PARTIAL DIFFERENTIAL EQUATIONS

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CONTENTS

ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \| \cdot \|)$ be a Banach space and $\mathscr{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. *An operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathscr{A})$ is a subspace of X and $\sup_{x \neq 0} \frac{\|\mathscr{A}x\|}{\|x\|} = +\infty$*

> **Exercise 1**
>
> Let $\mathscr{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathscr{A}) = \{f : \mathbb{R} \to \mathbb{R} : f, f' \in L^2\}$. Show $\mathscr{A}$ is an unbounded linear operator.

Notation: If $\mathscr{A}$ and $\mathscr{B}$ are unbounded linear operators, then $\mathscr{A} \supset \mathscr{B}$ if and only if $D(\mathscr{A}) \supset D(\mathscr{B})$ and for all $x \in D(\mathscr{B}), \mathscr{A}x = \mathscr{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be a UBLO.*

$$\rho(\mathscr{A}) = \text{Resolvent of } \mathscr{A} = \left\{ \lambda \in \mathbb{C} \left| \begin{array}{l} (\lambda I - \mathscr{A}) : D(\mathscr{A}) \to X \text{ is bijective, and} \\ (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X) \end{array} \right. \right\}$$

$$\sigma(\mathscr{A}) = \text{Spectrum of } \mathscr{A} = \mathbb{C} \setminus \rho(\mathscr{A}).$$

Definition 1.1.3. *$\mathscr{A}$ is closed if and only if the graph of $\mathscr{A}$, denoted as $G(\mathscr{A})$ is closed. Also $\mathscr{A}$ is closable if and only if there exists $\tilde{\mathscr{A}} \supset \mathscr{A}$ such that $G(\tilde{\mathscr{A}}) = \overline{G(\mathscr{A})}$.*

Note that $G(\mathscr{A}) = \{(x, \mathscr{A}x) | x \in D(\mathscr{A})\}$.

1. Prove that if it exists, $\tilde{\mathscr{A}}$ is unique, it then denoted by $\overline{\mathscr{A}}$ called closure of $\mathscr{A}$.

2. Let $\mathscr{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a,b],\mathbb{R}), \|.\| = \sup|f(x)|)$ and $D(\mathscr{A}_\ell) = C^\ell([a,b],\mathbb{R})$. Prove $\overline{\mathscr{A}_\ell} = \mathscr{A}_1$.

Lemma 1.1.1. *If $\mathscr{A}$ an unbounded linear operator is closable, then $\rho(\overline{\mathscr{A}}) = \rho(\mathscr{A})$. If $\mathscr{A}$ is closed then $\rho(\mathscr{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X)\}$.*

Hints (Exercise): If $\rho(\mathscr{A}) \neq \emptyset$ implies $\mathscr{A}$ is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathscr{L}(X)$ implies T is closed).

Corollary 1.1.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \cup \sigma_c(\mathscr{A})$ where*

1. *$\sigma_p(\mathscr{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) \neq \{0\}\}$ (punctual spectrum and λ's are the eigenvalue).*

2. *$\sigma_c(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) = \{0\}, \overline{Rg(\lambda I - \mathscr{A})} \subsetneq X\right\}$ (continuous spectrum).*

3. *$\sigma_r(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \lambda I - \mathscr{A} \text{ is injective}, \overline{Rg(\lambda I - \mathscr{A})} = X, Rg(\lambda I - \mathscr{A}) \neq X\right\}$ (residual spectrum).*

Let

$$X = \ell^2(\mathbb{C}) = \left\{(x_n)_{n\geq 0} : \sum_n |x_n|^2 < \infty\right\},$$

with $(\mathscr{A}x_n)_{n\geq 1} = \left(\frac{x_n}{1+n}\right)_{n\geq 0}$. Prove that $\mathscr{A}$ is a BLO, injective, $\overline{Rg(\mathscr{A})} = X$ and $Rg(\mathscr{A}) \subsetneq X$.

Theorem 1.1.1. *If $\mathscr{A}$ is a closed UBLO then $\rho(\mathscr{A})$ is open. If $\mu \in \rho(\mathscr{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathscr{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathscr{A})$ and*

$$(\lambda I - \mathscr{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathscr{A})^{-(n+1)}$$

To do Question: do you need $\mathscr{A}$ closed?

Theorem 1.1.2 (Resolvent Identity)**.** *Let $\mathscr{A}$ be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator*

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all $\lambda, \mu \in \rho(A)$,

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

Corollary 1.1.2. *The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathscr{A})$ into $\mathscr{L}(X)$ is analytic. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and $\mathscr{A}$ a closed UBLO with $\overline{D(\mathscr{A})} = X$ a dense UBLO.

If X and Y are Banach spaces with duals X^* and Y^*, then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x \rangle$.

Definition 1.1.4 (Dual Operator of $\mathscr{A}$). *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to Y$ (UBLO), be such that $\overline{D(\mathscr{A})} = X$. The dual operator $\mathscr{A}^* : D(\mathscr{A}^*) \subset Y^* \to X^*$ is a UBLO defined as follows:*

$$D(\mathscr{A}^*) := \{ y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x \rangle \; \forall x \in D(\mathscr{A}) \}.$$

and $y^* \in D(\mathscr{A}^*)$, the element z^* is unique and we define $A^* y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathscr{A} \in \mathscr{L}(X, Y)$. Then $\mathscr{A}^* \in \mathscr{L}(Y^*, X^*)$ and*

$$\| \mathscr{A}^* \|_{\mathscr{L}(Y^*, X^*)} = \| \mathscr{A} \|_{\mathscr{L}(X,Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closedly dense UBLO. Then $\overline{D(\mathscr{A}^*)} = X^* (\cong X)$, and $\mathscr{A}^*$ is closed.*

Theorem 1.1.3. *Let $\mathscr{A}$ be a closedly dense UBLO. Then $\rho(\mathscr{A}) = \rho(\mathscr{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$\left[(\lambda I - \mathscr{A})^{-1} \right]^* = (\lambda I - \mathscr{A}^*)^{-1}.$$

Exercise 4

1. Let $\mathscr{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathscr{A}) = \{ f \in X : f' \in L^2(\mathbb{R}) \}$. Show the following:

   a. $\rho(\mathscr{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathscr{A}) = i\mathbb{R}$.

   b. $\mathscr{A}$ is a closed unbounded linear operator.

   c. If $\lambda \in \rho(\mathscr{A})$ then $(\lambda I - \mathscr{A})^{-1} : X \longrightarrow D(\mathscr{A})$ is bounded.

   For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathscr{A})$ such that $(\lambda I - \mathscr{A}) f = g$.

   For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\| f_n \|_{\ell^2} = 1$ then $(i\omega I - \mathscr{A}) f_n \to 0$.

2. Do same for $\mathscr{A} = -i \frac{d}{dx}$.

1.2 Compact Operators

Let X and Y be Banach spaces on $\mathbb{K}$.

Definition 1.2.1. *Let $K : X \to Y$ be a BLO (in $\mathscr{L}(X, Y)$), then K compact iff $K(B_1^X(0))$ is relatively compact in Y (i.e. $\overline{K(B_1^X(0))}$ compact).*

$$\mathcal{K}(X, Y) = \{ K \in \mathscr{L}(X, Y) \mid K \text{ is compact} \}.$$

Let $X = C([a, b], \mathbb{C})$ and $k \in C^0([a, b] \times [c, d], \mathbb{C})$

Define $K \in \mathscr{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t, s)x(s)\, ds.$$

Show $K \in \mathcal{K}(X)$.

Theorem 1.2.1. $\mathcal{K}(X, Y)$ *is a closed subspace of* $\mathscr{L}(X, Y)$.

Proof. We shall show this in two steps

1. $\mathcal{K}(X, Y)$ is a vector space (Do it).

2. Closed: $K_n \to K$ (Prof. Yacine said he would send a different proof).

$\square$

Let $X = \ell^2(\mathbb{C})$,

$$\mathscr{A}((x_n)_{n \geq 0}) = \left(\frac{x_n}{n + 1} \right)_{n \geq 0}.$$

Show that $\mathscr{A}$ is compact.

Theorem 1.2.2. *Let* X, Y *and* Z *be Banach spaces on* $\mathbb{K}$.

$$X \xrightarrow{\mathscr{A}} Y \xrightarrow{\mathscr{B}} Z, \quad \mathscr{A} \in \mathscr{L}(X, Y), \; \mathscr{B} \in \mathscr{L}(Y, Z).$$

1. *If* $\mathscr{A}$ *is compact or* $\mathscr{B}$ *is compact, then* $\mathscr{B}\mathscr{A}$ *is compact.*

2. *If* $\mathscr{A}$ *is compact then* $\mathscr{A}^* \in \mathcal{K}(Y^*, X^*)$.

3. *If* $\mathscr{A}$ *is compact and* $\mathrm{Rg}(\mathscr{A})$ *is closed (in* Y *), then it is finite dimensional.*

To proceed with further results on compact operators, we need the following lemma

Lemma 1.2.1 (Riesz Lemma)**.** *Let* E *be a normed vector space,* $F = \overline{F} \subset E$. *Then* $\forall r \in (0, 1)$, $\exists x_r \in E$, *such that*

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

Proof. Since $F \neq E$ then this implies $\exists z \in E \setminus F$. Let $d = d(z, F) > 0$.

For $0 < r < 1, \exists y_r \in F$ s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

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For all $y \in F$,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|}\|z - (y_r + \|z - y_r\|y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

$\square$

Proposition 1.2.1. *Let $\mathscr{A} \in \mathcal{K}(X)$, such that X is a Banach space on $\mathbb{C}$. If $\lambda \in \mathbb{C}^*$, then $\ker((\lambda I - \mathscr{A})^n)$ has finite dimension.*

Proof. Only for $n = 1$. (do it for $n \geq 2$). Now, let

$$\tilde{K} := \ker(\lambda I - \mathscr{A}) = \{x \in X : \mathscr{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda}\mathscr{A}x\right\} \subset \mathrm{Rg}(\mathscr{A}).$$

So $\tilde{K}$ is closed in $\mathrm{Rg}(\mathscr{A})$. Suppose $\dim \tilde{K} = +\infty$. By Riesz lemma, $\exists (x_n)$ in $\tilde{K}$, such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|}\|\mathscr{A}x_n - \mathscr{A}x_m\| \geq \frac{1}{2}, \qquad \forall n \neq m$$

and so we have $\|\mathscr{A}x_n\| \leq \|\mathscr{A}\|$. So $(\mathscr{A}x_n)$ is not Cauchy, hence a contradiction.

$\square$

Exercise 7

Let X be a Banach space on $\mathbb{K}$. If $\mathscr{A} \in \mathcal{L}(X)$, assume $\exists n_0$ s.t. $\ker(\mathscr{A}^{n_0}) = \ker(\mathscr{A}^{n_0+1})$. Then $\forall n \geq n_0$,

$$\ker(\mathscr{A}^n) = \ker(\mathscr{A}^{n_0}).$$

Proposition 1.2.2. *Let $\mathscr{A} \in \mathcal{K}(X)$ and X be a Banach space on $\mathbb{C}$, $\lambda \neq 0$. Then $\exists n_0$ such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathscr{A})^n) = \ker((\lambda I - \mathscr{A})^{n_0}).$$

Proof. Using the previous exercise and arguing by contradiction, that for all $n \geq 1$ $\ker((\lambda I - \mathscr{A})^n) \subset \ker((\lambda I - \mathscr{A})^{n+1})$ and each of them is closed.

RL: with $r = \frac{1}{2}$ with $(x_n)_{n \geq 1} \in X$, such that $\|x_n\| = 1$. Then, $x_n \in \ker((\lambda I - \mathscr{A})^{n+1})$. Thus,

$$d(x_n, \ker((\lambda I - \mathscr{A})^n))) \geq \frac{1}{2}.$$

For $n = 1, x \in \ker(\lambda I - \mathscr{A}) \Rightarrow x = \frac{\mathscr{A}}{\lambda}x$. For all $1 \leq m < n$,

$$\begin{aligned}
\frac{\mathscr{A}x_n}{\lambda} - \frac{\mathscr{A}x_m}{\lambda} &= x_n - x_n + \frac{\mathscr{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathscr{A}x_m}{\lambda}\right) \\
&= x_n - \left[\frac{(\lambda I - \mathscr{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathscr{A})x_m}{\lambda}\right].
\end{aligned}$$

So,

$$\left\| \frac{\mathscr{A} x_n}{\lambda} - \frac{\mathscr{A} x_m}{\lambda} \right\| \geq d\left(x_n, \ker(\lambda I - \mathscr{A})^n\right) \geq \frac{1}{2}.$$

which is a contradiction. $\qquad\square$

Notice that if $\ker(\lambda I - \mathscr{A}) \neq \{0\}$, then $\lambda \in \sigma_p(\mathscr{A})$. Notice,

$$\dim \ker(\lambda I - \mathscr{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2 $\Rightarrow \exists n_0$ (smallest one) such that

$$\ker\left((\lambda I - \mathscr{A})^{n_0}\right) = \ker\left((\lambda I - \mathscr{A})^n\right), \qquad \forall n \geq n_0.$$

Note that,

$$\ker\left((\lambda I - \mathscr{A})^{n_0}\right) := \text{generalized eigenspace}.$$

$$\dim \ker\left((\lambda I - \mathscr{A})^{n_0}\right) := \text{algebraic multiplicity of } \lambda.$$

Proposition 1.2.3 (Fredholm aternative). *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$.*

$$\text{Rg}(\lambda I - \mathscr{A}) = X \iff \ker(\lambda I - \mathscr{A}) = \{0\}.$$

Proposition 1.2.4. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$, $\dim X = \infty$. If $\lambda_n \to \lambda$, $\lambda_n \in \sigma(\mathscr{A}) \setminus \{0\}$, pairwise distinct, then $\lambda = 0$. Hence every $\lambda \in \sigma(\mathscr{A}) \setminus \{0\}$ is isolated.*

Proof. Let $\lambda_n \in \sigma_p(\mathscr{A})$, $\exists \|x_n\| = 1$ such that $\mathscr{A} x_n = \lambda_n x_n$. Let

$$X_n = \text{span}(x_1, \ldots, x_n), \qquad X_n \subset X_{n+1}.$$

Let us prove that $\dim X_n = n$.

By induction: $n = 1$ is OK.

$$\dim X_n = n \implies \dim X_{n+1} = n + 1.$$

By contradiction, $x_{n+1} \in X_n$.

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i, \text{ which implies } \lambda_{n+1} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathscr{A} x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$

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Since (x_i) are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \qquad 1 \le i \le n.$$

which implies $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$, (Impossible).

Notice:

$$(\lambda_n I - \mathscr{A})X_n \subset X_{n-1}, \qquad \forall n \ge 2.$$

Recall:

$$\|y_n\| = 1, \qquad y_n \in X_n,$$
$$d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

For $2 \le m < n$,

$$\left\| \frac{\mathscr{A} y_n}{\lambda_n} - \frac{\mathscr{A} y_m}{\lambda_m} \right\| = \left\| y_n - \left[\frac{\lambda_n I - \mathscr{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathscr{A}}{\lambda_m} y_m \right] \right\|$$
$$\ge d(y_n, X_{n-1}) \ge \frac{1}{2}.$$

Assume that

$$\lambda_n \to \lambda \quad (n \to \infty).$$

Suppose $\lambda \ne 0$, then

$$\left| \frac{1}{\lambda_n} \right| \le C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left(\frac{\mathscr{A} y_n}{\lambda_n} \right)_{n \ge 1}$$

is a bounded sequence.

Then we have built a sequence in $\mathscr{A}(B_M^X(0))$, $M > 0$ which does not admit a convergent subsequence. Which is a Contradiction. $\qquad \square$

Theorem 1.2.3. *Let $\mathscr{A} \in \mathcal{K}(X)$, and let X be a Banach space on $\mathbb{C}$. Then $\sigma(\mathscr{A}) \setminus \{0\}$ is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in $\{0\}$.*

Main use of compact operators (in PDEs)

They appear as "inverse" of UBLO.

Definition 1.2.2. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ UBLO, closed, $\rho(\mathscr{A}) \ne \{0\}$. $\mathscr{A}$ is said to have compact resolvent if*
$$(\lambda I - \mathscr{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathscr{A}).$$

Main Example: $\mathscr{A} = -\Delta$ on Ω with $\mathscr{A}u = -u_{xx}$.

1.3 Adjoints, Symmetric and Self-adjoint Operators

Let $\mathscr{H}$ be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathscr{H}} : \mathscr{H} \times \mathscr{H} \to \mathbb{C}.$$

$\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ UBLO, $\overline{D(\mathscr{A})} = \mathscr{H}$.

Definition 1.3.1 (Adjoint Operator $\mathscr{A}^\circ$)**.**

$$D(\mathscr{A}^\circ) = \{x \in \mathscr{H} : v \mapsto \langle \mathscr{A}v, x \rangle_{\mathscr{H}} : D(\mathscr{A}) \to \mathbb{C} \text{ bdd operator}\}.$$

If $x \in D(\mathscr{A}^\circ)$, then there exists uniquely $z \in \mathscr{H}$ such that $\langle v, z \rangle_{\mathscr{H}} = \langle \mathscr{A}v, x \rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.

Observe, with Riesz representation and the fact that $\overline{D(\mathscr{A})} = \mathscr{H}$, we have that $z := \mathscr{A}^\circ x$ and $\langle v, \mathscr{A}^\circ \rangle_{\mathscr{H}} = \langle \mathscr{A}v, x \rangle_{\mathscr{H}}$ for all $v \in D(\mathscr{A})$.

Remark 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, $E : \mathscr{H} \to \mathscr{H}^*$, $x \mapsto \langle x, \cdot \rangle$. Linear isometry between $\mathscr{H}$ and $\mathscr{H}^*$. (One can identify $\mathscr{H}$ and $\mathscr{H}^*$). Now, we define the Dual operator as the following:*

$$\mathscr{A}^* : D(\mathscr{A}^*) \subset \mathscr{H}^* \to \mathscr{H}, \quad \mathscr{A}^\circ = E^{-1} \mathscr{A}^* E.$$

Definition 1.3.2 (Symmetric and Self-adjoint Operator). *Let $(\mathscr{H}, \langle, \rangle)$ be a Hilbert space and $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ be a UBLO, then*

1. ***Symmetric:** If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ \supset \mathscr{A}$ with $D(\mathscr{A}^\circ) \supset D(\mathscr{A})$ and for all $x, y \in D(\mathscr{A})$, $\langle \mathscr{A}x, y \rangle = \langle x, \mathscr{A}y \rangle$.*

2. ***Self-adjoint:** If $\overline{D(\mathscr{A})} = \mathscr{H}$ and $\mathscr{A}^\circ = \mathscr{A}$.*

Exercise 8

1. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. If $\mathscr{A}$ is closed, then $\overline{D(\mathscr{A}^\circ)} = \mathscr{H}$.

2. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\overline{D(\mathscr{A})} = \mathscr{H}$. Suppose $\mathscr{A}$ is symmetric and if $0 \in \sigma_p(\mathscr{A})$, then prove that $\lambda \in \mathbb{R}$ and

$$\inf_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A}x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A}x, x \rangle.$$

Proposition 1.3.1. *Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be an Hilbert space over $\mathbb{C}$, If $\mathscr{A}$ is self-adjoint, injective and $\overline{D(\mathscr{A})} = \mathscr{H}$. Then $\mathscr{A}^{-1} : \mathrm{Rg}(\mathscr{A}) \to \mathscr{H}$ is self-adjoint.*

Proof. Since $\mathscr{A}$ is injective then $\mathscr{A}^{-1}$ is well defined, and since $\mathscr{A} = \mathscr{A}^\circ$ then $\mathscr{A}$ is closed.

Now assume $(x_n) \subset D(\mathscr{A})$ such that $x_n \to x \in D(A)$ (because $\mathscr{A}$ is closed) and $\mathscr{A}x_n - \to y$ then for all $z \in D(\mathscr{A})$, $\langle \mathscr{A}x_n, z \rangle = \langle x_n, \mathscr{A}^\circ z \rangle$ which implies $\langle y, z \rangle = \langle x, \mathscr{A}^\circ z \rangle$ and so we have

$x \in D(\mathscr{A}^\circ) = D(\mathscr{A})$ and $y = \mathscr{A}^\circ$. Notice $\overline{\mathrm{Rg}(\mathscr{A})} = \ker(\mathscr{A})^\perp$ (because of self-adjointness). Injectivity implies $\overline{\mathrm{Rg}(\mathscr{A})} = \mathscr{H}$ which implies $\overline{D(\mathscr{A}^{-1})} = \mathscr{H}$. So $\mathscr{A}^{-1}$ is densely defined. Now, observe for all $u, v \in D(\mathscr{A}^{-1})$, $u = \mathscr{A}^\circ x$ and $v = \mathscr{A} y$ with $x, y \in D(\mathscr{A})$. Hence,

$$\langle \mathscr{A}^{-1} u, v \rangle = \langle x, \mathscr{A} y \rangle = \langle \mathscr{A} x, y \rangle = \langle x, \mathscr{A}^{-1} y \rangle.$$

To this end, $(\mathscr{A}^{-1})^\circ \subset \mathscr{A}^{-1}$, $\forall z \in D\big((\mathscr{A}^{-1})^\circ\big)$ $\exists w$, $\forall u \in D(\mathscr{A}^{-1}) = R(\mathscr{A})$ (i.e. $u = \mathscr{A} x$)

$$\langle \mathscr{A}^{-1} u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathscr{A}) \quad \langle x, z \rangle = \langle \mathscr{A} x, w \rangle$$

By definition $w \in D(\mathscr{A}^\circ)$ and $\mathscr{A}^\circ w = z$. $\mathscr{A} w = z \Rightarrow z \in Rg(\mathscr{A}) = D(\mathscr{A}^{-1})$. $\qquad \square$

Theorem 1.3.1. *Let $\mathscr{H}$ be a Hilbert space, suppose $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is symmetric and surjective then $\mathscr{A}$ is self-adjoint.*

Proof. $\mathscr{A}$ and $\mathscr{A}^\circ$ are injective. Do it only for $\mathscr{A}$, let $x \in D(\mathscr{A})$ and $\mathscr{A} x = 0$.

$$\forall y \in D(\mathscr{A}), \quad 0 = \langle \mathscr{A} x, y \rangle = \langle x, \mathscr{A} y \rangle \Rightarrow x \perp Rg(\mathscr{A}) = \mathscr{H}.$$

which implies $x = 0$.
Next, we show $\mathscr{A}$ closed.

$$(x_n)_{n \geq 1} \subset D(\mathscr{A}), \quad x_n \to x \text{ in } \mathscr{H}, \quad \mathscr{A} x_n \to y \text{ in } \mathscr{H}$$

We shall show $y = \mathscr{A} x$. Now, $\forall z \in D(\mathscr{A})$ then $\langle \mathscr{A} x_n, z \rangle = \langle x_n, \mathscr{A} z \rangle$, which implies $\langle y, z \rangle = \langle x, \mathscr{A} z \rangle$ which implies $x \in D(\mathscr{A}^\circ)$ and $y = \mathscr{A}^\circ x$. Since $\mathscr{A}$ surjective $\Rightarrow \exists w \in D(\mathscr{A})$ s.t. $\mathscr{A} w = y$ and $\mathscr{A}^\circ x = y$.
Since $\mathscr{A}$ is symmetric: $\mathscr{A}^\circ w = \mathscr{A} w$. Then $\mathscr{A}^\circ w = \mathscr{A}^\circ x$, $\mathscr{A}$ is injective $\Rightarrow w = x$. Hence $\mathscr{A} x = \mathscr{A} w = y \Rightarrow y = \mathscr{A} x \Rightarrow \mathscr{A}$ is closed.
By closed graph theorem both $\mathscr{A}$ and $\mathscr{A}^{-1} \in \mathscr{L}(X)$. We can conclude that $\mathscr{A}$ is a self-adjoint operator. $\qquad \square$

Exercise 9

Let $\mathscr{H} = L^2(0, \pi)$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ and

$$\mathscr{A} f = -f''$$

If $D(\mathscr{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?
Similarly, if $D(\mathscr{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$ is $\mathscr{A}$ a self-adjoint operator?

Theorem 1.3.2 (Fredrich's Extension). *Let $\mathscr{H}$ be a Hilbert space on $\mathbb{C}$ with $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, symmetric then $\mathscr{A}$ admits a unique self adjoint extension. If either*

a. $\inf_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A} x, x \rangle > -\infty$

b. $\sup_{\|x\|=1, x \in D(\mathscr{A})} \langle \mathscr{A}x, x \rangle < +\infty$

such that, $\mathscr{A} \subset \mathscr{A}^\circ \subset (\mathscr{A}^\circ)^\circ \subset \cdots$. *If (a) or (b) holds, then;* $\mathscr{A} \subset \mathscr{A}^\circ = (\mathscr{A}^\circ)^\circ$.

1.4 Dissipative Operator and Numerical range

Definition 1.4.1 (Duality Map). *Let X be a Banach space on $\mathbb{K}$. The duality map is defined as*
$J : X \longrightarrow 2^{X^*}, x \mapsto J(x) = \{x^* \in X^* \mid Re\langle x^*, x \rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$.
By the Hahn-Banach theorem, $J(x) \neq \emptyset$.

Question: What can you say about $J(X)$ when X is an Hilbert space or Reflexive?

Definition 1.4.2. *A map $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ (UBLO) is dissipative if for all $x \in D(\mathscr{A})$, there exists $x^* \in J(X)$ such that $Re\langle x^*, \mathscr{A}x \rangle \leq 0$.*

Lemma 1.4.1. *$\mathscr{A}$ is dissipative if and only if for all $\lambda > 0$, $x \in D(\mathscr{A})$ we have that*
$\|(\lambda I - \mathscr{A})x\| \geq \lambda \|x\|$.

Proof. let $x^* \in J(x)$. Then

$$\|(\lambda I - \mathscr{A})x\| \, \|x^*\| \geq |\langle x^*, (\lambda I - \mathscr{A})x \rangle| \geq \Re \langle x^*, (\lambda I - \mathscr{A})x \rangle,$$

$$= \lambda \Re \langle x^*, x \rangle - \Re \langle x^*, \mathscr{A}x \rangle \geq \lambda \|x\|^2.$$

Hence, if $\|x\| \neq 0$, then
$$\|(\lambda I - \mathscr{A})x\| \geq \lambda \|x\|.$$

($\Longleftarrow$) Let $x \in D(\mathscr{A})$, $x \neq 0$, and $\lambda > 0$. Let $y_\lambda^* \in J((\lambda I - \mathscr{A})x)$ and set $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$. Then

$$\|(\lambda I - \mathscr{A})x\|^2 = \|(\lambda I - \mathscr{A})x\| \, \|y_\lambda^*\| = \Re \langle y_\lambda^*, (\lambda I - \mathscr{A})x \rangle.$$

Since $y_\lambda^* \neq 0$, we have

$$\lambda \|x\| \leq \|(\lambda I - \mathscr{A})x\| = \Re \langle g_\lambda^*, (\lambda I - \mathscr{A})x \rangle = \lambda \langle g_\lambda^*, x \rangle - \Re \langle g_\lambda^*, \mathscr{A}x \rangle.$$

Hence,
$$\Re \langle g_\lambda^*, \mathscr{A}x \rangle \leq \lambda \langle g_\lambda^*, x \rangle - \lambda \|x\| \leq \|g_\lambda^*\| \, \|x\| = \|x\|.$$

Therefore,
$$\Re \langle g_\lambda^*, \mathscr{A}x \rangle \leq 0. \tag{**}$$

Idea: Let $\lambda \to +\infty$.

Unit ball in X^* is compact for weak* topology (Banach–Alaoglu).

(Up to subsequence)
$$g_\lambda^* \longrightarrow g^* \in X^*, \qquad \|g^*\| \leq 1.$$

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Then from $(**)$,
$$\Re\langle g^*, \mathscr{A}x\rangle \leq 0.$$

$(*)$ $\qquad \|x\| \leq \langle g_\lambda^*, x\rangle - \frac{1}{\lambda}\Re\langle g_\lambda^*, \mathscr{A}x\rangle.$

Let $\lambda \to +\infty$. Then $\|x\| \leq \langle g^*, x\rangle$. Hence, $\|g^*\| = 1$ $\quad$ and $\quad \langle g^*, x\rangle = \|x\|$.
Set $x^* = \|x\|g^*$. Then
$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x\rangle = \|x\|^2,$$

that is, $x^* \in J(x)$. $\hfill\square$

Theorem 1.4.1 (Lumer-Phillips). *Let X be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO. Assume that $\mathscr{A}$ is dissipative and that there exists $\lambda_0 > 0$ such that $\mathrm{Rg}(\lambda_0 I - \mathscr{A}) = X$.*

Then $\mathscr{A}$ is closed, $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$, and for all $\lambda > 0$,
$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}. \tag{1.4}$$

Proof. Let $\lambda_0 > 0$.

(1) To prove: $(\lambda_0 I - A)$ is bijective.

- surj: Assumption,

- inj: Lemma 1.4.1.

Hence,
$$(\lambda_0 I - \mathscr{A})^{-1} : X \longrightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any $y \in X$, there exists a unique $y \in X$ such that
$$x = (\lambda_0 I - \mathscr{A})^{-1}y, \qquad (\lambda_0 I - \mathscr{A})x = y.$$

By Lemma 1.4.1,
$$\frac{1}{\lambda_0}\|y\| \geq \|(\lambda_0 I - \mathscr{A})^{-1}y\|.$$

Hence,
$$\|(\lambda_0 I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_0}, \qquad \lambda_0 \in \rho(\mathscr{A}).$$

(2) $\mathscr{A}$ is closed.

Let $x_n \to x$, $x_n \in D(\mathscr{A})$, and $Ax_n \to y$.
Then
$$(\lambda_0 I - \mathscr{A})x_n \to \lambda_0 x - y.$$

Since $(\lambda_0 I - \mathscr{A})^{-1} \in \mathcal{L}(X)$, we have
$$x_n \longrightarrow (\lambda_0 I - \mathscr{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathscr{A})x \iff y = \mathscr{A}x.$$

Therefore, $\mathscr{A}$ is closed.

(3) $\rho(\mathscr{A}) \supset \mathbb{R}_+^*$ and (1.4).

Since $\mathscr{A}$ is closed and $\rho(\mathscr{A}) \neq \emptyset$, we know that $\rho(\mathscr{A})$ is open.

Let $\Lambda = \rho(\mathscr{A}) \cap \mathbb{R}_+^*$, which is open in $\mathbb{R}_+^*$. We show that it is closed.

Let $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda$ such that $\lambda_n \longrightarrow \lambda \in \mathbb{R}_+^*$. Note that since $\lambda_n \in \Lambda$, we have

$$\|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$(\lambda I - \mathscr{A}) = \big[I + u_n\big](\lambda_n I - \mathscr{A}) \implies \lambda I - \mathscr{A} = \lambda_n I - \mathscr{A} + u_n(\lambda_n I - \mathscr{A}),$$

$$\iff (\lambda - \lambda_n)I = u_n(\lambda_n I - \mathscr{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathscr{A})^{-1} = u_n.$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathscr{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For n large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that $\lambda \in \rho(\mathscr{A})$. Hence, Λ is closed, and therefore $\Lambda = \mathbb{R}_+^*$.

$\square$

Corollary 1.4.1. *Let X be a Banach space and $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a UBLO, closed, with $\overline{D(\mathscr{A})} = X$. Assume that $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative. Then*

$$\rho(\mathscr{A}) \supset \mathbb{R}_+^*, \qquad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathscr{A})^{-1}\| \leq 1.$$

Proof. It is easy to show that $\mathrm{Rg}(I - A) = X$ (i.e. $\lambda_0 = 1$ + Theorem 1.4.1).
$\mathscr{A}$ dissipative and closed implies

$$\mathrm{Rg}(I - \mathscr{A}) \text{ is a closed subspace of } X.$$

(give details!!!)

Let $x^* \in X^*$ such that

$$\langle x^*, (I - \mathscr{A})x \rangle = 0, \qquad \forall x \in D(\mathscr{A}). \tag{$**$}$$

Let us prove that $x^* = 0$.

Then $x^* \in D(\mathscr{A}^*)$ and

$$(I - \mathscr{A}^*)x^* = 0.$$

Since $\mathscr{A}^*$ is dissipative, by Lemma 1.4.1, we have $x^* = 0$. This implies that

$$\overline{\mathrm{Rg}(I - \mathscr{A})} = X.$$

Since $\mathrm{Rg}(I - \mathscr{A})$ is closed, we obtain

$$\mathrm{Rg}(I - \mathscr{A}) = X.$$

By contradiction and using Hahn–Banach. $\qquad\qquad\qquad\qquad\qquad\square$

Definition 1.4.3 (Numerical Range)**.** *Let* $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ *be UBLO. The numerical range of* $\mathscr{A}$, *denoted by* $W(\mathscr{A})$,

$$W(\mathscr{A}) = \{\langle x^*, \mathscr{A}x \rangle \mid x^* \in J(x),\ x \in D(\mathscr{A}),\ \|x\| = \|x^*\| = 1,\ \langle x^*, x \rangle = 1\}.$$

In case of a Hilbert space, we have that $W(\mathscr{A}) = \{\langle x, \mathscr{A}x \rangle \mid x \in D(\mathscr{A}),\ \|x\| = 1\}$.

Linear algebra in finite dimension $\mathscr{A} \in \mathcal{M}_n(\mathbb{K})$, *we have that* $W(\mathscr{A}) = \{\langle x, \mathscr{A}x \rangle \mid \|x\| = 1\}$.

Theorem 1.4.2 (Home-work)**.** *Let* $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ *be closed, with* $\overline{D(\mathscr{A})} = X$.

1) If $\lambda \notin \overline{W(\mathscr{A})}$, *then* $(\lambda I - \mathscr{A})$ *is injective, has closed image, and for all* $x \in D(\mathscr{A})$,

$$\|(\lambda I - \mathscr{A})x\| \geq d(\lambda, W(\mathscr{A}))\, \|x\|.$$

Moreover, if $\lambda \in \rho(\mathscr{A})$, *then*

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, W(\mathscr{A}))}. \qquad\qquad (\ast\ast)$$

2) If Λ *is a connected open subset of* $\mathbb{C} \setminus W(\mathscr{A})$ *such that* $\rho(\mathscr{A}) \cap \Lambda \neq \emptyset$, *then* $\rho(\mathscr{A}) \supset \Lambda$ *and* $(\ast\ast)$ *holds true.*

TWO

INTRODUCTION TO THE THEORY OF SEMI-GROUPS

2.1 Intro to the Introduction

Definition 2.1.1. *Let X be a Banach space over $\mathbb{K}$. A one-parameter family of bounded linear operators on X, $(T(t))_{t \geq 0}$, is a semigroup (SG) of bounded linear operators on X if:*

1. $T(0) = Id_X$,

2. $\forall (t, s) \in \mathbb{R}_+^2 : T(t + s) = T(t) \cdot T(s)$ *(SG property).*

Remark 2.1.1. *$T(t)$ and $T(s)$ commute.*

   2. **Infinitesimal generator of SG–LO** $(T(t))_{t \geq 0}$

Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be an unbounded linear operator with

$$D(\mathscr{A}) = \left\{ x \in X \;\middle|\; \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t \to 0^+} \frac{T(t)x - x}{t} = \mathscr{A}x, \qquad x \in D(\mathscr{A}).$$

where $D(\mathscr{A}) = $ domain of $\mathscr{A}$.

2.2 Uniformly Continuous SG–BLO

Definition 2.2.1. *A SG–BLO on X, $(T(t))_{t \geq 0}$ is uniformly continuous if*

$$\|T(t) - Id\|_{\mathscr{L}(X)} \to 0 \quad as \; t \to 0^+$$

Lemma 2.2.1. *Let $(T(t))_{t \geq 0}$ be a SG–BLO which is uniformly continuous. Then, $\forall t > 0$,*

$$\|T(s) - T(t)\| \xrightarrow[s \to t]{} 0$$

(continuity everywhere).

Proof. Let t be fixed. $T(s) = T(s - t + t)$, $\qquad s - t \geq 0$.

$$s \geq t \;\Rightarrow\; T(s) = T(s - t)T(t) \;\Rightarrow\; T(s) - T(t) = T(t)\left[T(s - t) - I_d\right]$$

$$\|T(s) - T(t)\| \leq \|T(t)\|\,\|T(s - t) - I_d\| \xrightarrow[s \to t]{} 0.$$

For $s \leq t$

$$T(t) = T(t - s)T(s) \;\Rightarrow\; T(t) - T(s) = T(s)\left[T(t - s) - I_d\right]$$

(Prove that $\sup_{[0,t]} \|T(t)\| < +\infty$)

Then

$$\|T(t) - T(s)\| \leq \|T(s)\|\,\|T(t - s) - I_d\|$$

$$\leq \sup \|T(s)\|\,\|T(t - s) - I_d\| \xrightarrow[s \to t]{} 0.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.2.1. *A linear operator $\mathscr{A}$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $\mathscr{A}$ is a bounded linear operator.*

Proof. Let $\mathscr{A}$ be a bounded linear operator on X and set

$$T(t) = e^{t\mathscr{A}} = \sum_{n=0}^{\infty} \frac{(t\mathscr{A})^n}{n!}. \tag{1.5}$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such t, a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t + s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq |t|\,\|\mathscr{A}\|\,e^{|t|\,\|\mathscr{A}\|}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathscr{A} \right\| \leq \|\mathscr{A}\| \cdot \max_{0 \leq s \leq t} \|T(s) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on X and that $\mathscr{A}$ is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on X. Fix $\rho > 0$, small enough, such that

$$\left\| I - \rho \int_0^\rho T(s)\,ds \right\| < 1.$$

This implies that $\rho^{-1} \int_0^\rho T(s)\,ds$ is invertible. Now,

$$h^{-1}(T(h) - I)\int_0^\rho T(s)\,ds = h^{-1}\left(\int_0^\rho T(s + h)\,ds - \int_0^\rho T(s)\,ds \right)$$

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$$= h^{-1} \left(\int_{\rho}^{\rho+h} T(s) \, ds - \int_{0}^{h} T(s) \, ds \right)$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1} \int_{\rho}^{\rho+h} T(s) \, ds - h^{-1} \int_{0}^{h} T(s) \, ds \right) \left(\int_{0}^{\rho} T(s) \, ds \right)^{-1}. \qquad (1.6)$$

Letting $h \to 0$ in (1.6) shows that $h^{-1}(T(h) - I)$ converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I) \left(\int_{0}^{\rho} T(s) \, ds \right)^{-1}$$

which is the infinitesimal generator of $T(t)$. $\qquad \square$

Remark 2.2.1. *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

Theorem 2.2.2. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t \to 0} \frac{T(t) - I}{t} = \mathscr{A} = \lim_{t \to 0} \frac{S(t) - I}{t}. \qquad (1.7)$$

then $T(t) = S(t)$ for $t \geq 0$.

Proof. We will show that given $T > 0$, $S(t) = T(t)$ for $0 \leq t \leq T$. Let $T > 0$ be fixed, since $t \mapsto \|T(t)\|$ and $t \mapsto \|S(t)\|$ are continuous there is a constant C such that

$$\|T(t)\| \|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given $\varepsilon > 0$ it follows from (1.7) that there is a $\delta > 0$ such that

$$h^{-1} \|T(h) - S(h)\| < \varepsilon / TC \qquad \text{for } 0 \leq h \leq \delta. \qquad (1.8)$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n \leq \delta$. From the semigroup property and (1.8) it then follows that

$$\|T(t) - S(t)\| = \left\| T\left(n\frac{t}{n}\right) - S\left(n\frac{t}{n}\right) \right\|$$

$$\leq \sum_{k=0}^{n-1} \left\| T\left((n-k)\frac{t}{n}\right) S\left(\frac{kt}{n}\right) - T\left((n-k-1)\frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\|$$

$$\leq \sum_{k=0}^{n-1} \left\| T\left((n-k-1)\frac{t}{n}\right) \right\| \left\| T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{t}{n}\right) \right\| \leq Cn\frac{\varepsilon}{TC}\frac{t}{n} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary $T(t) = S(t)$ for $0 \leq t \leq T$ and the proof is complete. $\qquad \square$

Corollary 2.2.1. *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

(a) *There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*

(b) *There exists a unique bounded linear operator $\mathscr{A}$ such that $T(t) = e^{t\mathscr{A}}$.*

(c) *The operator $\mathscr{A}$ in part (b) is the infinitesimal generator of $T(t)$.*

(d) *$t \mapsto T(t)$ is differentiable in norm and satisfies*

$$\frac{dT(t)}{dt} = \mathscr{A}T(t) = T(t)\mathscr{A}. \tag{1.9}$$

Proof. All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator $\mathscr{A}$. $\mathscr{A}$ is also the infinitesimal generator of $e^{t\mathscr{A}}$ defined by (1.5) and therefore, by Theorem 2.2.2, $T(t) = e^{t\mathscr{A}}$. $\qquad\square$

Remark 2.2.2. *The proofs above are from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 3. [Theorem 1.3 and Corollary 1.4].*

2.3 Strongly Continuos Semigroups (C_0-Semigroups)

Definition 2.3.1. *The SG-BLO $(T(t))_{t \geq 0}$ is strongly continous (SC or C_0) if $\forall x \in X$*

$$\|T(t)x - x\|_X \xrightarrow{t \to 0^+} 0 \tag{2.1}$$

Theorem 2.3.1. *Let $(T(t))_{t \geq 0}$, C_0-SG then $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0$ $\|T(t)\| \leq Me^{\omega t}$*

Proof. First we want to show that $\exists \eta > 0$, $\sup_{t \in [0, \eta]} \|T(t)\| < +\infty$.

By contradiction, assume that $\sup \|T(t)\| = +\infty$. Then $\exists (t_n)_{n \geq 0} \searrow 0$ such that $\|T(t_n)\| \geq n$ or $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive) $\exists x \in X$ such that $\sup \|T(t_n)x\| = +\infty$, but this contradicts the strong convergance 2.3.1.

Now take $M := \sup \|T(t)\| \geq 1$ (This is because $T(0) = Id$).

$\forall t \geq 0$ write $t = k\eta + \eta_t$ where $k = \left\lfloor \frac{t}{\eta} \right\rfloor$ and $\eta_t \in [0, \eta]$, then

$$
\begin{aligned}
\|T(t)\| &= \|T(k\eta + \eta_t)\| \\
&= \left\|[T(\eta)]^k \, T(\eta_t)\right\| && \text{by the SG property} \\
&= \|T(\eta)\|^k \, \|T(\eta_t)\| \\
&\leq M \cdot M^k && \text{M is upperbound} \\
&\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\
&= M(e^{\ln M})^{t/\eta} \\
&= Me^{t\frac{\ln M}{\eta}} = Me^{\omega t}
\end{aligned}
$$

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And since $\omega = \dfrac{\ln M}{\eta}$ and $M \geq 1$ then $\omega \geq 0$. $\qquad\square$

Corollary 2.3.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup then $\forall x \in X, t \mapsto T(t)x$ is continuous*

Proof. For $h > 0$ and $t > 0$:

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \to 0$ as $h \to 0$ by definition 2.3.1.

- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$. Since $\|T(t-h)\|$ is bounded by theorem 2.3.1, so this tends to 0 as $h \to 0$.

$\qquad\square$

Theorem. *Let $\mathscr{A}$ be the IG of C_0-SG $T(t)$, then:*

2.3.2 $\forall x \in X, \forall t \geq 0, \ \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x$ *for all $x \in X$.*

2.3.3 $\forall x \in X, \forall t \geq 0, \ \int_0^t T(s)x \, ds \in D(\mathscr{A})$ *and* $\mathscr{A} \int_0^t T(s)x \, ds = T(t)x - x$.

2.3.4 $\forall x \in D(\mathscr{A}), T(t)x \in D(\mathscr{A})$ *and* $\frac{d}{dt}T(t)x = \mathscr{A}T(t)x = T(t)\mathscr{A}x$.

2.3.5 $\forall x \in D(\mathscr{A}), \forall t \geq 0, \forall s \geq 0, \ T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x \, du$ *for $x \in D(\mathscr{A})$.*

Proof of 2.3.2. Consider the small interval $[t, t+h]$ relative to its value at t:

$$\frac{1}{h} \int_t^{t+h} T(s)x \, ds - T(t)x = \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] \, ds$$

By the continuity (corollary 2.3.1) of the map $s \mapsto T(s)x$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all s satisfying $|s - t| < \delta$, we have $\|T(s)x - T(t)x\| < \epsilon$.

Taking $0 < h < \delta$, we can estimate the norm of the integral:

$$\left\| \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] \, ds \right\| \leq \frac{1}{h} \int_t^{t+h} \|T(s)x - T(t)x\| \, ds < \frac{1}{h} \cdot h\epsilon = \epsilon$$

Then

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x$$

$\qquad\square$

Proof of 2.3.3. Let $h > 0$. Consider the difference quotient for the integral $y = \int_0^t T(s)x \, ds$:

$$\frac{T(h) - Id}{h} \int_0^t T(s)x \, ds = \frac{1}{h}\left[\int_0^t T(s+h)x \, ds - \int_0^t T(s)x \, ds \right]$$

$$= \frac{1}{h}\left[\int_h^{t+h} T(u)x \, du - \int_0^t T(u)x \, du \right]$$

$$= \frac{1}{h} \int_t^{t+h} T(u)x \, du - \frac{1}{h} \int_0^h T(u)x \, du$$

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As $h \to 0^+$, the first term converges to $T(t)x$ and the second to $T(0)x = x$ by **2.3.2**. Thus the limit exists, $y \in D(\mathscr{A})$, and $\mathscr{A}y = T(t)x - x$. $\qquad\square$

Proof of **2.3.4**. If $x \in D(\mathscr{A})$, then $T(t)\mathscr{A}x = T(t)\lim_{h\to 0}\frac{T(h)x-x}{h} = \lim_{h\to 0}\frac{T(h)T(t)x-T(t)x}{h}$. This limit exists and equals $\mathscr{A}(T(t)x)$, proving $T(t)x \in D(\mathscr{A})$ and $T(t)\mathscr{A}x = \mathscr{A}T(t)x$. This also shows the right-derivative of $T(t)x$ is $\mathscr{A}T(t)x$. A similar argument for the left-derivative completes the differentiability. $\qquad\square$

Proof of **2.3.5**. By Property **2.3.4**, the function $f(u) = T(u)x$ is differentiable with $f'(u) = T(u)\mathscr{A}x$. Since f' is continuous, we integrate f' over $[s, t]$ to obtain $f(t) - f(s) = \int_s^t f'(u)\,du$, which is $T(t)x - T(s)x = \int_s^t T(u)\mathscr{A}x\,du$. $\qquad\square$

Theorem 2.3.6. *The IG $\mathscr{A}$ of C_0-SG is a closed linear operator and $\overline{D(\mathscr{A})} = X$.*

Proof. For any $x \in X$, let $x_t = \frac{1}{t}\int_0^t T(s)x\,ds$. By **2.3.3**, $x_t \in D(\mathscr{A})$. By **2.3.2**, $x_t \to T(0)x = x$ as $t \to 0^+$, which shows $\overline{D(\mathscr{A})} = X$.

Let $x_n \in D(\mathscr{A})$ such that $x_n \to x$ and $\mathscr{A}x_n \to y$. From **2.3.3** take $s = 0$, we have $T(t)x_n - x_n = \int_0^t T(s)\mathscr{A}x_n\,ds$. Passing to the limit $n \to \infty$, we get $T(t)x - x = \int_0^t T(s)y\,ds$. Dividing by t and letting $t \to 0^+$, the Right-Hand Side (RHS) converges to y. Thus $x \in D(\mathscr{A})$ and $\mathscr{A}x = y$. $\qquad\square$

Theorem 2.3.7. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be two C_0-SG with infinitesimal generators $\mathscr{A}$ and B, respectively. If $\mathscr{A} = \mathscr{B}$, then $T(t) = S(t)$ for all $t \geq 0$.*

Proof. Assume $\mathscr{A} = \mathscr{B}$, then $D(\mathscr{A}) = D(\mathscr{B})$. Let $x \in D(\mathscr{A})$ be fixed, and for a fixed $t > 0$, define

$$\varphi : [0, t] \to X, \quad \varphi(s) = T(t - s)S(s)x$$

Since $x \in D(\mathscr{A})$, the map φ is of class C^1 on $[0, t]$. We differentiate φ with respect to s using the product rule and **2.3.4**, we get:

$$\begin{aligned}
\frac{d}{ds}\varphi(s) &= \frac{d}{ds}[T(t - s)]S(s)x + T(t - s)\frac{d}{ds}[S(s)x] \\
&= -\mathscr{A}T(t - s)S(s)x + T(t - s)BS(s)x
\end{aligned}$$

Because $T(t - s)$ commutes with its generator $\mathscr{A}$, and given $\mathscr{A} = B$, we have:

$$\frac{d}{ds}\varphi(s) = -T(t - s)\mathscr{A}S(s)x + T(t - s)\mathscr{A}S(s)x = 0$$

Since the derivative is zero for all $s \in [0, t]$, the function φ must be constant. Evaluating φ at the endpoints $s = 0$ and $s = t$ yields

$$\varphi(0) = T(t)S(0)x = T(t)x \quad \text{and} \quad \varphi(t) = T(0)S(t)x = S(t)x$$

Thus, $T(t)x = S(t)x$ for all $x \in D(\mathscr{A})$. Since $D(\mathscr{A})$ is dense in X and $T(t), S(t)$ are bounded linear operators, this identity extends to all $x \in X$ by continuity. Therefore, $T(t) = S(t)$ for all $t \geq 0$. $\square$

Theorem 2.3.8. *Let $\mathscr{A}$ be the IG of a C_0-SG $\{T(t)\}_{t \geq 0}$ on a Banach space X. Then the subspace*

$$X = \overline{\bigcap_{n \geq 1} D(\mathscr{A}^n)}$$

Proof. Let $\mathscr{D} = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \varphi \text{ has compact support in } \mathbb{R}_+^* \text{ and is smooth } C^\infty\}$. Let $x \in X$ and consider a test function $\varphi \in \mathscr{D}$. Define

$$x_\varphi = \int_0^\infty \varphi(s) T(s) x \, ds$$

First, we show that $x_\varphi \in D(\mathscr{A})$. Consider

$$
\begin{aligned}
\frac{T(h) - Id}{h} x_\varphi &= \frac{1}{h} \int_0^\infty \varphi(s)[T(s+h)x - T(s)x] \, ds \\
&= \frac{1}{h}\left[\int_h^\infty \varphi(u - h)T(u)x \, du - \int_0^\infty \varphi(u)T(u)x \, du \right] \\
&= \int_0^\infty \frac{\varphi(u - h) - \varphi(u)}{h} T(u)x \, du
\end{aligned}
$$

As $h \to 0$, the quotient $\frac{\varphi(u-h)-\varphi(u)}{h}$ converges uniformly to $-\varphi'(u)$ because φ is C^∞ and has compact support. Thus:

$$\mathscr{A} x_\varphi = -\int_0^\infty \varphi'(s) T(s) x \, ds$$

Since $\varphi' \in C_c^\infty(0, \infty)$, we can repeat this process inductively. For any $n \geq 1$, we find:

$$\mathscr{A}^n x_\varphi = (-1)^n \int_0^\infty \varphi^{(n)}(s) T(s) x \, ds$$

This proves that $x_\varphi \in D(\mathscr{A}^n)$ for all n.

To prove density, suppose $\overline{\bigcap_{n \geq 1} D(\mathscr{A}^n)} \neq X$. By the Hahn-Banach Theorem, there exists a non-zero functional $x^* \in X^*$ such that $\langle x^*, y \rangle = 0$ for all $y \in \bigcap_{n \geq 1} D(\mathscr{A}^n)$. Specifically, for any $x \in X$ and $\varphi \in C_c^\infty(0, +\infty)$:

$$\langle x^*, x_\varphi \rangle = \int_0^{+\infty} \varphi(s) \langle x^*, T(s)x \rangle \, ds = 0$$

This identity holds for all C^∞ functions φ with compact support. Then $\langle x^*, T(s)x \rangle$ must be zero for all $s > 0$.

By the strong continuity of the semigroup at $s = 0$:

$$\langle x^*, x \rangle = \lim_{s \to 0^+} \langle x^*, T(s)x \rangle = 0$$

Since this holds for all $x \in X$, it implies $x^* = 0$, which contradicts our assumption that x^* was non-zero. Thus, $\bigcap_{n \geq 1} D(\mathscr{A}^n)$ must be dense in X. $\qquad\square$

Let $X = \{f : \mathbb{R} \to \mathbb{C} \mid f$ is continuous and uniformly bounded$\}$ equipped with the supremum norm $\|f\|_\infty = \sup_{s \in \mathbb{R}} |f(s)|$. Define the family of operators $(T(t))_{t \geq 0}$ by:

$$(T(t)f)(s) = f(s + t), \quad s \in \mathbb{R}, t \geq 0$$

Prove that this family is C_0-SG, its IG is $\mathscr{A} f = f'$, and $\|T(t)\| = 1$.

2.4 Hille-Yosida Theorem

Definition 2.4.1. *A C_0-SG $\{T(t)\}_{t \geq 0}$ is called uniformly bounded semigroup if $\exists M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.*

Definition 2.4.2. *A C_0-SG $\{T(t)\}_{t \geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.*

Theorem 2.4.1 (Hille-Yosida Theorem). *A linear unbounded operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ is the IG of a C_0-SG of contractions if and only if:*

(i) $\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = X$.

(ii) $\mathbb{R}_+^* \subset \rho(\mathscr{A})$ and $\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

($\Rightarrow$). (i) follows directly from 2.3.6.

If $\mathscr{A}$ generates a contraction semigroup $\{T(t)\}_{t \geq 0}$, we define the resolvent for $\lambda > 0$ as follows

$$R(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t) \, dt$$

Taking the norm, we obtain:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)x\| \, dt \leq \int_0^{+\infty} e^{-\lambda t} \|x\| \, dt = \frac{1}{\lambda} \|x\|$$

$\qquad\square$

Remark 2.4.1. *Note for all real numbers λ, a with $\lambda > a$, we have the following:*

$$\frac{1}{\lambda - a} = \int\limits_0^{+\infty} e^{-(\lambda - a)t} \, dt$$

Extending this to the vector space we get the way of writing the resolvent operator from above.

2.4.1 The Yosida Approximation

To prove if ($\Leftarrow$), we introduce a family of bounded operators that approximate the unbounded generator A.

Definition 2.4.3. *For $\lambda > 0$, the Yosida Approximation of $\mathscr{A}$ is defined as:*

$$\mathscr{A}_\lambda := \lambda \mathscr{A} R(\lambda, \mathscr{A}) = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$$

Note that $\mathscr{A}_\lambda$ is a bounded linear operator for each $\lambda \in \rho(\mathscr{A})$.

Claim 2.4.1. *For $\lambda \in \rho(\mathscr{A})$ and $x \in D(\mathscr{A})$, the following identity holds:*

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x = R(\lambda, \mathscr{A})\mathscr{A} x$$

Proof. By the definition of the resolvent as the inverse of the operator $(\lambda I - \mathscr{A})$, we have:

$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A}) = Id_X$$

Applying this to any $x \in X$:
$$(\lambda I - \mathscr{A})R(\lambda, \mathscr{A})x = x$$

Distributing the operators on the left-hand side gives:

$$\lambda R(\lambda, \mathscr{A})x - \mathscr{A} R(\lambda, \mathscr{A})x = x$$

Rearranging the terms to isolate the $\mathscr{A} R(\lambda, \mathscr{A})x$ term yields:

$$\lambda R(\lambda, \mathscr{A})x - x = \mathscr{A} R(\lambda, \mathscr{A})x \tag{*}$$

This identity holds for all $x \in X$ because $R(\lambda, \mathscr{A})$ maps X into $D(\mathscr{A})$.

For the other side, let $x \in D(\mathscr{A})$. We use the fact that the resolvent also satisfies:

$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A}) = Id_{D(\mathscr{A})}$$

Applying this to $x \in D(\mathscr{A})$:
$$R(\lambda, \mathscr{A})(\lambda I - \mathscr{A})x = x$$

Distributing $R(\lambda, \mathscr{A})$ gives:

$$\lambda R(\lambda, \mathscr{A})x - R(\lambda, \mathscr{A})\mathscr{A} x = x$$

Rearranging the terms:
$$\lambda R(\lambda, \mathscr{A})x - x = R(\lambda, \mathscr{A})\mathscr{A} x \tag{**}$$

From (*) and (**) we get:

$$\mathscr{A} R(\lambda, \mathscr{A}) x = R(\lambda, \mathscr{A}) \mathscr{A} x$$

$\square$

Theorem 2.4.2. *For all* $x \in X$, $\displaystyle \lim_{\lambda \to +\infty} \mathscr{A}_\lambda x = \mathscr{A} x$.

Proof. Using the identity $\lambda R(\lambda, \mathscr{A}) x - x = \mathscr{A} R(\lambda, \mathscr{A}) x = R(\lambda, \mathscr{A}) \mathscr{A} x$ for $x \in D(\mathscr{A})$, we observe:

$$\|\lambda R(\lambda, \mathscr{A}) x - x\| = \|R(\lambda, \mathscr{A}) \mathscr{A} x\| \leq \frac{\|\mathscr{A} x\|}{\lambda} \xrightarrow{\lambda \to +\infty} 0$$

Since $\mathscr{A}_\lambda x = \lambda R(\lambda, \mathscr{A}) \mathscr{A} x$, and we just showed $\lambda R(\lambda, \mathscr{A}) x \to x$, $\forall x \in D(\mathscr{A})$, it follows that $\mathscr{A}_\lambda x \to \mathscr{A} x$. By density and the uniform bound $\|\lambda R(\lambda, \mathscr{A})\| \leq 1$, this convergence holds for all $x \in X$. $\square$

Lemma 2.4.1. *For each* $\lambda > 0$, $\mathscr{A}_\lambda$ *generates a uniformly continuous semigroup of contractions* $\{e^{t \mathscr{A}_\lambda}\}_{t \geq 0}$.

Proof. Since $\mathscr{A}_\lambda = \lambda^2 R(\lambda, \mathscr{A}) - \lambda I$, we have:

$$\left\| e^{t \mathscr{A}_\lambda} \right\| = \left\| e^{-t\lambda} e^{t\lambda^2 R(\lambda, \mathscr{A})} \right\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda, \mathscr{A})\|} \leq e^{-t\lambda} e^{t\lambda^2 \frac{1}{\lambda}} = e^{-t\lambda} e^{t\lambda} = 1$$

This confirms the contraction property for the approximating semigroups. $\square$

Lemma 2.4.2. *For any* $x \in X$ *and* $\lambda, \mu > 0$, *the following estimate holds:*

$$\left\| e^{t \mathscr{A}_\lambda} x - e^{t \mathscr{A}_\mu} x \right\| \leq t \left\| \mathscr{A}_\lambda x - \mathscr{A}_\mu x \right\|$$

Proof. Fix $x \in X$, consider the function $\phi : [0, 1] \to X$ defined by:

$$\phi(s) = e^{st \mathscr{A}_\lambda} e^{(1-s)t \mathscr{A}_\mu} x$$

Since $\mathscr{A}_\lambda$ and $\mathscr{A}_\mu$ commute (as resolvents commute), the semigroups $e^{t \mathscr{A}_\lambda}$ and $e^{t \mathscr{A}_\mu}$ also commute. The function ϕ is differentiable with respect to s:

$$\begin{aligned}
\frac{d}{ds}\phi(s) &= t \mathscr{A}_\lambda e^{st \mathscr{A}_\lambda} e^{(1-s)t \mathscr{A}_\mu} x - e^{st \mathscr{A}_\lambda} t \mathscr{A}_\mu e^{(1-s)t \mathscr{A}_\mu} x \\
&= t e^{st \mathscr{A}_\lambda} e^{(1-s)t \mathscr{A}_\mu} (\mathscr{A}_\lambda - \mathscr{A}_\mu) x
\end{aligned}$$

Integrating from 0 to 1:

$$\phi(1) - \phi(0) = e^{t \mathscr{A}_\lambda} x - e^{t \mathscr{A}_\mu} x = \int_0^1 t e^{st \mathscr{A}_\lambda} e^{(1-s)t \mathscr{A}_\mu} (\mathscr{A}_\lambda - \mathscr{A}_\mu) x \, ds$$

Taking the norm and using the contraction property $\|e^{t\mathscr{A}_\lambda}\| \leq e^{t\|\mathscr{A}_\lambda\|} \leq 1$ (since $\mathscr{A}_\lambda$ is dissipative):

$$\left\|e^{t\mathscr{A}_\lambda}x - e^{t\mathscr{A}_\mu}x\right\| \leq \int_0^1 t \left\|e^{st\mathscr{A}_\lambda}\right\| \left\|e^{(1-s)t\mathscr{A}_\mu}\right\| \left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\| ds$$
$$\leq t \left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\|$$

$\square$

Now let's start proving the only if direction of Hille-Yosida ($\Longleftarrow$).

Proof. We need the sequence $(e^{t\mathscr{A}_\lambda}x)_{\lambda>0}$ to be converging. But doing so is hard, what we can do is show that it is a Cauchy sequence in X, since X is complete.

Taking $x \in D(\mathscr{A})$ we have the following:

$$\left\|e^{t\mathscr{A}_\lambda}x - e^{t\mathscr{A}_\mu}x\right\| \leq t \left\|\mathscr{A}_\lambda x - \mathscr{A}_\mu x\right\| \leq t(\|\mathscr{A}_\lambda x - \mathscr{A}x\| + \|\mathscr{A}x - \mathscr{A}_\mu x\|) \xrightarrow[\mu\to+\infty]{\lambda\to+\infty} 0$$

This is possible because $\lim_{\lambda\to+\infty} \mathscr{A}_\lambda x = \mathscr{A}x$.

Define:

$$T(t)x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}x$$

This limit is well defined by the argument above for all $x \in D(\mathscr{A})$. Moreover, since $D(A)$ is dense in X and $\|e^{tA_\lambda}\| \leq 1$, we can extend $T(t)$ to a bounded linear operator on all of X by density.

Now we show that the family $(T(t))_{t\geq 0}$ defined above is a C_0-semigroup of contractions.

We verify the semigroup properties:

1. Identity: $T(0)x = \lim_{\lambda\to+\infty} e^{0\cdot\mathscr{A}_\lambda}x = x$.

2. Semigroup Property: For $x \in X$,

$$T(t+s)x = \lim_{\lambda\to+\infty} e^{(t+s)\mathscr{A}_\lambda}x = \lim_{\lambda\to+\infty} e^{t\mathscr{A}_\lambda}e^{s\mathscr{A}_\lambda}x = T(t)T(s)x.$$

3. Strong Continuity: For $x \in D(\mathscr{A})$, convergence is uniform on compact intervals of t, because of lemma 2.4.2. Thus $t \mapsto T(t)x$ is continuous. By the density of $D(A)$ and uniform boundedness $\|T(t)\| \leq 1$, continuity extends to all $x \in X$.

We must show that the generator of the constructed semigroup $T(t)$ is indeed $\mathscr{A}$. Let $\mathscr{B}$ be the generator of $T(t)$, we show that $\mathscr{A} = \mathscr{B}$.

For any $x \in D(\mathscr{A})$, we have the identity:

$$e^{t\mathscr{A}_\lambda}x - x = \int_0^t e^{s\mathscr{A}_\lambda}\mathscr{A}_\lambda x \, ds$$

As $\lambda \to +\infty$, $e^{s\mathscr{A}_\lambda} \to T(s)$ strongly and uniformly on compact sets, and $\mathscr{A}_\lambda x \to \mathscr{A} x$. Passing to the limit:

$$T(t)x - x = \int_0^t T(s)\mathscr{A} x \, ds$$

Dividing by t and taking $t \to 0^+$:

$$\lim_{t \to 0^+} \frac{T(t)x - x}{t} = \lim_{t \to 0^+} \frac{1}{t} \int_0^t T(s)\mathscr{A} x \, ds = T(0)\mathscr{A} x = \mathscr{A} x$$

Thus, $x \in D(\mathscr{B})$ and $\mathscr{B} x = \mathscr{A} x$, implying $\mathscr{A} \subset \mathscr{B}$. Since $\mathscr{B}$ is the generator of a C_0-semigroup of contractions, $1 \in \rho(\mathscr{B})$. By hypothesis, $1 \in \rho(\mathscr{A})$. Since $\mathscr{A} \subset \mathscr{B}$ and both $(I - \mathscr{A})$ and $(I - \mathscr{B})$ are surjective (mapping onto X), it follows that $\mathscr{A} = \mathscr{B}$. $\qquad\square$

Corollary 2.4.1. *Let $\mathscr{A}$ be the IG of a C_0-SG of contractions $(T(t))_{t\geq 0}$. Then for every $x \in X$, the semigroup is given by the limit:*

$$T(t)x = \lim_{\lambda \to +\infty} e^{t\mathscr{A}_\lambda} x$$

Proof. In the previous proof, we constructed a SG, let us call it $S(t)$, defined by $S(t)x = \lim_{\lambda \to +\infty} e^{t\mathscr{A}_\lambda} x$. We proved that the generator of $S(t)$ is exactly the operator $\mathscr{A}$.

Since $\mathscr{A}$ is the generator of the original semigroup $T(t)$ by hypothesis, and we know that a C_0-SG is uniquely determined by its generator (Uniqueness Theorem), it follows that:

$$T(t) = S(t), \quad \forall t \geq 0$$

$\qquad\square$

Corollary 2.4.2. *Let $\mathscr{A}$ be IG of a C_0-SG of contractions $(T(t))_{t\geq 0}$. Then the resolvent set $\rho(\mathscr{A})$ contains the open right half-plane:*

$$\rho(\mathscr{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$$

Furthermore, for all λ with $\Re(\lambda) > 0$, the following estimate holds:

$$\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\Re(\lambda)}$$

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. We define the operator $R(\lambda)$ on X by the Laplace transform of the semigroup:

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since $T(t)$ is a contraction semigroup ($\|T(t)\| \leq 1$) and $\Re(\lambda) > 0$, the integrand is exponentially bounded:

$$\left\| e^{-\lambda t} T(t)x \right\| = e^{-\Re(\lambda)t} \|T(t)x\| \leq e^{-\Re(\lambda)t} \|x\|$$

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Thus, the integral converges absolutely, defining a bounded linear operator. We calculate its norm:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\Re(\lambda)t} \|x\| \, dt = \|x\| \left[\frac{-e^{-\Re(\lambda)t}}{\Re(\lambda)} \right]_0^{+\infty} = \frac{1}{\Re(\lambda)} \|x\|$$

This proves the bound $\|R(\lambda)\| \leq \dfrac{1}{\Re(\lambda)}$.

It remains to show that this integral operator $R(\lambda)$ is indeed the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$. For any $x \in X$ and $h > 0$:

$$\begin{aligned}
\frac{T(h) - I}{h} R(\lambda)x &= \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t + h)x \, dt - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \\
&= \frac{e^{\lambda h}}{h} \int_h^{+\infty} e^{-\lambda s} T(s)x \, ds - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T(t)x \, dt
\end{aligned}$$

Taking the limit as $h \to 0^+$, the first term converges to $\lambda R(\lambda)x$ and the second term converges to $-x$. Thus, for any $x \in X$, $R(\lambda)x \in D(\mathscr{A})$ and $\mathscr{A} R(\lambda)x = \lambda R(\lambda)x - x$, which implies $(\lambda I - \mathscr{A})R(\lambda)x = x$. Similarly, one can show $R(\lambda)(\lambda I - \mathscr{A})x = x$ for $x \in D(\mathscr{A})$.

Therefore, $R(\lambda) = (\lambda I - \mathscr{A})^{-1}$. $\qquad\square$

Exercise 11

Let $X = BVC(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{C} \mid f \text{ is bounded and uniformly continuous}\}$. Equipped with the supremum norm $\|f\|_\infty = \sup_{t \geq 0} |f(t)|$, $(X, \|\cdot\|_\infty)$ is a Banach space.

For $t \geq 0$, define the operator $T(t)$ by the left shift:

$$(T(t)f)(s) = f(s + t), \quad \forall s \geq 0$$

Check the following:

1. $(T(t))_{t \geq 0}$ is a C_0-semigroup of contractions (i.e., $\|T(t)\| \leq 1$).

2. Show that $\|T(t)\| = 1$.

3. The infinitesimal generator is the differentiation operator $\mathscr{A} f = f'$, with an appropriate domain $D(\mathscr{A})$.

4. Verify that $\rho(\mathscr{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$.

2.4.2 The General Hille-Yosida Theorem

We now consider the general case where the semigroup is not necessarily a contraction.

Theorem 2.4.3. *A linear operator $\mathscr{A}$ generates a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq M e^{\omega t}$ if and only if:*

1. *A is closed and densely defined.*

2. $(\omega, +\infty) \subset \rho(\mathscr{A})$.

3. *For all $\lambda > \omega$ and $n \geq 1$:*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}.$$

2.4.3 Reduction to the Case $\omega = 0$

If $\|T(t)\| \leq Me^{\omega t}$, consider the rescaled semigroup $S(t) = e^{-\omega t} T(t)$. The generator of $S(t)$ is $\mathscr{A} - \omega I$, and $\|S(t)\| \leq M$. Conversely, if we prove the theorem for $\omega = 0$, the general case follows by applying the result to $\mathscr{A} - \omega I$.

Now we have the following corollary:

Corollary 2.4.3 (Hille-Yosida for $(1, \omega)$). *A linear operator $\mathscr{A}$ is the IG of a C_0-SG satisfying $\|T(t)\| \leq e^{\omega t}$ if and only if:*

1. *$\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = X$.*

2. *$\rho(\mathscr{A}) \supset (\omega, +\infty)$.*

3. *For all $\lambda > \omega$, the following estimate holds:*

$$\|R(\lambda, \mathscr{A})\| \leq \frac{1}{\lambda - \omega}.$$

Proof. ($\Rightarrow$): Suppose $\mathscr{A}$ generates a semigroup $T(t)$ such that $\|T(t)\| \leq e^{\omega t}$. Consider the rescaled family of operators $S(t) = e^{-\omega t} T(t)$. It is easy to verify that $S(t)$ is a C_0-SG. Furthermore, it is a contraction:

$$\|S(t)\| = e^{-\omega t} \|T(t)\| \leq e^{-\omega t} e^{\omega t} = 1$$

Let $\mathscr{B}$ be the generator of $S(t)$. By the definition of the generator:

$$\mathscr{B}x = \lim_{t \to 0^+} \frac{e^{-\omega t} T(t)x - x}{t} = \lim_{t \to 0^+} \left(e^{-\omega t} \frac{T(t)x - x}{t} + \frac{e^{-\omega t} - 1}{t} x \right) = \mathscr{A}x - \omega x$$

Thus, $\mathscr{B} = \mathscr{A} - \omega I$. Since $\mathscr{B}$ generates a contraction semigroup, by the Hille-Yosida Theorem for contractions (Case $M = 1, \omega = 0$), we know that for any $\mu > 0$, $\mu \in \rho(\mathscr{B})$ and $\|R(\mu, \mathscr{B})\| \leq \frac{1}{\mu}$.

Let $\lambda = \mu + \omega$. Then $\lambda > \omega$. Since $R(\mu, \mathscr{B}) = (\mu I - \mathscr{B})^{-1} = (\mu I - (\mathscr{A} - \omega I))^{-1} = ((\mu + \omega)I - \mathscr{A})^{-1}$, we have:

$$R(\lambda, \mathscr{A}) = R(\lambda - \omega, \mathscr{B})$$

Substituting the norm bound:

$$\|R(\lambda, \mathscr{A})\| = \|R(\lambda - \omega, \mathscr{B})\| \leq \frac{1}{\lambda - \omega}$$

($\Longleftarrow$): Conversely, suppose $\mathscr{A}$ satisfies conditions (1)-(3). Define $\mathscr{B} = \mathscr{A} - \omega I$. Clearly, $\mathscr{B}$ is closed and densely defined. For any $\mu > 0$, let $\lambda = \mu + \omega > \omega$. Then $\lambda \in \rho(\mathscr{A})$, which implies $\mu \in \rho(\mathscr{B})$. The resolvent satisfies:

$$\|R(\mu, \mathscr{B})\| = \|R(\mu + \omega, \mathscr{A})\| \leq \frac{1}{(\mu + \omega) - \omega} = \frac{1}{\mu}$$

Thus, $\mathscr{B}$ satisfies the Hille-Yosida conditions for the contraction case ($M = 1, \omega = 0$). Therefore, $\mathscr{B}$ generates a contraction semigroup $S(t)$ with $\|S(t)\| \leq 1$. Defining $T(t) = e^{\omega t} S(t)$, we see that $T(t)$ is a C_0-semigroup generated by $\mathscr{A} = \mathscr{B} + \omega I$, and it satisfies:

$$\|T(t)\| = e^{\omega t} \|S(t)\| \leq e^{\omega t}$$

$\square$

This corollary provides us with a method to rescale the bound. Thus, we focus on the case $(M, 0)$, i.e., $\|T(t)\| \leq M$. The condition on the resolvent becomes $\|R(\lambda, \mathscr{A})^n\| \leq \dfrac{M}{\lambda^n}$.

Lemma 2.4.3. *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup satisfying $\|T(t)\| \leq M$ for all $t \geq 0$. Let $\mathscr{A}$ be its infinitesimal generator. Then for all $\lambda > 0$ and all integers $n \geq 0$:*

$$\|R(\lambda, \mathscr{A})^n\| \leq \frac{M}{\lambda^n}$$

Equivalently, $\|\lambda^n R(\lambda, \mathscr{A})^n\| \leq M$.

Proof. For $\lambda > 0$, the resolvent is given by the Laplace transform of the semigroup:

$$R(\lambda, \mathscr{A})x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since the integral converges absolutely (due to the exponential decay $e^{-\lambda t}$ and bounded $T(t)$), we can differentiate this expression with respect to λ inside the integral sign. Differentiating $n - 1$ times:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathscr{A})x = \int_0^{+\infty} (-t)^{n-1} e^{-\lambda t} T(t)x \, dt$$

On the other hand, from the general theory of resolvents, we have the identity:

$$\frac{d^k}{d\lambda^k} R(\lambda, \mathscr{A}) = (-1)^k k! R(\lambda, \mathscr{A})^{k+1}$$

Setting $k = n - 1$, we get:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathscr{A}) = (-1)^{n-1} (n - 1)! R(\lambda, \mathscr{A})^n$$

Equating the two expressions for the derivative:

$$(-1)^{n-1}(n-1)! R(\lambda, \mathscr{A})^n x = \int_0^{+\infty} (-1)^{n-1} t^{n-1} e^{-\lambda t} T(t) x \, dt$$

Simplifying and solving for $R(\lambda, \mathscr{A})^n x$:

$$R(\lambda, \mathscr{A})^n x = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t) x \, dt$$

Now, we take the norm and use the bound $\|T(t)\| \leq M$:

$$\|R(\lambda, \mathscr{A})^n x\| \leq \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \|T(t) x\| \, dt$$

$$\leq \frac{M \|x\|}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt$$

The integral on the right is the Gamma function definition. Substituting $u = \lambda t$ ($dt = du/\lambda$):

$$\int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt = \frac{1}{\lambda^n} \int_0^{+\infty} u^{n-1} e^{-u} \, du = \frac{\Gamma(n)}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

Substituting this back into the inequality:

$$\|R(\lambda, \mathscr{A})^n x\| \leq \frac{M \|x\|}{(n-1)!} \cdot \frac{(n-1)!}{\lambda^n} = \frac{M}{\lambda^n} \|x\|$$

Thus, $\|R(\lambda, \mathscr{A})^n\| \leq \dfrac{M}{\lambda^n}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.4.4 Renorming Lemma

The idea is to construct an equivalent norm on X under which $\mathscr{A}$ becomes dissipative (generating a contraction semigroup), allowing us to apply the contraction case result.

Lemma 2.4.4. *Let $A : D(A) \subset X \to X$ be a linear operator with $\rho(A) \supset \mathbb{R}_+^*$ such that for all $\lambda > 0$ and $n \geq 0$, $\|\lambda^n R(\lambda, A)^n\| \leq M$. Then, there exists a norm $\|\cdot\|_\mu$ on X such that:*

1. *The norms are equivalent: $\|x\| \leq \|x\|_\mu \leq M \|x\|$ for all $x \in X$.*

2. *For all $\lambda > 0$, the operator $\lambda R(\lambda, A)$ is a contraction in the new norm: $\|\lambda R(\lambda, A) x\|_\mu \leq \|x\|_\mu$.*

Proof. Fix $\mu > 0$. We define the new norm $\|\cdot\|_\mu$ by:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\|$$

To show that the norms are equivalent, let $n = 0$, the term is $\|I x\| = \|x\|$ so $\|x\| \leq \|x\|_\mu$.

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Using the hypothesis $\|\mu^n R(\mu, A)^n\| \leq M$, we have:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\| \leq \sup_{n \geq 0} M \|x\| = M \|x\|$$

Thus, $\|x\| \leq \|x\|_\mu \leq M \|x\|$.

Now to show that $\mu R(\mu, A)$ is a contraction, we check the contraction property for the specific value μ:

$$\begin{aligned}
\|\mu R(\mu, A)x\|_\mu &= \sup_{n \geq 0} \|\mu^n R(\mu, A)^n (\mu R(\mu, A)x)\| \\
&= \sup_{n \geq 0} \|\mu^{n+1} R(\mu, A)^{n+1} x\| \\
&= \sup_{k \geq 1} \|\mu^k R(\mu, A)^k x\| \\
&\leq \sup_{k \geq 0} \|\mu^k R(\mu, A)^k x\| = \|x\|_\mu.
\end{aligned}$$

So, $\|\mu R(\mu, A)\|_\mu \leq 1$.

Finally, we show the contraction for $0 < \lambda \leq \mu$:

Let $x \in X$ and define $y = R(\lambda, \mathscr{A})x$. We want to show $\lambda \|y\|_\mu \leq \|x\|_\mu$. Recall the Resolvent Identity:

$$R(\lambda, \mathscr{A}) - R(\mu, \mathscr{A}) = (\mu - \lambda) R(\mu, \mathscr{A}) R(\lambda, \mathscr{A})$$

Applying this to x:

$$y = R(\mu, \mathscr{A})x + (\mu - \lambda) R(\mu, \mathscr{A})y = R(\mu, \mathscr{A})[x + (\mu - \lambda)y].$$

Taking the $\|\cdot\|_\mu$ norm and using the fact that $\|\mu R(\mu, A)z\|_\mu \leq \|z\|_\mu \implies \|R(\mu, A)z\|_\mu \leq \frac{1}{\mu} \|z\|_\mu$:

$$\begin{aligned}
\|y\|_\mu &= \|R(\mu, A)[x + (\mu - \lambda)y]\|_\mu \\
&\leq \frac{1}{\mu} \|x + (\mu - \lambda)y\|_\mu \\
&\leq \frac{1}{\mu} \left(\|x\|_\mu + (\mu - \lambda) \|y\|_\mu \right)
\end{aligned}$$

Multiplying by μ:

$$\mu \|y\|_\mu \leq \|x\|_\mu + (\mu - \lambda) \|y\|_\mu$$

Subtracting $(\mu - \lambda) \|y\|_\mu$ from both sides (since $\mu - \lambda \geq 0$):

$$\lambda \|y\|_\mu \leq \|x\|_\mu.$$

Thus, $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$ for $0 < \lambda \leq \mu$. Since this holds for any sufficiently large μ, and the definition of the norm can be adjusted, this property extends to all $\lambda > 0$. $\qquad\square$