of \mathcal{A}). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow Y$ (UBLO), be such that $\overline{D(\mathcal{A})} = X$. The dual operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y^* \rightarrow X^*$ is a UBLO defined as follows:*

$$D(\mathcal{A}^*) := \{y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x \rangle \forall x \in D(\mathcal{A})\}.$$

and $y^* \in D(\mathcal{A}^*)$, the element z^* is unique and we define $\mathcal{A}^* y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathcal{A} \in \mathcal{L}(X, Y)$. Then $\mathcal{A}^* \in \mathcal{L}(Y^*, X^*)$ and*

$$\|\mathcal{A}^*\|_{\mathcal{L}(Y^*, X^*)} = \|\mathcal{A}\|_{\mathcal{L}(X, Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a closedly dense UBLO. Then $\overline{D(\mathcal{A}^*)} = X^*(\cong X)$, and \mathcal{A}^* is closed.*

Theorem 1.1.3. *Let \mathcal{A} be a closedly dense UBLO. Then $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$[(\lambda I - \mathcal{A})^{-1}]^* = (\bar{\lambda} I - \mathcal{A}^*)^{-1}.$$

Exercise 4: 2

1. Let $\mathcal{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathcal{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:
 - a. $\rho(\mathcal{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathcal{A}) = i\mathbb{R}$.
 - b. \mathcal{A} is a closed unbounded linear operator.
 - c. If $\lambda \in \rho(\mathcal{A})$ then $(\lambda I - \mathcal{A})^{-1} : X \rightarrow D(\mathcal{A})$ is bounded.

For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathcal{A})$ such that $(\lambda I - \mathcal{A})f = g$.

For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathcal{A})f_n \rightarrow 0$.

2. Do same for $\mathcal{A} = -i \frac{d}{dx}$.

1.2 Compact Operators

Let X and Y be Banach spaces on \mathbb{K} .

Definition 1.2.1. *Let $K : X \rightarrow Y$ be a BLO (in $\mathcal{L}(X, Y)$), then K compact iff $K(B_1^X(0))$ is relatively compact in Y (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) \mid K \text{ is compact}\}.$$

Exercise 5: 1

Let $X = C([a, b], \mathbb{C})$ and $k \in C^0([a, b] \times [c, d], \mathbb{C})$

Define $K \in \mathcal{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t, s)x(s) ds.$$

Show $K \in \mathcal{K}(X)$.

Theorem 1.2.1. $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.

Proof. We shall show this in two steps

1. $\mathcal{K}(X, Y)$ is a vector space (Do it).
2. Closed: $K_n \rightarrow K$ (Prof. Yacine said he would send a different proof).

□

Exercise 6: 1

Let $X = \ell^2(\mathbb{C})$,

$$\mathcal{A}((x_n)_{n \geq 0}) = \left(\frac{x_n}{n+1} \right)_{n \geq 0}.$$

Show that \mathcal{A} is compact.

Theorem 1.2.2. Let X, Y and Z be Banach spaces on \mathbb{K} .

$$X \xrightarrow{\mathcal{A}} Y \xrightarrow{\mathcal{B}} Z, \quad \mathcal{A} \in \mathcal{L}(X, Y), \quad \mathcal{B} \in \mathcal{L}(Y, Z).$$

1. If \mathcal{A} is compact or \mathcal{B} is compact, then $\mathcal{B}\mathcal{A}$ is compact.
2. If \mathcal{A} is compact then $\mathcal{A}^* \in \mathcal{K}(Y^*, X^*)$.
3. If \mathcal{A} is compact and $\text{Rg}(\mathcal{A})$ is closed (in Y), then it is finite dimensional.

To proceed with further results on compact operators, we need the following lemma

Lemma 1.2.1 (Riesz Lemma). Let E be a normed vector space, $F = \overline{F} \subset E$. Then $\forall r \in (0, 1)$, $\exists x_r \in E$, such that

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

Proof. Since $F \neq E$ then this implies $\exists z \in E \setminus F$. Let $d = d(z, F) > 0$.

For $0 < r < 1$, $\exists y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|} \|z - (y_r + \|z - y_r\|y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

□

Proposition 1.2.1. *Let $\mathcal{A} \in \mathcal{K}(X)$, such that X is a Banach space on \mathbb{C} . If $\lambda \in \mathbb{C}^*$, then $\ker((\lambda I - \mathcal{A})^n)$ has finite dimension.*

Proof. Only for $n = 1$. (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathcal{A}) = \{x \in X : \mathcal{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda} \mathcal{A}x\right\} \subset \text{Rg}(\mathcal{A}).$$

So \tilde{K} is closed in $\text{Rg}(\mathcal{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists (x_n)$ in \tilde{K} , such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|} \|\mathcal{A}x_n - \mathcal{A}x_m\| \geq \frac{1}{2}, \quad \forall n \neq m$$

and so we have $\|\mathcal{A}x_n\| \leq \|\mathcal{A}\|$. So $(\mathcal{A}x_n)$ is not Cauchy, hence a contradiction.

□

Exercise 7: 1

Let X be a Banach space on \mathbb{K} . If $\mathcal{A} \in \mathcal{L}(X)$, assume $\exists n_0$ s.t. $\ker(\mathcal{A}^{n_0}) = \ker(\mathcal{A}^{n_0+1})$. Then $\forall n \geq n_0$,

$$\ker(\mathcal{A}^n) = \ker(\mathcal{A}^{n_0}).$$

Proposition 1.2.2. *Let $\mathcal{A} \in \mathcal{K}(X)$ and X be a Banach space on \mathbb{C} , $\lambda \neq 0$. Then $\exists n_0$ such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathcal{A})^n) = \ker((\lambda I - \mathcal{A})^{n_0}).$$

Proof. Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathcal{A})^n) \subset \ker((\lambda I - \mathcal{A})^{n+1})$ and each of them is closed.

RL: with $r = \frac{1}{2}$ with $(x_n)_{n \geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathcal{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathcal{A})^n)) \geq \frac{1}{2}.$$

For $n = 1, x \in \ker(\lambda I - \mathcal{A}) \Rightarrow x = \frac{\mathcal{A}}{\lambda}x$. For all $1 \leq m < n$,

$$\begin{aligned} \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} &= x_n - x_m + \frac{\mathcal{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathcal{A}x_m}{\lambda}\right) \\ &= x_n - \left[\frac{(\lambda I - \mathcal{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathcal{A})x_m}{\lambda}\right]. \end{aligned}$$

So,

$$\left\| \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} \right\| \geq d(x_n, \ker(\lambda I - \mathcal{A})^n) \geq \frac{1}{2}.$$

which is a contradiction. □

Notice that if $\ker(\lambda I - \mathcal{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathcal{A})$. Notice,

$$\dim \ker(\lambda I - \mathcal{A}) = \text{geometric multiplicity.}$$

With Proposition 1.2.2 $\Rightarrow \exists n_0$ (smallest one) such that

$$\ker((\lambda I - \mathcal{A})^{n_0}) = \ker((\lambda I - \mathcal{A})^n), \quad \forall n \geq n_0.$$

Note that,

$$\ker((\lambda I - \mathcal{A})^{n_0}) := \text{generalized eigenspace.}$$

$$\dim \ker((\lambda I - \mathcal{A})^{n_0}) := \text{algebraic multiplicity of } \lambda.$$

Proposition 1.2.3 (Fredholm alternative). *Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} .*

$$\text{Rg}(\lambda I - \mathcal{A}) = X \iff \ker(\lambda I - \mathcal{A}) = \{0\}.$$

Proposition 1.2.4. *Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} , $\dim X = \infty$. If $\lambda_n \rightarrow \lambda$, $\lambda_n \in \sigma(\mathcal{A}) \setminus \{0\}$, pairwise distinct, then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathcal{A}) \setminus \{0\}$ is isolated.*

Proof. Let $\lambda_n \in \sigma_p(\mathcal{A})$, $\exists \|x_n\| = 1$ such that $\mathcal{A}x_n = \lambda_n x_n$. Let

$$X_n = \text{span}(x_1, \dots, x_n), \quad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \Rightarrow \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^n \alpha_i x_i, \text{ which implies } \lambda_{n+1} x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathcal{A}x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^n \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

Since (x_i) are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \quad 1 \leq i \leq n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathcal{A})X_n \subset X_{n-1}, \quad \forall n \geq 2.$$

Recall:

$$\|y_n\| = 1, \quad y_n \in X_n,$$

$$d(y_n, X_{n-1}) \geq \frac{1}{2}.$$

For $2 \leq m < n$,

$$\begin{aligned} \left\| \frac{\mathcal{A}y_n}{\lambda_n} - \frac{\mathcal{A}y_m}{\lambda_m} \right\| &= \left\| y_n - \left[\frac{\lambda_n I - \mathcal{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathcal{A}}{\lambda_m} y_m \right] \right\| \\ &\geq d(y_n, X_{n-1}) \geq \frac{1}{2}. \end{aligned}$$

Assume that

$$\lambda_n \rightarrow \lambda \quad (n \rightarrow \infty).$$

Suppose $\lambda \neq 0$, then

$$\left| \frac{1}{\lambda_n} \right| \leq C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathcal{A}y_n}{\lambda_n} \right)_{n \geq 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathcal{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. \square

Theorem 1.2.3. *Let $\mathcal{A} \in \mathcal{K}(X)$, and let X be a Banach space on \mathbb{C} . Then $\sigma(\mathcal{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.*

Main use of compact operators (in PDEs)

They appear as “inverse” of UBLO.

Definition 1.2.2. *Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ UBLO, closed, $\rho(\mathcal{A}) \neq \{0\}$. \mathcal{A} is said to have compact resolvent if*

$$(\lambda I - \mathcal{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathcal{A}).$$

Main Example: $\mathcal{A} = -\Delta$ on Ω with $\mathcal{A}u = -u_{xx}$.

1.3 Adjoints, Symmetric and Self-adjoint Operators

Let \mathcal{H} be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}.$$

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ UBLO, } \overline{D(\mathcal{A})} = \mathcal{H}.$$

Definition 1.3.1 (Adjoint Operator \mathcal{A}°).

$$D(\mathcal{A}^\circ) = \{x \in \mathcal{H} : v \mapsto \langle \mathcal{A}v, x \rangle_{\mathcal{H}} : D(\mathcal{A}) \rightarrow \mathbb{C} \text{ bdd operator}\}.$$

If $x \in D(\mathcal{A}^\circ)$, then there exists uniquely $z \in \mathcal{H}$ such that $\langle v, z \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$ for all $v \in D(\mathcal{A})$.

Observe, with Riesz representation and the fact that $\overline{D(\mathcal{A})} = \mathcal{H}$, we have that $z := \mathcal{A}^\circ x$ and $\langle v, \mathcal{A}^\circ x \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$ for all $v \in D(\mathcal{A})$.

Remark 1.3.1. Let \mathcal{H} be a Hilbert space, $E : \mathcal{H} \rightarrow \mathcal{H}^*$, $x \mapsto \langle x, \cdot \rangle$. Linear isometry between \mathcal{H} and \mathcal{H}^* . (One can identify \mathcal{H} and \mathcal{H}^*). Now, we define the Dual operator as the following:

$$\mathcal{A}^* : D(\mathcal{A}^*) \subset \mathcal{H}^* \rightarrow \mathcal{H}, \quad \mathcal{A}^\circ = E^{-1} \mathcal{A}^* E.$$

Definition 1.3.2 (Symmetric and Self-adjoint Operator). Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a UBLO, then

1. **Symmetric:** If $\overline{D(\mathcal{A})} = \mathcal{H}$ and $\mathcal{A}^\circ \supset \mathcal{A}$ with $D(\mathcal{A}^*) \supset D(\mathcal{A})$ and for all $x, y \in D(\mathcal{A})$, $\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle$.
2. **Self-adjoint:** If $\overline{D(\mathcal{A})} = \mathcal{H}$ and $\mathcal{A}^\circ = \mathcal{A}$.

Exercise 8: 2

1. Let \mathcal{H} be a Hilbert space, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, $\overline{D(\mathcal{A})} = \mathcal{H}$. If \mathcal{A} is closed, then $\overline{D(\mathcal{A}^\circ)} = \mathcal{H}$.
2. Let \mathcal{H} be a Hilbert space, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, $\overline{D(\mathcal{A})} = \mathcal{H}$. Suppose \mathcal{A} is symmetric and if $0 \in \rho(\mathcal{A})$, then prove that $\lambda \in \mathbb{R}$ and

$$\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle.$$

Proposition 1.3.1. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an Hilbert space over \mathbb{C} , If \mathcal{A} is self-adjoint, injective and $\overline{D(\mathcal{A})} = \mathcal{H}$. Then $\mathcal{A}^{-1} : \text{Rg}(\mathcal{A}) \rightarrow \mathcal{H}$ is self-adjoint.

Proof. Since \mathcal{A} is injective then \mathcal{A}^{-1} is well defined. Now, for all $z \in D(\mathcal{A})$, $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}^\circ z \rangle$ which implies $\langle y, z \rangle = \langle x, \mathcal{A}^\circ z \rangle$ and so we have $x \in D(\mathcal{A}^\circ) = D(\mathcal{A})$ and $y = \mathcal{A}^\circ$. Notice $\overline{\text{Rg}(\mathcal{A})} = \ker(\mathcal{A})^\perp$ (because of self-adjointness).

Injectivity implies $\overline{\text{Rg}(\mathcal{A})} = \mathcal{H}$ which implies $\overline{D(\mathcal{A}^{-1})} = \mathcal{H}$. So \mathcal{A}^{-1} is densely defined.

Now, observe for all $u, v \in D(\mathcal{A}^{-1})$, $u = \mathcal{A}^\circ x$ and $v = \mathcal{A} y$ with $x, y \in D(\mathcal{A})$. Hence,

$$\langle \mathcal{A}^{-1}u, v \rangle = \langle x, \mathcal{A} y \rangle = \langle \mathcal{A} x, y \rangle = \langle x, \mathcal{A}^{-1}y \rangle.$$

To this end, $(\mathcal{A}^{-1})^\circ \subset \mathcal{A}^{-1}$, $\forall z \in D((\mathcal{A}^{-1})^\circ) \exists w, \forall u \in D(\mathcal{A}^{-1}) = R(\mathcal{A})$ (i.e. $u = \mathcal{A} x$)

$$\langle \mathcal{A}^{-1}u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathcal{A}) \quad \langle x, z \rangle = \langle \mathcal{A} x, w \rangle$$

By definition $w \in D(\mathcal{A}^\circ)$ and $\mathcal{A}^\circ w = z$. $\mathcal{A} w = z \Rightarrow z \in Rg(\mathcal{A}) = D(\mathcal{A}^{-1})$. \square

Theorem 1.3.1. *Let \mathcal{H} be a Hilbert space, suppose $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is symmetric and surjective then \mathcal{A} is self-adjoint.*

Proof. \mathcal{A} and \mathcal{A}° are injective. Do it only for \mathcal{A} , let $x \in D(\mathcal{A})$ and $\mathcal{A} x = 0$.

$$\forall y \in D(\mathcal{A}), \quad 0 = \langle \mathcal{A} x, y \rangle = \langle x, \mathcal{A} y \rangle \Rightarrow x \perp Rg(\mathcal{A}) = \mathcal{H}.$$

which implies $x = 0$.

Next, we show \mathcal{A} closed.

$$(x_n)_{n \geq 1} \subset D(\mathcal{A}), \quad x_n \rightarrow x \text{ in } \mathcal{H}, \quad \mathcal{A} x_n \rightarrow y \text{ in } \mathcal{H}$$

We shall show $y = \mathcal{A} x$. Now, $\forall z \in D(\mathcal{A})$ then $\langle \mathcal{A} x_n, z \rangle = \langle x_n, \mathcal{A} z \rangle$, which implies $\langle y, z \rangle = \langle x, \mathcal{A} z \rangle$ which implies $x \in D(\mathcal{A}^\circ)$ and $y = \mathcal{A}^\circ x$. Since \mathcal{A} surjective $\Rightarrow \exists w \in D(\mathcal{A})$ s.t. $\mathcal{A} w = y$ and $\mathcal{A}^\circ x = y$.

Since \mathcal{A} is symmetric: $\mathcal{A}^\circ w = \mathcal{A} w$. Then $\mathcal{A}^\circ w = \mathcal{A}^\circ x$, \mathcal{A} is injective $\Rightarrow w = x$. Hence $\mathcal{A} x = \mathcal{A} w = y \Rightarrow y = \mathcal{A} x \Rightarrow \mathcal{A}$ is closed.

By closed graph theorem both \mathcal{A} and $\mathcal{A}^{-1} \in \mathcal{L}(X)$. We can conclude that \mathcal{A} is a self-adjoint operator. \square

Exercise 9: 2

Let $\mathcal{H} = L^2(0, \pi)$ with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ and

$$\mathcal{A} f = -f''$$

If $D(\mathcal{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is \mathcal{A} a self-adjoint operator?

Similarly, if $D(\mathcal{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is \mathcal{A} a self-adjoint operator?

Theorem 1.3.2 (Fredrich's Extension). *Let \mathcal{H} be a Hilbert space on \mathbb{C} with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, symmetric then \mathcal{A} admits a unique self adjoint extension. If either*

$$a. \inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A} x, x \rangle > -\infty$$

$$b. \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A} x, x \rangle < +\infty$$

such that, $\mathcal{A} \subset \mathcal{A}^\circ \subset (\mathcal{A}^\circ)^\circ \subset \dots$. If (a) or (b) holds, then; $\mathcal{A} \subset \mathcal{A}^\circ = (\mathcal{A}^\circ)^\circ$.

1.4 Dissipative Operator and Numerical range

Definition 1.4.1 (Duality Map). Let X be a Banach space on \mathbb{K} . The duality map is defined as $J : X \longrightarrow 2^{X^*}$, $x \mapsto J(x) = \{x^* \in X^* \mid \operatorname{Re}\langle x^*, x \rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$. By the Hahn-Banach theorem, $J(x) \neq \emptyset$.

Question: What can you say about $J(X)$ when X is an Hilbert space or Reflexive?

Definition 1.4.2. A map $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ (UBLO) is dissipative if for all $x \in D(\mathcal{A})$, there exists $x^* \in J(X)$ such that $\operatorname{Re}\langle x^*, \mathcal{A}x \rangle \leq 0$.

Lemma 1.4.1. \mathcal{A} is dissipative if and only if for all $\lambda > 0$, $x \in D(\mathcal{A})$ we have that $\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|$.

Proof. let $x^* \in J(x)$. Then

$$\begin{aligned} \|(\lambda I - \mathcal{A})x\| \|x^*\| &\geq |\langle x^*, (\lambda I - \mathcal{A})x \rangle| \geq \Re\langle x^*, (\lambda I - \mathcal{A})x \rangle, \\ &= \lambda \Re\langle x^*, x \rangle - \Re\langle x^*, \mathcal{A}x \rangle \geq \lambda \|x\|^2. \end{aligned}$$

Hence, if $\|x\| \neq 0$, then

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

(\Leftarrow) Let $x \in D(\mathcal{A})$, $x \neq 0$, and $\lambda > 0$. Let $y_\lambda^* \in J((\lambda I - \mathcal{A})x)$ and set $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$. Then

$$\|(\lambda I - \mathcal{A})x\|^2 = \|(\lambda I - \mathcal{A})x\| \|y_\lambda^*\| = \Re\langle y_\lambda^*, (\lambda I - \mathcal{A})x \rangle.$$

Since $y_\lambda^* \neq 0$, we have

$$\lambda \|x\| \leq \|(\lambda I - \mathcal{A})x\| = \Re\langle g_\lambda^*, (\lambda I - \mathcal{A})x \rangle = \lambda \langle g_\lambda^*, x \rangle - \Re\langle g_\lambda^*, \mathcal{A}x \rangle.$$

Hence,

$$\Re\langle g_\lambda^*, \mathcal{A}x \rangle \leq \lambda \langle g_\lambda^*, x \rangle - \lambda \|x\| \leq \|g_\lambda^*\| \|x\| = \|x\|.$$

Therefore,

$$\Re\langle g_\lambda^*, \mathcal{A}x \rangle \leq 0. \quad (**)$$

Idea: Let $\lambda \rightarrow +\infty$.

Unit ball in X^* is compact for weak* topology (Banach–Alaoglu).

(Up to subsequence)

$$g_\lambda^* \longrightarrow g^* \in X^*, \quad \|g^*\| \leq 1.$$

Then from (**),

$$\Re\langle g^*, \mathcal{A}x \rangle \leq 0.$$

$$(*) \quad \|x\| \leq \langle g_\lambda^*, x \rangle - \frac{1}{\lambda} \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Let $\lambda \rightarrow +\infty$. Then $\|x\| \leq \langle g^*, x \rangle$. Hence, $\|g^*\| = 1$ and $\langle g^*, x \rangle = \|x\|$.
Set $x^* = \|x\|g^*$. Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x \rangle = \|x\|^2,$$

that is, $x^* \in J(x)$. □

Theorem 1.4.1 (Lumer-Phillips). *Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be a UBLO. Assume that \mathcal{A} is dissipative and that there exists $\lambda_0 > 0$ such that $\text{Rg}(\lambda_0 I - \mathcal{A}) = X$.*

Then \mathcal{A} is closed, $\rho(\mathcal{A}) \supset \mathbb{R}_+$, and for all $\lambda > 0$,

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}. \quad (1.4)$$

Proof. Let $\lambda_0 > 0$.

(1) To prove: $(\lambda_0 I - \mathcal{A})$ is bijective.

- surj: Assumption,
- inj: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathcal{A})^{-1} : X \rightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any $y \in X$, there exists a unique $x \in X$ such that

$$x = (\lambda_0 I - \mathcal{A})^{-1}y, \quad (\lambda_0 I - \mathcal{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0} \|y\| \geq \|(\lambda_0 I - \mathcal{A})^{-1}y\|.$$

Hence,

$$\|(\lambda_0 I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_0}, \quad \lambda_0 \in \rho(\mathcal{A}).$$

(2) \mathcal{A} is closed.

Let $x_n \rightarrow x$, $x_n \in D(\mathcal{A})$, and $Ax_n \rightarrow y$.

Then

$$(\lambda_0 I - \mathcal{A})x_n \rightarrow \lambda_0 x - y.$$

Since $(\lambda_0 I - \mathcal{A})^{-1} \in \mathcal{L}(X)$, we have

$$x_n \rightarrow (\lambda_0 I - \mathcal{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathcal{A})x \iff y = \mathcal{A}x.$$

Therefore, \mathcal{A} is closed.

(3) $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$ and (1.4).

Since \mathcal{A} is closed and $\rho(\mathcal{A}) \neq \emptyset$, we know that $\rho(\mathcal{A})$ is open.

Let $\Lambda = \rho(\mathcal{A}) \cap \mathbb{R}_+^*$, which is open in \mathbb{R}_+^* . We show that it is closed.

Let $(\lambda_n)_{n \in \mathbb{N}} \subset \Lambda$ such that $\lambda_n \longrightarrow \lambda \in \mathbb{R}_+^*$. Note that since $\lambda_n \in \Lambda$, we have

$$\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$\begin{aligned} (\lambda I - \mathcal{A}) &= [I + u_n](\lambda_n I - \mathcal{A}) \implies \lambda I - \mathcal{A} = \lambda_n I - \mathcal{A} + u_n(\lambda_n I - \mathcal{A}), \\ &\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathcal{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathcal{A})^{-1} = u_n. \end{aligned}$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For n large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that $\lambda \in \rho(\mathcal{A})$. Hence, Λ is closed, and therefore $\Lambda = \mathbb{R}_+^*$.

□

Corollary 1.4.1. *Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ be a UBLO, closed, with $\overline{D(\mathcal{A})} = X$. Assume that \mathcal{A} and \mathcal{A}^* are dissipative. Then*

$$\rho(\mathcal{A}) \supset \mathbb{R}_+^*, \quad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathcal{A})^{-1}\| \leq 1.$$

Proof. It is easy to show that $\text{Rg}(I - \mathcal{A}) = X$ (i.e. $\lambda_0 = 1$ + Theorem 1.4.1).

\mathcal{A} dissipative and closed implies

$$\text{Rg}(I - \mathcal{A}) \text{ is a closed subspace of } X.$$

(give details!!!)

Let $x^* \in X^*$ such that

$$\langle x^*, (I - \mathcal{A})x \rangle = 0, \quad \forall x \in D(\mathcal{A}). \quad (**)$$

Let us prove that $x^* = 0$.

Then $x^* \in D(\mathcal{A}^*)$ and

$$(I - \mathcal{A}^*)x^* = 0.$$

Since \mathcal{A}^* is dissipative, by Lemma 1.4.1, we have $x^* = 0$. This implies that

$$\overline{\text{Rg}(I - \mathcal{A})} = X.$$

Since $\text{Rg}(I - \mathcal{A})$ is closed, we obtain

$$\text{Rg}(I - \mathcal{A}) = X.$$

By contradiction and using Hahn–Banach. □

Definition 1.4.3 (Numerical Range). *Let $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ be UBLO. The numerical range of \mathcal{A} , denoted by $W(\mathcal{A})$,*

$$W(\mathcal{A}) = \{\langle x^*, \mathcal{A}x \rangle \mid x^* \in J(x), x \in D(\mathcal{A}), \|x\| = \|x^*\| = 1, \langle x^*, x \rangle = 1\}.$$

In case of a Hilbert space, we have that $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid x \in D(\mathcal{A}), \|x\| = 1\}$.

Linear algebra in finite dimension $\mathcal{A} \in \mathcal{M}_n(\mathbb{K})$, we have that $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid \|x\| = 1\}$.

Theorem 1.4.2 (Home-work). *Let $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$ be closed, with $\overline{D(\mathcal{A})} = X$.*

1) If $\lambda \notin \overline{W(\mathcal{A})}$, then $(\lambda I - \mathcal{A})$ is injective, has closed image, and for all $x \in D(\mathcal{A})$,

$$\|(\lambda I - \mathcal{A})x\| \geq d(\lambda, W(\mathcal{A})) \|x\|.$$

Moreover, if $\lambda \in \rho(\mathcal{A})$, then

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{d(\lambda, W(\mathcal{A}))}. \quad (**)$$

*2) If Λ is a connected open subset of $\mathbb{C} \setminus W(\mathcal{A})$ such that $\rho(\mathcal{A}) \cap \Lambda \neq \emptyset$, then $\rho(\mathcal{A}) \supset \Lambda$ and $(**)$ holds true.*

INTRODUCTION TO THE THEORY OF SEMI-GROUPS

2.1 Intro to the Introduction

Definition 2.1.1. Let X be a Banach space over \mathbb{K} . A one-parameter family of bounded linear operators on X , $(T(t))_{t \geq 0}$, is a semigroup (SG) of bounded linear operators on X if:

1. $T(0) = Id_X$,
2. $\forall (t, s) \in \mathbb{R}_+^2 : T(t + s) = T(t) \cdot T(s)$ (SG property).

Remark 2.1.1. $T(t)$ and $T(s)$ commute.

2. Infinitesimal generator of SG–LO $(T(t))_{t \geq 0}$

Let $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ be an unbounded linear operator with

$$D(\mathcal{A}) = \left\{ x \in X \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \mathcal{A}x, \quad x \in D(\mathcal{A}).$$

where $D(\mathcal{A}) = \text{domain of } \mathcal{A}$.

2.2 Uniformly Continuous SG–BLO

Definition 2.2.1. A SG–BLO on X , $(T(t))_{t \geq 0}$ is uniformly continuous if

$$\|T(t) - Id\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as } t \rightarrow 0^+$$

Lemma 2.2.1. Let $(T(t))_{t \geq 0}$ be a SG–BLO which is uniformly continuous. Then, $\forall t > 0$,

$$\|T(s) - T(t)\| \xrightarrow{s \rightarrow t} 0$$

(continuity everywhere).

Proof. Let t be fixed. $T(s) = T(s - t + t)$, $s - t \geq 0$.

$$s \geq t \Rightarrow T(s) = T(s - t)T(t) \Rightarrow T(s) - T(t) = T(t) [T(s - t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\| \|T(s - t) - I_d\| \xrightarrow{s \rightarrow t} 0.$$

For $s \leq t$

$$T(t) = T(t - s)T(s) \Rightarrow T(t) - T(s) = T(s) [T(t - s) - I_d]$$

(Prove that $\sup_{[0,t]} \|T(t)\| < +\infty$)

Then

$$\|T(t) - T(s)\| \leq \|T(s)\| \|T(t - s) - I_d\|$$

$$\leq \sup \|T(s)\| \|T(t - s) - I_d\| \xrightarrow{s \rightarrow t} 0.$$

□

Theorem 2.2.1. *A linear operator \mathcal{A} is the infinitesimal generator of a uniformly continuous semigroup if and only if \mathcal{A} is a bounded linear operator.*

Proof. Let \mathcal{A} be a bounded linear operator on X and set

$$T(t) = e^{t\mathcal{A}} = \sum_{n=0}^{\infty} \frac{(t\mathcal{A})^n}{n!}. \quad (1.5)$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such t , a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t + s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq |t| \|\mathcal{A}\| e^{|t| \|\mathcal{A}\|}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathcal{A} \right\| \leq \|\mathcal{A}\| \cdot \max_{0 \leq s \leq t} \|T(s) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on X and that \mathcal{A} is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on X . Fix $\rho > 0$, small enough, such that

$$\left\| I - \rho \int_0^\rho T(s) ds \right\| < 1.$$

This implies that $\rho^{-1} \int_0^\rho T(s) ds$ is invertible. Now,

$$h^{-1}(T(h) - I) \int_0^\rho T(s) ds = h^{-1} \left(\int_0^\rho T(s + h) ds - \int_0^\rho T(s) ds \right)$$

$$= h^{-1} \left(\int_{\rho}^{\rho+h} T(s) ds - \int_0^h T(s) ds \right)$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1} \int_{\rho}^{\rho+h} T(s) ds - h^{-1} \int_0^h T(s) ds \right) \left(\int_0^{\rho} T(s) ds \right)^{-1}. \quad (1.6)$$

Letting $h \rightarrow 0$ in (1.6) shows that $h^{-1}(T(h) - I)$ converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I) \left(\int_0^{\rho} T(s) ds \right)^{-1}$$

which is the infinitesimal generator of $T(t)$. □

Remark 2.2.1. *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

Theorem 2.2.2. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t \rightarrow 0} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t \rightarrow 0} \frac{S(t) - I}{t}. \quad (1.7)$$

then $T(t) = S(t)$ for $t \geq 0$.

Proof. We will show that given $T > 0$, $S(t) = T(t)$ for $0 \leq t \leq T$. Let $T > 0$ be fixed, since $t \mapsto \|T(t)\|$ and $t \mapsto \|S(t)\|$ are continuous there is a constant C such that

$$\|T(t)\| \|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given $\varepsilon > 0$ it follows from (1.7) that there is a $\delta > 0$ such that

$$h^{-1} \|T(h) - S(h)\| < \varepsilon / TC \quad \text{for } 0 \leq h \leq \delta. \quad (1.8)$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n \leq \delta$. From the semigroup property and (1.8) it then follows that

$$\begin{aligned} \|T(t) - S(t)\| &= \left\| T\left(n \frac{t}{n}\right) - S\left(n \frac{t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k) \frac{t}{n}\right) S\left(\frac{kt}{n}\right) - T\left((n-k-1) \frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k-1) \frac{t}{n}\right) \right\| \left\| T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{t}{n}\right) \right\| \leq C n \frac{\varepsilon}{TC} \frac{t}{n} \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $T(t) = S(t)$ for $0 \leq t \leq T$ and the proof is complete. □

Corollary 2.2.1. *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

- (a) *There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*
- (b) *There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.*
- (c) *The operator A in part (b) is the infinitesimal generator of $T(t)$.*
- (d) *$t \mapsto T(t)$ is differentiable in norm and satisfies*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A. \quad (1.9)$$

Proof. All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator A . A is also the infinitesimal generator of e^{tA} defined by (1.5) and therefore, by Theorem 2.2.2, $T(t) = e^{tA}$. \square

Remark 2.2.2. *The proofs above are from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 3. [Theorem 1.3 and Corollary 1.4].*

2.3 Strongly Continuous Semigroups (C_0 -Semigroups)

Definition 2.3.1. *The SG-BLO $(T(t))_{t \geq 0}$ is strongly continuous (SC or C_0) if $\forall x \in X$*

$$\|T(t)x - x\|_X \xrightarrow{t \rightarrow 0^+} 0 \quad (2.1)$$

Theorem 2.3.1. *Let $(T(t))_{t \geq 0}$, C_0 -SG then $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0 \ \|T(t)\| \leq Me^{\omega t}$*

Proof. First we want to show that $\exists \eta > 0, \sup_{t \in [0, \eta]} \|T(t)\| < +\infty$.

By contradiction, assume that $\sup \|T(t)\| = +\infty$. Then $\exists (t_n)_{n \geq 0} \searrow 0$ such that $\|T(t_n)\| \geq n$ or $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive) $\exists x \in X$ such that $\sup \|T(t_n)x\| = +\infty$, but this contradicts the strong convergence 2.3.1.

Now take $M := \sup \|T(t)\| \geq 1$ (This is because $T(0) = Id$).

$\forall t \geq 0$ write $t = k\eta + \eta_t$ where $k = \lfloor \frac{t}{\eta} \rfloor$ and $\eta_t \in [0, \eta]$, then

$$\begin{aligned} \|T(t)\| &= \|T(k\eta + \eta_t)\| \\ &= \|[T(\eta)]^k T(\eta_t)\| && \text{by the SG property} \\ &= \|T(\eta)\|^k \|T(\eta_t)\| \\ &\leq M \cdot M^k && M \text{ is upperbound} \\ &\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\ &= M(e^{\ln M})^{t/\eta} \\ &= Me^{t \frac{\ln M}{\eta}} = Me^{\omega t} \end{aligned}$$

And since $\omega = \frac{\ln M}{\eta}$ and $M \geq 1$ then $\omega \geq 0$. □

Corollary 2.3.1. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup then $\forall x \in X, t \mapsto T(t)x$ is continuous*

Proof. For $h > 0$ and $t > 0$:

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \rightarrow 0$ as $h \rightarrow 0$ by definition [2.3.1](#).
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$. Since $\|T(t-h)\|$ is bounded by theorem [2.3.1](#), so this tends to 0 as $h \rightarrow 0$.

□