Dinur's Proof of the PCP Theorem

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- Dinur presents a new proof that $\text{NP} = \text{PCP}_{\frac{1}{2}}[c \log n, q, 1]$ where c and q are constants.

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  - $\forall (u, v) \in E$,

$$c_{(u,v)}(\sigma(u), \sigma(v)) = \begin{cases} 1, & \sigma(u) \neq \sigma(v) \\ 0, & \text{otherwise} \end{cases}$$

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  - CG $\in$ NP. Coloring is the proof, verification complexity is linear in number of edges

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- Boost the accuracy of the solution to show that NP$\subseteq$ $\mathrm{PCP}_{(1-\epsilon)^c}[O(\log n), 8c, 1]$

Regarding Expanders

- Edge expansion property for a graph $G = \langle V, E \rangle$:
  $\varphi(G) = \min_{|S| \leq \frac{n}{2}} \left\{ \frac{E(S, \bar{s})}{|S|} \right\}$ such that $S \subseteq V$.

The Gap Amplification Lemma

Lemma: Gap Amplification

There exists a constant $0 < \alpha < 1$, an alphabet Σ, and a polynomial time reduction mapping the CG $G = \langle\langle V, E\rangle, \Sigma, \mathcal{C}\rangle$ to $G = \langle\langle V', E'\rangle, \Sigma', \mathcal{C}'\rangle$ such that:

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Step 2: Power Step

Assuming that the degree of G is constant, we amplify $\mathrm{UNSAT}(G)$ while blowing up $|G|$ and $|\Sigma|$ by a constant factor.

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  - Now, we have that $|V'| = \sum_{v \in V} d_v = 2|E|$ and $|E'| = frac(d+1)|V'|2 = (d+1)|E|$, so $|G'|$ is $\Theta(|G|)$.

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  - If $\text{UNSAT}(G) = 0$, then $UNSAT(G') = 0$ by assigning each vertex in G_{d_v} to the color of v
  - Now, we need to show that if $\text{UNSAT}(G) \neq 0$, then $\text{UNSAT}(G')$ is not much smaller

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- $S = \{v \in V' \mid \sigma'(v)$ is not the popular color$\}$

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$\mathrm{UNSAT}(G') = \frac{|B'|}{|E'|} \geq \frac{\mu|E|}{2|E'|} = \frac{\mu}{2(d+1)} = \frac{\mathrm{UNSAT}(G)}{2(d+1)}$

In both cases, since $\mathrm{UNSAT}(G')$ is optimal, we have proven that $\mathrm{UNSAT}(G) \leq k \cdot \mathrm{UNSAT}(G')$ for some constant k

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There exists as constant $\beta > 0$ such that if UNSAT$(G) \le \frac{1}{t}$ then UNSAT$(G') \ge \beta\sqrt{t}UNSAT(G)$

Proof:

- Let $\sigma' : V \mapsto \Sigma'$ be the best coloring for G', hence $\alpha' =$UNSAT(G') is the fraction of walk constraints violated by σ'
- Let $\sigma : V \mapsto \Sigma$ be $\sigma(v) = \arg\max_{a\in\Sigma} \Pr[X_{v,t/2} = a]$ where $X_{v,i}$ is the opinion that a vertex has of v that is i random steps away from v
- If $\sigma(v) = a$ then $\Pr[X_{v,t/2} = a] \ge \frac{1}{\Sigma}$
- Let B be the set of edges that are violated by σ in G, then $\frac{|B|}{|E|} \ge$UNSAT$(G) = \alpha$

Proof of Main Lemma

We want to show that bad walks do not overlap very much

- Let w be a random t-walk in G', we now define the following random variable:

$$N = \begin{cases} \text{number of bad edges in } I & C_w \text{ is violated by } \sigma' \\ 0 & \text{otherwise} \end{cases}$$

- $\mathrm{UNSAT}(G') \geq \Pr[N > 0]$, we wish to lower bound $\Pr[N > 0]$
- <u>Claim 1:</u> $\exists \mu > 0$ such that $E[N] \geq \frac{2\mu\sqrt{t}|B|}{|E|}$
- <u>Claim 2:</u> $\exists C > 0$ such that $E[N^2] \leq \frac{C\sqrt{t}|B|}{|E|}$
- Chosing $\beta = \frac{4\mu^2}{C}$ completes the proof:

$$\Pr[N > 0] \geq \frac{E[N]^2}{E[N^2]} \text{ (By the second moments inequality)}$$

$$= \frac{\frac{4\mu^2 t|B|^2}{|E|^2}}{\frac{C\sqrt{t}|B|}{|E|}} = \frac{4\mu^2\sqrt{t}}{C}\frac{|B|}{|E|} \geq \beta \mathrm{UNSAT}(G)\sqrt{t}$$

Proof of Claim 1

- For a random walk w, define 2 random variables:

$$Z_i = \begin{cases} 1, & i^{\text{th}} \text{ edge of } w \text{ is in } B \\ 0, & \text{otherwise} \end{cases}$$

$$Y_i = \begin{cases} 1, & w \text{ is a rejecting } t\text{-walk and } Z_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

- Note: $\forall i, Y_i \leq Z_i$

Proof of Claim 1

- Let $N = \sum_{i \in I} Y_i$, then we have:

$$\begin{aligned} E[N] &= \sum_{i \in I} E[Y_i] \\ &= \sum_{i \in I} \Pr[Y_i = 1] \\ &= \sum_{i \in I} \Pr[Y_i = 1 \mid Z_i = 1] \Pr[Z_i = 1] \end{aligned}$$

We know that $\Pr[Z_i = 1] = \frac{|B|}{|E|}$ and $\Pr[Y_i = 1 \mid Z_i = 1] \geq \frac{\tau^2}{|\Sigma|^2} = \mu$, so we get:

$$\begin{aligned} &\geq \sum_{i \in I} \mu \frac{|B|}{|E|} \\ &= \frac{2\mu\sqrt{t}|B|}{|E|} \end{aligned}$$

Proof of Claim 2

- Since $Y_i \le Z_i$, we know that $N \le \sum_{i \in I} Z_i$, therefore:

$$
\begin{aligned}
E[N^2] &\le E[(\sum_{i \in I} Z_i)^2] \\
&\le 2 \sum_{i \in I} \sum_{j \ge i} E[Z_i Z_j] \\
&= 2 \sum_{i \in I} \Pr[Z_i = 1] \sum_{j \ge i} \Pr[Z_j = 1 \mid Z_i = 1] \\
&= 2 \frac{|B|}{|E|} \sum_{i \in I} \sum_{j \ge i} \underbrace{\Pr[Z_j = 1 \mid Z_i = 1]}_{\downarrow}
\end{aligned}
$$

Probability that a random walk has its $(j - i + 1)^{\text{th}}$ edge in B given that the first edge in the walk is in B

Proof of Claim 2

$$
\begin{aligned}
E[N^2] &\leq 2\frac{|B|}{|E|}\sum_{i\in I}\sum_{j\geq i}\Pr[Z_j = 1 \mid Z_i = 1] \\
&\leq 2\frac{|B|}{|E|}\sum_{i\in I}\sum_{j\geq i}\left(\frac{|B|}{|E|} + \lambda^{j-i}\right) \\
&\leq C\frac{|B|}{|E|}2\sqrt{t}\sqrt{t}\left(\frac{1}{\sqrt{t}}\right) \\
&\leq \frac{C\sqrt{t}|B|}{|E|}
\end{aligned}
$$