

# 1 Problems

Write  $\text{Succ}(s, t) = \text{Succ}'(s, t) = \text{Succ}''(s, t)$  for the successors and  $\text{Pred}(s, t) = \text{Pred}'(s, t) = \text{Pred}''(s, t)$  for the predecessors (ancestors) of the pair in  $D$ ,  $D'$ ,  $D''$ , respectively.

Since  $D = D' \cup D''$ , we also have  $\text{Succ}(s, t) = \text{Succ}'(s, t) \cup \text{Succ}''(s, t)$  and  $\text{Pred}(s, t) = \text{Pred}'(s, t) \cup \text{Pred}''(s, t)$ .

## 1.1 Lemma 3.1.

Suppose  $(x, y)$  and  $(a, b)$  are birth-death pairs of  $f : X \rightarrow R$ ,  $a, x$  are consecutive in the ordering of the cells by  $f$ , and the transposition  $a, x$  is a switch. Then

$$\text{Succ}'(a, y) = \{(x, b)\} \cup \text{Succ}'(a, b) \cup \{(s, t) \in \text{Succ}'(x, y) | f(t) < f(b)\}$$

$$\text{Succ}'(x, b) = \{(s, t) \in \text{Succ}'(x, y) \cup \text{Succ}'(a, b) | f(t) > f(b)\}$$

## 1.2 Lemma 3.2.

Suppose  $(a, b)$  and  $(x, y)$  are birth-death pairs of  $f : X \rightarrow R$ ,  $y, b$  are consecutive in the ordering by  $f$ , and transposition of  $y, b$  is a switch. Then

$$\text{Succ}''(x, b) = \{(a, y)\} \cup \text{Succ}''(a, b) \cup \{(s, t) \in \text{Succ}''(x, y) | f(a) < f(s)\}$$

$$\text{Succ}''(a, y) = \{(s, t) \in \text{Succ}''(a, b) \cup \text{Succ}''(x, y) | f(s) < f(a)\}$$

## 1.3 Lemma 3.3.

Suppose  $(a, b)$  and  $(x, y)$  are birth-death pairs of  $f : X \rightarrow R$ ,  $b, x$  are consecutive in the ordering by  $f$ , and the transposition of  $b, x$  is a switch. Then

$$\text{Succ}(a, x) = \text{Succ}(a, b) \text{ and } \text{Succ}(b, y) = \text{Succ}(x, y)$$

## 1.4 Hypothesis 1

Suppose  $a$  and  $b$  are 2-simplices consecutive in the ordering by  $f$ . And there is another Morse function  $f^*$ :

$$f^*(s) = \begin{cases} f(s), & \text{if } s \neq a, b \\ f(b), & \text{if } s = a \\ f(a), & \text{if } s = b \end{cases}$$

Let's denote  $DP_f^{\min}$  the transitive reduction of the Depth Poset defined by the filtration  $f$ . And let's denote  $DP_f^{\min}(s)$  the set of nodes in  $DP_f(s)$  which are pairs containing cell  $s$  and the set of edges with these nodes.

**Hypothesis:** if the cell  $s$  has no faces and cofaces with  $a$  and  $b$ , then  $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$ .

## 2 Model and Experiments

The probabilistic model is simple. The first we just generate the cloud of  $n$  points uniformly distributed in  $[0, 1]^d$ . After this we calculate the Alpha Complex with these points, and then find its Depth Poset. Then we iterate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

As we know, an Alpha Complex is a Simplicial Complex, which can be represented as Lefschetz Complex. We also study the dual complexed transposing the border matrices over minor diagonal.

We can see the calculated cases in the table the given:

case	n	complex dim	alpha	dual	case	n	complex dim	alpha	dual
1	6	2	0	1	33	16	2	64	65
2	6	2	2	3	34	16	2	66	67
3	6	2	4	5	35	8	3	68	69
4	6	2	6	7	36	8	3	70	71
5	6	2	8	9	37	8	3	72	73
6	6	2	10	11	38	8	3	74	75
7	8	2	12	13	39	8	3	76	77
8	8	2	14	15	40	8	3	78	79
9	8	2	16	17	41	12	3	80	81
10	8	2	18	19	42	12	3	82	83
11	8	2	20	21	43	12	3	84	85
12	8	2	22	23	44	12	3	86	87
13	8	2	24	25	45	12	3	88	89
14	8	2	26	27	46	12	3	90	91
15	12	2	28	29	47	12	3	92	93
16	12	2	30	31	48	12	3	94	95
17	12	2	32	33	49	16	3	96	97
18	12	2	34	35	50	16	3	98	99
19	12	2	36	37	51	16	3	100	101
20	12	2	38	39	52	16	3	102	103
21	12	2	40	41	53	16	3	104	105
22	12	2	42	43	54	16	3	106	107
23	12	2	44	45	55	8	4	108	109
24	12	2	46	47	56	8	4	110	111
25	12	2	48	49	57	12	4	112	113
26	12	2	50	51	58	12	4	114	115
27	16	2	52	53	59	12	4	116	117
28	16	2	54	55	60	12	4	118	119
29	16	2	56	57	61	12	4	120	121
30	16	2	58	59	62	12	4	122	123
31	16	2	60	61	63	16	4	124	125
32	16	2	62	63	64	16	4	126	127

We can see the distribution of transposition types in each complex in the given table:

	switch forward		switch backward		no switch			
	birth- birth	death- death	birth- death	birth- birth	death- death	birth- birth	death- death	birth- death
complex								
Total	95	95	1776	96	96	1533	1533	4470
0	0	0	2	1	0	3	4	5
1	0	0	2	0	1	4	3	5
2	0	1	0	1	0	3	3	6
3	1	0	0	0	1	3	3	6
4	0	0	2	1	0	3	4	5
5	0	0	2	0	1	4	3	5
6	1	1	2	0	1	4	3	3
7	1	1	2	1	0	3	4	3
8	1	0	2	0	0	3	4	3
9	0	1	2	0	0	4	3	3
10	1	0	3	0	0	3	4	5
11	0	1	3	0	0	4	3	5
12	0	1	2	1	1	6	5	6
13	1	0	2	1	1	5	6	6
14	0	1	0	2	1	5	5	9
15	1	0	0	1	2	5	5	9
16	0	1	4	1	1	6	5	3
17	1	0	4	1	1	5	6	3
18	1	0	2	0	0	5	6	9
19	0	1	2	0	0	6	5	9
20	1	1	1	0	1	6	5	9
21	1	1	1	1	0	5	6	9
22	0	0	0	0	1	6	5	8
23	0	0	0	1	0	5	6	8
24	0	0	5	1	1	6	6	4
25	0	0	5	1	1	6	6	4
26	0	1	0	0	0	7	6	8
27	1	0	0	0	0	6	7	8
28	1	4	1	1	0	11	9	12
29	4	1	1	0	1	9	11	12
30	1	1	2	1	0	9	10	14
31	1	1	2	0	1	10	9	14
32	2	0	0	2	1	7	10	14
33	0	2	0	1	2	10	7	14

	switch forward		switch backward		no switch			
	birth- birth	death- death	birth- death	birth- birth	death- death	birth- birth	death- death	birth- death
complex								
34	1	1	3	0	1	10	9	10
35	1	1	3	1	0	9	10	10
36	1	0	0	1	0	8	10	19
37	0	1	0	0	1	10	8	19
38	1	2	2	0	0	10	9	16
39	2	1	2	0	0	9	10	16
40	1	0	2	2	1	7	9	14
41	0	1	2	1	2	9	7	14
42	0	0	1	2	1	9	10	17
43	0	0	1	1	2	10	9	17
44	0	3	4	0	0	12	9	9
45	3	0	4	0	0	9	12	9
46	0	1	2	1	2	10	8	13
47	1	0	2	2	1	8	10	13
48	1	2	1	1	1	11	10	14
49	2	1	1	1	1	10	11	14
50	0	0	2	0	0	10	10	14
51	0	0	2	0	0	10	10	14
52	0	0	2	3	2	14	15	21
53	0	0	2	2	3	15	14	21
54	1	0	2	0	1	15	15	21
55	0	1	2	1	0	15	15	21
56	0	3	1	2	0	13	12	23
57	3	0	1	0	2	12	13	23
58	3	2	0	0	1	14	14	23
59	2	3	0	1	0	14	14	23
60	2	1	2	2	1	15	17	17
61	1	2	2	1	2	17	15	17
62	0	1	0	1	0	15	15	24
63	1	0	0	0	1	15	15	24
64	0	2	2	1	1	16	14	18
65	2	0	2	1	1	14	16	18
66	0	0	1	1	0	14	15	24
67	0	0	1	0	1	15	14	24
68	0	0	7	1	0	6	7	17

	switch forward		switch backward		no switch			
	birth- birth	death- death	birth- death	birth- birth	death- death	birth- birth	death- death	birth- death
complex								
69	0	0	7	0	1	7	6	17
70	0	0	8	0	0	7	7	19
71	0	0	8	0	0	7	7	19
72	1	1	9	1	0	7	8	13
73	1	1	9	0	1	8	7	13
74	1	1	7	0	0	6	6	17
75	1	1	7	0	0	6	6	17
76	1	3	6	2	1	8	7	15
77	3	1	6	1	2	7	8	15
78	0	0	9	0	1	8	7	18
79	0	0	9	1	0	7	8	18
80	0	1	18	0	0	15	14	41
81	1	0	18	0	0	14	15	41
82	1	3	13	3	2	14	13	40
83	3	1	13	2	3	13	14	40
84	2	1	13	1	1	10	11	39
85	1	2	13	1	1	11	10	39
86	0	0	16	1	1	14	14	44
87	0	0	16	1	1	14	14	44
88	0	0	22	1	0	12	13	30
89	0	0	22	0	1	13	12	30
90	1	1	20	1	0	10	11	42
91	1	1	20	0	1	11	10	42
92	0	0	18	1	2	13	12	44
93	0	0	18	2	1	12	13	44
94	1	0	16	0	1	10	10	47
95	0	1	16	1	0	10	10	47
96	1	1	20	2	0	21	23	64
97	1	1	20	0	2	23	21	64
98	1	1	17	0	0	19	19	72
99	1	1	17	0	0	19	19	72
100	0	1	22	0	0	19	18	67
101	1	0	22	0	0	18	19	67
102	0	0	22	2	1	22	23	75
103	0	0	22	1	2	23	22	75

complex	switch forward			switch backward		no switch		
	birth- birth	death- death	birth- death	birth- birth	death- death	birth- birth	death- death	birth- death
104	0	0	18	2	0	22	24	65
105	0	0	18	0	2	24	22	65
106	1	3	17	0	0	27	25	61
107	3	1	17	0	0	25	27	61
108	2	0	28	1	0	6	9	34
109	0	2	28	0	1	9	6	34
110	0	1	21	0	1	9	7	32
111	1	0	21	1	0	7	9	32
112	1	0	43	0	0	18	19	98
113	0	1	43	0	0	19	18	98
114	0	1	47	1	1	21	20	90
115	1	0	47	1	1	20	21	90
116	1	1	41	1	1	23	23	103
117	1	1	41	1	1	23	23	103
118	0	0	44	1	0	20	21	101
119	0	0	44	0	1	21	20	101
120	0	1	62	2	2	17	16	93
121	1	0	62	2	2	16	17	93
122	2	0	60	2	2	14	16	103
123	0	2	60	2	2	16	14	103
124	1	0	106	1	2	29	29	174
125	0	1	106	2	1	29	29	174
126	4	2	83	1	0	40	43	157
127	2	4	83	0	1	43	40	157

### 3 Conclusions and Unexpected Cases

#### 3.1 Lemma 3.1

Lemma 3.1 can be measured by 2 scores: **jacard\_l31a** and **jacard\_l31b**. And there are 0 of 95 switch-forward birth-birth transpositions found where these scores are not 1.

#### 3.2 Lemma 3.2

Lemma 3.2 can be measured by 2 scores: **jacard\_l32a** and **jacard\_l32b**. And there are 0 of 95 switch-forward death-death transpositions found where these scores are not 1.

### 3.3 Lemma 3.3

Lemma 3.3 can be measured by 2 scores: **jacard\_l33a** and **jacard\_l33b**. And there are 0 of 1776 switch-forward birth-death transpositions found where these scores are not 1.

### 3.4 Hypothesis

The Hypothesis can be measured by 2 scores: **jacard\_nn.nodes** and **jacard\_nn.edges**. And there are 0 of 9822 transpositions found where these scores are not 1.