Остановились на формуле Лиувилля для однородной системы.

$\dot{\vec{X}} = \dot{A}(t)\vec{X} \implies W(t) = \det\Phi(T) : \dot{W}(t) = (\mathrm{Tr}A(t))W \; W(t) = W(t_0)e^{\int_{t_0}^{t}\mathrm{Tr}A(\tau)d\tau}$

$L_n[y] = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = 0$
$x_1(t) = y(t)$
$x_2(t) = y'(t)$
$x_n(t) = y^{(n-1)}(t)$

$$A(t) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & & \\ & & 0 & 1 \\ -a_0 & \ldots & & -a_{n-1} \end{pmatrix} \implies \mathrm{Tr}A(t) = -a_{n-1}(t)$$

$\implies W(t) = W(t_0)e^{\int_{t_0}^{t}a_{n-1}(\tau)d\tau}$

$y'' + a_1(t)y' + a_0(t)y(t) = 0$. Знаем $\phi_1(t)$ - нетривиальное решение. Не знаем ϕ_2

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{vmatrix} = \phi_1(t)\dot{\phi}_2(t) - \dot{\phi}_1(t)\phi_2(t) = \underbrace{A}_{\neq 0}e^{-\int a_1(\tau)d\tau}$$

$\phi_2' - \frac{\phi_1'(t)}{\phi_1(t)}\phi_2 = e^{-\int a_1(\tau)d\tau}$

ЛНСДУ

$\dot{\vec{x}} = A(t)\vec{x} + \vec{b}(t) \quad \vec{x_ч} = ?$

Матрица Коши:

$$\left.\begin{matrix} K(t,\tau) = \Phi(t)\Phi^{-1}(t) \\ \begin{cases} \dot{\Phi} = A(t)\Phi \\ \Phi(\tau) = I(\text{единичная}) \end{cases} \end{matrix}\right\} \implies$$

$\implies \vec{x_ч}(t) = \int_{t_0}^{t}K(t,\tau)\vec{b}(\tau)d\tau = \int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)\vec{b}(\tau)d\tau$

$\underline{\dot{\vec{x}}} = \Phi(t)\Phi^{-1}(t)\vec{b}(t) + \int_{t_0}^{t}\dot{\Phi}(t)\Phi^{-1}(\tau)\vec{b}(\tau)d\tau = \vec{b}(t) + \int_{t_0}^{t}A(t)\Phi(t)\Phi^{-1}(\tau)\vec{b}(\tau)d\tau =$
$\underline{\vec{b}(t) + A(t)\vec{x}}$

$A(t) \equiv A \implies K(t,\tau) = e^{A(t-\tau)}$
$\Phi(t) = e^{At} \quad x_ч(t) = \int_{t_0}^{t}e^{A(t-\tau)}\vec{b}(\tau)d\tau$

$$\begin{cases} \dot{\vec{x}} = A(t)\vec{x} + b(\tau) \\ \vec{x}(t_0) = \vec{C}_0 \end{cases}$$

$\widetilde{\Phi}(t) = \Phi(t)\Phi^{-1}(t_0) = K(t,t_0) = K(t,t_0)\vec{C}_0 + \int_{t_0}^{t}\vec{b}(\tau)d\tau$

$\widetilde{\Phi}(t_0) = I$

Ответ: $x(t) = \widetilde{\Phi}(t)\vec{C}_0 + \int_{t_0}^{t} K(t,\tau)\vec{b}(\tau)d\tau$

Периодическая задача

Коэффициенты - периодические функции

$P = P(t) \in \mathbb{C}^{n\times n} \quad w$ - период, $P(t+w) = P(t)$

$\dot{\vec{x}} = P(t)x$

Theorem. o $\Phi(t)$ sistemy s period koeff.

$\Phi(t) = G(t)e^{tR} \quad R \in \mathbb{R}^{n\times n}$

$G(t) - w$-Периодическая, $\det G(t) \neq 0$

Proof. $\dot{\Phi} = P(t)\Phi \implies \Psi(t) = \Phi(t+w)$ - тоже фундаментальная матрица

$\dot{\Psi} = \dot{\Phi}(t+w) = P(t+w)\Phi(t+w) = P(t)\Psi(t)$

$\Psi(t) = \Phi(t+w) = \Phi(t)B, \det B \neq 0, B$ - какая-то матрица, которая называется матрицой МонодромиИ.

$B = \Phi^{-1}(t)\Phi(t+w)$

$B = \Phi^{-1}(0)\Phi(w)$

Положим $R = \frac{1}{w}\ln B, \quad B = e^{wR}$

Если $\det B \neq 0 \implies \exists \ln B$
$\ln z = \ln|z| + i\arg z + 2\pi k i$

$\ln J_r = \ln \lambda I_r + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(\frac{1}{\lambda}z_r)^k$

$\boxed{\Phi_1(t+w) = \Phi_1(t)B_1}$

$$\Pi_1(t+w) = \begin{matrix} \Phi(t)SB_1 = \Phi_1(t)BS \\ \Phi_1(t)B_1 = \Phi(t+w)S = \Phi_1(t)BS \end{matrix}$$

$S^{-1}|SB_1 = BS$

$B_1 = S^{-1}BS$

$B_1 \sim B$

$\widetilde{\Phi}(0) = I$

$\widetilde{\Phi}(w) = \widetilde{\Phi}(0)B$

$B = \widetilde{\Phi}(w)$

$\mu_1, \ldots, \mu_n$ - собственные числа B, мультипликаторы

$\lambda_1, \ldots, \lambda_n$ собственные числа R (характеристические показатели)

$$\boxed{\lambda_j = \frac{1}{w}\ln \mu_j}$$

$\det B = \det \widetilde{\Phi}(w) \overset{\text{лиувиль}}{=} e^{\int_0^w \mathrm{Tr}P(\tau)d\tau} = \mu_1 \ldots \mu_n$

$y'' + P(t)y = 0$

$y = x_1, y' = x_2$

$\implies \begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -P(t)x_1 \end{cases}, P(t) = \begin{pmatrix} 0 & 1 \\ -P(t) & 0 \end{pmatrix}$

$\mathrm{Tr}P(t) = 0 \overset{\text{liuville}}{\implies} \mu_1\mu_2 = 1$

$\mu_2 - 2a\mu + 1 = 0$ - характеристический полином

$\Phi(w) = \begin{pmatrix} \phi_1(w) & \phi_2(w) \\ \dot{\phi}_1(w) & \dot{\phi}_2(w) \end{pmatrix}$

Условия

$\begin{cases} \phi_1(t) = 1 \\ \dot{\phi}_1(0) = 0 \end{cases} \begin{cases} \phi_2(0) = 0 \\ \dot{\phi}_2(0) = 1 \end{cases}$

$a = \frac{1}{2}(\phi_1(w) + \dot{\phi}_2(w))$

Theorem. μ - мультипликатор $\iff$

$\exists\vec{\phi}$ решение $\dot{\vec{\phi}} = \Phi(t)\vec{\phi} : \forall t \quad \phi(t+w) = \mu\phi(t)$

Proof. μ - мультипликатор $\exists\vec{x_\mu}$ собственный вектор, $\widetilde{\Phi}(w)\vec{x_\mu} = \mu\vec{x_\mu}$

$\begin{cases} \dot{\vec{\pi}} = P(t)\vec{\phi} \\ \overline{\phi}(0) = \vec{x_\mu} \end{cases} \implies \begin{matrix} \overline{\phi}(t) = \widetilde{\Phi}(t)\vec{x_\mu} \\ \vec{\phi}(t+w) = \underbrace{\widetilde{\Phi}(t+w)}_{=\widetilde{\Phi}(t)B=\widetilde{\Phi}(w)}x_\mu = \widetilde{\Phi}(t)\boxed{\widetilde{\Phi}(w)\vec{x_\mu}} = \mu\widetilde{\Phi}(t)\vec{x_\mu} = \mu\vec{\phi}(t) \end{matrix}$

Замена переменных в $x' = P(t)x$

$\Phi(t) = G(t)e^{tR} = \underbrace{G(t)S}_{=G_1(t) \implies \Phi_1(t)e^{tJ}} \quad e^{tJ}S^{-1} \mid \cdot S$

$R = Se^{tJ}S^{-1}$

$G_1(t) = \{\vec{g_1}(t)\dots\vec{g_n}(t)\}$

$\vec{g_j}(t+w) = \vec{g}(t)$ лин. незав.

$\Phi_1(t) = \{\vec{\phi_1}(t)\dots\vec{\phi_n}(t)\}, e^{tJ} = \mathrm{diag}\{e^{tJ_0}\dots e^{tJ_a}\}$

$J_k = \lambda_{p+k}I + Z_{rk} \quad \lambda_{p+k} = \frac{1}{w}\ln\mu_{p+k}$

$\vec{\phi_j}(t) = \vec{g_j}(t)e^{t\lambda_j} \quad j = 1, \dots,$

$\vec{\phi}_{p+r_1+\dots+r_{p-1}}(t) = g_{-||-}(t)e^{t\lambda_{p+k}}$

$\vec{\phi}_{p+r_1+\dots+r_k}(t) = (\frac{t^{r_n-1}}{(r_k-1)!}\vec{g}_{p+r_1+\dots+r_{k-1}+1}(t) + \dots + \vec{g}_{p+r_1+\dots+r_{k-1}+r_k}(t))e^{t\lambda_{p+k}}$

Приводимость

$\Phi(t) = G(t)e^{tK} \mid \cdot e^{-tK} \quad G(t) = \Phi e^{-tK} \mid \cdot\frac{d}{dt}$

$\implies \dot{G}(t) = P(t)\Phi(t)e^{-tR} - \Phi(t)e^{-tR}R = P(t)G(t) - G(t)R$

$\dot{\vec{x}} = P(t)x \quad x(t) = G(t)\vec{y}(t) \mid \cdot\frac{d}{dt}$

$\dot{G}(t)\vec{y}(t) + G(t)\dot{\vec{y}}(t) = P(t)G(t)y(t) - G(t)Ry(t) + G(t)\dot{y}(t) = P(t)G(t)y(t)$
$\implies G(t)\dot{\vec{y}}(t) = G(t)R\vec{y}(t)$
$\implies \dot{\vec{y}} = R\vec{y}$ существует линейная периодическая замена