1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x,y) and (a,b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

$$\mathrm{Succ}'(a,y) = \{(x,b)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$$

$$\mathrm{Succ}'(x,b) = \{(s,t) \in \mathrm{Succ}'(x,y) \cup \mathrm{Succ}'(a,b) | f(t) > f(b)\}$$

1.2 Lemma 3.2.

Suppose (a,b) and (x,y) are birth-death pairs of $f : X \to R$, y, b are consecutive in the ordering by f, and transposition of y, b is a switch. Then

$$\mathrm{Succ}''(x,b) = \{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s)\}$$

$$\mathrm{Succ}''(a,y) = \{(s,t) \in \mathrm{Succ}''(a,b) \cup \mathrm{Succ}''(x,y) | f(s) < f(a)\}$$

1.3 Lemma 3.3.

Suppose (a,b) and (x,y) are birth-death pairs of f: $X \to R$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then

$$\mathrm{Succ}(a,x) = \mathrm{Succ}(a,b) \text{ and } \mathrm{Succ}(b,y) = \mathrm{Succ}(x,y)$$

1.4 Hypothesis 1

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Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), \text{ if } s \neq a, b \\ f(b), \text{ if } s = a \\ f(a), \text{ if } s = b \end{cases}$$

Let's denote $DP_f^{\min}$ the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f^{\min}(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$.

2 Model

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0,1]^d$. After this we calculate the Alpha complex with these points, and then find its depth poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

The number of points in the cloud n and the dimension $d = \dim$ for each generated cloud and coresponding complex are given in the table:

complex	n	dim	simplices
0	6	2	21
1	6	2	23
2	6	2	23
3	6	2	21
4	8	2	29
5	8	2	31
6	8	2	31
7	8	2	33
8	8	2	31
9	8	2	33
10	8	2	33
11	8	2	33
12	12	2	51
13	12	2	53

complex	n	dim	simplices
13	12	2	53
14	12	2	51
15	12	2	55
16	12	2	55
17	12	2	53
18	16	2	79
19	16	2	79
20	8	3	67
21	8	3	77
22	8	3	67
23	8	3	65
24	8	3	69
25	8	3	73

3 Scores

- **jacard_nodes_filtration**: The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.
- **jacard_nodes_simplex**: The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_edges_filtration**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.
- **jacard_edges_simplex**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_l31a**: Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(a,y)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$
- **jacard_l31b**: Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(s,t) \in \mathrm{Succ}'(x,y) \cup \mathrm{Succ}'(a,b) | f(t) > f(b)\}$

- **jacard_l32a**: Jacard Index of $\text{Succ}''(x, b)$ and $\{(a, y)\} \cup \text{Succ}''(a, b) \cup \{(s, t) \in \text{Succ}''(x, y) | f(a) < f(s) < f(x)\}$
- **jacard_l32b**: Jacard Index of $\text{Succ}''(a, y)$ and $\{(s, t) \in \text{Succ}''(x, y) \cup \text{Succ}''(a, b) | f(s) < f(a)\}$
- **jacard_l33a**: Jacard Index of $\text{Succ}(a, x)$ and $\text{Succ}(a, b)$
- **jacard_l33b**: Jacard Index of $\text{Succ}(b, y)$ and $\text{Succ}(x, y)$
- **jacard_nn_nodes**: The Jacard index of subsets of nodes (s, t) from 2 depth posets, s.t. $s, t \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.
- **jacard_nn_edges**: The Jacard index of subsets of edges $((s_0, t_0), (s_1, t_1))$ from 2 depth posets, s.t. $s_0, t_0, s_1, t_1 \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Results

4.1 Unexpected results

The Hypothesis problem can be measured by 2 scores with expected values: **jacard_nn_edges**, **jacard_nn_nodes** There are 4 cases, which does not corespond the expectations:

complex	simplex 0	simplex 1	type	jacard_nn_edges	jacard_nn_nodes	Figure
1	[0, 4]	[0, 2]	birth-birth	0.00	1.00	Figure 1
8	[5, 6]	[0, 6]	birth-birth	0.00	1.00	Figure 2
9	[0, 1]	[0, 6]	birth-birth	0.00	1.00	
10	[1, 3]	[0, 1]	death-death	0.50	1.00	

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The Lemma 3.1 problem can be measured by 2 scores with expected values: **jacard_l31a**, **jacard_l31b** There are 2 cases, which does not corespond the expectations:

complex	simplex 0	simplex 1	type	jacard_l31a	jacard_l31b	Figure
9	[5]	[6]	birth-birth	1.00	0.50	Figure 3
25	[5]	[6]	birth-birth	1.00	0.00	

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The Lemma 3.2 problem can be measured by 2 scores with expected values: **jacard_l32a**, **jacard_l32b** There are 0 cases, which does not corespond the expectations.

The Lemma 3.3 problem can be measured by 2 scores with expected values: **jacard_l33a**, **jacard_l33b** There are 0 cases, which does not corespond the expectations.

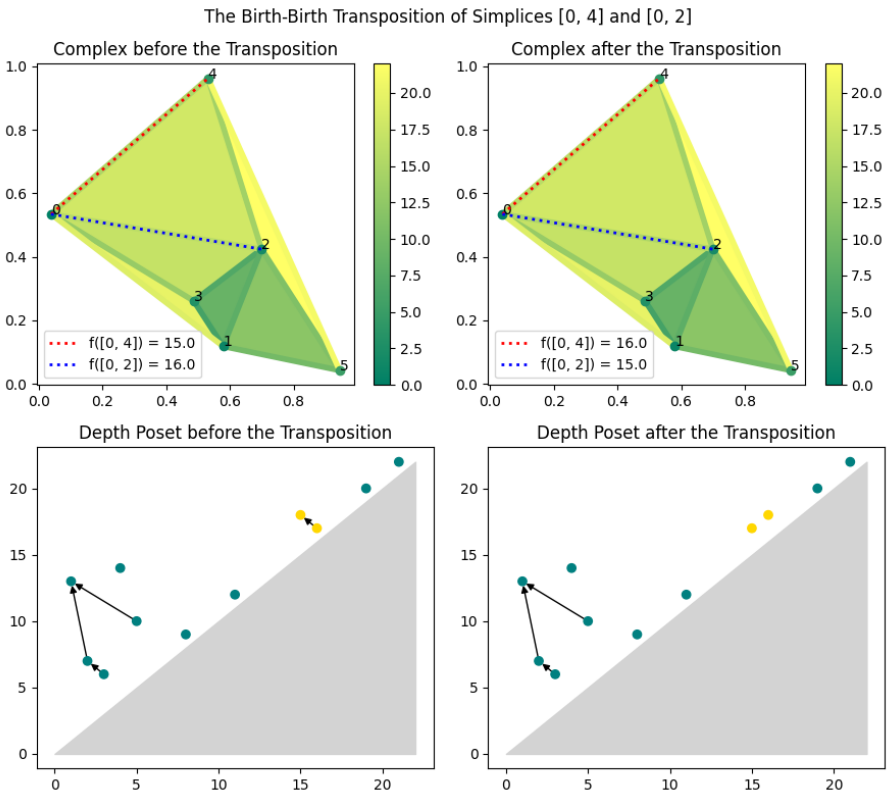


Figure 1: The Birth-Birth Transposition of simplices [0, 4] and [0, 2]

4.1.1 Complex 1

The switch transposition and unexpected scores are given in the table:

simplex 0	simplex 1	type
(1)	(2)	birth-birth
(1, 3)	(2, 3)	death-death
(2, 3)	(1, 2)	birth-death
(1, 5)	(2, 5)	birth-death
(2, 4)	(0, 4)	birth-death
(0, 2, 3)	(0, 2, 4)	death-death

4.1.2 Complex 8

The switch transposition and unexpected scores are given in the table:

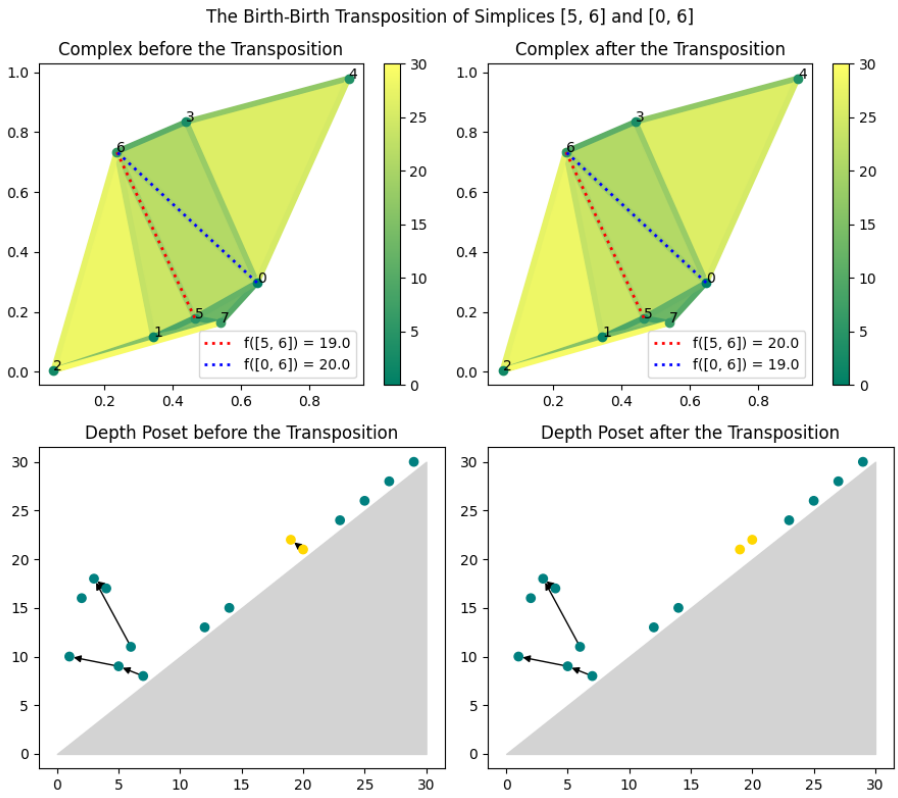


Figure 2: The Birth-Birth Transposition of simplices [5, 6] and [0, 6]

simplex 0	simplex 1	type
(3)	(4)	birth-birth
(0, 3)	(5, 6)	birth-death
(0, 3, 6)	(0, 5, 6)	death-death

4.1.3 Complex 9

The switch transposition and unexpected scores are given in the table:

simplex 0	simplex 1	type	jacard_l31b
(5)	(6)	birth-birth	0.50
(0, 6, 7)	(0, 1, 7)	death-death	0.00

4.1.4 Complex 10

The switch transposition and unexpected scores are given in the table:

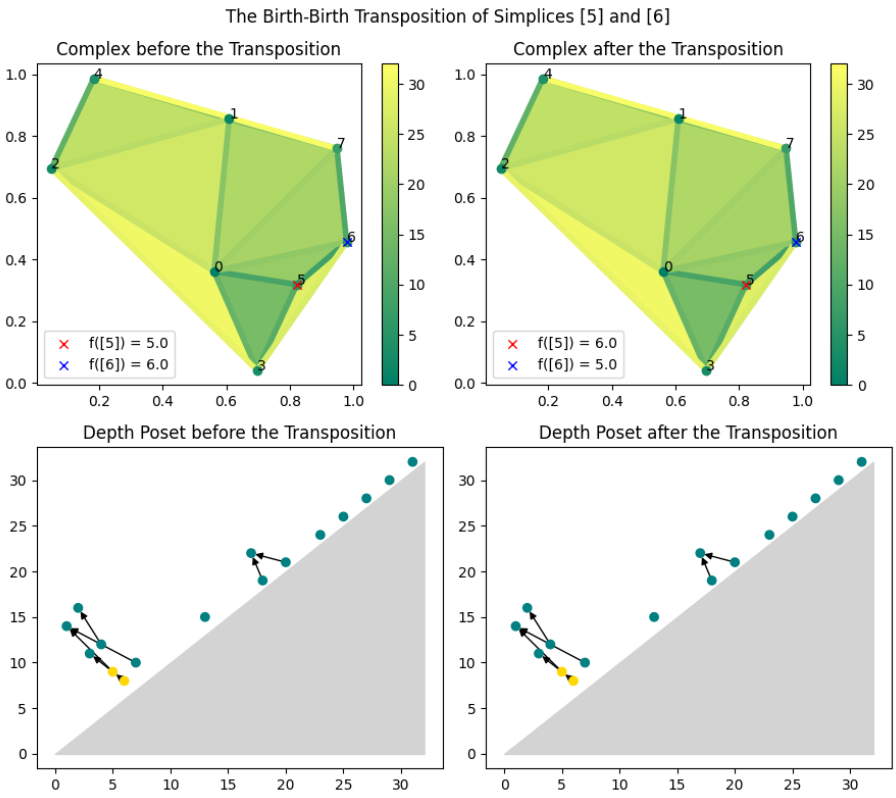
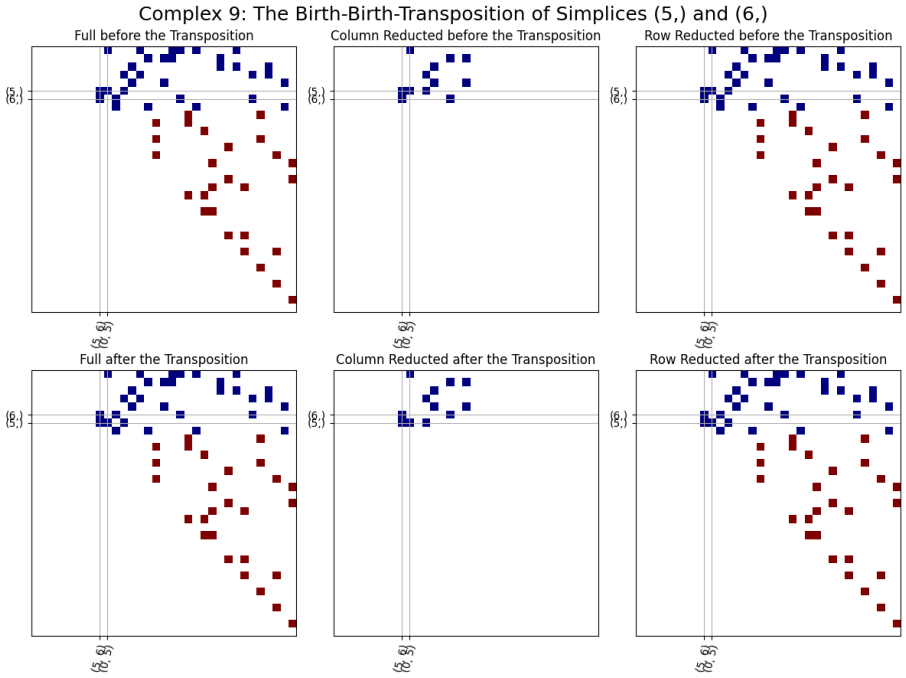


Figure 3: The Birth-Birth Transposition of simplices [5] and [6]

simplex 0	simplex 1	type
(1, 2, 5)	(1, 2, 6)	death-death

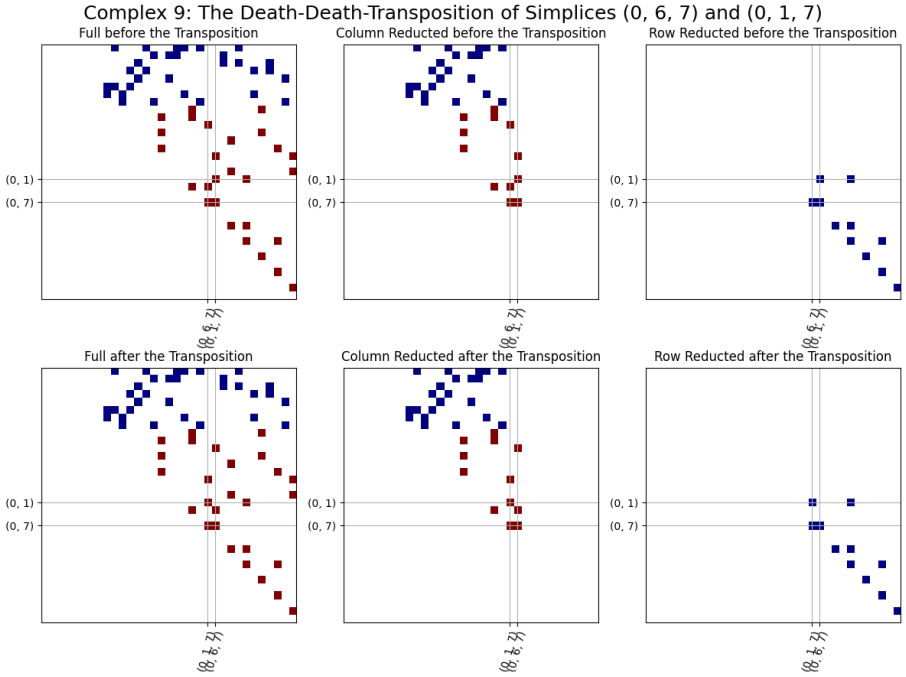
4.1.5 Complex 25

The switch transposition and unexpected scores are given in the table:

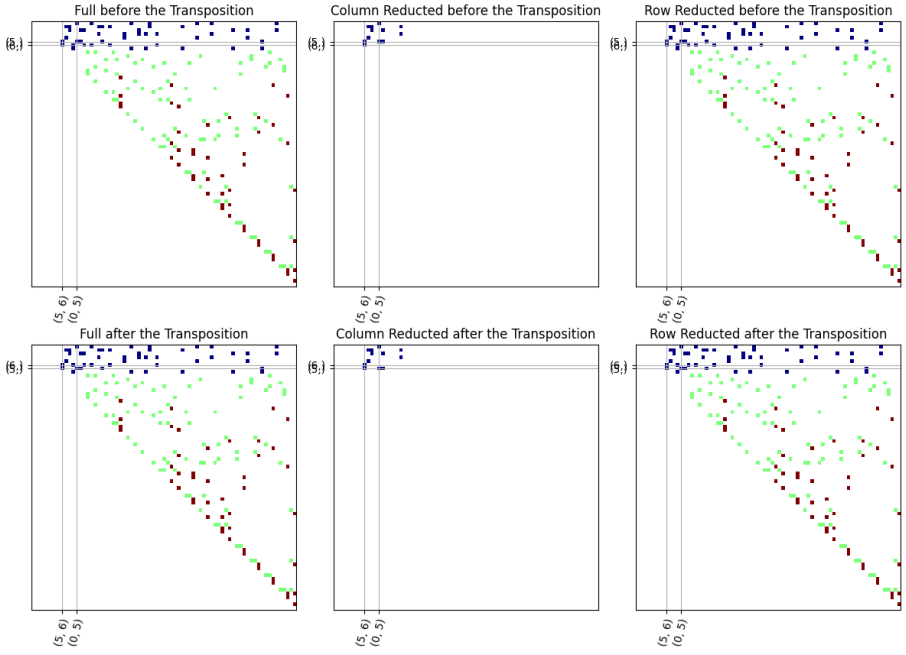


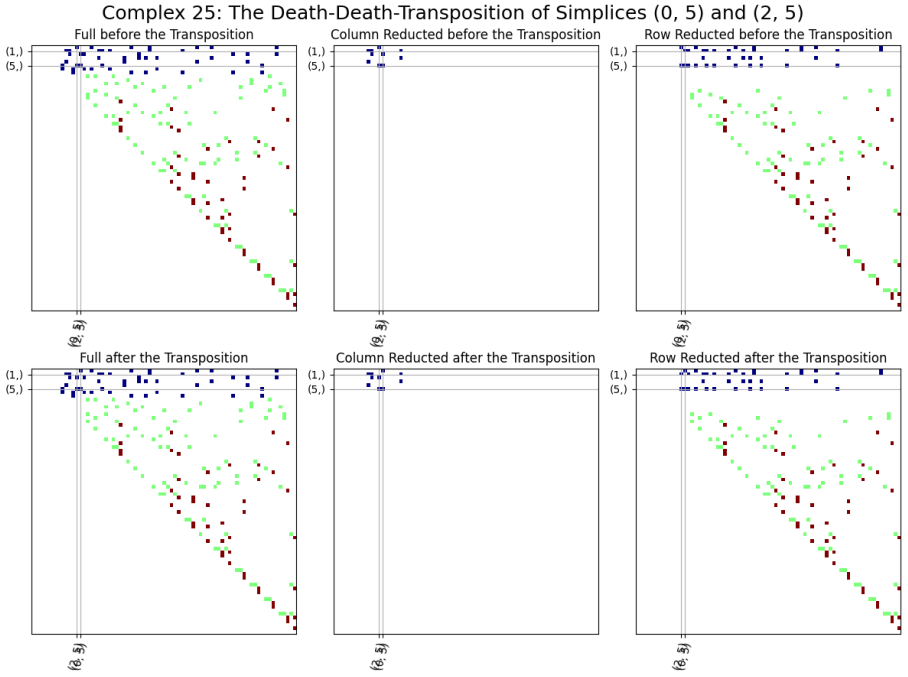
simplex 0	simplex 1	type	jacard_l31b
(5)	(6)	birth-birth	0.00
(0, 5)	(2, 5)	death-death	0.00
(2, 5)	(2, 6)	birth-death	NaN
(1, 5, 6)	(1, 2, 6)	birth-death	NaN
(1, 3, 6)	(3, 6, 7)	death-death	0.00
(1, 3, 7)	(1, 6, 7)	birth-death	NaN
(0, 1, 3)	(0, 3, 7)	birth-death	NaN
(2, 3, 7)	(2, 6, 7)	birth-death	NaN
(1, 2, 4)	(2, 3, 4)	birth-death	NaN
(0, 5, 7)	(5, 6, 7)	birth-death	NaN
(0, 1, 2)	(0, 2, 5)	birth-death	NaN

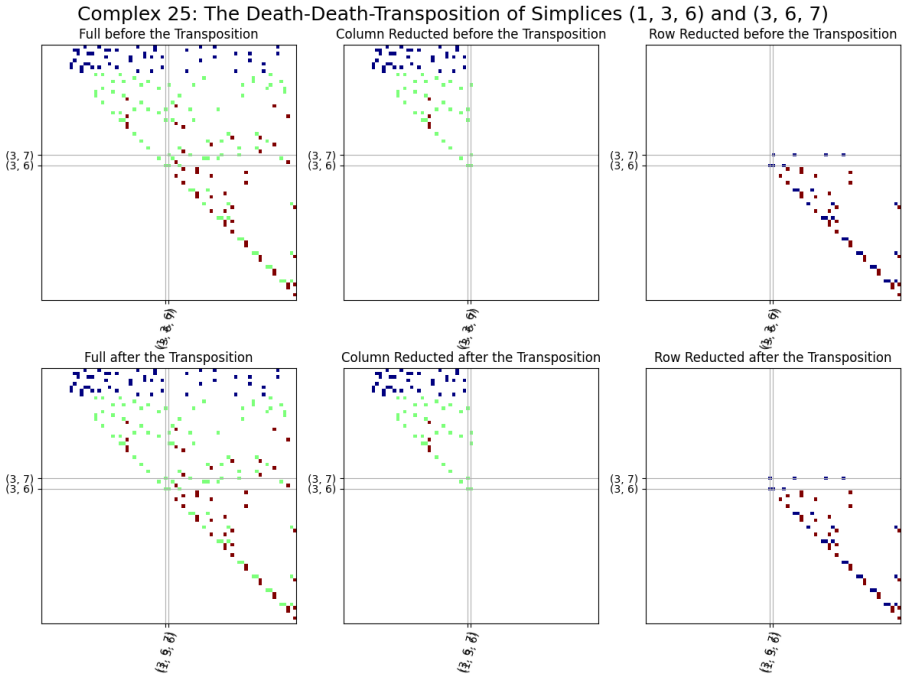
Complex 9: The Death-Death-Transposition of Simplices (0, 6, 7) and (0, 1, 7)



Complex 25: The Birth-Birth-Transposition of Simplices (5,) and (6,)




Complex 25: The Death-Death-Transposition of Simplices (0, 5) and (2, 5)
Full before the Transposition
Column Reducted before the Transposition
Row Reducted before the Transposition
Full after the Transposition
Column Reducted after the Transposition
Row Reducted after the Transposition
(1,)
(5,)
⟨2, 5⟩
⟨0, 5⟩


Complex 25: The Death-Death-Transposition of Simplices (1, 3, 6) and (3, 6, 7)
Full before the Transposition
Column Reducted before the Transposition
Row Reducted before the Transposition
Full after the Transposition
Column Reducted after the Transposition
Row Reducted after the Transposition
(3, 7)
(3, 6)
⟨3, 6, 7⟩
⟨1, 3, 6⟩