Analytical Number Theory: Lecture 02

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1 Additional Properties

Corollary 1. *Let f and g be arithmetic functions. If both g and $f * g$ are multiplicative, then f is multiplicative.*

Proof. Since g is multiplicative, then g is Dirichlet invertible. Furthermore, the inverse g^{-1} is also multiplicative. We can express f as:

$$f = f * u = f * (g * g^{-1}) = (f * g) * g^{-1}$$

We are given that $f*g$ is multiplicative. Since the convolution of two multiplicative functions is multiplicative, and both $f * g$ and g^{-1} are multiplicative, their convolution f must be multiplicative. □

Theorem 2. *Let f be a multiplicative function. Then f is completely multiplicative if and only if*

$$f^{-1}(n) = \mu(n)f(n) \quad \forall n \geq 1$$

Proof. $\Rightarrow$ Assume f is completely multiplicative.
We want to verify that the inverse is given by $(\mu \cdot f)$. We compute the convolution $(\mu \cdot f) * f$ evaluated at n:

$$\begin{aligned}
((\mu \cdot f) * f)(n) &= \sum_{d|n} (\mu \cdot f)(d) f\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \mu(d) f(d) f\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \mu(d) f(n) && \text{since } f \text{ is completely multiplicative} \\
&= f(n) \sum_{d|n} \mu(d) && f(n) \text{ does not depend on the sum} \\
&= f(n) \cdot u(n) && \text{by the identity } \sum_{d|n} \mu(d) = u(n) \\
&= u(n)
\end{aligned}$$

Thus, $(\mu \cdot f) * f = u$, which implies $f^{-1} = \mu \cdot f$.

$\Leftarrow$ Assume $f^{-1} = \mu \cdot f$.
Since f^{-1} is the product of two multiplicative functions (μ and f), and f is multiplicative, we need to show complete multiplicativity. It is enough to show that for any prime p and positive integer k:

$$f(p^k) = [f(p)]^k.$$

We use the definition of the inverse: $(f * f^{-1})(n) = u(n)$. For $n = p^k$ with $k \geq 1$, we have:

$$0 = u(p^k) = (f * f^{-1})(p^k) = \sum_{d|p^k} f(d) f^{-1}\left(\frac{p^k}{d}\right).$$

Substitute $f^{-1}(m) = \mu(m)f(m)$:

$$\sum_{j=0}^{k} f(p^j)\mu(p^{k-j})f(p^{k-j}) = 0.$$

In this sum, $\mu(p^{k-j})$ is non-zero only when the exponent is 0 or 1.

- When $j = k$: The term is $f(p^k)\mu(1)f(1) = f(p^k) \cdot 1 \cdot 1 = f(p^k)$.
- When $j = k-1$: The term is $f(p^{k-1})\mu(p)f(p) = f(p^{k-1})(-1)f(p) = -f(p^{k-1})f(p)$.

All other terms vanish because $\mu(p^r) = 0$ for $r \geq 2$. Thus:

$$f(p^k) - f(p^{k-1})f(p) = 0 \implies f(p^k) = f(p^{k-1})f(p).$$

By induction on k, this implies $f(p^k) = [f(p)]^k$. Therefore, f is completely multiplicative. $\square$

Example 1. The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ is a completely multiplicative function. Therefore, by theorem 2:

$$\lambda^{-1}(n) = \mu(n)\lambda(n)$$

Calculating the values:

$$\lambda^{-1}(n) = \mu(n)(-1)^{\Omega(n)} = (-1)^{\omega(n)}(-1)^{\Omega(n)}$$

For square-free n, $\omega(n) = \Omega(n)$, so $(-1)^{2\omega(n)} = 1 = \mu^2(n)$. If n has a square factor, $\mu(n) = 0$. Thus:

$$\lambda^{-1}(n) = \mu^2(n)$$

Example 2. We know that $\varphi = \mu * Id$. Taking the inverse:

$$\varphi^{-1} = (\mu * \mathrm{Id})^{-1} = \mu^{-1} * \mathrm{Id}^{-1}.$$

We know $\mu^{-1} = \mathbb{1}$. Since Id is completely multiplicative, $\mathrm{Id}^{-1}(n) = \mu(n)\mathrm{Id}(n) = (Id\mu)(n)$. Therefore:

$$\varphi^{-1}(n) = ((Id\mu) * \mathbb{1})(n) = (\mathbb{1} * (Id\mu))(n) = \sum_{d|n} d\mu(d)$$

Theorem 3. *If f is completely multiplicative, and g, h are arithmetic functions, then:*

$$f \cdot (g * h) = (f \cdot g) * (f \cdot h).$$

Proof. We evaluate the right-hand side at n:

$$\begin{aligned}((f \cdot g) * (f \cdot h))(n) &= \sum_{d|n} (f \cdot g)(d)(f \cdot h)\left(\frac{n}{d}\right) \\ &= \sum_{d|n} f(d)g(d)f\left(\frac{n}{d}\right)h\left(\frac{n}{d}\right).\end{aligned}$$

Since f is completely multiplicative, $f(d)f(n/d) = f(n)$. We factor this out:

$$\begin{aligned}&= f(n)\sum_{d|n} g(d)h\left(\frac{n}{d}\right) \\ &= f(n)(g * h)(n) \\ &= (f \cdot (g * h))(n).\end{aligned}$$

□

Remark 1. This theorem does not prevent the equality if we don't have all the conditions for example take $f = \mu, g = u$, h any arithmetic function then we have:

$$\mu \cdot (u * h) = \mu \cdot h$$

And

$$(\mu \cdot u) * (\mu \cdot h) = u * (\mu \cdot h) = \mu \cdot h$$

2 Exercises

Exercise 1. If f is multiplicative, prove that:

$$f^{-1}(n) = \mu(n)f(n)$$

for every square-free integer n.

Exercise 2. If f is multiplicative, prove that for a prime p:

$$f^{-1}(p^2) = f(p)^2 - f(p^2).$$

Exercise 3. Let f be a completely multiplicative function and g be an invertible arithmetic function. Show that:

$$(f \cdot g)^{-1} = f \cdot g^{-1}.$$