1 Möbius Inversion Formula

Theorem 1. *Let f and g be two arithmetic functions satisfying for every integer $n \geq 1$:*

$$g(n) = \sum_{d|n} f(d)$$

Then, for every integer $n \geq 1$:

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = (\mu * g)(n)$$

Proof. Left as an exercise. $\square$

2 Dirichlet Series

Definition 1. *Let f be an arithmetic function and $s = \sigma + it \in \mathbb{C}$. The Dirichlet series associated to f is defined as:*

$$L(f, s) = \sum_{n \geq 1} \frac{f(n)}{n^s}$$

Lemma 1. *[Comparison of a Sum with an Integral] Let M, N be two real numbers $M < N$. Let $x_1, x_2, \ldots, x_r$ be real numbers with $M \leq x_1 < x_2 < \cdots < x_r \leq N$. Let $a(x_i)$ be a complex number for $i = 1, \ldots, r$ and put:*

$$A(t) = \sum_{x_i \leq t} a(x_i)$$

Let $g : [M, N] \to \mathbb{C}$ be a continuously differentiable function. Then:

$$\sum_{i=1}^{r} a(x_i) g(x_i) = A(N) g(N) - \int_M^N A(t) g'(t) dt$$

Proof. Put $A(x_0) = 0$ if $x_0 < M$.

$$\begin{aligned}
\sum_{j=1}^{r} a(x_j) g(x_j) &= \sum_{j=1}^{r} \left[A(x_j) - A(x_{j-1})\right] g(x_j) \\
&= \sum_{j=1}^{r} A(x_j) g(x_j) - \sum_{j=1}^{r} A(x_{j-1}) g(x_j) \\
&= \sum_{j=1}^{r} A(x_j) g(x_j) - \sum_{j=1}^{r-1} A(x_j) g(x_{j+1}) \\
&= A(x_r) g(x_r) - \sum_{j=1}^{r-1} A(x_j) \left[g(x_{j+1}) - g(x_j)\right]
\end{aligned}$$

Now since $A(t) = A(x_k)$ for $x_k \le t < x_{k+1}$, we have:

$$A(x_k)\left[g(x_{k+1}) - g(x_k)\right] = \int_{x_k}^{x_{k+1}} A(t)g'(t)dt$$

Hence:

$$\begin{aligned}\sum_{j=1}^{r} a(x_j)g(x_j) &= A(x_r)g(x_r) - \sum_{j=1}^{r-1} \int_{x_j}^{x_{j+1}} A(t)g'(t)dt \\ &= A(x_r)g(x_r) - \int_{x_1}^{x_r} A(t)g'(t)dt \\ &= A(N)g(x_r) - \int_{M}^{N} A(t)g'(t)dt + \int_{M}^{x_1} A(t)g'(t)dt + \int_{x_r}^{N} A(t)g'(t)dt\end{aligned}$$

- If $x_1 = M$ then $\int_{M}^{x_1} A(t)g'(t)dt = 0$
- If $x_1 > M$ then $A(t) = 0$, $M \le t < x$ hence $\int_{M}^{x_1} A(t)g'(t)dt = 0$
- If $x_r = N$ we get $\int_{x_r}^{N} A(t)g'(t)dt = 0$ and we get the result.
- If $x_r < N$ then $A(t) = A(x_r)$ for $t \ge x_r$ hence:

$$\begin{aligned}\sum_{j=1}^{r} a(x_j)g(x_j) &= A(N)g(x_r) - \int_{M}^{N} A(t)g'(t)dt + A(N)\int_{x_r}^{N} g'(t)dt \\ &= A(N)g(N) - \int_{M}^{N} A(t)g'(t)dt\end{aligned}$$

□

Theorem 2. *[Weierstrass's Convergence Theorem] Let $U \subseteq \mathbb{C}$ be a non-empty open set and $\{f_n\}$ a sequence of analytical functions $U \to \mathbb{C}$ converging pointwise to a function f on U. Assume that for every compact subset K of U, there is a constant C_K such that $|f_n(z)| < C_K$ for all $z \in K$ and $n \ge 1$. Then:*

1. *f is analytical on U.*
2. *$f_n^{(k)} \to f^{(k)}$ pointwise on U for all $k \ge 1$ and is analytical.*

Theorem 3. *Let f be an arithmetic function such that there exists a constant where $\sum_{n=1}^{N} f(n)$ is bounded for every $N \geq 1$. Then $L(s, f)$ converges for every $s \in \mathbb{C}$ with $Re(s) > 0$.*

More precisely, on $\{s \in \mathbb{C} | Re(s) > 0\}$, the function $L(f, s)$ is analytical and:

$$L^{(k)}(f, s) = (-1)^k \sum_{n=1}^{\infty} \frac{f(n)(\ln n)^k}{n^s}$$

Proof. On the half open plan $\Re(s) > 0$, we consider the partial sums:

$$\begin{aligned} L_N(s, f) &= \sum_{n=1}^{N} \frac{f(n)}{n^s} \qquad\qquad N = 1, 2, \ldots \\ &= \sum_{n=1}^{N} f(n) \exp(-s \ln n) \end{aligned}$$

Which is analytical and:

$$L^(k)_N(s, f) = (-1)^k \sum_{n=1}^{N} \frac{f(n)(\ln(n))^k}{n^s} \qquad k \geq 0$$

Let $s \in \mathbb{C}, \Re(s) > 0$, using lemma 1 with $x_j = j,\ 1 \leq j \leq N$ define:

$$F(t) = \sum_{1 \leq n \leq t} f(n), \qquad g(t) = t^{-s}$$

Then:

$$\begin{aligned} L_N(s, f) &= F(N)N^{-s} - \int_1^N F(t)(-s)t^{-s-1}dt \\ &= F(N)N^{-s} + s \int_1^N F(t)t^{-s-1}dt \end{aligned}$$

Now taking the norm we find that the bound does not depend on N hence $L_N(s, f)$ is uniformally bounded then by theorem 2 we are done. □

Corollary 1. *Let f be an arithmetic function and let s_0 be a complex number such that $\sum f(n)/n^{s_0}$ is converging. Then for $s \in \mathbb{C}$ with $Re(s) > Re(s_0)$, the function $L(s, f)$ converges and is analytical.*

Proof. Write $s = s' + s_0$ then if $\Re(s) > \Re(s_0)$ then $\Re(s') > 0$, we have:

$$\sum_{n \geq 1} \frac{f(n)}{n^{s'+s_0}} = \sum_{n \geq 1} \frac{f(n)}{n^{s'} n^{s_0}}$$

Then this series is bounded (why?). By applying theorem 3, we are done. □

Theorem 4. *There exists $\sigma(f)$ with $-\infty \leq \sigma(f) \leq \infty$ such that $L(s, f)$ converges for all $s \in \mathbb{C}$ with $Re(s) > \sigma(f)$. $\sigma(f)$ is called the abscissa of convergence. Similarly, $\sigma_a(f)$ is the abscissa of absolute convergence.*

Proof. Left as an exercise. $\square$

Definition 2. *The number $\sigma(|f|)$ is called the abcissa of absolute convergence of $L(s, f)$ and denoted by $\sigma_a(f)$.*

Theorem 5. *For any arithmetic function, we have:*

$$\sigma(f) \leq \sigma_a(f) \leq \sigma(f) + 1$$

Proof. Left as an exercise. $\square$

3 Euler Product of Dirichlet Series

Theorem 6. *Let f be a multiplicative arithmetic function. If the series $L(s, f)$ converges absolutely for a complex number s, then:*

$$L(f, s) = \prod_{p\ prime} \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right)$$

Moreover, if f is completely multiplicative, then:

$$L(f, s) = \prod_{p} \left(\frac{1}{1 - f(p)p^{-s}} \right)$$

Proof. Left as an exercise. $\square$

Example 1. *For $f(n) = \mathbb{1}(n)$, $L(s, \mathbb{1}) = \zeta(s) = \sum_{n \geq 1} 1/n^s$. Since $\mathbb{1}$ is completely multiplicative:*

$$\zeta(s) = \prod_{p} \left(\frac{1}{1 - p^{-s}} \right) \text{ for } Re(s) > 1$$

4 Exercises

Exercise 1. *Write the following Dirichlet series in terms of the Riemann Zeta function $\zeta(s)$:*

1. $\displaystyle\sum_{n \geq 1} \frac{\tau(n^2)}{n^s}$

2. $\displaystyle\sum_{n \geq 1} \frac{\omega(n)}{n^s}$

3. $\displaystyle\sum_{n \geq 1} \frac{2^{\omega(n)}}{n^s}$

4. $\displaystyle\sum_{n \geq 1} \frac{2^{\omega(n)}\lambda(n)}{n^s}$

5. $\displaystyle\sum_{n \geq 1} \frac{\kappa(n)}{n^s}$

6. $\sum_{n\geq 1} \frac{3^{\omega(n)}\kappa(n)}{n^s}$

7. $\sum_{n\geq 1} \frac{3^{\omega(n)}\kappa(n)\lambda(n)}{n^s}$

Where:

$$\kappa(n) = \begin{cases} 1 & \text{if } n = 1 \\ a_1 \cdots a_k & \text{if } n = p_1^{a_1} \cdots p_k^{a_k} \end{cases}$$

Exercise 2. *If $f(n)$ is a completely multiplicative arithmetic function prove that:*

$$\frac{L'(f,s)}{L(f,s)} = -\sum_{n\geq 1} \frac{f(n)\Lambda(n)}{n^s}$$