PARTIAL DIFFERENTIAL EQUATIONS

International Mathematics Masters
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CONTENTS

CHAPTER

ONE

THEORY OF (UNBOUNDED) OPERATORS

1.1 Preliminaries on Operators

Let $(X, \|\cdot\|)$ be a Banach space and $\mathscr{L}(X)$ be the Banach space of bounded linear operators.

Definition 1.1.1. *An operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ is called an unbounded linear operator (UBLO) if $D(\mathscr{A})$ is a subspace of X and* $\sup_{x\neq 0} \frac{\|\mathscr{A}x\|}{\|x\|} = +\infty$

Exercise 1: L

Let $\mathscr{A} : H^1 \longrightarrow L^2$, such that $f \mapsto f'$ and $D(\mathscr{A}) = \{f : \mathbb{R} \to \mathbb{R} : f, f' \in L^2\}$. Show $\mathscr{A}$ is an unbounded linear operator.

Notation: If $\mathscr{A}$ and $\mathscr{B}$ are unbounded linear operators, then $\mathscr{A} \supset \mathscr{B}$ if and only if $D(\mathscr{A}) \supset D(\mathscr{B})$ That is, for all $x \in D(\mathscr{B})$, $\mathscr{A}x = \mathscr{B}x$.

1.1.1 Resolvent Operator

Definition 1.1.2. *Let $\mathscr{A} : D(\mathscr{A}) \longrightarrow X$ be a UBLO.*

$$\rho(\mathscr{A}) = \textit{ Resolvent of } \mathscr{A} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} (\lambda I - \mathscr{A}) : D(\mathscr{A}) \to X \textit{ is bijective, and} \\ (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X) \end{array} \right\}$$

$$\sigma(\mathscr{A}) = \textit{ Spectrum of } \mathscr{A} = \mathbb{C} \setminus \rho(\mathscr{A}).$$

Definition 1.1.3. *$\mathscr{A}$ is closed if and only if the graph of $\mathscr{A}$, denoted as $G(\mathscr{A})$ is closed. Also $\mathscr{A}$ is closable if and only if there exists $\tilde{\mathscr{A}} \supset \mathscr{A}$ such that $G(\tilde{\mathscr{A}}) = \overline{G(\mathscr{A})}$.*

Note that $G(\mathscr{A}) = \big\{(x, \mathscr{A}x) \big| x \in D(\mathscr{A})\big\}$.

Exercise 2: 3

1. Prove that if it exists, $\tilde{\mathscr{A}}$ is unique, it then denoted by $\overline{\mathscr{A}}$ called closure of $\mathscr{A}$.
2. Let $\mathscr{A}_\ell = \frac{d}{dx}$ with $(X = C^0([a,b],\mathbb{R}), \|.\| = \sup|f(x)|)$ and $D(\mathscr{A}_\ell) = C^\ell([a,b],\mathbb{R})$. Prove $\overline{\mathscr{A}_\ell} = \mathscr{A}_1$.

Lemma 1.1.1. *If $\mathscr{A}$ an unbounded linear operator is closable, then $\rho(\overline{\mathscr{A}}) = \rho(\mathscr{A})$. If $\mathscr{A}$ is closed then $\rho(\mathscr{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(X)\}$.*
Hints (Exercise): If $\rho(\mathscr{A}) \neq 0$ implies $\mathscr{A}$ is closed. (Show first that if T is a UBLO with $T^{-1} \in \mathscr{L}(X)$ implies T is closed).

Corollary 1.1.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset X \longrightarrow X$ be a closed UBLO then $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \cup \sigma_c(\mathscr{A})$ where*

1. *$\sigma_p(\mathscr{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) \neq \{0\}\}$ (punctual spectrum and λ's are the eigenvalue).*
2. *$\sigma_c(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \ker(\lambda I - \mathscr{A}) = \{0\}, \overline{Rg(\lambda I - \mathscr{A})} \subset X\right\}$ (continuous spectrum).*
3. *$\sigma_r(\mathscr{A}) = \left\{\lambda \in \mathbb{C} : \lambda I - \mathscr{A} \text{ is injective}, \overline{Rg(\lambda I - \mathscr{A})} = X,\ Rg(\lambda I - \mathscr{A}) \neq X\right\}$ (residual spectrum).*

Exercise 3: L

Let

$$X = \ell^2(\mathbb{C}) = \left\{(x_n)_{n\geq 0} : \sum_n |x_n|^2 < \infty\right\},$$

with $(\mathscr{A}x_n)_{n\geq 1} = \left(\frac{x_n}{1+n}\right)_{n\geq 0}$. Prove that $\mathscr{A}$ is a BLO, injective, $\overline{\mathrm{Rg}(\mathscr{A})} = X$ and $\mathrm{Rg}(\mathscr{A}) \subset X$.

Theorem 1.1.1. *If $\mathscr{A}$ is a closed UBLO then $\rho(\mathscr{A})$ is open. If $\mu \in \rho(\mathscr{A})$, then for all $\lambda \in \mathbb{C}$ with $r := |\mu - \lambda|, \|(\mu I - \mathscr{A})^{-1}\| < 1$ then $\lambda \in \rho(\mathscr{A})$ and*

$$(\lambda I - \mathscr{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathscr{A})^{-(n+1)}$$

To do Question: do you need $\mathscr{A}$ closed?

Theorem 1.1.2 (Resolvent Identity)**.** *Let $\mathscr{A}$ be a UBLO. For $\lambda \in \rho(A)$, define the resolvent operator*

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all $\lambda, \mu \in \rho(A)$,

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

Corollary 1.1.2. *The mapping $\lambda \mapsto R(\lambda)$ from $\rho(\mathscr{A})$ into $\mathscr{L}(X)$ is analytic. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

1.1.2 Dual Operators

Let $X \cong X^*$ and $\mathscr{A}$ a closed UBLO with $\overline{D(\mathscr{A})} = X$ a dense UBLO.
If X and Y are Banach spaces with duals X^* and Y^*, then for $x \in X$ and $x^* \in X^*$, we define the duality product as $\langle x^*, x\rangle$.

Definition 1.1.4 (Dual Operator of $\mathscr{A}$). *Let $\mathscr{A} : D(\mathscr{A}) \subset X \to Y$ (UBLO), be such that $\overline{D(\mathscr{A})} = X$. The* dual operator *$\mathscr{A}^* : D(\mathscr{A}^*) \subset Y^* \to X^*$ is a UBLO defined as follows:*

$$D(\mathscr{A}^*) := \{y^* \in Y^* \,|\, \exists z^* \in X^* \text{ such that } \langle y^*, Ax\rangle = \langle z^*, x^*\rangle \ \forall x \in D(\mathscr{A})\}\,.$$

and $y^* \in D(\mathscr{A}^*)$, the element z^* is unique and we define $A^* y^* := z^*$.

Lemma 1.1.2. *Let X, Y be Banach spaces and let $\mathscr{A} \in \mathscr{L}(X, Y)$. Then $\mathscr{A}^* \in \mathscr{L}(Y^*, X^*)$ and*

$$\|\mathscr{A}^*\|_{\mathscr{L}(Y^*,X^*)} = \|\mathscr{A}\|_{\mathscr{L}(X,Y)}.$$

Lemma 1.1.3. *Let X be a reflexive Banach space with $X = X^*$ and let $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ be a closedly dense UBLO. Then $\overline{D(\mathscr{A}^*)} = X^*(\cong X)$, and $\mathscr{A}^*$ is closed.*

Theorem 1.1.3. *Let $\mathscr{A}$ be a closedly dense UBLO. Then $\rho(\mathscr{A}) = \rho(\mathscr{A}^*)$ and for all $\lambda \in \rho(A)$,*

$$\left[(\lambda I - \mathscr{A})^{-1}\right]^* = (\lambda I - \mathscr{A}^*).$$

Exercise 4: l

1. Let $\mathscr{A} = \frac{d}{dx}$ on $X = L^2(\mathbb{R})$ and $D(\mathscr{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$. Show the following:

   a. $\rho(\mathscr{A}) = \mathbb{C} \setminus i\mathbb{R}$ which implies $\sigma(\mathscr{A}) = i\mathbb{R}$.

   b. $\mathscr{A}$ is a closed unbounded linear operator.

   c. If $\lambda \in \rho(\mathscr{A})$ then $(\lambda I - \lambda)^{-1} : X \longrightarrow D(\mathscr{A})$ is bounded.

   For $\Re(\lambda) \neq 0$; show for all $g \in X$, there exists uniquely $f \in D(\mathscr{A})$ such that $(\lambda I - \mathscr{A})f = g$.
   For $\Re(\lambda) = 0$; show for all $f_n \in X$ with $\|f_n\|_{\ell^2} = 1$ then $(i\omega I - \mathscr{A})f_n \to 0$.

2. Do same for $\mathscr{A} = -i\frac{d}{dx}$.