

1 Problems

Write $\text{Succ}(s, t) = \text{Succ}'(s, t) = \text{Succ}''(s, t)$ for the successors and $\text{Pred}(s, t) = \text{Pred}'(s, t) = \text{Pred}''(s, t)$ for the predecessors (ancestors) of the pair in D , D' , D'' , respectively.

Since $D = D' \cup D''$, we also have $\text{Succ}(s, t) = \text{Succ}'(s, t) \cup \text{Succ}''(s, t)$ and $\text{Pred}(s, t) = \text{Pred}'(s, t) \cup \text{Pred}''(s, t)$.

1.1 Lemma 3.1.

Suppose (x, y) and (a, b) are birth-death pairs of $f : X \rightarrow R$, a, x are consecutive in the ordering of the cells by f , and the transposition a, x is a switch. Then

$$\text{a } W'_{\text{after}}[y, t] = \begin{cases} W'_{\text{before}}[y, t] + W'_{\text{before}}[b, t], & \text{if } t \neq b \\ W'_{\text{before}}[y, b] + 1, & \text{otherwise} \end{cases}$$

$$\text{b } W'_{\text{after}}[b, t] = W'_{\text{before}}[b, t]$$

These 2 formulas can be described in archaic way:

a $(y, t) \in B'_{\text{after}}$ iff one of the sentences is true:

- $t \neq b$ and $(y, t) \in B'_{\text{before}}$ and $(b, t) \notin B'_{\text{before}}$
- $t \neq b$ and $(y, t) \notin B'_{\text{before}}$ and $(b, t) \in B'_{\text{before}}$
- $t = b$ and $(y, b) \notin B'_{\text{before}}$

b $(b, t) \in B'_{\text{after}}$ iff $(b, t) \in B'_{\text{before}}$

And they have implications:

$$\text{c } \text{Succ}'_{\text{after}}(a, y) = \begin{cases} \text{Succ}'_{\text{before}}(x, y) \cup \text{Succ}'_{\text{before}}(a, b) \cup \{(x, b)\}, & \text{if } (a, b) \notin \text{Succ}'_{\text{before}}(x, y) \\ \text{Succ}'_{\text{before}}(x, y) / (\{a, b\} \cup \{(s, t) \in \text{Succ}'_{\text{before}}(a, b) : \kappa(t, a) = 0\}) \end{cases}$$

$$\text{d } \text{Succ}'_{\text{after}}(x, b) = \text{Succ}'_{\text{before}}(a, b)$$

1.2 Hypothesis 1

Suppose a and b are 2-simplices consecutive in the ordering by f . And there is another Morse function f^* :

$$f^*(s) = \begin{cases} f(s), & \text{if } s \neq a, b \\ f(b), & \text{if } s = a \\ f(a), & \text{if } s = b \end{cases}$$

Let's denote DP_f^{\min} the transitive reduction of the Depth Poset defined by the filtration f . And let's denote $DP_f^{\min}(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b , then $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$

2 Model and Experiments

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0, 1]^d$. After this we calculate the Alpha Complex with these points, and then find its Depth Poset. Then we iterate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

As we know, an Alpha Complex is a Simplicial Complex, which can be represented as Lefschetz Complex. We also study the dual complexed transposing the border matrices over minor diagonal.

We can see the calculated cases in the table the given:

case	n	complex dim	alpha	dual	case	n	complex dim	alpha	dual
1	6	2	2	3	11	8	2	22	23
2	6	2	4	5	12	8	2	24	25
3	6	2	6	7	13	8	2	26	27
4	6	2	8	9	14	8	2	28	29
5	6	2	10	11	15	8	3	30	31
6	6	2	12	13	16	8	3	32	33
7	8	2	14	15	17	8	3	34	35
8	8	2	16	17	18	8	3	36	37
9	8	2	18	19	19	8	3	38	39
10	8	2	20	21	20	8	3	40	41

We also check few specific transpositions for cases, with given border matrices:

complex	dim	matrix	transpositions
0	2	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	1
1	1	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	1

We can see the distribution of transposition types in each complex in the given table:

complex	switch forward		switch backward		no switch		birth-death	birth-death
	birth-death	birth-death	birth-death	birth-death	birth-death	birth-death		
Total	36	34	138	28	28	244	204	396
0	1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	0	1	2	2	1	3	2	5
3	1	0	2	1	2	3	2	5
4	0	0	1	0	0	6	5	4
5	0	0	1	0	0	6	5	4
6	0	1	2	1	1	4	2	5
7	1	0	2	1	1	3	3	5
8	2	1	2	0	0	5	5	3
9	1	2	2	0	0	6	4	3
10	1	1	2	0	0	5	4	4
11	1	1	2	0	0	5	4	4
12	0	0	2	2	1	4	4	4
13	0	0	2	1	2	5	3	4
14	2	0	1	1	0	6	8	5
15	0	2	1	0	1	9	5	5
16	0	2	0	0	0	7	4	11
17	2	0	0	0	0	5	6	11
18	2	0	2	0	0	6	7	8
19	0	2	2	0	0	8	5	8
20	2	1	0	0	0	6	6	9
21	1	2	0	0	0	7	5	9
22	1	0	1	2	2	5	5	9
23	0	1	1	2	2	6	4	9
24	4	1	1	0	0	4	6	8
25	1	4	1	0	0	7	3	8
26	0	0	1	0	0	7	6	9
27	0	0	1	0	0	7	6	9
28	2	1	1	1	1	4	4	8
29	1	2	1	1	1	5	3	8
30	1	0	8	0	2	9	7	16
31	0	1	8	2	0	8	8	16
32	0	2	9	1	0	8	6	18
33	2	0	9	0	1	7	7	18

complex	switch forward		switch backward		no switch		birth-death	birth-death
	birth-death	death-death	birth-death	birth-death	birth-death	death-death		
34	0	1	8	1	1	8	6	17
35	1	0	8	1	1	7	7	17
36	0	0	5	3	1	6	7	16
37	0	0	5	1	3	8	5	16
38	1	2	8	1	1	8	6	20
39	2	1	8	1	1	7	7	20
40	1	1	13	1	1	7	6	19
41	1	1	13	1	1	7	6	19

3 Scores

jacard_nodes_filtration - The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_nodes_simplex - The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.

jacard_edges_filtration - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_edges_simplex - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.

l31a - Returns the portion of simplices t which are births or deaths in both depth posets such that $W'_{\text{after}}[y, t] = \begin{cases} W'_{\text{before}}[y, t] + W'_{\text{before}}[b, t], & \text{if } t \neq b \\ W'_{\text{before}}[y, b] + 1, & \text{otherwise} \end{cases}$

l31b - Returns the portion of simplices t which are births or deaths in both depth posets such that $W'_{\text{after}}[b, t] = W'_{\text{before}}[b, t]$

l31c - If $(a, b) \notin \text{Desc}'(x, y)$ will return the Jacard index of sets: $\text{Desc}'_{\text{after}}(a, y)$ and $\text{Desc}'_{\text{before}}(x, y) \cup \text{Desc}'_{\text{before}}(a, b) \cup \{(x, b)\}$. And will return the Jacard index of sets $\text{Desc}'_{\text{after}}(a, y)$ and $\text{Desc}'_{\text{before}}(x, y) / (\{a, b\} \cup \{(s, t) \in \text{Desc}'_{\text{before}}(a, b) : \kappa(t, a) = 0\})$ otherwise.

l31d - The Jacard index of sets $\text{Desc}'_{\text{after}}(x, b)$ and $\text{Desc}'_{\text{before}}(a, b)$

l33a - The Jacard index of sets $\text{Succ}_{\text{after}}(a, x)$ and $\text{Succ}_{\text{before}}(a, b)$

l33b - The Jacard index of sets $\text{Succ}_{\text{after}}(b, y)$ and $\text{Succ}_{\text{before}}(x, y)$

jacard_nn_nodes - The Jacard index of subsets of nodes (s, t) from 2 depth posets, s.t. $s, t \notin \nabla \partial \sigma_0 \cup \partial \nabla \sigma_0 \cup \nabla \partial \sigma_1 \cup \partial \nabla \sigma_1$, where σ_0 and σ_1 are transposing simplices.

jacard_nn_edges - The Jacard index of subsets of edges $((s_0, t_0), (s_1, t_1))$ from 2 depth posets, s.t. $s_0, t_0, s_1, t_1 \notin \nabla \partial \sigma_0 \cup \partial \nabla \sigma_0 \cup \nabla \partial \sigma_1 \cup \partial \nabla \sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Conclusions and Unexpected Cases

4.1 Lemma 3.1

Lemma 3.1 can be measured by 2 scores: **l31a**, **l31b**, **l31c**, **l31d**. And there are 0 of 36 switch forward birth-birth transpositions found where these scores are not 1.

4.2 Lemma 3.3

Lemma 3.3 can be measured by 2 scores: **l33a**, **l33b**. And there are 0 of 138 switch forward birth-death transpositions found where these scores are not 1.

4.3 Hypothesis

Hypothesis can be measured by 2 scores: **jacard_nn_nodes**, **jacard_nn_edges**. And there are 2 of 1108 transpositions found where these scores are not 1.

We can see these situations in the table:

case	complex	simplex 0	simplex 1	jacard_nn_nodes	jacard_nn_edges	Figure
	0	1	2	NaN	NaN	Figure 1
	1	1	2	NaN	NaN	Figure 2

The Birth-Birth Switch-Forward Transposition of Simplices 1 and 2 in Special Complex 0

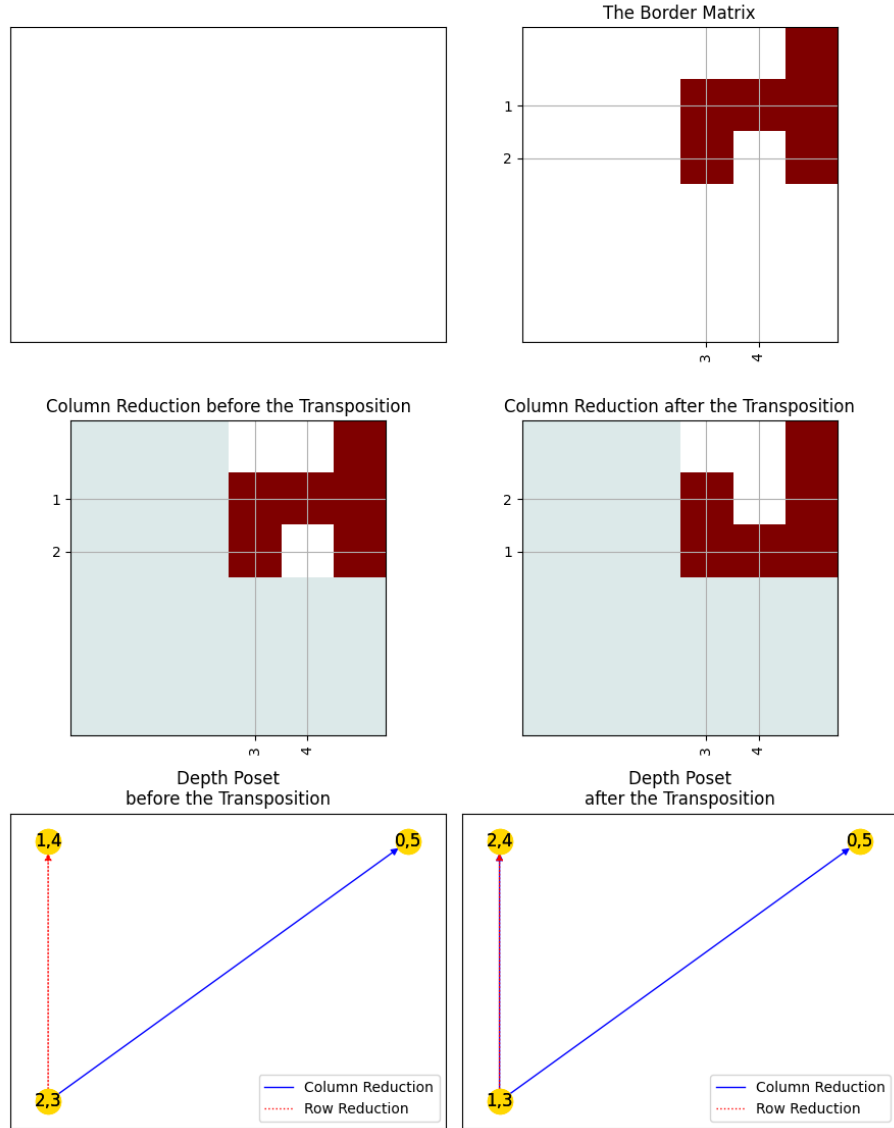


Figure 1: The birth-birth Transposition of simplices 1 and 2 in complex 0

The Birth-Birth Switch-Forward Transposition of Simplices 1 and 2 in Special Complex 1

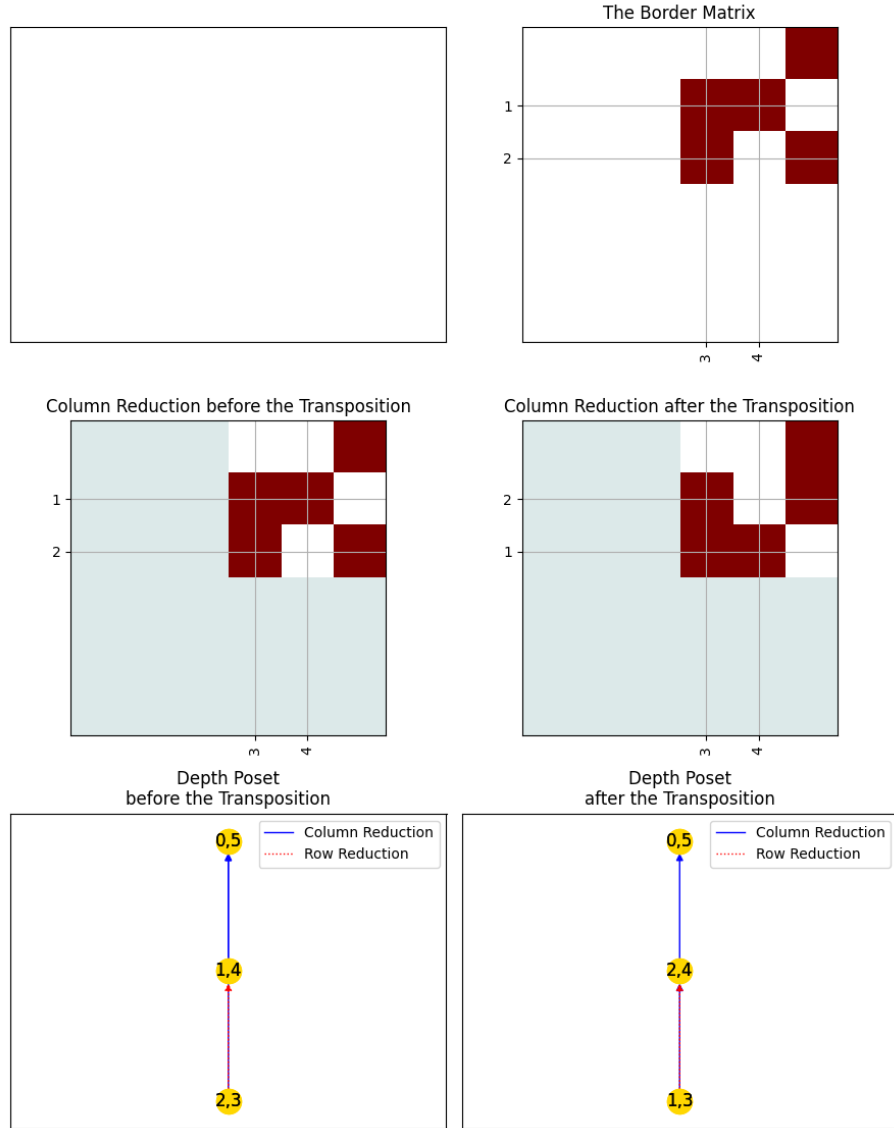


Figure 2: The birth-birth Transposition of simplices 1 and 2 in complex 1