

# PARTIAL DIFFERENTIAL EQUATIONS

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## THEORY OF (UNBOUNDED) OPERATORS

## 1.1 Preliminaries on Operators

Let  $(X, \|\cdot\|)$  be a Banach space and  $\mathcal{L}(X)$  be the Banach space of bounded linear operators.

**Definition 1.1.1.** An operator  $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$  is called an unbounded linear operator (UBLO) if  $D(\mathcal{A})$  is a subspace of  $X$  and  $\sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|} = +\infty$

**Exercise 1**

Let  $\mathcal{A} : H^1 \longrightarrow L^2$ , such that  $f \mapsto f'$  and  $D(\mathcal{A}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f, f' \in L^2\}$ . Show  $\mathcal{A}$  is an unbounded linear operator.

**Notation:** If  $\mathcal{A}$  and  $\mathcal{B}$  are unbounded linear operators, then  $\mathcal{A} \supset \mathcal{B}$  if and only if  $D(\mathcal{A}) \supset D(\mathcal{B})$  and for all  $x \in D(\mathcal{B})$ ,  $\mathcal{A}x = \mathcal{B}x$ .

## 1.1.1 Resolvent Operator

**Definition 1.1.2.** Let  $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$  be a UBLO.

$$\rho(\mathcal{A}) = \text{Resolvent of } \mathcal{A} = \left\{ \lambda \in \mathbb{C} \left| \begin{array}{l} (\lambda I - \mathcal{A}) : D(\mathcal{A}) \rightarrow X \text{ is bijective, and} \\ (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X) \end{array} \right. \right\}$$

$$\sigma(\mathcal{A}) = \text{Spectrum of } \mathcal{A} = \mathbb{C} \setminus \rho(\mathcal{A}).$$

**Definition 1.1.3.**  $\mathcal{A}$  is closed if and only if the graph of  $\mathcal{A}$ , denoted as  $G(\mathcal{A})$  is closed. Also  $\mathcal{A}$  is closable if and only if there exists  $\tilde{\mathcal{A}} \supset \mathcal{A}$  such that  $G(\tilde{\mathcal{A}}) = \overline{G(\mathcal{A})}$ .

Note that  $G(\mathcal{A}) = \{(x, \mathcal{A}x) \mid x \in D(\mathcal{A})\}$ .

## Exercise 2

1. Prove that if it exists,  $\tilde{\mathcal{A}}$  is unique, it then denoted by  $\overline{\mathcal{A}}$  called closure of  $\mathcal{A}$ .
2. Let  $\mathcal{A}_\ell = \frac{d}{dx}$  with  $(X = C^0([a, b], \mathbb{R}), \|\cdot\|_\infty = \sup |f(x)|$  and  $D(\mathcal{A}_\ell) = C^\ell([a, b], \mathbb{R})$ .  
Prove  $\overline{\mathcal{A}_\ell} = \mathcal{A}_1$ .

**Lemma 1.1.1.** If  $\mathcal{A}$  an unbounded linear operator is closable, then  $\rho(\overline{\mathcal{A}}) = \rho(\mathcal{A})$ . If  $\mathcal{A}$  is closed then  $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$ .

*Hints (Exercise): If  $\rho(\mathcal{A}) \neq \emptyset$  implies  $\mathcal{A}$  is closed. (Show first that if  $T$  is a UBLO with  $T^{-1} \in \mathcal{L}(X)$  implies  $T$  is closed).*

**Corollary 1.1.1.** Let  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  be a closed UBLO then  $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$  where

1.  $\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) \neq \{0\}\}$  (punctual spectrum and  $\lambda$ 's are the eigenvalue).
2.  $\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - \mathcal{A}) = \{0\}, \overline{\text{Rg}(\lambda I - \mathcal{A})} \subsetneq X \right\}$  (continuous spectrum).
3.  $\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is injective, } \overline{\text{Rg}(\lambda I - \mathcal{A})} = X, \text{Rg}(\lambda I - \mathcal{A}) \neq X \right\}$  (residual spectrum).

## Exercise 3

Let

$$X = \ell^2(\mathbb{C}) = \left\{ (x_n)_{n \geq 0} : \sum_n |x_n|^2 < \infty \right\},$$

with  $(\mathcal{A} x_n)_{n \geq 0} = \left( \frac{x_n}{1+n} \right)_{n \geq 0}$ . Prove that  $\mathcal{A}$  is a BLO, injective,  $\overline{\text{Rg}(\mathcal{A})} = X$  and  $\text{Rg}(\mathcal{A}) \subsetneq X$ .

**Theorem 1.1.1.** If  $\mathcal{A}$  is a closed UBLO then  $\rho(\mathcal{A})$  is open. If  $\mu \in \rho(\mathcal{A})$ , then for all  $\lambda \in \mathbb{C}$  with  $r := |\mu - \lambda|$ ,  $\|(\mu I - \mathcal{A})^{-1}\| < 1$  then  $\lambda \in \rho(\mathcal{A})$  and

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu I - \mathcal{A})^{-(n+1)}$$

*To do Question: do you need  $\mathcal{A}$  closed?*

**Theorem 1.1.2** (Resolvent Identity). Let  $\mathcal{A}$  be a UBLO. For  $\lambda \in \rho(A)$ , define the resolvent operator

$$R(\lambda) := (\lambda I - A)^{-1}.$$

Then for all  $\lambda, \mu \in \rho(A)$ ,

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\mu) R(\lambda).$$

**Corollary 1.1.2.** The mapping  $\lambda \mapsto R(\lambda)$  from  $\rho(\mathcal{A})$  into  $\mathcal{L}(X)$  is analytic. Moreover,

$$\frac{d^n}{d\lambda^n} (\lambda I - A)^{-1} = (-1)^n n! [(\lambda I - A)^{-1}]^{(n+1)}.$$

### 1.1.2 Dual Operators

Let  $X \cong X^*$  and  $\mathcal{A}$  a closed UBLO with  $\overline{D(\mathcal{A})} = X$  a dense UBLO.

If  $X$  and  $Y$  are Banach spaces with duals  $X^*$  and  $Y^*$ , then for  $x \in X$  and  $x^* \in X^*$ , we define the duality product as  $\langle x^*, x \rangle$ .

**Definition 1.1.4** (Dual Operator of  $\mathcal{A}$ ). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow Y$  (UBLO), be such that  $\overline{D(\mathcal{A})} = X$ . The dual operator  $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y^* \rightarrow X^*$  is a UBLO defined as follows:*

$$D(\mathcal{A}^*) := \{y^* \in Y^* \mid \exists z^* \in X^* \text{ such that } \langle y^*, Ax \rangle = \langle z^*, x \rangle \forall x \in D(\mathcal{A})\}.$$

and  $y^* \in D(\mathcal{A}^*)$ , the element  $z^*$  is unique and we define  $\mathcal{A}^* y^* := z^*$ .

**Lemma 1.1.2.** *Let  $X, Y$  be Banach spaces and let  $\mathcal{A} \in \mathcal{L}(X, Y)$ . Then  $\mathcal{A}^* \in \mathcal{L}(Y^*, X^*)$  and*

$$\|\mathcal{A}^*\|_{\mathcal{L}(Y^*, X^*)} = \|\mathcal{A}\|_{\mathcal{L}(X, Y)}.$$

**Lemma 1.1.3.** *Let  $X$  be a reflexive Banach space with  $X = X^*$  and let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a closedly dense UBLO. Then  $\overline{D(\mathcal{A}^*)} = X^*(\cong X)$ , and  $\mathcal{A}^*$  is closed.*

**Theorem 1.1.3.** *Let  $\mathcal{A}$  be a closedly dense UBLO. Then  $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$  and for all  $\lambda \in \rho(A)$ ,*

$$[(\lambda I - \mathcal{A})^{-1}]^* = (\lambda I - \mathcal{A}^*)^{-1}.$$

#### Exercise 4

1. Let  $\mathcal{A} = \frac{d}{dx}$  on  $X = L^2(\mathbb{R})$  and  $D(\mathcal{A}) = \{f \in X : f' \in L^2(\mathbb{R})\}$ . Show the following:
  - a.  $\rho(\mathcal{A}) = \mathbb{C} \setminus i\mathbb{R}$  which implies  $\sigma(\mathcal{A}) = i\mathbb{R}$ .
  - b.  $\mathcal{A}$  is a closed unbounded linear operator.
  - c. If  $\lambda \in \rho(\mathcal{A})$  then  $(\lambda I - \mathcal{A})^{-1} : X \rightarrow D(\mathcal{A})$  is bounded.

For  $\Re(\lambda) \neq 0$ ; show for all  $g \in X$ , there exists uniquely  $f \in D(\mathcal{A})$  such that  $(\lambda I - \mathcal{A})f = g$ .

For  $\Re(\lambda) = 0$ ; show for all  $f_n \in X$  with  $\|f_n\|_{\ell^2} = 1$  then  $(i\omega I - \mathcal{A})f_n \rightarrow 0$ .

2. Do same for  $\mathcal{A} = -i \frac{d}{dx}$ .

## 1.2 Compact Operators

Let  $X$  and  $Y$  be Banach spaces on  $\mathbb{K}$ .

**Definition 1.2.1.** *Let  $K : X \rightarrow Y$  be a BLO (in  $\mathcal{L}(X, Y)$ ), then  $K$  compact iff  $K(B_1^X(0))$  is relatively compact in  $Y$  (i.e.  $\overline{K(B_1^X(0))}$  compact).*

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) \mid K \text{ is compact}\}.$$

### Exercise 5

Let  $X = C([a, b], \mathbb{C})$  and  $k \in C^0([a, b] \times [c, d], \mathbb{C})$

Define  $K \in \mathcal{L}(X)$  by

$$(Kx)(t) = \int_a^b k(t, s)x(s) ds.$$

Show  $K \in \mathcal{K}(X)$ .

**Theorem 1.2.1.**  $\mathcal{K}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ .

*Proof.* We shall show this in two steps

1.  $\mathcal{K}(X, Y)$  is a vector space (Do it).
2. Closed:  $K_n \rightarrow K$  (Prof. Yacine said he would send a different proof).

□

### Exercise 6

Let  $X = \ell^2(\mathbb{C})$ ,

$$\mathcal{A}((x_n)_{n \geq 0}) = \left( \frac{x_n}{n+1} \right)_{n \geq 0}.$$

Show that  $\mathcal{A}$  is compact.

**Theorem 1.2.2.** Let  $X, Y$  and  $Z$  be Banach spaces on  $\mathbb{K}$ .

$$X \xrightarrow{\mathcal{A}} Y \xrightarrow{\mathcal{B}} Z, \quad \mathcal{A} \in \mathcal{L}(X, Y), \quad \mathcal{B} \in \mathcal{L}(Y, Z).$$

1. If  $\mathcal{A}$  is compact or  $\mathcal{B}$  is compact, then  $\mathcal{B}\mathcal{A}$  is compact.
2. If  $\mathcal{A}$  is compact then  $\mathcal{A}^* \in \mathcal{K}(Y^*, X^*)$ .
3. If  $\mathcal{A}$  is compact and  $\text{Rg}(\mathcal{A})$  is closed (in  $Y$ ), then it is finite dimensional.

To proceed with further results on compact operators, we need the following lemma

**Lemma 1.2.1** (Riesz Lemma). Let  $E$  be a normed vector space,  $F = \overline{F} \subset E$ . Then  $\forall r \in (0, 1)$ ,  $\exists x_r \in E$ , such that

$$\|x_r\| = 1, \quad d(x_r, F) \geq r.$$

*Proof.* Since  $F \neq E$  then this implies  $\exists z \in E \setminus F$ . Let  $d = d(z, F) > 0$ .

For  $0 < r < 1$ ,  $\exists y_r \in F$  s.t.

$$0 < d \leq \|z - y_r\| < \frac{d}{r}.$$

Normalize:

$$x_r = \frac{z - y_r}{\|z - y_r\|}, \quad \|x_r\| = 1.$$

For all  $y \in F$ ,

$$\|x_r - y\| = \frac{1}{\|z - y_r\|} \|z - (y_r + \|z - y_r\|y)\| \geq \frac{d}{\|z - y_r\|} > r.$$

□

**Proposition 1.2.1.** *Let  $\mathcal{A} \in \mathcal{K}(X)$ , such that  $X$  is a Banach space on  $\mathbb{C}$ . If  $\lambda \in \mathbb{C}^*$ , then  $\ker((\lambda I - \mathcal{A})^n)$  has finite dimension.*

*Proof.* Only for  $n = 1$ . (do it for  $n \geq 2$ ). Now, let

$$\tilde{K} := \ker(\lambda I - \mathcal{A}) = \{x \in X : \mathcal{A}x = \lambda x\} = \left\{x \in X : x = \frac{1}{\lambda} \mathcal{A}x\right\} \subset \text{Rg}(\mathcal{A}).$$

So  $\tilde{K}$  is closed in  $\text{Rg}(\mathcal{A})$ . Suppose  $\dim \tilde{K} = +\infty$ . By Riesz lemma,  $\exists (x_n)$  in  $\tilde{K}$ , such that

$$\|x_n\| = 1, \quad \|x_n - x_m\| \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{|\lambda|} \|\mathcal{A}x_n - \mathcal{A}x_m\| \geq \frac{1}{2}, \quad \forall n \neq m$$

and so we have  $\|\mathcal{A}x_n\| \leq \|\mathcal{A}\|$ . So  $(\mathcal{A}x_n)$  is not Cauchy, hence a contradiction.

□

### Exercise 7

Let  $X$  be a Banach space on  $\mathbb{K}$ . If  $\mathcal{A} \in \mathcal{L}(X)$ , assume  $\exists n_0$  s.t.  $\ker(\mathcal{A}^{n_0}) = \ker(\mathcal{A}^{n_0+1})$ . Then  $\forall n \geq n_0$ ,

$$\ker(\mathcal{A}^n) = \ker(\mathcal{A}^{n_0}).$$

**Proposition 1.2.2.** *Let  $\mathcal{A} \in \mathcal{K}(X)$  and  $X$  be a Banach space on  $\mathbb{C}$ ,  $\lambda \neq 0$ . Then  $\exists n_0$  such that*

$$\forall n \geq n_0, \quad \ker((\lambda I - \mathcal{A})^n) = \ker((\lambda I - \mathcal{A})^{n_0}).$$

*Proof.* Using the previous exercise and arguing by contradiction, that for all  $n \geq 1$   $\ker((\lambda I - \mathcal{A})^n) \subset \ker((\lambda I - \mathcal{A})^{n+1})$  and each of them is closed.

**RL:** with  $r = \frac{1}{2}$  with  $(x_n)_{n \geq 1} \in X$ , such that  $\|x_n\| = 1$ . Then,  $x_n \in \ker((\lambda I - \mathcal{A})^{n+1})$ . Thus,

$$d(x_n, \ker((\lambda I - \mathcal{A})^n)) \geq \frac{1}{2}.$$

For  $n = 1$ ,  $x \in \ker(\lambda I - \mathcal{A}) \Rightarrow x = \frac{\mathcal{A}}{\lambda}x$ . For all  $1 \leq m < n$ ,

$$\begin{aligned} \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} &= x_n - x_m + \frac{\mathcal{A}x_n}{\lambda} - \left(x_m - x_m - \frac{\mathcal{A}x_m}{\lambda}\right) \\ &= x_n - \left[\frac{(\lambda I - \mathcal{A})x_n}{\lambda} + x_m - \frac{(\lambda I - \mathcal{A})x_m}{\lambda}\right]. \end{aligned}$$

So,

$$\left\| \frac{\mathcal{A}x_n}{\lambda} - \frac{\mathcal{A}x_m}{\lambda} \right\| \geq d(x_n, \ker(\lambda I - \mathcal{A})^n) \geq \frac{1}{2}.$$

which is a contradiction. □

Notice that if  $\ker(\lambda I - \mathcal{A}) \neq \{0\}$ , then  $\lambda \in \sigma_p(\mathcal{A})$ . Notice,

$$\dim \ker(\lambda I - \mathcal{A}) = \text{geometric multiplicity}.$$

With Proposition 1.2.2  $\Rightarrow \exists n_0$  (smallest one) such that

$$\ker((\lambda I - \mathcal{A})^{n_0}) = \ker((\lambda I - \mathcal{A})^n), \quad \forall n \geq n_0.$$

Note that,

$$\ker((\lambda I - \mathcal{A})^{n_0}) := \text{generalized eigenspace}.$$

$$\dim \ker((\lambda I - \mathcal{A})^{n_0}) := \text{algebraic multiplicity of } \lambda.$$

**Proposition 1.2.3** (Fredholm alternative). *Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ .*

$$\text{Rg}(\lambda I - \mathcal{A}) = X \iff \ker(\lambda I - \mathcal{A}) = \{0\}.$$

**Proposition 1.2.4.** *Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ ,  $\dim X = \infty$ . If  $\lambda_n \rightarrow \lambda$ ,  $\lambda_n \in \sigma(\mathcal{A}) \setminus \{0\}$ , pairwise distinct, then  $\lambda = 0$ . Hence every  $\lambda \in \sigma(\mathcal{A}) \setminus \{0\}$  is isolated.*

*Proof.* Let  $\lambda_n \in \sigma_p(\mathcal{A})$ ,  $\exists \|x_n\| = 1$  such that  $\mathcal{A}x_n = \lambda_n x_n$ . Let

$$X_n = \text{span}(x_1, \dots, x_n), \quad X_n \subset X_{n+1}.$$

Let us prove that  $\dim X_n = n$ .

By induction:  $n = 1$  is OK.

$$\dim X_n = n \Rightarrow \dim X_{n+1} = n + 1.$$

By contradiction,  $x_{n+1} \in X_n$ .

$$x_{n+1} = \sum_{i=1}^n \alpha_i x_i, \text{ which implies } \lambda_{n+1} x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_{n+1} x_i.$$

Thus,

$$\mathcal{A}x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_i x_i.$$

Hence,

$$0 = \sum_{i=1}^n \alpha_i (\lambda_{n+1} - \lambda_i) x_i.$$



Since  $(x_i)$  are linearly independent,

$$\alpha_i(\lambda_{n+1} - \lambda_i) = 0, \quad 1 \leq i \leq n.$$

which implies  $\Rightarrow \alpha_i = 0. \Rightarrow x_{n+1} = 0$ , (Impossible).

Notice:

$$(\lambda_n I - \mathcal{A})X_n \subset X_{n-1}, \quad \forall n \geq 2.$$

**Recall:**

$$\|y_n\| = 1, \quad y_n \in X_n,$$

$$d(y_n, X_{n-1}) \geq \frac{1}{2}.$$

For  $2 \leq m < n$ ,

$$\begin{aligned} \left\| \frac{\mathcal{A} y_n}{\lambda_n} - \frac{\mathcal{A} y_m}{\lambda_m} \right\| &= \left\| y_n - \left[ \frac{\lambda_n I - \mathcal{A}}{\lambda_n} y_n + y_m + \frac{\lambda_m I - \mathcal{A}}{\lambda_m} y_m \right] \right\| \\ &\geq d(y_n, X_{n-1}) \geq \frac{1}{2}. \end{aligned}$$

Assume that

$$\lambda_n \rightarrow \lambda \quad (n \rightarrow \infty).$$

Suppose  $\lambda \neq 0$ , then

$$\left| \frac{1}{\lambda_n} \right| \leq C_0 \quad \text{for } n \text{ large enough.}$$

Then

$$\left( \frac{\mathcal{A} y_n}{\lambda_n} \right)_{n \geq 1}$$

is a bounded sequence.

Then we have built a sequence in  $\mathcal{A}(B_M^X(0))$ ,  $M > 0$  which does not admit a convergent subsequence. Which is a Contradiction.  $\square$

**Theorem 1.2.3.** *Let  $\mathcal{A} \in \mathcal{K}(X)$ , and let  $X$  be a Banach space on  $\mathbb{C}$ . Then  $\sigma(\mathcal{A}) \setminus \{0\}$  is made of eigenvalues, contains a countable number of points and the set of accumulation points contained in  $\{0\}$ .*

## Main use of compact operators (in PDEs)

They appear as “inverse” of UBLO.

**Definition 1.2.2.** *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  UBLO, closed,  $\rho(\mathcal{A}) \neq \{0\}$ .  $\mathcal{A}$  is said to have compact resolvent if*

$$(\lambda I - \mathcal{A})^{-1} \in \mathcal{K}(X), \quad \forall \lambda \in \rho(\mathcal{A}).$$

**Main Example:**  $\mathcal{A} = -\Delta$  on  $\Omega$  with  $\mathcal{A}u = -u_{xx}$ .

### 1.3 Adjoints, Symmetric and Self-adjoint Operators

Let  $\mathcal{H}$  be a Hilbert space, with inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}.$$

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ UBLO, } \overline{D(\mathcal{A})} = \mathcal{H}.$$

**Definition 1.3.1** (Adjoint Operator  $\mathcal{A}^\circ$ ).

$$D(\mathcal{A}^\circ) = \{x \in \mathcal{H} : v \mapsto \langle \mathcal{A}v, x \rangle_{\mathcal{H}} : D(\mathcal{A}) \rightarrow \mathbb{C} \text{ bdd operator}\}.$$

If  $x \in D(\mathcal{A}^\circ)$ , then there exists uniquely  $z \in \mathcal{H}$  such that  $\langle v, z \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$  for all  $v \in D(\mathcal{A})$ .

Observe, with Riesz representation and the fact that  $\overline{D(\mathcal{A})} = \mathcal{H}$ , we have that  $z := \mathcal{A}^\circ x$  and  $\langle v, \mathcal{A}^\circ x \rangle_{\mathcal{H}} = \langle \mathcal{A}v, x \rangle_{\mathcal{H}}$  for all  $v \in D(\mathcal{A})$ .

**Remark 1.3.1.** Let  $\mathcal{H}$  be a Hilbert space,  $E : \mathcal{H} \rightarrow \mathcal{H}^*$ ,  $x \mapsto \langle x, \cdot \rangle$ . Linear isometry between  $\mathcal{H}$  and  $\mathcal{H}^*$ . (One can identify  $\mathcal{H}$  and  $\mathcal{H}^*$ ). Now, we define the Dual operator as the following:

$$\mathcal{A}^* : D(\mathcal{A}^*) \subset \mathcal{H}^* \rightarrow \mathcal{H}, \quad \mathcal{A}^\circ = E^{-1} \mathcal{A}^* E.$$

**Definition 1.3.2** (Symmetric and Self-adjoint Operator). Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  be a UBLO, then

1. **Symmetric:** If  $\overline{D(\mathcal{A})} = \mathcal{H}$  and  $\mathcal{A}^\circ \supset \mathcal{A}$  with  $D(\mathcal{A}^\circ) \supset D(\mathcal{A})$  and for all  $x, y \in D(\mathcal{A})$ ,  $\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle$ .
2. **Self-adjoint:** If  $\overline{D(\mathcal{A})} = \mathcal{H}$  and  $\mathcal{A}^\circ = \mathcal{A}$ .

#### Exercise 8

1. Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ ,  $\overline{D(\mathcal{A})} = \mathcal{H}$ . If  $\mathcal{A}$  is closed, then  $\overline{D(\mathcal{A}^\circ)} = \mathcal{H}$ .
2. Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ ,  $\overline{D(\mathcal{A})} = \mathcal{H}$ . Suppose  $\mathcal{A}$  is symmetric and if  $\lambda \in \sigma_p(\mathcal{A})$ , then prove that  $\lambda \in \mathbb{R}$  and

$$\inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle \leq \lambda \leq \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle.$$

**Proposition 1.3.1.** Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be an Hilbert space over  $\mathbb{C}$ , If  $\mathcal{A}$  is self-adjoint, injective and  $\overline{D(\mathcal{A})} = \mathcal{H}$ . Then  $\mathcal{A}^{-1} : \text{Rg}(\mathcal{A}) \rightarrow \mathcal{H}$  is self-adjoint.

*Proof.* Since  $\mathcal{A}$  is injective then  $\mathcal{A}^{-1}$  is well defined, and since  $\mathcal{A} = \mathcal{A}^\circ$  then  $\mathcal{A}$  is closed.

Now assume  $(x_n) \subset D(\mathcal{A})$  such that  $x_n \rightarrow x \in D(\mathcal{A})$  (because  $\mathcal{A}$  is closed) and  $\mathcal{A}x_n \rightarrow y$  then for all  $z \in D(\mathcal{A})$ ,  $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}^\circ z \rangle$  which implies  $\langle y, z \rangle = \langle x, \mathcal{A}^\circ z \rangle$  and so we have

$x \in D(\mathcal{A}^\circ) = D(\mathcal{A})$  and  $y = \mathcal{A}^\circ x$ . Notice  $\overline{\text{Rg}(\mathcal{A})} = \ker(\mathcal{A})^\perp$  (because of self-adjointness). Injectivity implies  $\overline{\text{Rg}(\mathcal{A})} = \mathcal{H}$  which implies  $\overline{D(\mathcal{A}^{-1})} = \mathcal{H}$ . So  $\mathcal{A}^{-1}$  is densely defined. Now, observe for all  $u, v \in D(\mathcal{A}^{-1})$ ,  $u = \mathcal{A}^\circ x$  and  $v = \mathcal{A} y$  with  $x, y \in D(\mathcal{A})$ . Hence,

$$\langle \mathcal{A}^{-1}u, v \rangle = \langle x, \mathcal{A} y \rangle = \langle \mathcal{A} x, y \rangle = \langle x, \mathcal{A}^{-1}y \rangle.$$

To this end,  $(\mathcal{A}^{-1})^\circ \subset \mathcal{A}^{-1}$ ,  $\forall z \in D((\mathcal{A}^{-1})^\circ) \exists w, \forall u \in D(\mathcal{A}^{-1}) = R(\mathcal{A})$  (i.e.  $u = \mathcal{A}x$ )

$$\langle \mathcal{A}^{-1}u, z \rangle = \langle u, w \rangle \Rightarrow \forall x \in D(\mathcal{A}) \quad \langle x, z \rangle = \langle \mathcal{A}x, w \rangle$$

By definition  $w \in D(\mathcal{A}^\circ)$  and  $\mathcal{A}^\circ w = z$ .  $\mathcal{A}w = z \Rightarrow z \in R(\mathcal{A}) = D(\mathcal{A}^{-1})$ . □

**Theorem 1.3.1.** *Let  $\mathcal{H}$  be a Hilbert space, suppose  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is symmetric and surjective then  $\mathcal{A}$  is self-adjoint.*

*Proof.*  $\mathcal{A}$  and  $\mathcal{A}^\circ$  are injective. Do it only for  $\mathcal{A}$ , let  $x \in D(\mathcal{A})$  and  $\mathcal{A}x = 0$ .

$$\forall y \in D(\mathcal{A}), \quad 0 = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle \Rightarrow x \perp \text{Rg}(\mathcal{A}) = \mathcal{H}.$$

which implies  $x = 0$ .

Next, we show  $\mathcal{A}$  closed.

$$(x_n)_{n \geq 1} \subset D(\mathcal{A}), \quad x_n \rightarrow x \text{ in } \mathcal{H}, \quad \mathcal{A}x_n \rightarrow y \text{ in } \mathcal{H}$$

We shall show  $y = \mathcal{A}x$ . Now,  $\forall z \in D(\mathcal{A})$  then  $\langle \mathcal{A}x_n, z \rangle = \langle x_n, \mathcal{A}z \rangle$ , which implies  $\langle y, z \rangle = \langle x, \mathcal{A}z \rangle$  which implies  $x \in D(\mathcal{A}^\circ)$  and  $y = \mathcal{A}^\circ x$ . Since  $\mathcal{A}$  surjective  $\Rightarrow \exists w \in D(\mathcal{A})$  s.t.  $\mathcal{A}w = y$  and  $\mathcal{A}^\circ x = y$ .

Since  $\mathcal{A}$  is symmetric:  $\mathcal{A}^\circ w = \mathcal{A}w$ . Then  $\mathcal{A}^\circ w = \mathcal{A}^\circ x$ ,  $\mathcal{A}$  is injective  $\Rightarrow w = x$ . Hence  $\mathcal{A}x = \mathcal{A}w = y \Rightarrow y = \mathcal{A}x \Rightarrow \mathcal{A}$  is closed.

By closed graph theorem both  $\mathcal{A}$  and  $\mathcal{A}^{-1} \in \mathcal{L}(X)$ . We can conclude that  $\mathcal{A}$  is a self-adjoint operator. □

### Exercise 9

Let  $\mathcal{H} = L^2(0, \pi)$  with  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  and

$$\mathcal{A}f = -f''$$

If  $D(\mathcal{A}) = \{u \in C^2 : u(0) = u(\pi) = 0\}$  is  $\mathcal{A}$  a self-adjoint operator?

Similarly, if  $D(\mathcal{A}) = \{u \in C^2 \mid u'(0) = u'(\pi) = 0\}$  is  $\mathcal{A}$  a self-adjoint operator?

**Theorem 1.3.2** (Fredrich's Extension). *Let  $\mathcal{H}$  be a Hilbert space on  $\mathbb{C}$  with  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ , symmetric then  $\mathcal{A}$  admits a unique self adjoint extension. If either*

$$a. \inf_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle > -\infty$$

$$b. \sup_{\|x\|=1, x \in D(\mathcal{A})} \langle \mathcal{A}x, x \rangle < +\infty$$

such that,  $\mathcal{A} \subset \mathcal{A}^\circ \subset (\mathcal{A}^\circ)^\circ \subset \dots$ . If (a) or (b) holds, then;  $\mathcal{A} \subset \mathcal{A}^\circ = (\mathcal{A}^\circ)^\circ$ .

## 1.4 Dissipative Operator and Numerical range

**Definition 1.4.1** (Duality Map). Let  $X$  be a Banach space on  $\mathbb{K}$ . The duality map is defined as  $J : X \longrightarrow 2^{X^*}$ ,  $x \mapsto J(x) = \{x^* \in X^* \mid \operatorname{Re} \langle x^*, x \rangle = \|x\|^2, \|x^*\|_{X^*} = \|x\|_X\}$ .

By the Hahn-Banach theorem,  $J(x) \neq \emptyset$ .

**Question:** What can you say about  $J(X)$  when  $X$  is an Hilbert space or Reflexive?

**Definition 1.4.2.** A map  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  (UBLO) is dissipative if for all  $x \in D(\mathcal{A})$ , there exists  $x^* \in J(X)$  such that  $\operatorname{Re} \langle x^*, \mathcal{A}x \rangle \leq 0$ .

**Lemma 1.4.1.**  $\mathcal{A}$  is dissipative if and only if for all  $\lambda > 0$ ,  $x \in D(\mathcal{A})$  we have that  $\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|$ .

*Proof.* let  $x^* \in J(x)$ . Then

$$\begin{aligned} \|(\lambda I - \mathcal{A})x\| \|x^*\| &\geq |\langle x^*, (\lambda I - \mathcal{A})x \rangle| \geq \Re \langle x^*, (\lambda I - \mathcal{A})x \rangle, \\ &= \lambda \Re \langle x^*, x \rangle - \Re \langle x^*, \mathcal{A}x \rangle \geq \lambda \|x\|^2. \end{aligned}$$

Hence, if  $\|x\| \neq 0$ , then

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|.$$

( $\Leftarrow$ ) Let  $x \in D(\mathcal{A})$ ,  $x \neq 0$ , and  $\lambda > 0$ . Let  $y_\lambda^* \in J((\lambda I - \mathcal{A})x)$  and set  $g_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$ . Then

$$\|(\lambda I - \mathcal{A})x\|^2 = \|(\lambda I - \mathcal{A})x\| \|y_\lambda^*\| = \Re \langle y_\lambda^*, (\lambda I - \mathcal{A})x \rangle.$$

Since  $y_\lambda^* \neq 0$ , we have

$$\lambda \|x\| \leq \|(\lambda I - \mathcal{A})x\| = \Re \langle g_\lambda^*, (\lambda I - \mathcal{A})x \rangle = \lambda \langle g_\lambda^*, x \rangle - \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Hence,

$$\Re \langle g_\lambda^*, \mathcal{A}x \rangle \leq \lambda \langle g_\lambda^*, x \rangle - \lambda \|x\| \leq \|g_\lambda^*\| \|x\| = \|x\|.$$

Therefore,

$$\Re \langle g_\lambda^*, \mathcal{A}x \rangle \leq 0. \quad (**)$$

**Idea:** Let  $\lambda \rightarrow +\infty$ .

Unit ball in  $X^*$  is compact for weak\* topology (Banach–Alaoglu).

(Up to subsequence)

$$g_\lambda^* \longrightarrow g^* \in X^*, \quad \|g^*\| \leq 1.$$

Then from (\*\*),

$$\Re \langle g^*, \mathcal{A}x \rangle \leq 0.$$

$$(*) \quad \|x\| \leq \langle g_\lambda^*, x \rangle - \frac{1}{\lambda} \Re \langle g_\lambda^*, \mathcal{A}x \rangle.$$

Let  $\lambda \rightarrow +\infty$ . Then  $\|x\| \leq \langle g^*, x \rangle$ . Hence,  $\|g^*\| = 1$  and  $\langle g^*, x \rangle = \|x\|$ .

Set  $x^* = \|x\|g^*$ . Then

$$\|x^*\| = \|x\| \quad \text{and} \quad \langle x^*, x \rangle = \|x\|^2,$$

that is,  $x^* \in J(x)$ . □

**Theorem 1.4.1** (Lumer-Phillips). *Let  $X$  be a Banach space and  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO. Assume that  $\mathcal{A}$  is dissipative and that there exists  $\lambda_0 > 0$  such that  $\text{Rg}(\lambda_0 I - \mathcal{A}) = X$ .*

*Then  $\mathcal{A}$  is closed,  $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$ , and for all  $\lambda > 0$ ,*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}. \quad (1.4)$$

*Proof.* Let  $\lambda_0 > 0$ .

(1) To prove:  $(\lambda_0 I - \mathcal{A})$  is bijective.

- Surjective: Assumption,
- Injective: Lemma 1.4.1.

Hence,

$$(\lambda_0 I - \mathcal{A})^{-1} : X \rightarrow X$$

is well-defined and linear.

It is bounded: since bijective, for any  $y \in X$ , there exists a unique  $x \in X$  such that

$$x = (\lambda_0 I - \mathcal{A})^{-1}y, \quad (\lambda_0 I - \mathcal{A})x = y.$$

By Lemma 1.4.1,

$$\frac{1}{\lambda_0} \|y\| \geq \|(\lambda_0 I - \mathcal{A})^{-1}y\|.$$

Hence,

$$\|(\lambda_0 I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_0}, \quad \lambda_0 \in \rho(\mathcal{A}).$$

(2)  $\mathcal{A}$  is closed.

Let  $x_n \rightarrow x$ ,  $x_n \in D(\mathcal{A})$ , and  $\mathcal{A}x_n \rightarrow y$ .

Then

$$(\lambda_0 I - \mathcal{A})x_n \rightarrow \lambda_0 x - y.$$

Since  $(\lambda_0 I - \mathcal{A})^{-1} \in \mathcal{L}(X)$ , we have

$$x_n \rightarrow (\lambda_0 I - \mathcal{A})^{-1}(\lambda_0 x - y) = x.$$

Hence,

$$\lambda_0 x - y = (\lambda_0 I - \mathcal{A})x \iff y = \mathcal{A}x.$$

Therefore,  $\mathcal{A}$  is closed.

(3)  $\rho(\mathcal{A}) \supset \mathbb{R}_+^*$  and (1.4).

Since  $\mathcal{A}$  is closed and  $\rho(\mathcal{A}) \neq \emptyset$ , we know that  $\rho(\mathcal{A})$  is open.

Let  $\Lambda = \rho(\mathcal{A}) \cap \mathbb{R}_+^*$ , which is open in  $\mathbb{R}_+^*$ . We show that it is closed.

Let  $(\lambda_n)_{n \in \mathbb{N}} \subset \Lambda$  such that  $\lambda_n \longrightarrow \lambda \in \mathbb{R}_+^*$ . Note that since  $\lambda_n \in \Lambda$ , we have

$$\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda_n}.$$

We write

$$\begin{aligned} (\lambda I - \mathcal{A}) &= [I + u_n](\lambda_n I - \mathcal{A}) \implies \lambda I - \mathcal{A} = \lambda_n I - \mathcal{A} + u_n(\lambda_n I - \mathcal{A}), \\ \iff (\lambda - \lambda_n)I &= u_n(\lambda_n I - \mathcal{A}) \iff (\lambda - \lambda_n)(\lambda_n I - \mathcal{A})^{-1} = u_n. \end{aligned}$$

Hence,

$$\|u_n\| \leq |\lambda - \lambda_n| \|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{|\lambda - \lambda_n|}{\lambda_n}.$$

For  $n$  large enough,

$$\frac{|\lambda - \lambda_n|}{\lambda_n} \leq \frac{1}{2}.$$

It follows that  $\lambda \in \rho(\mathcal{A})$ . Hence,  $\Lambda$  is closed, and therefore  $\Lambda = \mathbb{R}_+^*$ .

□

**Corollary 1.4.1.** *Let  $X$  be a Banach space and  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  be a UBLO, closed, with  $\overline{D(\mathcal{A})} = X$ . Assume that  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative. Then*

$$\rho(\mathcal{A}) \supset \mathbb{R}_+^*, \quad \forall \lambda > 0, \quad \lambda \|(\lambda I - \mathcal{A})^{-1}\| \leq 1.$$

*Proof.* It is easy to show that  $\text{Rg}(I - \mathcal{A}) = X$  (i.e.  $\lambda_0 = 1$  + Theorem 1.4.1).

$\mathcal{A}$  dissipative and closed implies

$$\text{Rg}(I - \mathcal{A}) \text{ is a closed subspace of } X.$$

(give details!!!)

Let  $x^* \in X^*$  such that

$$\langle x^*, (I - \mathcal{A})x \rangle = 0, \quad \forall x \in D(\mathcal{A}). \quad (**)$$

Let us prove that  $x^* = 0$ .

Then  $x^* \in D(\mathcal{A}^*)$  and

$$(I - \mathcal{A}^*)x^* = 0.$$

Since  $\mathcal{A}^*$  is dissipative, by Lemma 1.4.1, we have  $x^* = 0$ . This implies that

$$\overline{\text{Rg}(I - \mathcal{A})} = X.$$

Since  $\text{Rg}(I - \mathcal{A})$  is closed, we obtain

$$\text{Rg}(I - \mathcal{A}) = X.$$

By contradiction and using Hahn–Banach. □

**Definition 1.4.3** (Numerical Range). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  be UBLO. The numerical range of  $\mathcal{A}$ , denoted by  $W(\mathcal{A})$ ,*

$$W(\mathcal{A}) = \{\langle x^*, \mathcal{A}x \rangle \mid x^* \in J(x), x \in D(\mathcal{A}), \|x\| = \|x^*\| = 1, \langle x^*, x \rangle = 1\}.$$

*In case of a Hilbert space, we have that  $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid x \in D(\mathcal{A}), \|x\| = 1\}$ .*

*Linear algebra in finite dimension  $\mathcal{A} \in \mathcal{M}_n(\mathbb{K})$ , we have that  $W(\mathcal{A}) = \{\langle x, \mathcal{A}x \rangle \mid \|x\| = 1\}$ .*

**Theorem 1.4.2** (Home-work). *Let  $\mathcal{A} : D(\mathcal{A}) \longrightarrow X$  be closed, with  $\overline{D(\mathcal{A})} = X$ .*

*1) If  $\lambda \notin \overline{W(\mathcal{A})}$ , then  $(\lambda I - \mathcal{A})$  is injective, has closed image, and for all  $x \in D(\mathcal{A})$ ,*

$$\|(\lambda I - \mathcal{A})x\| \geq d(\lambda, W(\mathcal{A})) \|x\|.$$

*Moreover, if  $\lambda \in \rho(\mathcal{A})$ , then*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{d(\lambda, W(\mathcal{A}))}. \quad (**)$$

*2) If  $\Lambda$  is a connected open subset of  $\mathbb{C} \setminus W(\mathcal{A})$  such that  $\rho(\mathcal{A}) \cap \Lambda \neq \emptyset$ , then  $\rho(\mathcal{A}) \supset \Lambda$  and  $(**)$  holds true.*

## INTRODUCTION TO THE THEORY OF SEMI-GROUPS

### 2.1 Intro to the Introduction

**Definition 2.1.1.** Let  $X$  be a Banach space over  $\mathbb{K}$ . A one-parameter family of bounded linear operators on  $X$ ,  $(T(t))_{t \geq 0}$ , is a semigroup (SG) of bounded linear operators on  $X$  if:

1.  $T(0) = Id_X$ ,
2.  $\forall (t, s) \in \mathbb{R}_+^2 : T(t + s) = T(t) \cdot T(s)$  (SG property).

**Remark 2.1.1.**  $T(t)$  and  $T(s)$  commute.

#### 2. Infinitesimal generator of SG-LO $(T(t))_{t \geq 0}$

Let  $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$  be an unbounded linear operator with

$$D(\mathcal{A}) = \left\{ x \in X \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

and

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \mathcal{A}x, \quad x \in D(\mathcal{A}).$$

where  $D(\mathcal{A}) = \text{domain of } \mathcal{A}$ .

### 2.2 Uniformly Continuous SG-BLO

**Definition 2.2.1.** A SG-BLO on  $X$ ,  $(T(t))_{t \geq 0}$  is uniformly continuous if

$$\|T(t) - Id\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as } t \rightarrow 0^+$$

**Lemma 2.2.1.** Let  $(T(t))_{t \geq 0}$  be a SG-BLO which is uniformly continuous. Then,  $\forall t > 0$ ,

$$\|T(s) - T(t)\| \xrightarrow{s \rightarrow t} 0$$

(continuity everywhere).



*Proof.* Let  $t$  be fixed.  $T(s) = T(s - t + t)$ ,  $s - t \geq 0$ .

$$s \geq t \Rightarrow T(s) = T(s - t)T(t) \Rightarrow T(s) - T(t) = T(t) [T(s - t) - I_d]$$

$$\|T(s) - T(t)\| \leq \|T(t)\| \|T(s - t) - I_d\| \xrightarrow{s \rightarrow t} 0.$$

For  $s \leq t$

$$T(t) = T(t - s)T(s) \Rightarrow T(t) - T(s) = T(s) [T(t - s) - I_d]$$

(Prove that  $\sup_{[0,t]} \|T(t)\| < +\infty$ )

Then

$$\|T(t) - T(s)\| \leq \|T(s)\| \|T(t - s) - I_d\|$$

$$\leq \sup \|T(s)\| \|T(t - s) - I_d\| \xrightarrow{s \rightarrow t} 0.$$

□

**Theorem 2.2.1.** *A linear operator  $\mathcal{A}$  is the infinitesimal generator of a uniformly continuous semigroup if and only if  $\mathcal{A}$  is a bounded linear operator.*

*Proof.* Let  $\mathcal{A}$  be a bounded linear operator on  $X$  and set

$$T(t) = e^{t\mathcal{A}} = \sum_{n=0}^{\infty} \frac{(t\mathcal{A})^n}{n!}. \quad (1.5)$$

The right-hand side of (1.5) converges in norm for every  $t \geq 0$  and defines, for each such  $t$ , a bounded linear operator  $T(t)$ . It is clear that  $T(0) = I$  and a straightforward computation with the power series shows that  $T(t + s) = T(t)T(s)$ . Estimating the power series yields

$$\|T(t) - I\| \leq |t| \|\mathcal{A}\| e^{|t| \|\mathcal{A}\|}$$

and

$$\left\| \frac{T(t) - I}{t} - \mathcal{A} \right\| \leq \|\mathcal{A}\| \cdot \max_{0 \leq s \leq t} \|T(s) - I\|$$

which imply that  $T(t)$  is a uniformly continuous semigroup of bounded linear operators on  $X$  and that  $\mathcal{A}$  is its infinitesimal generator.

Let  $T(t)$  be a uniformly continuous semigroup of bounded linear operators on  $X$ . Fix  $\rho > 0$ , small enough, such that

$$\left\| I - \rho \int_0^\rho T(s) ds \right\| < 1.$$

This implies that  $\rho^{-1} \int_0^\rho T(s) ds$  is invertible. Now,

$$h^{-1}(T(h) - I) \int_0^\rho T(s) ds = h^{-1} \left( \int_0^\rho T(s + h) ds - \int_0^\rho T(s) ds \right)$$

$$= h^{-1} \left( \int_{\rho}^{\rho+h} T(s) ds - \int_0^h T(s) ds \right)$$

and therefore

$$h^{-1}(T(h) - I) = \left( h^{-1} \int_{\rho}^{\rho+h} T(s) ds - h^{-1} \int_0^h T(s) ds \right) \left( \int_0^{\rho} T(s) ds \right)^{-1}. \quad (1.6)$$

Letting  $h \rightarrow 0$  in (1.6) shows that  $h^{-1}(T(h) - I)$  converges in norm and therefore strongly to the bounded linear operator

$$(T(\rho) - I) \left( \int_0^{\rho} T(s) ds \right)^{-1}$$

which is the infinitesimal generator of  $T(t)$ . □

**Remark 2.2.1.** *The proof above was from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 2. [Theorem 1.2].*

**Theorem 2.2.2.** *Let  $T(t)$  and  $S(t)$  be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t \rightarrow 0} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t \rightarrow 0} \frac{S(t) - I}{t}. \quad (1.7)$$

*then  $T(t) = S(t)$  for  $t \geq 0$ .*

*Proof.* We will show that given  $T > 0$ ,  $S(t) = T(t)$  for  $0 \leq t \leq T$ . Let  $T > 0$  be fixed, since  $t \mapsto \|T(t)\|$  and  $t \mapsto \|S(t)\|$  are continuous there is a constant  $C$  such that

$$\|T(t)\| \|S(s)\| \leq C \quad \text{for } 0 \leq s, t \leq T.$$

Given  $\varepsilon > 0$  it follows from (1.7) that there is a  $\delta > 0$  such that

$$h^{-1} \|T(h) - S(h)\| < \varepsilon / TC \quad \text{for } 0 \leq h \leq \delta. \quad (1.8)$$

Let  $0 \leq t \leq T$  and choose  $n \geq 1$  such that  $t/n \leq \delta$ . From the semigroup property and (1.8) it then follows that

$$\begin{aligned} \|T(t) - S(t)\| &= \left\| T\left(n \frac{t}{n}\right) - S\left(n \frac{t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k) \frac{t}{n}\right) S\left(\frac{kt}{n}\right) - T\left((n-k-1) \frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| T\left((n-k-1) \frac{t}{n}\right) \right\| \left\| T\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{t}{n}\right) \right\| \leq C n \frac{\varepsilon}{TC} \frac{t}{n} \leq \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary  $T(t) = S(t)$  for  $0 \leq t \leq T$  and the proof is complete. □

**Corollary 2.2.1.** *Let  $T(t)$  be a uniformly continuous semigroup of bounded linear operators. Then*

- (a) *There exists a constant  $\omega \geq 0$  such that  $\|T(t)\| \leq e^{\omega t}$ .*
- (b) *There exists a unique bounded linear operator  $\mathcal{A}$  such that  $T(t) = e^{t\mathcal{A}}$ .*
- (c) *The operator  $\mathcal{A}$  in part (b) is the infinitesimal generator of  $T(t)$ .*
- (d)  *$t \mapsto T(t)$  is differentiable in norm and satisfies*

$$\frac{dT(t)}{dt} = \mathcal{A}T(t) = T(t)\mathcal{A}. \quad (1.9)$$

*Proof.* All the assertions of Corollary 2.2.1 follow easily from (b). To prove (b) note that the infinitesimal generator of  $T(t)$  is a bounded linear operator  $\mathcal{A}$ .  $\mathcal{A}$  is also the infinitesimal generator of  $e^{t\mathcal{A}}$  defined by (1.5) and therefore, by Theorem 2.2.2,  $T(t) = e^{t\mathcal{A}}$ .  $\square$

**Remark 2.2.2.** *The proofs above are from the recommended text (Semigroups of Linear Operators and Applications to Partial Differential Equations) Page 3. [Theorem 1.3 and Corollary 1.4].*

## 2.3 Strongly Continuous Semigroups ( $C_0$ -Semigroups)

**Definition 2.3.1.** *The SG-BLO  $(T(t))_{t \geq 0}$  is strongly continuous (SC or  $C_0$ ) if  $\forall x \in X$*

$$\|T(t)x - x\|_X \xrightarrow{t \rightarrow 0^+} 0$$

**Theorem 2.3.1.** *Let  $(T(t))_{t \geq 0}$ ,  $C_0$ -SG then  $\exists \omega \geq 0, \exists M \geq 1, \forall t \geq 0 \ \|T(t)\| \leq Me^{\omega t}$*

*Proof.* First we want to show that  $\forall \eta > 0, \sup_{t \in [0, \eta]} \|T(t)\| < +\infty$ .

By contradiction, assume that  $\sup \|T(t)\| = +\infty$ . Then  $\exists (t_n)_{n \geq 0} \searrow 0$  such that  $\|T(t_n)\| \geq n$  or  $\|T(t_n)\| \nearrow \infty$

By Banach-Steinhaus (the contrapositive)  $\exists x \in X$  such that  $\sup \|T(t_n)x\| = +\infty$ , but this contradicts the strong convergence 2.3.1.

Now take  $M := \sup \|T(t)\| \geq 1$  (This is because  $T(0) = Id$ ).

$\forall t \geq 0$  write  $t = k\eta + \eta_t$  where  $k = \left\lfloor \frac{t}{\eta} \right\rfloor$  and  $\eta_t \in [0, \eta]$ , then

$$\begin{aligned} \|T(t)\| &= \|T(k\eta + \eta_t)\| \\ &= \left\| [T(\eta)]^k T(\eta_t) \right\| && \text{by the SG property} \\ &= \|T(\eta)\|^k \|T(\eta_t)\| \\ &\leq M \cdot M^k && \text{M is upperbound} \\ &\leq M \cdot M^{t/\eta} && \text{since } k \leq \frac{t}{\eta} \\ &= M(e^{\ln M})^{t/\eta} \\ &= Me^{t \frac{\ln M}{\eta}} = Me^{\omega t} \end{aligned}$$

And since  $\omega = \frac{\ln M}{\eta}$  and  $M \geq 1$  then  $\omega \geq 0$ . □

**Corollary 2.3.1.** *Let  $\{T(t)\}_{t \geq 0}$  be a  $C_0$ -semigroup then  $\forall x \in X, t \mapsto T(t)x$  is continuous*

*Proof.* For  $h > 0$  and  $t > 0$ :

- $T(t+h)x - T(t)x = T(t)[T(h) - Id]x \rightarrow 0$  as  $h \rightarrow 0$  by definition 2.3.1.
- $T(t-h)x - T(t)x = T(t-h)[Id - T(h)]x$ . Since  $\|T(t-h)\|$  is bounded by theorem 2.3.1, so this tends to 0 as  $h \rightarrow 0$ . □

**Theorem.** *Let  $\mathcal{A}$  be the IG of  $C_0$ -SG  $T(t)$ , then:*

2.3.2  $\forall x \in X, \forall t \geq 0, \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$  for all  $x \in X$ .

2.3.3  $\forall x \in X, \forall t \geq 0, \int_0^t T(s)x ds \in D(\mathcal{A})$  and  $\mathcal{A} \int_0^t T(s)x ds = T(t)x - x$ .

2.3.4  $\forall x \in D(\mathcal{A}), T(t)x \in D(\mathcal{A})$  and  $\frac{d}{dt}T(t)x = \mathcal{A}T(t)x = T(t)\mathcal{A}x$ .

2.3.5  $\forall x \in D(\mathcal{A}), \forall t \geq 0, \forall s \geq 0, T(t)x - T(s)x = \int_s^t T(u)\mathcal{A}x du$  for  $x \in D(\mathcal{A})$ .

*Proof of 2.3.2.* Consider the small interval  $[t, t+h]$  relative to its value at  $t$ :

$$\frac{1}{h} \int_t^{t+h} T(s)x ds - T(t)x = \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] ds$$

By the continuity (corollary 2.3.1) of the map  $s \mapsto T(s)x$ , for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $s$  satisfying  $|s - t| < \delta$ , we have  $\|T(s)x - T(t)x\| < \epsilon$ .

Taking  $0 < h < \delta$ , we can estimate the norm of the integral:

$$\left\| \frac{1}{h} \int_t^{t+h} [T(s)x - T(t)x] ds \right\| \leq \frac{1}{h} \int_t^{t+h} \|T(s)x - T(t)x\| ds < \frac{1}{h} \cdot h\epsilon = \epsilon$$

Then

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$$
□

*Proof of 2.3.3.* Let  $h > 0$ . Consider the difference quotient for the integral  $y = \int_0^t T(s)x ds$ :

$$\begin{aligned} \frac{T(h) - Id}{h} \int_0^t T(s)x ds &= \frac{1}{h} \left[ \int_0^t T(s+h)x ds - \int_0^t T(s)x ds \right] \\ &= \frac{1}{h} \left[ \int_h^{t+h} T(u)x du - \int_0^t T(u)x du \right] \\ &= \frac{1}{h} \int_t^{t+h} T(u)x du - \frac{1}{h} \int_0^h T(u)x du \end{aligned}$$

As  $h \rightarrow 0^+$ , the first term converges to  $T(t)x$  and the second to  $T(0)x = x$  by 2.3.2. Thus the limit exists,  $y \in D(\mathcal{A})$ , and  $\mathcal{A}y = T(t)x - x$ .  $\square$

*Proof of 2.3.4.* If  $x \in D(\mathcal{A})$ , then  $T(t)\mathcal{A}x = T(t) \lim_{h \rightarrow 0} \frac{T(h)x - x}{h} = \lim_{h \rightarrow 0} \frac{T(h)T(t)x - T(t)x}{h}$ . This limit exists and equals  $\mathcal{A}(T(t)x)$ , proving  $T(t)x \in D(\mathcal{A})$  and  $T(t)\mathcal{A}x = \mathcal{A}T(t)x$ . This also shows the right-derivative of  $T(t)x$  is  $\mathcal{A}T(t)x$ . A similar argument for the left-derivative completes the differentiability.  $\square$

*Proof of 2.3.5.* By Property 2.3.4, the function  $f(u) = T(u)x$  is differentiable with  $f'(u) = T(u)\mathcal{A}x$ . Since  $f'$  is continuous, we integrate  $f'$  over  $[s, t]$  to obtain  $f(t) - f(s) = \int_s^t f'(u) du$ , which is  $T(t)x - T(s)x = \int_s^t T(u)\mathcal{A}x du$ .  $\square$

**Theorem 2.3.6.** *The IG  $\mathcal{A}$  of  $C_0$ -SG is a closed linear operator and  $\overline{D(\mathcal{A})} = X$ .*

*Proof.* For any  $x \in X$ , let  $x_t = \frac{1}{t} \int_0^t T(s)x ds$ . By 2.3.3,  $x_t \in D(\mathcal{A})$ . By 2.3.2,  $x_t \rightarrow T(0)x = x$  as  $t \rightarrow 0^+$ , which shows  $\overline{D(\mathcal{A})} = X$ .

Let  $x_n \in D(\mathcal{A})$  such that  $x_n \rightarrow x$  and  $\mathcal{A}x_n \rightarrow y$ . From 2.3.3 take  $s = 0$ , we have  $T(t)x_n - x_n = \int_0^t T(s)\mathcal{A}x_n ds$ . Passing to the limit  $n \rightarrow \infty$ , we get  $T(t)x - x = \int_0^t T(s)y ds$ . Dividing by  $t$  and letting  $t \rightarrow 0^+$ , the Right-Hand Side (RHS) converges to  $y$ . Thus  $x \in D(\mathcal{A})$  and  $\mathcal{A}x = y$ .  $\square$

**Theorem 2.3.7.** *Let  $\{T(t)\}_{t \geq 0}$  and  $\{S(t)\}_{t \geq 0}$  be two  $C_0$ -SG with infinitesimal generators  $\mathcal{A}$  and  $B$ , respectively. If  $\mathcal{A} = B$ , then  $T(t) = S(t)$  for all  $t \geq 0$ .*

*Proof.* Assume  $\mathcal{A} = B$ , then  $D(\mathcal{A}) = D(B)$ . Let  $x \in D(\mathcal{A})$  be fixed, and for a fixed  $t > 0$ , define

$$\varphi : [0, t] \rightarrow X, \quad \varphi(s) = T(t-s)S(s)x$$

Since  $x \in D(\mathcal{A})$ , the map  $\varphi$  is of class  $C^1$  on  $[0, t]$ . We differentiate  $\varphi$  with respect to  $s$  using the product rule and 2.3.4, we get:

$$\begin{aligned} \frac{d}{ds}\varphi(s) &= \frac{d}{ds}[T(t-s)]S(s)x + T(t-s)\frac{d}{ds}[S(s)x] \\ &= -\mathcal{A}T(t-s)S(s)x + T(t-s)BS(s)x \end{aligned}$$

Because  $T(t-s)$  commutes with its generator  $\mathcal{A}$ , and given  $\mathcal{A} = B$ , we have:

$$\frac{d}{ds}\varphi(s) = -T(t-s)\mathcal{A}S(s)x + T(t-s)\mathcal{A}S(s)x = 0$$

Since the derivative is zero for all  $s \in [0, t]$ , the function  $\varphi$  must be constant. Evaluating  $\varphi$  at the endpoints  $s = 0$  and  $s = t$  yields

$$\varphi(0) = T(t)S(0)x = T(t)x \quad \text{and} \quad \varphi(t) = T(0)S(t)x = S(t)x$$

Thus,  $T(t)x = S(t)x$  for all  $x \in D(\mathcal{A})$ . Since  $D(\mathcal{A})$  is dense in  $X$  and  $T(t), S(t)$  are bounded linear operators, this identity extends to all  $x \in X$  by continuity. Therefore,  $T(t) = S(t)$  for all  $t \geq 0$ .  $\square$

**Theorem 2.3.8.** *Let  $\mathcal{A}$  be the IG of a  $C_0$ -SG  $\{T(t)\}_{t \geq 0}$  on a Banach space  $X$ . Then the subspace*

$$X = \overline{\bigcap_{n \geq 1} D(\mathcal{A}^n)}$$

*Proof.* Let  $\mathcal{D} = \{\varphi : \mathbb{R} \rightarrow \mathbb{C} \mid \varphi \text{ has compact support in } \mathbb{R}_+^* \text{ and is smooth } C^\infty\}$ . Let  $x \in X$  and consider a test function  $\varphi \in \mathcal{D}$ . Define

$$x_\varphi = \int_0^\infty \varphi(s) T(s)x \, ds$$

First, we show that  $x_\varphi \in D(\mathcal{A})$ . Consider

$$\begin{aligned} \frac{T(h) - Id}{h} x_\varphi &= \frac{1}{h} \int_0^\infty \varphi(s) [T(s+h)x - T(s)x] \, ds \\ &= \frac{1}{h} \left[ \int_h^\infty \varphi(u-h) T(u)x \, du - \int_0^\infty \varphi(u) T(u)x \, du \right] \\ &= \int_0^\infty \frac{\varphi(u-h) - \varphi(u)}{h} T(u)x \, du \end{aligned}$$

As  $h \rightarrow 0$ , the quotient  $\frac{\varphi(u-h) - \varphi(u)}{h}$  converges uniformly to  $-\varphi'(u)$  because  $\varphi$  is  $C^\infty$  and has compact support. Thus:

$$\mathcal{A} x_\varphi = - \int_0^\infty \varphi'(s) T(s)x \, ds$$

Since  $\varphi' \in C_c^\infty(0, \infty)$ , we can repeat this process inductively. For any  $n \geq 1$ , we find:

$$\mathcal{A}^n x_\varphi = (-1)^n \int_0^\infty \varphi^{(n)}(s) T(s)x \, ds$$

This proves that  $x_\varphi \in D(\mathcal{A}^n)$  for all  $n$ .

To prove density, suppose  $\overline{\bigcap_{n \geq 1} D(\mathcal{A}^n)} \neq X$ . By the Hahn-Banach Theorem, there exists a non-zero functional  $x^* \in X^*$  such that  $\langle x^*, y \rangle = 0$  for all  $y \in \bigcap_{n \geq 1} D(\mathcal{A}^n)$ . Specifically, for any  $x \in X$  and  $\varphi \in C_c^\infty(0, +\infty)$ :

$$\langle x^*, x_\varphi \rangle = \int_0^{+\infty} \varphi(s) \langle x^*, T(s)x \rangle \, ds = 0$$

This identity holds for all  $C^\infty$  functions  $\varphi$  with compact support. Then  $\langle x^*, T(s)x \rangle$  must be zero for all  $s > 0$ .

By the strong continuity of the semigroup at  $s = 0$ :

$$\langle x^*, x \rangle = \lim_{s \rightarrow 0^+} \langle x^*, T(s)x \rangle = 0$$

Since this holds for all  $x \in X$ , it implies  $x^* = 0$ , which contradicts our assumption that  $x^*$  was non-zero. Thus,  $\bigcap_{n \geq 1} D(\mathcal{A}^n)$  must be dense in  $X$ .  $\square$

### Exercise 10

Let  $X = \{f : \mathbb{R} \rightarrow \mathbb{C} \mid f \text{ is continuous and uniformly bounded}\}$  equipped with the supremum norm  $\|f\|_\infty = \sup_{s \in \mathbb{R}} |f(s)|$ . Define the family of operators  $(T(t))_{t \geq 0}$  by:

$$(T(t)f)(s) = f(s+t), \quad s \in \mathbb{R}, t \geq 0$$

Prove that this family is  $C_0$ -SG, its IG is  $\mathcal{A}f = f'$ , and  $\|T(t)\| = 1$ .

## 2.4 Hille-Yosida Theorem

**Definition 2.4.1.** A  $C_0$ -SG  $\{T(t)\}_{t \geq 0}$  is called *uniformly bounded semigroup* if  $\exists M \geq 1$  such that  $\|T(t)\| \leq M$  for all  $t \geq 0$ .

**Definition 2.4.2.** A  $C_0$ -SG  $\{T(t)\}_{t \geq 0}$  is called a *contraction semigroup* if  $\|T(t)\| \leq 1$  for all  $t \geq 0$ .

**Theorem 2.4.1** (Hille-Yosida Theorem (Contraction Case)). A linear unbounded operator  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  is the IG of a  $C_0$ -SG of contractions if and only if:

- (i)  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A})} = X$ .
- (ii)  $\mathbb{R}_+^* \subset \rho(\mathcal{A})$  and  $\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$  for all  $\lambda > 0$ .

( $\Rightarrow$ ). (i) follows directly from 2.3.6.

If  $\mathcal{A}$  generates a contraction semigroup  $\{T(t)\}_{t \geq 0}$ , we define the resolvent for  $\lambda > 0$  as follows

$$R(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t) dt$$

Taking the norm, we obtain:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)x\| dt \leq \int_0^{+\infty} e^{-\lambda t} \|x\| dt = \frac{1}{\lambda} \|x\|$$

$\square$

**Remark 2.4.1.** Note for all real numbers  $\lambda, a$  with  $\lambda > a$ , we have the following:

$$\frac{1}{\lambda - a} = \int_0^{+\infty} e^{-(\lambda - a)t} dt$$

Extending this to the vector space we get the way of writing the resolvent operator from above.

### 2.4.1 The Yosida Approximation

To prove if  $(\Leftarrow)$ , we introduce a family of bounded operators that approximate the unbounded generator  $A$ .

**Definition 2.4.3.** For  $\lambda > 0$ , the Yosida Approximation of  $\mathcal{A}$  is defined as:

$$\mathcal{A}_\lambda := \lambda \mathcal{A} R(\lambda, \mathcal{A}) = \lambda^2 R(\lambda, \mathcal{A}) - \lambda I$$

Note that  $\mathcal{A}_\lambda$  is a bounded linear operator for each  $\lambda \in \rho(\mathcal{A})$ .

**Claim 2.4.1.** For  $\lambda \in \rho(\mathcal{A})$  and  $x \in D(\mathcal{A})$ , the following identity holds:

$$\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A} R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$$

*Proof.* By the definition of the resolvent as the inverse of the operator  $(\lambda I - \mathcal{A})$ , we have:

$$(\lambda I - \mathcal{A})R(\lambda, \mathcal{A}) = Id_X$$

Applying this to any  $x \in X$ :

$$(\lambda I - \mathcal{A})R(\lambda, \mathcal{A})x = x$$

Distributing the operators on the left-hand side gives:

$$\lambda R(\lambda, \mathcal{A})x - \mathcal{A} R(\lambda, \mathcal{A})x = x$$

Rearranging the terms to isolate the  $\mathcal{A} R(\lambda, \mathcal{A})x$  term yields:

$$\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A} R(\lambda, \mathcal{A})x \tag{*}$$

This identity holds for all  $x \in X$  because  $R(\lambda, \mathcal{A})$  maps  $X$  into  $D(\mathcal{A})$ .

For the other side, let  $x \in D(\mathcal{A})$ . We use the fact that the resolvent also satisfies:

$$R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}) = Id_{D(\mathcal{A})}$$

Applying this to  $x \in D(\mathcal{A})$ :

$$R(\lambda, \mathcal{A})(\lambda I - \mathcal{A})x = x$$

Distributing  $R(\lambda, \mathcal{A})$  gives:

$$\lambda R(\lambda, \mathcal{A})x - R(\lambda, \mathcal{A})\mathcal{A}x = x$$

Rearranging the terms:

$$\lambda R(\lambda, \mathcal{A})x - x = R(\lambda, \mathcal{A})\mathcal{A}x \tag{**}$$



From (\*) and (\*\*) we get:

$$\mathcal{A}R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$$

□

**Theorem 2.4.2.** For all  $x \in X$ ,  $\lim_{\lambda \rightarrow +\infty} \mathcal{A}_\lambda x = \mathcal{A}x$ .

*Proof.* Using the identity  $\lambda R(\lambda, \mathcal{A})x - x = \mathcal{A}R(\lambda, \mathcal{A})x = R(\lambda, \mathcal{A})\mathcal{A}x$  for  $x \in D(\mathcal{A})$ , we observe:

$$\|\lambda R(\lambda, \mathcal{A})x - x\| = \|R(\lambda, \mathcal{A})\mathcal{A}x\| \leq \frac{\|\mathcal{A}x\|}{\lambda} \xrightarrow{\lambda \rightarrow +\infty} 0$$

Since  $\mathcal{A}_\lambda x = \lambda R(\lambda, \mathcal{A})\mathcal{A}x$ , and we just showed  $\lambda R(\lambda, \mathcal{A})x \rightarrow x$ ,  $\forall x \in D(\mathcal{A})$ , it follows that  $\mathcal{A}_\lambda x \rightarrow \mathcal{A}x$ . By density and the uniform bound  $\|\lambda R(\lambda, \mathcal{A})\| \leq 1$ , this convergence holds for all  $x \in X$ . □

**Lemma 2.4.1.** For each  $\lambda > 0$ ,  $\mathcal{A}_\lambda$  generates a uniformly continuous semigroup of contractions  $\{e^{t\mathcal{A}_\lambda}\}_{t \geq 0}$ .

*Proof.* Since  $\mathcal{A}_\lambda = \lambda^2 R(\lambda, \mathcal{A}) - \lambda I$ , we have:

$$\|e^{t\mathcal{A}_\lambda}\| = \left\| e^{-t\lambda} e^{t\lambda^2 R(\lambda, \mathcal{A})} \right\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda, \mathcal{A})\|} \leq e^{-t\lambda} e^{t\lambda^2 \frac{1}{\lambda}} = e^{-t\lambda} e^{t\lambda} = 1$$

This confirms the contraction property for the approximating semigroups. □

**Lemma 2.4.2.** For any  $x \in X$  and  $\lambda, \mu > 0$ , the following estimate holds:

$$\|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| \leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\|$$

*Proof.* Fix  $x \in X$ , consider the function  $\phi : [0, 1] \rightarrow X$  defined by:

$$\phi(s) = e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} x$$

Since  $\mathcal{A}_\lambda$  and  $\mathcal{A}_\mu$  commute (as resolvents commute), the semigroups  $e^{t\mathcal{A}_\lambda}$  and  $e^{t\mathcal{A}_\mu}$  also commute. The function  $\phi$  is differentiable with respect to  $s$ :

$$\begin{aligned} \frac{d}{ds}\phi(s) &= t\mathcal{A}_\lambda e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} x - e^{st\mathcal{A}_\lambda} t\mathcal{A}_\mu e^{(1-s)t\mathcal{A}_\mu} x \\ &= t e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} (\mathcal{A}_\lambda - \mathcal{A}_\mu)x \end{aligned}$$

Integrating from 0 to 1:

$$\phi(1) - \phi(0) = e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x = \int_0^1 t e^{st\mathcal{A}_\lambda} e^{(1-s)t\mathcal{A}_\mu} (\mathcal{A}_\lambda - \mathcal{A}_\mu)x ds$$

Taking the norm and using the contraction property  $\|e^{t\mathcal{A}_\lambda}\| \leq e^{t\|\mathcal{A}_\lambda\|} \leq 1$  (since  $\mathcal{A}_\lambda$  is dissipative):

$$\begin{aligned} \|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| &\leq \int_0^1 t \|e^{st\mathcal{A}_\lambda}\| \|e^{(1-s)t\mathcal{A}_\mu}\| \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\| ds \\ &\leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\| \end{aligned}$$

□

Now let's start proving the only if direction of Hille-Yosida (  $\Leftarrow$  ).

*Proof.* We need the sequence  $(e^{t\mathcal{A}_\lambda}x)_{\lambda>0}$  to be converging. But doing so is hard, what we can do is show that it is a Cauchy sequence in  $X$ , since  $X$  is complete.

Taking  $x \in D(\mathcal{A})$  we have the following:

$$\|e^{t\mathcal{A}_\lambda}x - e^{t\mathcal{A}_\mu}x\| \leq t \|\mathcal{A}_\lambda x - \mathcal{A}_\mu x\| \leq t(\|\mathcal{A}_\lambda x - \mathcal{A}x\| + \|\mathcal{A}x - \mathcal{A}_\mu x\|) \xrightarrow[\mu \rightarrow +\infty]{\lambda \rightarrow +\infty} 0$$

This is possible because  $\lim_{\lambda \rightarrow +\infty} \mathcal{A}_\lambda x = \mathcal{A}x$ .

Define:

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda}x$$

This limit is well defined by the argument above for all  $x \in D(\mathcal{A})$ . Moreover, since  $D(\mathcal{A})$  is dense in  $X$  and  $\|e^{t\mathcal{A}_\lambda}\| \leq 1$ , we can extend  $T(t)$  to a bounded linear operator on all of  $X$  by density.

Now we show that the family  $(T(t))_{t \geq 0}$  defined above is a  $C_0$ -semigroup of contractions.

We verify the semigroup properties:

1. Identity:  $T(0)x = \lim_{\lambda \rightarrow +\infty} e^{0 \cdot \mathcal{A}_\lambda}x = x$ .
2. Semigroup Property: For  $x \in X$ ,

$$T(t+s)x = \lim_{\lambda \rightarrow +\infty} e^{(t+s)\mathcal{A}_\lambda}x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda}e^{s\mathcal{A}_\lambda}x = T(t)T(s)x.$$

3. Strong Continuity: For  $x \in D(\mathcal{A})$ , convergence is uniform on compact intervals of  $t$ , because of lemma 2.4.2. Thus  $t \mapsto T(t)x$  is continuous. By the density of  $D(\mathcal{A})$  and uniform boundedness  $\|T(t)\| \leq 1$ , continuity extends to all  $x \in X$ .

We must show that the generator of the constructed semigroup  $T(t)$  is indeed  $\mathcal{A}$ . Let  $\mathcal{B}$  be the generator of  $T(t)$ , we show that  $\mathcal{A} = \mathcal{B}$ .

For any  $x \in D(\mathcal{A})$ , we have the identity:

$$e^{t\mathcal{A}_\lambda}x - x = \int_0^t e^{s\mathcal{A}_\lambda}\mathcal{A}_\lambda x ds$$

As  $\lambda \rightarrow +\infty$ ,  $e^{s\mathcal{A}_\lambda} \rightarrow T(s)$  strongly and uniformly on compact sets, and  $\mathcal{A}_\lambda x \rightarrow \mathcal{A}x$ . Passing to the limit:

$$T(t)x - x = \int_0^t T(s)\mathcal{A}x \, ds$$

Dividing by  $t$  and taking  $t \rightarrow 0^+$ :

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t T(s)\mathcal{A}x \, ds = T(0)\mathcal{A}x = \mathcal{A}x$$

Thus,  $x \in D(\mathcal{B})$  and  $\mathcal{B}x = \mathcal{A}x$ , implying  $\mathcal{A} \subset \mathcal{B}$ . Since  $\mathcal{B}$  is the generator of a  $C_0$ -semigroup of contractions,  $1 \in \rho(\mathcal{B})$ . By hypothesis,  $1 \in \rho(\mathcal{A})$ . Since  $\mathcal{A} \subset \mathcal{B}$  and both  $(I - \mathcal{A})$  and  $(I - \mathcal{B})$  are surjective (mapping onto  $X$ ), it follows that  $\mathcal{A} = \mathcal{B}$ .  $\square$

**Corollary 2.4.1.** *Let  $\mathcal{A}$  be the IG of a  $C_0$ -SG of contractions  $(T(t))_{t \geq 0}$ . Then for every  $x \in X$ , the semigroup is given by the limit:*

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda} x$$

*Proof.* In the previous proof, we constructed a SG, let us call it  $S(t)$ , defined by  $S(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda} x$ . We proved that the generator of  $S(t)$  is exactly the operator  $\mathcal{A}$ .

Since  $\mathcal{A}$  is the generator of the original semigroup  $T(t)$  by hypothesis, and we know that a  $C_0$ -SG is uniquely determined by its generator (Uniqueness Theorem), it follows that:

$$T(t) = S(t), \quad \forall t \geq 0$$

$\square$

**Corollary 2.4.2.** *Let  $\mathcal{A}$  be IG of a  $C_0$ -SG of contractions  $(T(t))_{t \geq 0}$ . Then the resolvent set  $\rho(\mathcal{A})$  contains the open right half-plane:*

$$\rho(\mathcal{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$$

Furthermore, for all  $\lambda$  with  $\Re(\lambda) > 0$ , the following estimate holds:

$$\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\Re(\lambda)}$$

*Proof.* Let  $\lambda \in \mathbb{C}$  with  $\Re(\lambda) > 0$ . We define the operator  $R(\lambda)$  on  $X$  by the Laplace transform of the semigroup:

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since  $T(t)$  is a contraction semigroup ( $\|T(t)\| \leq 1$ ) and  $\Re(\lambda) > 0$ , the integrand is exponentially bounded:

$$\|e^{-\lambda t} T(t)x\| = e^{-\Re(\lambda)t} \|T(t)x\| \leq e^{-\Re(\lambda)t} \|x\|$$

Thus, the integral converges absolutely, defining a bounded linear operator. We calculate its norm:

$$\|R(\lambda)x\| \leq \int_0^{+\infty} e^{-\Re(\lambda)t} \|x\| dt = \|x\| \left[ \frac{-e^{-\Re(\lambda)t}}{\Re(\lambda)} \right]_0^{+\infty} = \frac{1}{\Re(\lambda)} \|x\|$$

This proves the bound  $\|R(\lambda)\| \leq \frac{1}{\Re(\lambda)}$ .

It remains to show that this integral operator  $R(\lambda)$  is indeed the resolvent  $R(\lambda, A) = (\lambda I - A)^{-1}$ . For any  $x \in X$  and  $h > 0$ :

$$\begin{aligned} \frac{T(h) - I}{h} R(\lambda)x &= \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t+h)x dt - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt \\ &= \frac{e^{\lambda h}}{h} \int_h^{+\infty} e^{-\lambda s} T(s)x ds - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_0^{+\infty} e^{-\lambda t} T(t)x dt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T(t)x dt \end{aligned}$$

Taking the limit as  $h \rightarrow 0^+$ , the first term converges to  $\lambda R(\lambda)x$  and the second term converges to  $-x$ . Thus, for any  $x \in X$ ,  $R(\lambda)x \in D(\mathcal{A})$  and  $\mathcal{A}R(\lambda)x = \lambda R(\lambda)x - x$ , which implies  $(\lambda I - \mathcal{A})R(\lambda)x = x$ . Similarly, one can show  $R(\lambda)(\lambda I - \mathcal{A})x = x$  for  $x \in D(\mathcal{A})$ .

Therefore,  $R(\lambda) = (\lambda I - \mathcal{A})^{-1}$ . □

### Exercise 11

Let  $X = BVC(\mathbb{R}_+) = \{f : \mathbb{R}_+ \rightarrow \mathbb{C} \mid f \text{ is bounded and uniformly continuous}\}$ . Equipped with the supremum norm  $\|f\|_\infty = \sup_{t \geq 0} |f(t)|$ ,  $(X, \|\cdot\|_\infty)$  is a Banach space.

For  $t \geq 0$ , define the operator  $T(t)$  by the left shift:

$$(T(t)f)(s) = f(s+t), \quad \forall s \geq 0$$

Check the following:

1.  $(T(t))_{t \geq 0}$  is a  $C_0$ -semigroup of contractions (i.e.,  $\|T(t)\| \leq 1$ ).
2. Show that  $\|T(t)\| = 1$ .
3. The infinitesimal generator is the differentiation operator  $\mathcal{A}f = f'$ , with an appropriate domain  $D(\mathcal{A})$ .
4. Verify that  $\rho(\mathcal{A}) \supset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$ .

## 2.4.2 The General Hille-Yosida Theorem

We now consider the general case where the semigroup is not necessarily a contraction.

**Theorem 2.4.3.** *A linear operator  $\mathcal{A}$  generates a  $C_0$ -semigroup  $T(t)$  satisfying  $\|T(t)\| \leq Me^{\omega t}$  if and only if:*

(i)  $A$  is closed and densely defined.

(ii)  $(\omega, +\infty) \subset \rho(\mathcal{A})$  and for all  $\lambda > \omega$  and  $n \geq 1$ :

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}.$$

### 2.4.3 Reduction to the Case $\omega = 0$

If  $\|T(t)\| \leq Me^{\omega t}$ , consider the rescaled semigroup  $S(t) = e^{-\omega t}T(t)$ . The generator of  $S(t)$  is  $\mathcal{A} - \omega I$  where  $\mathcal{A}$  is the IG of  $(T(t))_{t \geq 0}$ , and  $\|S(t)\| \leq M$ . Conversely, if we prove the theorem for  $\omega = 0$ , the general case follows by applying the result to  $\mathcal{A} - \omega I$ .

Now we have the following corollary:

**Corollary 2.4.3** (Hille-Yosida for  $(1, \omega)$ ). *A linear operator  $\mathcal{A}$  is the IG of a  $C_0$ -SG satisfying  $\|T(t)\| \leq e^{\omega t}$  if and only if:*

(i)  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A})} = X$ .

(ii)  $\rho(\mathcal{A}) \supset (\omega, +\infty)$  and for all  $\lambda > \omega$ , the following estimate holds:

$$\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda - \omega}.$$

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathcal{A}$  generates a semigroup  $T(t)$  such that  $\|T(t)\| \leq e^{\omega t}$ . Consider the rescaled family of operators  $S(t) = e^{-\omega t}T(t)$ . It is easy to verify that  $S(t)$  is a  $C_0$ -SG. Furthermore, it is a contraction:

$$\|S(t)\| = e^{-\omega t} \|T(t)\| \leq e^{-\omega t} e^{\omega t} = 1$$

Let  $\mathcal{B}$  be the generator of  $S(t)$ . By the definition of the generator:

$$\mathcal{B}x = \lim_{t \rightarrow 0^+} \frac{e^{-\omega t}T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \left( e^{-\omega t} \frac{T(t)x - x}{t} + \frac{e^{-\omega t} - 1}{t} x \right) = \mathcal{A}x - \omega x$$

Thus,  $\mathcal{B} = \mathcal{A} - \omega I$ . Since  $\mathcal{B}$  generates a contraction semigroup, by the Hille-Yosida Theorem for contractions (Case  $M = 1, \omega = 0$ ), we know that for any  $\mu > 0$ ,  $\mu \in \rho(\mathcal{B})$  and  $\|R(\mu, \mathcal{B})\| \leq \frac{1}{\mu}$ .

Let  $\lambda = \mu + \omega$ . Then  $\lambda > \omega$ . Since  $R(\mu, \mathcal{B}) = (\mu I - \mathcal{B})^{-1} = (\mu I - (\mathcal{A} - \omega I))^{-1} = ((\mu + \omega)I - \mathcal{A})^{-1}$ , we have:

$$R(\lambda, \mathcal{A}) = R(\lambda - \omega, \mathcal{B})$$

Substituting the norm bound:

$$\|R(\lambda, \mathcal{A})\| = \|R(\lambda - \omega, \mathcal{B})\| \leq \frac{1}{\lambda - \omega}$$

( $\Leftarrow$ ): Conversely, suppose  $\mathcal{A}$  satisfies conditions (1)-(3). Define  $\mathcal{B} = \mathcal{A} - \omega I$ . Clearly,  $\mathcal{B}$  is closed and densely defined. For any  $\mu > 0$ , let  $\lambda = \mu + \omega > \omega$ . Then  $\lambda \in \rho(\mathcal{A})$ , which implies

$\mu \in \rho(\mathcal{B})$ . The resolvent satisfies:

$$\|R(\mu, \mathcal{B})\| = \|R(\mu + \omega, \mathcal{A})\| \leq \frac{1}{(\mu + \omega) - \omega} = \frac{1}{\mu}$$

Thus,  $\mathcal{B}$  satisfies the Hille-Yosida conditions for the contraction case ( $M = 1, \omega = 0$ ). Therefore,  $\mathcal{B}$  generates a contraction semigroup  $S(t)$  with  $\|S(t)\| \leq 1$ . Defining  $T(t) = e^{\omega t} S(t)$ , we see that  $T(t)$  is a  $C_0$ -semigroup generated by  $\mathcal{A} = \mathcal{B} + \omega I$ , and it satisfies:

$$\|T(t)\| = e^{\omega t} \|S(t)\| \leq e^{\omega t}$$

□

This corollary provides us with a method to rescale the bound. Thus, we focus on the case  $(M, 0)$ , i.e.,  $\|T(t)\| \leq M$ . The condition on the resolvent becomes  $\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$ .

**Lemma 2.4.3.** *Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup satisfying  $\|T(t)\| \leq M$  for all  $t \geq 0$ . Let  $\mathcal{A}$  be its infinitesimal generator. Then for all  $\lambda > 0$  and all integers  $n \geq 0$ :*

$$\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$$

Equivalently,  $\|\lambda^n R(\lambda, \mathcal{A})^n\| \leq M$ .

*Proof.* For  $\lambda > 0$ , the resolvent is given by the Laplace transform of the semigroup:

$$R(\lambda, \mathcal{A})x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad \forall x \in X$$

Since the integral converges absolutely (due to the exponential decay  $e^{-\lambda t}$  and bounded  $T(t)$ ), we can differentiate this expression with respect to  $\lambda$  inside the integral sign. Differentiating  $n - 1$  times:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathcal{A})x = \int_0^{+\infty} (-t)^{n-1} e^{-\lambda t} T(t)x \, dt$$

On the other hand, from the general theory of resolvents, we have the identity:

$$\frac{d^k}{d\lambda^k} R(\lambda, \mathcal{A}) = (-1)^k k! R(\lambda, \mathcal{A})^{k+1}$$

Setting  $k = n - 1$ , we get:

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, \mathcal{A}) = (-1)^{n-1} (n-1)! R(\lambda, \mathcal{A})^n$$

Equating the two expressions for the derivative:

$$(-1)^{n-1} (n-1)! R(\lambda, \mathcal{A})^n x = \int_0^{+\infty} (-1)^{n-1} t^{n-1} e^{-\lambda t} T(t)x \, dt$$

Simplifying and solving for  $R(\lambda, \mathcal{A})^n x$ :

$$R(\lambda, \mathcal{A})^n x = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t)x \, dt$$

Now, we take the norm and use the bound  $\|T(t)\| \leq M$ :

$$\begin{aligned} \|R(\lambda, \mathcal{A})^n x\| &\leq \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \|T(t)x\| \, dt \\ &\leq \frac{M \|x\|}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt \end{aligned}$$

The integral on the right is the Gamma function definition. Substituting  $u = \lambda t$  ( $dt = du/\lambda$ ):

$$\int_0^{+\infty} t^{n-1} e^{-\lambda t} \, dt = \frac{1}{\lambda^n} \int_0^{+\infty} u^{n-1} e^{-u} \, du = \frac{\Gamma(n)}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

Substituting this back into the inequality:

$$\|R(\lambda, \mathcal{A})^n x\| \leq \frac{M \|x\|}{(n-1)!} \cdot \frac{(n-1)!}{\lambda^n} = \frac{M}{\lambda^n} \|x\|$$

Thus,  $\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{\lambda^n}$ . □

## 2.4.4 Renorming Lemma

The idea is to construct an equivalent norm on  $X$  under which  $\mathcal{A}$  becomes dissipative (generating a contraction semigroup), allowing us to apply the contraction case result.

**Lemma 2.4.4.** *Let  $A : D(A) \subset X \rightarrow X$  be a linear operator with  $\rho(A) \supset \mathbb{R}_+^*$  such that for all  $\lambda > 0$  and  $n \geq 0$ ,  $\|\lambda^n R(\lambda, A)^n\| \leq M$ . Then, there exists a norm  $\|\cdot\|_\mu$  on  $X$  such that:*

1. *The norms are equivalent:  $\|x\| \leq \|x\|_\mu \leq M \|x\|$  for all  $x \in X$ .*
2. *For all  $\lambda > 0$ , the operator  $\lambda R(\lambda, A)$  is a contraction in the new norm:  $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$ .*

*Proof.* Fix  $\mu > 0$ . We define the new norm  $\|\cdot\|_\mu$  by:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\|$$

To show that the norms are equivalent, let  $n = 0$ , the term is  $\|Ix\| = \|x\|$  so  $\|x\| \leq \|x\|_\mu$ . Using the hypothesis  $\|\mu^n R(\mu, A)^n\| \leq M$ , we have:

$$\|x\|_\mu = \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\| \leq \sup_{n \geq 0} M \|x\| = M \|x\|$$

Thus,  $\|x\| \leq \|x\|_\mu \leq M \|x\|$ .

Now to show that  $\mu R(\mu, A)$  is a contraction, we check the contraction property for the specific value  $\mu$ :

$$\begin{aligned}
\|\mu R(\mu, A)x\|_\mu &= \sup_{n \geq 0} \|\mu^n R(\mu, A)^n (\mu R(\mu, A)x)\| \\
&= \sup_{n \geq 0} \|\mu^{n+1} R(\mu, A)^{n+1} x\| \\
&= \sup_{k \geq 1} \|\mu^k R(\mu, A)^k x\| \\
&\leq \sup_{k \geq 0} \|\mu^k R(\mu, A)^k x\| = \|x\|_\mu.
\end{aligned}$$

So,  $\|\mu R(\mu, A)\|_\mu \leq 1$ .

Finally, we show the contraction for  $0 < \lambda \leq \mu$ :

Let  $x \in X$  and define  $y = R(\lambda, \mathcal{A})x$ . We want to show  $\lambda \|y\|_\mu \leq \|x\|_\mu$ . Recall the Resolvent Identity:

$$R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}) = (\mu - \lambda)R(\mu, \mathcal{A})R(\lambda, \mathcal{A})$$

Applying this to  $x$ :

$$y = R(\mu, \mathcal{A})x + (\mu - \lambda)R(\mu, \mathcal{A})y = R(\mu, \mathcal{A})[x + (\mu - \lambda)y].$$

Taking the  $\|\cdot\|_\mu$  norm and using the fact that  $\|\mu R(\mu, A)z\|_\mu \leq \|z\|_\mu \implies \|R(\mu, A)z\|_\mu \leq \frac{1}{\mu} \|z\|_\mu$ :

$$\begin{aligned}
\|y\|_\mu &= \|R(\mu, A)[x + (\mu - \lambda)y]\|_\mu \\
&\leq \frac{1}{\mu} \|x + (\mu - \lambda)y\|_\mu \\
&\leq \frac{1}{\mu} (\|x\|_\mu + (\mu - \lambda) \|y\|_\mu)
\end{aligned}$$

Multiplying by  $\mu$ :

$$\mu \|y\|_\mu \leq \|x\|_\mu + (\mu - \lambda) \|y\|_\mu$$

Subtracting  $(\mu - \lambda) \|y\|_\mu$  from both sides (since  $\mu - \lambda \geq 0$ ):

$$\lambda \|y\|_\mu \leq \|x\|_\mu.$$

Thus,  $\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$  for  $0 < \lambda \leq \mu$ . Since this holds for any sufficiently large  $\mu$ , and the definition of the norm can be adjusted, this property extends to all  $\lambda > 0$ .  $\square$

**Remark 2.4.2.** *In equivalent norm, the following are preserved:*

1. *The closeness of an operator.*
2. *The density of the image of an operator.*



### 3. The strong continuity of the $C_0$ -SG.

**Theorem 2.4.4** (Hille-Yosida for  $(M, 0)$ ). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a linear operator on a Banach space  $X$ . Then  $\mathcal{A}$  is the IG of a  $C_0$ -SG  $T(t)$  satisfying  $\|T(t)\| \leq M$  for all  $t \geq 0$  if and only if:*

- (i)  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A})} = X$ .
- (ii)  $\rho(\mathcal{A}) \supset (0, \infty)$  and for every  $\lambda > 0$ :

$$\|(\lambda^n R(\lambda, \mathcal{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

*Proof.*  $\Rightarrow$

Let  $\mathcal{A}$  be the generator of a  $C_0$ -semigroup  $T(t)$  with  $\|T(t)\| \leq M$ . By theorem 2.3.6 part (i) is done.

For part (ii) is direct from lemma 2.4.3

$\Leftarrow$

By using lemma 2.4.4 and apply it on the space the assumptions change to the following:

- (i)  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A})} = X$ .
- (ii)  $\mathbb{R}_+^* \subset \rho(\mathcal{A})$  and  $\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$  for all  $\lambda > 0$ .

Which are just the conditions on Hille-Yosida for contractions and we get the desired result.  $\square$

**Theorem 2.4.5** (Hille-Yosida for the Generale Case  $(M, \omega)$ ). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a linear operator on a Banach space  $X$ . Then  $\mathcal{A}$  is the IG of a  $C_0$ -SG  $T(t)$  satisfying  $\|T(t)\| \leq M \cdot e^{\omega t}$  for all  $t \geq 0$  if and only if:*

- (i)  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A})} = X$ .
- (ii)  $\rho(\mathcal{A}) \supset (0, \infty)$  and for every  $\lambda > 0$ :

$$\|(\lambda^n R(\lambda, \mathcal{A}))^n\| \leq M, \quad \forall n \in \mathbb{N}$$

**Theorem 2.4.6.** *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO and the IG of a  $C_0$ -SG  $T(t)_{t \geq 0}$  then for all  $x \in X$ :*

$$T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t\mathcal{A}_\lambda} x$$

Where  $\mathcal{A}_\lambda$  is the Yosida Approximation of  $\mathcal{A}$ .

*Proof.* This can have 2 cases depending on  $\omega$ .

First case, let  $\omega = 0$ . Then by applying lemma 2.4.4 we get a contraction and it is direct by corollary 2.4.1

Second case, let  $\omega > 0$ . We want to get a bound on the approximation.

$$\begin{aligned}
\|e^{t\mathcal{A}_\lambda}\| &= \|e^{t(\lambda^2 R(\lambda, \mathcal{A}) - \lambda I)}\| \\
&= e^{-\lambda t} \|e^{t\lambda^2 R(\lambda, \mathcal{A})}\| \\
&\leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(t\lambda^2)^k}{k!} \|R(\lambda, \mathcal{A})^k\| \\
&\leq e^{-\lambda t} M \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t\lambda^2}{\lambda - \omega}\right)^k \quad \text{since } \|R(\lambda, \mathcal{A})^k\| \leq \frac{M}{(\lambda - \omega)^k} \\
&= M e^{-\lambda t} \exp\left(\frac{t\lambda^2}{\lambda - \omega}\right) \\
&= M \exp\left(-\lambda t + \frac{t\lambda^2}{\lambda - \omega}\right)
\end{aligned}$$

We can do the following calculation in the exponent:

$$\begin{aligned}
-\lambda + \frac{\lambda^2}{\lambda - \omega} &= \left(\frac{-\lambda(\lambda - \omega) + \lambda^2}{\lambda - \omega}\right) \\
&= \frac{\lambda\omega}{\lambda - \omega} \\
&= \omega \left(\frac{\lambda - \omega + \omega}{\lambda - \omega}\right) \\
&= \omega \left(1 + \frac{\omega}{\lambda - \omega}\right) \\
&= \omega + \frac{\omega^2}{\lambda - \omega}
\end{aligned}$$

Now for all  $\lambda \geq 2\omega$  we have  $\lambda - \omega \geq \omega > 0$  and  $\frac{1}{\lambda - \omega} \leq \frac{1}{\omega}$ , hence:

$$\omega + \frac{\omega^2}{\lambda - \omega} \leq \omega + \frac{\omega^2}{\omega} = 2\omega$$

And we get the bound:

$$\|e^{t\mathcal{A}_\lambda}\| \leq M \cdot e^{2t\omega} \quad (*)$$

Let  $S(t) = e^{-\omega t} T(t)$ , this  $C_0$ -SG is generated by  $\mathcal{A} - \omega I$  (rescaling of  $T(t)$ ).

Since  $S(t)$  is  $(M, 0)$  then by the first case we have:

$$\forall t \geq 0, \forall x \in X \quad S(t)x = \lim_{\lambda \rightarrow +\infty} e^{t(\mathcal{A} - \lambda I)\lambda}$$

Equivalently:

$$e^{-\omega t} T(t)x = \lim_{\lambda \rightarrow +\infty} e^{t(\mathcal{A} - \lambda I)\lambda}$$

Now we have to show that  $\forall t \geq 0, \forall x \in X$  the following:

$$e^{t[(\mathcal{A}-\omega I)_\lambda + \omega I]}x - e^{t\mathcal{A}_\lambda}x = 0$$

We have the following:

$$\begin{aligned} e^{t[(\mathcal{A}-\omega I)_\lambda + \omega I]}x - e^{t\mathcal{A}_\lambda}x &= e^{t[(\mathcal{A}-\omega I)_\lambda + \omega I - \mathcal{A}_\lambda + \mathcal{A}_\lambda]}x - e^{t\mathcal{A}_\lambda}x \\ &= e^{t(A_\lambda + H(\lambda))}x - e^{tA_\lambda}x && \text{by setting } H(\lambda) := (\mathcal{A} - \omega I)_\lambda + \omega I - \mathcal{A}_\lambda \\ &= e^{tA_\lambda}e^{tH(\lambda)}x - e^{tA_\lambda}x \\ &= e^{tA_\lambda} \left( e^{tH(\lambda)} - I \right) x && \mathcal{A}_\lambda \text{ and } H(\lambda) \text{ commutes (why?)} \end{aligned}$$

Now, we aim to show that for any fixed  $x \in D(A)$ ,  $H(\lambda)x \rightarrow 0$  as  $\lambda \rightarrow +\infty$ .

By theorem 2.4.2 we have:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} H(\lambda)x &= \lim_{\lambda \rightarrow \infty} ((A - \omega I)_\lambda x + \omega x - A_\lambda x) \\ &= (Ax - \omega x) + \omega x - Ax \\ &= 0 \end{aligned}$$

Thus,  $H(\lambda)x \rightarrow 0$  for all  $x \in D(A)$ .

Now we have:

$$e^{t[(\mathcal{A}-\omega I)_\lambda + \omega I]}x - e^{t\mathcal{A}_\lambda}x = e^{tA_\lambda} \left( e^{tH(\lambda)} - I \right) x$$

By taking the norm:

$$\left\| e^{t[(\mathcal{A}-\omega I)_\lambda + \omega I]}x - e^{t\mathcal{A}_\lambda}x \right\| \leq \|e^{tA_\lambda}\| \left\| \left( e^{tH(\lambda)} - I \right) x \right\|$$

From (\*), we have that  $\|e^{tA_\lambda}\|$  is uniformly bounded hence we need to work on  $\|(e^{tH(\lambda)} - I)x\|$ .

Note that  $e^{tH(\lambda)} = e^{t(B_\lambda + \omega I)}e^{-tA_\lambda}$ . From the bound estimates shown earlier, both  $\|e^{s(B_\lambda + \omega I)}\|$  and  $\|e^{-sA_\lambda}\|$  are uniformly bounded for  $s$  in a compact interval  $[0, T]$  and sufficiently large  $\lambda$ . Let  $C$  be this bound. Then:

$$\|(e^{tH(\lambda)} - I)x\| \leq t \cdot C \cdot \|H(\lambda)x\|$$

Since we proved that  $\lim_{\lambda \rightarrow \infty} \|H(\lambda)x\| = 0$ , it follows immediately that:

$$\lim_{\lambda \rightarrow \infty} \|(e^{tH(\lambda)} - I)x\| = 0$$

Finally, combining this with the uniform bound on  $\|e^{tA_\lambda}\|$ , we obtain:

$$\lim_{\lambda \rightarrow \infty} \|e^{t(B_\lambda + \omega I)}x - e^{tA_\lambda}x\| = 0$$

This proves that the limit is the same, regardless of the shift. This completed our proof.  $\square$

## 2.5 Inverse Laplace Transform and Resolvent Formula

We have established that for a  $C_0$ -SG  $(T(t))_{t \geq 0}$  with  $\|T(t)\| \leq Me^{t\omega}$ , the resolvent is given using the following formula:

$$R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x dt \quad \text{for } \Re(\lambda) > \omega.$$

which is just the Laplace Transform of the semigroup.

A major problem in semigroup theory is recovering  $T(t)$  from  $R(\lambda, \mathcal{A})x$  (Inverting the Laplace Transform).

**Lemma 2.5.1.** *Let  $X$  be BS and  $\mathcal{B} \in \mathcal{L}(X)$  then  $\forall \gamma > \|\mathcal{B}\|$  we have:*

$$\begin{aligned} e^{t\mathcal{B}} &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda, \mathcal{B}) d\lambda \\ &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{(\gamma+is)t} R(\gamma+is, \mathcal{B}) i ds \end{aligned}$$

*Proof.* To make sense and motivate this proof we start by showing that this is correct for the scalar case. Let  $b \in \mathbb{C}$  and the exponential function  $f(t) = e^{tb}$ . The Laplace transform of  $f(t)$  is:

$$\mathcal{L}\{e^{tb}\}(\lambda) = \int_0^\infty e^{-\lambda t} e^{tb} dt = \frac{1}{\lambda - b}, \quad \text{for } \Re(\lambda) > \Re(b)$$

Now if we choose  $|\lambda| > |b|$ . We can expand  $(\lambda - b)^{-1}$  as a convergent geometric series: Now if we choose  $\gamma$  such that  $\Re(\gamma) > |b|$ . Let show that:

$$e^{tb} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{1}{\lambda - b} d\lambda$$

For a circle  $C(0, r)$  with radius  $r > |b|$ , we can expand the term  $(\lambda - b)^{-1}$  as a geometric series for  $|\lambda| > |b|$ :

$$\frac{1}{\lambda - b} = \frac{1}{\lambda(1 - \frac{b}{\lambda})} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{b^k}{\lambda^k} = \sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}$$

Plugging this into the formula above we get:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{1}{\lambda - b} d\lambda &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}} d\lambda \quad (\text{the choice of } \gamma \text{ allow for this}) \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} b^k \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^{k+1}} d\lambda \quad (\text{The sum is uniformly convergent}) \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} 2\pi i \frac{b^k t^k}{k!} \quad (\text{Cauchy Integral Formula}) \\ &= e^{bt} \end{aligned}$$

This shows that the formula works for the scalar case.

The case of the operator follows the same steps and same reasoning.

Again let  $C(0, r)$  with radius  $r$  such that  $r > \|\mathcal{B}\|$ . For any  $\lambda \in C(0, r)$ , we have  $|\lambda| > \|\mathcal{B}\|$ , which implies  $\|\frac{\mathcal{B}}{\lambda}\| < 1$ .

In this region, the resolvent  $R(\lambda, \mathcal{B}) = (\lambda I - \mathcal{B})^{-1}$  admits a uniformly convergent Neumann series expansion:

$$R(\lambda, \mathcal{B}) = \frac{1}{\lambda} \left( I - \frac{\mathcal{B}}{\lambda} \right)^{-1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}^k}{\lambda^{k+1}}.$$

We substitute this series into the contour integral over  $C(0, r)$ :

$$\begin{aligned} \frac{1}{2\pi i} \int_{C(0, r)} e^{\lambda t} R(\lambda, \mathcal{B}) d\lambda &= \frac{1}{2\pi i} \int_{C(0, r)} e^{\lambda t} \left( \sum_{k=0}^{\infty} \frac{\mathcal{B}^k}{\lambda^{k+1}} \right) d\lambda \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \mathcal{B}^k \left[ \int_{C_r} \frac{e^{\lambda t}}{\lambda^{k+1}} d\lambda \right] \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \mathcal{B}^k \left[ 2\pi i \frac{t^k}{k!} \right] \\ &= \sum_{k=0}^{\infty} \frac{t^k \mathcal{B}^k}{k!} \\ &= e^{t\mathcal{B}} \end{aligned}$$

□

## 2.6 Spectral Mapping Theorem

Let  $(T(t))_{t \geq 0}$  be  $C_0$ -SG with  $\mathcal{A}$  as its IG, then what is the relation between  $\sigma(T(t))$  and  $\sigma(\mathcal{A})$  ?

### Exercise 12

Let  $(T(t))_{t \geq 0}$  be UC-SG with  $\mathcal{A}$  as its IG then  $e^{t\sigma(\mathcal{A})} = \sigma(T(t))$ .

### Exercise 13

Take  $X = \{f : [0, 1] \rightarrow \mathbb{C} \mid f \text{ is continuous and } f(1) = 0\}$  then  $(X, \|\cdot\|_\infty)$  is BS. Define:

$$T(t)f(x) = \begin{cases} f(x+t) & x+t \leq 1 \\ 0 & x+t > 1 \end{cases}$$

Then:

1.  $(T(t))_{t \geq 0}$  is a  $C_0$ -SG.
2.  $\mathcal{A}$  the IG and  $\mathcal{A}f = f'$ . Determine  $D(\mathcal{A})$ .
3. Check  $\forall \lambda \in \mathbb{C}, \exists! f \in X$  such that  $(\lambda I - \mathcal{A})f = g$  for all  $g \in X$ .
4.  $\sigma(\mathcal{A}) = \emptyset$ .
5.  $\sigma(T(t)) \neq \emptyset$ .

**Lemma 2.6.1.** [Preparatory Lemma] Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -SG and let  $\mathcal{A}$  be its IG. For all  $t \geq 0$  and for all  $x \in X$  define:

$$\mathcal{B}_\lambda(t)x = \int_0^t e^{\lambda(t-s)} T(s)x \, ds$$

then  $(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$ .

*Proof.* It is enough to show that:

$$\mathcal{A}\mathcal{B}_\lambda(t)x = T(t)x + \lambda\mathcal{B}_\lambda(t)x - e^{\lambda t}x$$

Recall that:

$$\mathcal{A} = \lim_{h \rightarrow 0^+} \frac{T(h) - Id}{h}$$

Then we have:

$$\begin{aligned} \frac{T(h) - Id}{h} \mathcal{B}_\lambda(t)x &= \frac{1}{h} \int_0^t e^{\lambda(t-s)} (T(h)T(s) - T(s)) x \, ds \\ &= \frac{1}{h} \left[ \int_0^t e^{\lambda(t-s)} T(s+h)x \, ds - \int_0^t e^{\lambda(t-s)} T(s)x \, ds \right] \\ &= \frac{1}{h} \left[ \int_h^{t+h} e^{\lambda(t-s+h)} T(s)x \, ds - \int_0^t e^{\lambda(t-s)} T(s)x \, ds \right] \\ &= \frac{1}{h} \left[ e^{\lambda h} \int_h^{t+h} e^{\lambda(t-s)} T(s)x \, ds - \int_0^t e^{\lambda(t-s)} T(s)x \, ds \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h} \left[ (e^{\lambda h} - 1) \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds + \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds - \int_0^t e^{\lambda(t-s)} T(s) x \, ds \right] \\
&= \frac{1}{h} \left[ (e^{\lambda h} - 1) \int_h^{t+h} e^{\lambda(t-s)} T(s) x \, ds + \int_t^{t+h} e^{\lambda(t-s)} T(s) x \, ds - \int_0^h e^{\lambda(t-s)} T(s) x \, ds \right]
\end{aligned}$$

Now taking  $\lim_{h \rightarrow 0^+}$  we get:

$$\mathcal{A} \mathcal{B}_\lambda(t)x = \lambda \mathcal{B}_\lambda(t)x + T(t)x - e^{\lambda t}x$$

□

**Remark 2.6.1.** In the lemma above  $\mathcal{A}$  commutes if  $x \in D(\mathcal{A})$ , and we get:

$$\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = (\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)x = e^{\lambda t}x - T(t)x$$

This is true because of the fact that  $\mathcal{A}$  is the limit.

**Theorem 2.6.1.** Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -SG generated by  $\mathcal{A}$ . Then for all  $t \geq 0$ :

$$e^{t\sigma(\mathcal{A})} \subset \sigma(T(t))$$

*Proof.* We prove the equivalent statement: if  $e^{\lambda t} \in \rho(T(t))$ , then  $\lambda \in \rho(\mathcal{A})$ .

Assume  $e^{\lambda t} \in \rho(T(t))$ . Then the operator  $(e^{\lambda t}I - T(t))$  is invertible with a bounded inverse  $Q = (e^{\lambda t}I - T(t))^{-1} \in \mathcal{L}(X)$ .

From Lemma 2.6.1, we have:

$$(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t) = e^{\lambda t}I - T(t).$$

**Claim 2.6.1.**  $Q$  and  $\mathcal{B}_\lambda(t)$  commute. (Prove it)

Multiplying by  $Q$  on the right:

$$(\lambda I - \mathcal{A})\mathcal{B}_\lambda(t)Q = I.$$

Similarly, for  $x \in D(\mathcal{A})$ :

$$Q\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = x.$$

Defining  $R(\lambda) = \mathcal{B}_\lambda(t)Q$ , we see that  $R(\lambda)$  acts as a two-sided inverse for  $(\lambda I - \mathcal{A})$ . Since  $\mathcal{B}_\lambda(t)$  and  $Q$  are bounded,  $R(\lambda)$  is bounded. Thus  $\lambda \in \rho(\mathcal{A})$ .

By contraposition,  $\lambda \in \sigma(\mathcal{A}) \implies e^{\lambda t} \in \sigma(T(t))$ .

□

While the full spectral mapping theorem fails, a precise relationship holds for the point spectrum  $\sigma_p$  and the residual spectrum  $\sigma_r$ .

Recall the decomposition of the spectrum:

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where  $\sigma_p$  denotes point spectrum (eigenvalues),  $\sigma_c$  continuous spectrum, and  $\sigma_r$  residual spectrum.

**Theorem 2.6.2.** *Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -SG generated by  $\mathcal{A}$ . Then:*

$$e^{t\sigma_p(\mathcal{A})} \subseteq \sigma_p(T(t)) \subseteq e^{t\sigma_p(\mathcal{A})} \cup \{0\}$$

*Precisely, if  $e^{\lambda t} \in \sigma_p(T(t))$ , then there exists  $k \in \mathbb{Z}$  such that:*

$$\lambda_k = \lambda + \frac{2\pi i k}{t} \in \sigma_p(\mathcal{A})$$

*Proof.* First inclusion,  $\lambda \in \sigma_p(\mathcal{A}) \implies e^{\lambda t} \in \sigma_p(T(t))$ .

Let  $\lambda \in \sigma_p(\mathcal{A})$ . Then there exists  $x_0 \neq 0$  such that  $\mathcal{A}x_0 = \lambda x_0$  and by lemma 2.6.1 we have:

$$\mathcal{B}_\lambda(t)(\lambda I - \mathcal{A})x = \left(e^{\lambda t} I - T(t)\right)x \quad \forall x \in D(\mathcal{A})$$

Now take  $x = x_0$  and note the fact that  $(\lambda I - \mathcal{A})x_0 = 0$ :

$$\left(e^{\lambda t} I - T(t)\right)x_0 = 0$$

Hence  $\ker(e^{\lambda t} I - T(t)) \neq \{0\}$  then  $e^{\lambda t} \in \sigma_p(T(t))$ .

For the second inclusion, let  $\mu \in \sigma_p(T(t))$  with  $\mu \neq 0$ . Write  $\mu = e^{\lambda t}$  for some  $\lambda \in \mathbb{C}$ .

There exists  $x_0 \neq 0$  such that  $T(t)x_0 = e^{\lambda t}x_0$ .

Consider the function  $\phi_{x_0}(s)$  defined by:

$$\phi_{x_0}(s) = e^{-\lambda s} T(s)x_0$$

Clearly:

$$\phi_{x_0}(0) = x_0 = \phi_{x_0}(t)$$

We check if  $\phi_{x_0}$  is periodic with period  $t$ .

$$\phi_{x_0}(s+t) = e^{-\lambda(s+t)} T(s+t)x_0 = e^{-\lambda s} e^{-\lambda t} T(s)T(t)x_0$$

Using the eigen-property  $T(t)x_0 = e^{\lambda t}x_0$ :

$$\phi_{x_0}(s+t) = e^{-\lambda s} e^{-\lambda t} T(s)(e^{\lambda t}x_0) = e^{-\lambda s} T(s)x_0 = \phi_{x_0}(s)$$

Since  $\phi_{x_0}(s)$  is a  $t$ -periodic continuous function (and assuming  $x_0 \in X$ ), we can expand it into a



Fourier series. The  $k$ -th Fourier coefficient is:

$$\hat{x}_k = \frac{1}{t} \int_0^t e^{-2\pi i k s/t} \phi_{x_0}(s) ds$$

Substituting  $\phi_{x_0}(s) = e^{-\lambda s} T(s) x_0$ :

$$\hat{x}_k = \frac{1}{t} \int_0^t e^{-(\lambda + \frac{2\pi i k}{t})s} T(s) x_0 ds$$

Let  $\lambda_k = \lambda + \frac{2\pi i k}{t}$ . Since  $\phi_{x_0}$  is not identically zero (as  $\phi_{x_0}(0) = x_0 \neq 0$ ), at least one coefficient  $\hat{x}_k$  must be non-zero.

We will show that if  $\hat{x}_k \neq 0$ , then  $\lambda_k \in \sigma_p(\mathcal{A})$ .

We use the resolvent formula. For  $\Re(\gamma)$  sufficiently large,  $R(\gamma, \mathcal{A}) = \int_0^\infty e^{-\gamma s} T(s) ds$ . Applying this to  $x_0$ :

$$R(\gamma, \mathcal{A})x_0 = \int_0^\infty e^{-\gamma s} T(s) x_0 ds$$

Decompose the integral over intervals  $[nt, (n+1)t]$ :

$$\begin{aligned} R(\gamma, \mathcal{A})x_0 &= \sum_{n=0}^\infty \int_{nt}^{(n+1)t} e^{-\gamma s} T(s) x_0 ds \\ &= \sum_{n=0}^\infty \int_0^t e^{-\gamma(nt+\tau)} T(nt+\tau) x_0 d\tau \quad (\text{let } s = nt + \tau) \end{aligned}$$

Using the semigroup property  $T(nt + \tau)x_0 = T(\tau)T(t)^n x_0 = T(\tau)e^{n\lambda t} x_0$ :

$$\begin{aligned} R(\gamma, \mathcal{A})x_0 &= \sum_{n=0}^\infty e^{-n\gamma t} e^{n\lambda t} \int_0^t e^{-\gamma \tau} T(\tau) x_0 d\tau \\ &= \left( \sum_{n=0}^\infty e^{-n(\gamma-\lambda)t} \right) \int_0^t e^{-\gamma \tau} T(\tau) x_0 d\tau \end{aligned}$$

The geometric series converges if  $\Re(\gamma) > \Re(\lambda)$ :

$$\sum_{n=0}^\infty (e^{-(\gamma-\lambda)t})^n = \frac{1}{1 - e^{-(\gamma-\lambda)t}}$$

Thus:

$$R(\gamma, \mathcal{A})x_0 = \frac{1}{1 - e^{(\lambda-\gamma)t}} \int_0^t e^{-\gamma \tau} T(\tau) x_0 d\tau$$

The function  $\gamma \mapsto R(\gamma, \mathcal{A})x_0$  is meromorphic. The poles of the resolvent indicate the spectrum. The denominator vanishes when:

$$e^{(\lambda-\gamma)t} = 1 \iff (\lambda - \gamma)t = 2\pi i k \iff \gamma = \lambda - \frac{2\pi i k}{t}$$

Let  $\mu_k = \lambda + \frac{2\pi i k}{t}$ . The resolvent has a pole at  $\mu_k$ . Specifically, the residue near the pole relates to the existence of an eigenvector. If we analyze the limit:

$$\lim_{\gamma \rightarrow \lambda_k} (\gamma - \lambda_k) R(\gamma, \mathcal{A}) x_0 \neq 0 \implies \lambda_k \in \sigma_p(\mathcal{A}).$$

This confirms that the eigenvalues of  $\mathcal{A}$  are exactly the logarithms of the eigenvalues of  $T(t)$  (modulo  $2\pi i/t$ ). □

## APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Let  $X$  be a Banach space and  $A : D(A) \subset X \rightarrow X$  be a linear operator. We define the homogeneous and inhomogeneous abstract Cauchy problems as follows:

- Homogeneous Cauchy Problem  $h - (CP)_{0,x_0}$ :

$$\frac{du}{dt} = Au, \quad t \geq 0$$

$$u(0) = x_0$$

- Inhomogeneous Cauchy Problem  $inh - (CP)_{0,x_0}$ :

$$\frac{du}{dt} = Au + f(t), \quad t \geq 0$$

$$u(0) = x_0$$

where  $f : \mathbb{R}_+ \rightarrow X$ .

### 3.1 The Cauchy Problem

**Definition 3.1.1.** We distinguish three types of solutions for the Cauchy problem:

1. **Strong Solution:** A function  $x(\cdot) \in C^1(\mathbb{R}_+, D(\mathcal{A}))$  such that for all  $t \geq 0$ ,  $x(t) \in D(\mathcal{A})$  and  $\frac{dx}{dt} = \mathcal{A}x(t)$ .
2. **Mild Solution:** A function  $x(\cdot) \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+^*, D(\mathcal{A}))$  with  $x(0) = x_0$ , and for all  $t > 0$ ,  $x(t) \in D(\mathcal{A})$  such that  $\frac{dx}{dt} = \mathcal{A}x(t)$ .
3. **Weak Solution:** A function  $x(\cdot) \in C^0(\mathbb{R}_+, X)$  such that  $x(0) = x_0$ , and for all  $x^* \in D(A^*)$ , the map  $t \mapsto \langle x^*, x(t) \rangle$  is  $C^1$  and satisfies:

$$\frac{d}{dt} \langle x^*, x(t) \rangle = \langle A^* x^*, x(t) \rangle$$

**Remark 3.1.1.** *Some properties of the solutions and constraints on the initial value:*

1. If  $x_0 \notin D(A)$ , there is no valid strong solution for  $(CP)_{0,x_0}$ .
2. For all  $x_0 \in X$ ,  $t \mapsto T(t)x_0$  is the unique solution of  $(CP)_{0,x_0}$ .

**Example 3.1.1.** *Consider the transport equation:*

$$u_t = u_x \quad x \in [0, 1], \quad t \geq 0$$

subject to boundary conditions at  $t = 0$ ,  $u(0, \cdot) = u_0(\cdot) \in X$ .

We associate this problem with the operator  $\mathcal{A}$  defined by  $\mathcal{A}u = \frac{du}{dx}$ . To define the weak solution, we need to determine the action of the adjoint operator  $\mathcal{A}^*$ .

Let  $\varphi$  be a test function (of  $D(A^*)$ ). We compute the pairing  $\langle \mathcal{A}u, \varphi \rangle$  using the inner product in  $L^2([0, 1])$ :

$$\langle \mathcal{A}u, \varphi \rangle = \int_0^1 (\mathcal{A}u)(x) \varphi(x) dx = \int_0^1 \frac{du}{dx}(x) \varphi(x) dx$$

By integration by parts:

$$\int_0^1 u'(x) \varphi(x) dx = [u(x) \varphi(x)]_0^1 - \int_0^1 u(x) \varphi'(x) dx$$

Using the Dirichlet boundary conditions ( $u(t, 1) = u(t, 0) = 0$ ), the boundary term  $[u(x) \varphi(x)]_0^1$  vanishes. Thus, we obtain:

$$\langle \mathcal{A}u, \varphi \rangle = - \int_0^1 u(x) \varphi'(x) dx = \int_0^1 u(x) \left( -\frac{d\varphi}{dx} \right) dx$$

This can be rewritten in terms of the inner product as:

$$\langle \mathcal{A}u, \varphi \rangle = \langle u, -\varphi' \rangle$$

From the definition of the adjoint operator  $\langle \mathcal{A}u, \varphi \rangle = \langle u, \mathcal{A}^* \varphi \rangle$ , we identify:

$$\mathcal{A}^* \varphi = -\frac{d\varphi}{dx}$$

Therefore, the condition for  $u(t)$  to be a weak solution:

$$\frac{d}{dt} \langle \varphi, u(t) \rangle = \langle \mathcal{A}^* \varphi, u(t) \rangle$$

becomes:

$$\frac{d}{dt} \int_0^1 u(t, x) \varphi(x) dx = - \int_0^1 u(t, x) \varphi'(x) dx$$

## 3.2 Homogeneous Cauchy Problem

**Theorem 3.2.1.** *Let  $\mathcal{A}$  be the IG of a  $C_0$ -SG  $(T(t))_{t \geq 0}$ , then:*

1. *For all  $x_0 \in D(\mathcal{A})$ , there exists a unique strong solution of  $(CP)_{0,x_0}$  given by  $u(t) = T(t)x_0$ , which is also a mild and weak solution.*
2. *For all  $x_0 \in X$ ,  $t \mapsto T(t)x_0$  is a weak solution of  $(CP)_{0,x_0}$ .*

*Proof.* The proof of part 1 is direct.

For part 2: Let  $x_0 \in X$  and choose a sequence  $x_n \in D(\mathcal{A})$  such that  $x_n \rightarrow x_0$  (why is this possible?). For all  $x^* \in D(\mathcal{A}^*)$  and  $t \geq 0$ , the function  $t \mapsto \langle x^*, T(t)x_n \rangle$  is  $C^1$ , and we have:

$$\frac{d}{dt} \langle x^*, T(t)x_n \rangle = \langle x^*, \mathcal{A}T(t)x_n \rangle = \langle \mathcal{A}^*x^*, T(t)x_n \rangle$$

Integrating yields:

$$\langle x^*, T(t)x_n \rangle = \langle x^*, x_n \rangle + \int_0^t \langle \mathcal{A}^*x^*, T(s)x_n \rangle ds$$

By passing to the limit as  $n \rightarrow \infty$  (show how), we extend this formula for all  $x_0 \in X$ , showing  $t \mapsto T(t)x_0$  is a weak solution.

Now to show the uniqueness of the solution, consider two weak solutions  $x_1(\cdot)$  and  $x_2(\cdot)$  of  $(CP)_{0,x_0}$ . Set  $u = x_1 - x_2$ . Then  $u$  is a weak solution of the problem with initial condition 0. For all  $t \geq 0$  and  $x^* \in D(\mathcal{A}^*)$ , we have:

$$\langle x^*, u(t) \rangle = \int_0^t \langle \mathcal{A}^*x^*, u(s) \rangle ds$$

Let  $U(t) = \int_0^t u(s) ds$ , with  $U(0) = 0$ . Then,  $\langle x^*, u(t) \rangle = \langle \mathcal{A}^*x^*, U(t) \rangle$ , which implies:

$$\frac{d}{dt} \langle x^*, U(t) \rangle = \langle \mathcal{A}^*x^*, U(t) \rangle \quad (***)$$

**Claim 3.2.1.** *If  $(T(t))_{t \geq 0}$  is a  $C_0$ -SG with  $\mathcal{A}$  as its IG then  $T(t)^*D(\mathcal{A}^*) \subset D(\mathcal{A}^*)$  and  $T(t)^*$  and  $\mathcal{A}^*$  commute, that is:*

$$T(t)^*\mathcal{A}^* = \mathcal{A}^*T(t)^*$$

*Proof of claim.* We have to show that  $\forall x^* \in D(\mathcal{A}^*), \forall x \in D(\mathcal{A}), \forall t \geq 0$ , the following:

$$\exists \xi^* \in X^*, \langle T(t)^*x^*, \mathcal{A}x \rangle = \langle \xi^*, x \rangle$$

We have the following:

$$\begin{aligned}
\langle T(t)^* x^*, \mathcal{A}x \rangle &= \langle x^*, T(t)\mathcal{A}x \rangle \\
&= \langle x^*, \mathcal{A}T(t)x \rangle \\
&= \langle \mathcal{A}^* x^*, T(t)x \rangle \\
&= \langle T(t)^* \mathcal{A}^* x^*, x \rangle
\end{aligned}$$

Then  $T(t)^* \mathcal{A}^* x^* \in D(\mathcal{A}^*)$  and  $T(t)^* \mathcal{A}^* = \mathcal{A}^* T(t)^*$ . □

Apply (\*\*\*) replacing  $x^*$  with  $T^*(t^* - t)x^*$ , where  $0 \leq t \leq t^*$ :

$$\frac{d}{dt} \langle T^*(t^* - t)x^*, U(t) \rangle = \langle -\mathcal{A}^* T^*(t^* - t)x^*, U(t) \rangle + \langle T^*(t^* - t)x^*, u(t) \rangle$$

Since  $\langle T^*(t^* - t)x^*, u(t) \rangle = \langle \mathcal{A}^* T^*(t^* - t)x^*, U(t) \rangle$ , the derivative is 0.

Integrating from  $t = 0$  to  $t = t^*$  yields:

$$\langle x^*, U(t^*) \rangle - \langle T^*(t^*)x^*, U(0) \rangle = 0$$

Since  $U(0) = 0$ , we get  $\langle x^*, U(t^*) \rangle = 0$  for all  $t^* \geq 0$ . Since this holds for all  $x^* \in D(\mathcal{A}^*)$  and  $D(\mathcal{A}^*)$  is weak\*-dense,  $U(t) \equiv 0$ .

Differentiating gives  $u(t) = 0$ , proving uniqueness. □

**Theorem 3.2.2.** *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a closed, densely defined operator. If there exists  $\omega$  such that for  $\lambda > \omega$ ,  $\lambda \in \rho(\mathcal{A})$  and  $\|R(\lambda, \mathcal{A})\| = o(e^{\sigma\lambda})$ , then  $(CP)_{0,x_0}$  admits a unique mild solution.*

We first prove a necessary lemma for uniqueness:

**Lemma 3.2.1.** *Let  $u : [0, T] \rightarrow X$  be continuous with  $u(0) = 0$ . Assume there exists  $M \geq 0$  such that for all  $n \geq 0$ ,*

$$\left\| \int_0^T e^{ns} u(s) ds \right\| \leq M$$

*Then  $u(t) = 0$  for all  $t \in [0, T]$ .*

*Proof.* For any  $x^* \in X^*$ , define the continuous function  $\varphi(s) := \langle x^*, u(s) \rangle$ . By assumption,  $\left\| \int_0^T e^{ns} \varphi(s) ds \right\| \leq M$  for all  $n$ . Define the sequence of functions:

$$\psi_n(t) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k!} e^{knt} = 1 - e^{-e^{nt}}$$

We consider the integral:

$$\int_0^T \psi_n(t - T + s) \varphi(s) ds$$

By substituting the sum definition of  $\psi_n$  and using the bound on  $\varphi$ , we have:

$$\begin{aligned} \left\| \int_0^T \sum_{k \geq 1} \frac{(-1)^{k+1}}{k!} e^{kn(t-T+s)} \varphi(s) ds \right\| &\leq \sum_{k \geq 1} \frac{1}{k!} e^{kn(t-T)} \left\| \int_0^T e^{kns} \varphi(s) ds \right\| \\ &\leq M \sum_{k \geq 1} \frac{(e^{n(t-T)})^k}{k!} \\ &= M(e^{e^{n(t-T)}} - 1) \end{aligned}$$

For  $t < T$ , as  $n \rightarrow \infty$ ,  $n(t-T) \rightarrow -\infty$ , thus  $e^{n(t-T)} \rightarrow 0$ , meaning the integral evaluates to 0.

On the other hand, analyzing the behavior of  $\psi_n(t-T+s)$  directly as  $n \rightarrow \infty$ :

- If  $s > T-t$ , then  $t-T+s > 0$ , so  $\psi_n \rightarrow 1$ .
- If  $s < T-t$ , then  $t-T+s < 0$ , so  $\psi_n \rightarrow 0$ .

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^T \psi_n(t-T+s) \varphi(s) ds = \int_{T-t}^T \varphi(s) ds$$

Equating the two limits, we deduce  $\int_{T-t}^T \varphi(s) ds = 0$  for all  $t$ . Differentiating with respect to  $t$  gives  $\varphi(T-t) = 0$ , implying  $\varphi \equiv 0$ , and thus  $u \equiv 0$ .  $\square$

*Proof of theorem 3.2.2.* Without loss of generality we can assume  $\omega = 0$  since if it is not then the transform  $\mathcal{A} \mapsto \mathcal{A} - zI$ , shifts the coefficient by  $e^{zt}$ , and:

$$\begin{aligned} \frac{du_z}{dt} &= (\mathcal{A} + zI)u_z \\ \frac{du}{dt} &= \mathcal{A}u \end{aligned}$$

Assume  $u$  is a mild solution of  $(CP)_{0,0}$ , meaning  $u(0) = 0$  and  $\frac{du}{dt} = \mathcal{A}u$ . For  $\lambda > 0$ , define  $v(t) = R(\lambda, \mathcal{A})u(t)$ . Since

$$\frac{d}{dt} R(\lambda, \mathcal{A})u(t) = R(\lambda, \mathcal{A})\mathcal{A}u(t) = R(\lambda, \mathcal{A})(\mathcal{A} - \lambda I + \lambda I)u(t) = -u(t) + \lambda R(\lambda, \mathcal{A})u(t)$$

we get  $\frac{dv}{dt} - \lambda v(t) = -u(t)$ . Solving this ODE yields:

$$R(\lambda, \mathcal{A})u(t) = \int_0^t e^{\lambda(t-s)} u(s) ds$$

Multiplying by  $e^{-\sigma\lambda}$ , we get:

$$e^{-\sigma\lambda} R(\lambda, \mathcal{A})u(t) = \int_0^t e^{\lambda(t-\sigma-s)} u(s) ds$$

**Claim 3.2.2.**  $\lim_{\lambda \rightarrow +\infty} \int_0^t e^{\lambda(t-\sigma-s)} u(s) ds = 0.$

*Proof.* Do it. (Hint: use the bond from the assumption and lemma 3.2.1) □

By the claim we get  $u \equiv 0.$  □

#### Exercise 14: Homework

Let  $X$  be a Hilbert Space and  $\mathcal{A}$  be defined by

$$\mathcal{A}y = \sum \lambda_n \langle y, e_n \rangle e_n$$

over an orthonormal basis  $\{e_n\}$ , with real numbers  $\lambda_n \nearrow +\infty$ . Its domain is  $D(\mathcal{A}) = \{y \in X \mid \sum \lambda_n^2 \langle y, e_n \rangle^2 < +\infty\}.$

1. Prove  $\mathcal{A}$  is self-adjoint.
2. Prove it generates a  $C_0$ -SG.

(Hint:  $\lambda I - \mathcal{A}$  is formally bounded if and only if  $\lim_{n \geq 1} |\lambda - \lambda_n| > 0$ )

What can we say without  $\mathcal{A}$  being the IG of some  $C_0$ -SG?

**Theorem 3.2.3** (Existence of Solution). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO with  $\overline{D(\mathcal{A})} = X$  such that  $\rho(\mathcal{A}) \supset (\lambda_0, \infty)$ ,  $\lambda_0 \in \mathbb{R}$  and*

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \ln \|R(\lambda, \mathcal{A})\| = 0$$

*then  $\forall x \in X$   $(CP)_{0,x}$  admits a mild solution.*

Note the difference between the assumption here and Hille-Yosida assumption, so one can not use Hille-Yosida directly.

**Lemma 3.2.2.** *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO then:*

1.  $\forall \lambda \in \rho(\mathcal{A}) \ D(\mathcal{A}^2) = (\lambda I - \mathcal{A})^{-1} D(\mathcal{A}).$
2. *If  $\rho(\mathcal{A}) \neq \emptyset$  and  $\overline{D(\mathcal{A})} = X$  then  $\overline{D(\mathcal{A}^2)} = X.$*

*Proof.* Left as an exercise. □

**Theorem 3.2.4** (Uniqueness of Solution). *Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be a UBLO with  $\overline{D(\mathcal{A})} = X$  and  $\rho(\mathcal{A}) \neq \emptyset$ .  $\forall x \in D(\mathcal{A})$ ,  $(CP)_{0,x}$  admits a mild solution if and only if  $\mathcal{A}$  is the IG of some  $C_0$ -SG.*

*Proof.* Let's start with ( $\Leftarrow$ ). Assume  $\mathcal{A}$  is the IG of some  $C_0$ -SG then by theorem 3.2.1 part 1 we have a unique strong solution which is also mild.



For (  $\implies$  ). We need to construct a  $C_0$ -SG with  $\mathcal{A}$  as its generator. For this we use the flow of the solution. For all  $x \in D(\mathcal{A})$  we have:

$$\begin{aligned}\frac{du}{dt} &= \mathcal{A}u \\ u(0) &= x\end{aligned}$$

Since  $u$  is the unique mild solution of  $(CP)_{0,x}$  (actually it is a strong solution as well, why?). Then the mapping:

$$\begin{aligned}u : \mathbb{R}_+ \times D(\mathcal{A}) &\rightarrow D(\mathcal{A}) \\ (t, x) &\mapsto u(t, x)\end{aligned}$$

is well defined. We can now consider the flow associated with  $(CP)_{0,x}$ . For each  $t \geq 0$ , define the operator  $T(t) : D(\mathcal{A}) \rightarrow D(\mathcal{A})$  by:

$$T(t)x = u(t, x)$$

We must prove that the family  $(T(t))_{t \geq 0}$  forms a  $C_0$ -SG.

1. Identity:  $T(0)x = u(0, x) = x$ . Thus,  $T(0) = Id_{D(\mathcal{A})}$ .
2. Linearity: The operator  $T(t)$  is linear due to the linearity of the derivative and the operator  $\mathcal{A}$ .
3. Semigroup Property: We need to show  $T(t + s) = T(t)T(s)$ .

By definition,  $T(t)T(s)x = u(t, T(s)x)$ , which is the solution evaluated at time  $t$  for the initial value problem starting at  $T(s)x$ . Similarly,  $T(t + s)x = u(t + s, x)$ , which is the solution evaluated at time  $t + s$  for the initial value problem starting at  $x$ . Due to the time-invariance of the equation  $\frac{du}{dt} = \mathcal{A}u$  and the uniqueness of the mild solution, advancing the flow by  $s$  and then by  $t$  is equivalent to advancing it by  $t + s$ . Hence,  $T(t + s) = T(t)T(s)$ .

Now we need to find the bound on  $T(t)$  to show it is  $C_0$ -SG. To do this we need to equip the domain  $D(\mathcal{A})$  with a suitable Banach space structure (put a complete norm on it).

We put the graph norm on  $D(\mathcal{A})$ :

$$|x|_G = \|x\| + \|\mathcal{A}x\|$$

With this and the fact that  $\mathcal{A}$  is closed (why?),  $(D(\mathcal{A}), |\cdot|_G)$  is a Banach space. This will be denoted by  $[D(\mathcal{A})]$ .

Let  $t_0 > 0$ , define  $X_{t_0}$  as follows:

$$X_{t_0} = C^0([0, t_0], [D(\mathcal{A})])$$

Clearly,  $(X_{t_0}, \|\cdot\|_\infty)$  is a Banach space.

Define the operator  $S_{t_0}$  by:

$$\begin{aligned} S_{t_0} : [D(A)] &\rightarrow X_{t_0} \\ x &\mapsto (t \mapsto u(t, x) = T(t)x) \end{aligned}$$

$S_{t_0}$  is well-defined and linear because of the uniqueness of solutions of  $(CP)_{0,x}$ .

**Claim 3.2.3.** *The operator  $S_{t_0}$  is closed.*

*Proof.* Suppose  $x_n \rightarrow x$  in  $[D(\mathcal{A})]$  and  $S_{t_0}(x_n) \rightarrow v$  in  $X_{t_0}$ . We aim to show that  $S_{t_0}(x) = v$ .

By definition,  $S_{t_0}(x_n) = u(\cdot, x_n)$ . The convergence in  $X_{t_0}$  implies:

$$\|S_{t_0}(x_n) - v\|_\infty = \sup_{t \in [0, t_0]} |u(t, x_n) - v(t)|_G \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Because the map is continuous, and  $u$  is the unique solution, we must have  $v(t) = u(t, x)$ . Therefore,  $S_{t_0}(x) = v$ .  $\square$

Because  $S_{t_0}$  is a closed operator acting between two Banach spaces, the Closed Graph Theorem implies that  $S_{t_0}$  is bounded. Thus, there exists a constant  $C_{t_0} > 0$  such that for all  $x \in [D(\mathcal{A})]$ :

$$\|S_{t_0}(x)\|_\infty \leq C_{t_0}|x|_G$$

Consequently, for all  $t \in [0, t_0]$ , we have  $|T(t)x|_G \leq C_{t_0}|x|_G$ . This demonstrates that  $T(t) \in \mathcal{L}([D(\mathcal{A})])$ , meaning  $(T(t))_{t \geq 0}$  is a  $C_0$ -SG on  $[D(\mathcal{A})]$ .

Now, we prove that the semigroup commutes with its generator on appropriate domains.

**Claim 3.2.4.** *For all  $y \in D(\mathcal{A}^2)$ ,  $T(t)\mathcal{A}y = \mathcal{A}T(t)y$ .*

*Proof.* Let  $y \in D(\mathcal{A}^2)$ . We define the function  $w(t)$ :

$$w(t) = y + \int_0^t u(s, \mathcal{A}y) ds$$

Since  $u(s, \mathcal{A}y)$  is continuous,  $w(t)$  is continuously differentiable ( $C^1$ ) with respect to  $t$ , and:

$$w'(t) = u(t, \mathcal{A}y)$$

We can also pull the closed operator  $\mathcal{A}$  outside the integral (since  $u(s, \mathcal{A}y) \in D(\mathcal{A})$ ):

$$w'(t) = \mathcal{A}y + \int_0^t \frac{d}{ds} u(s, \mathcal{A}y) ds = \mathcal{A} \left[ y + \int_0^t u(s, \mathcal{A}y) ds \right] = \mathcal{A}w(t)$$

At  $t = 0$ ,  $w(0) = y$ .

This shows that  $w(t)$  is a mild solution to  $(CP)_{0,y}$ . Because solutions are unique, it must coincide with our defined flow:

$$\forall t \geq 0, \quad w(t) = u(t, y) = T(t)y$$

Differentiating both sides with respect to  $t$  yields:

$$w'(t) = \frac{d}{dt}[u(t, y)] = \mathcal{A}u(t, y) = \mathcal{A}T(t)y$$

However, from our earlier calculation, we also know  $w'(t) = u(t, \mathcal{A}y) = T(t)\mathcal{A}y$ . Equating the two expressions for  $w'(t)$  gives the desired result:

$$T(t)\mathcal{A}y = \mathcal{A}T(t)y$$

□

Now since  $\rho(\mathcal{A}) \neq \emptyset$ , then  $\exists \lambda_0 \in \rho(\mathcal{A})$ ,  $\forall y \in D(\mathcal{A}^2)$  set  $x = (\lambda_0 I - \mathcal{A})y$  then:

$$T(t)x = T(t)(\lambda_0 I - \mathcal{A})y = (\lambda_0 I - \mathcal{A})T(t)y$$

then

$$\|T(t)x\| \leq |\lambda_0| \|T(t)y\| + \|\mathcal{A}T(t)y\| \leq C_{\lambda_0} \|T(t)y\| \leq C_{\lambda_0} M e^{\omega t} |y|_G$$

Where

$$\begin{aligned} |y|_G &= \|y\| + \|\mathcal{A}y\| = \|y\| + \|(\mathcal{A} - \lambda_0 I)y + \lambda_0 y\| \\ &\leq C_1 \|x\| + \|x\| + |\lambda_0| C_2 \|x\| \\ &\leq C_3 \|x\| \end{aligned}$$

Finally we get:

$$\|T(t)x\| \leq M' e^{\omega t} \|x\|$$

Now by lemma 3.2.2 and the fact that  $\rho(\mathcal{A}) \neq \emptyset$  this bound can be extended to all  $X$ . And the operator commutes as well.

Finally, we need to show that  $\mathcal{A}$  is the IG of  $(T(t))_{t \geq 0}$ . Let  $\mathcal{A}_1$  be the IG of  $(T(t))_{t \geq 0}$  then by definition of  $T(t)$  we have  $T(t)x = u(t, x) \forall x \in D(\mathcal{A})$  then:

$$\frac{d}{dt}T(t)x = \mathcal{A}T(t)x \quad \text{for } t \geq 0$$

Which means in particular that  $\frac{d}{dt}T(t)x|_{t=0} = \mathcal{A}x$  then  $\mathcal{A}_1 \supset \mathcal{A}$ .

Let  $\Re(\lambda) > \omega$  and let  $y \in D(\mathcal{A}^2)$ . Then:

$$e^{-\lambda t} \mathcal{A}T(t)y = e^{-\lambda t} T(t)\mathcal{A}y = e^{-\lambda t} T(t)\mathcal{A}_1 y$$

By integrating both sides from 0 to  $\infty$  we get:

$$\mathcal{A}R(\lambda, \mathcal{A}_1)y = R(\lambda, \mathcal{A}_1)\mathcal{A}y = \mathcal{A}_1R(\lambda, \mathcal{A}_1)y$$

Since  $R(\lambda, \mathcal{A}_1)\mathcal{A}_1y = \mathcal{A}_1R(\lambda, \mathcal{A}_1)y$  then  $R(\lambda, \mathcal{A}_1)\mathcal{A}y = R(\lambda, \mathcal{A}_1)\mathcal{A}_1y \forall y \in D(\mathcal{A}^2)$ . Since  $R(\lambda, \mathcal{A}_1)\mathcal{A}_1$  is uniformly bounded,  $\mathcal{A}$  is closed and  $\overline{D(\mathcal{A}^2)} = X$  we get  $R(\lambda, \mathcal{A}_1)\mathcal{A}y = R(\lambda, \mathcal{A}_1)\mathcal{A}_1y \forall y \in X$ . This implies that  $D(\mathcal{A}) \supset Rg(R(\lambda, \mathcal{A}_1)) = D(\mathcal{A}_1)$  hence  $\mathcal{A}_1 \subset \mathcal{A}$  then  $\mathcal{A}_1 = \mathcal{A}$  and the proof is complete.  $\square$