1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x,y) and (a,b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

$$\mathrm{Succ}'(a,y) = \{(x,b)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$$

$$\mathrm{Succ}'(x,b) = \{(s,t) \in \mathrm{Succ}'(x,y) \cup \mathrm{Succ}'(a,b) | f(t) > f(b)\}$$

1.2 Lemma 3.2.

Suppose (a,b) and (x,y) are birth-death pairs of $f : X \to R$, y, b are consecutive in the ordering by f, and transposition of y, b is a switch. Then

$$\mathrm{Succ}''(x,b) = \{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s)\}$$

$$\mathrm{Succ}''(a,y) = \{(s,t) \in \mathrm{Succ}''(a,b) \cup \mathrm{Succ}''(x,y) | f(s) < f(a)\}$$

1.3 Lemma 3.3.

Suppose (a,b) and (x,y) are birth-death pairs of $f\colon X \to R$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then

$$\mathrm{Succ}(a,x) = \mathrm{Succ}(a,b) \;\text{ and }\; \mathrm{Succ}(b,y) = \mathrm{Succ}(x,y)$$

1.4 Hypothesis 1

Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), & \text{if } s \neq a, b \\ f(b), & \text{if } s = a \\ f(a), & \text{if } s = b \end{cases}$$

Let's denote $DP_f^{\min}$ the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f^{\min}(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$.

2 Model and Experiments

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0,1]^d$. After this we calculate the Alpha Complex with these points, and then find its Depth Poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

As we know, an Alpha Complex is a Simplicial Complex, which can be represented as Lefschetz Complex. We also study the dual complexed transposing the border matrices over minor diagonal.

We can see the calculated cases in the table the given:

case	n	complex dim	alpha	dual	case	n	complex dim	alpha	dual
1	6	2	0	1	17	8	2	32	33
2	6	2	2	3	18	8	2	34	35
3	6	2	4	5	19	12	2	36	37
4	6	2	6	7	20	12	2	38	39
5	6	2	8	9	21	12	2	40	41
6	6	2	10	11	22	12	2	42	43
7	8	2	12	13	23	12	2	44	45
8	8	2	14	15	24	12	2	46	47
9	8	2	16	17	25	16	2	48	49
10	8	2	18	19	26	16	2	50	51
11	8	2	20	21	27	8	3	52	53
12	8	2	22	23	28	8	3	54	55
13	8	2	24	25	29	8	3	56	57
14	8	2	26	27	30	8	3	58	59
15	8	2	28	29	31	8	3	60	61
16	8	2	30	31	32	8	3	62	63

3 Unexpected Cases

3.1 Lemma 3.1

Lemma 3.1 can be measured by 2 scores: **jacard_l31a** and **jacard_l31b**. And there are 7 of 38 switch-forward birth-birth transpositions found where these scores are not 1.

We can see these situations in the table:

case	complex	simplex 0	simplex 1	jacard_l31a	jacard_l31b	Figure
3	4	(4,)	(5,)	1.00	0.00	Figure 1
12	23	(4, 5)	(1, 5)	1.00	NaN	Figure 6
17	33	(3, 5)	(2, 5)	1.00	NaN	Figure 7
18	35	(5, 7)	(4, 7)	1.00	NaN	
24	47	(3, 7)	(2, 7)	1.00	NaN	
25	48	(10,)	(11,)	1.00	NaN	
27	53	(6, 7)	(1, 7)	1.00	NaN	

3.2 Lemma 3.2

Lemma 3.2 can be measured by 2 scores: **jacard_l32a** and **jacard_l32b**. And there are 7 of 38 switch-forward death-death transpositions found where these scores are not 1.

We can see these situations in the table:

case	complex	simplex 0	simplex 1	jacard_l32a	jacard_l32b	Figure
3	5	(5,)	(4,)	1.00	NaN	Figure 2
12	22	(1, 5)	(4, 5)	1.00	NaN	Figure 5
17	32	(2, 5)	(3, 5)	1.00	NaN	
18	34	(4, 7)	(5, 7)	1.00	NaN	
24	46	(2, 7)	(3, 7)	1.00	NaN	
25	49	(11,)	(10,)	1.00	NaN	
27	52	(1, 7)	(6, 7)	1.00	NaN	

3.3 Lemma 3.3

Lemma 3.3 can be measured by 2 scores: **jacard_l33a** and **jacard_l33b**. And there are 0 of 190 switch-forward birth-death transpositions found where these scores are not 1.

3.4 Hypothesis

The Hypothesis can be measured by 2 scores: **jacard_nn_nodes** and **jacard_nn_edges**. And there are 10 of 2016 transpositions found where these scores are not 1.

We can see these situations in the table:

case	complex	simplex 0	simplex 1	jacard_nn_nodes	jacard_nn_edges	Figure
5	8	(0, 1)	(1, 2)	1.00	0.50	Figure 3
5	9	(1, 2)	(0, 1)	1.00	0.50	
6	10	(3, 5)	(0, 5)	1.00	0.00	Figure 4
6	11	(0, 5)	(3, 5)	1.00	0.00	
18	34	(0, 7)	(0, 5)	1.00	0.50	
18	35	(0, 5)	(0, 7)	1.00	0.50	
25	48	(5,)	(6,)	1.00	0.91	Figure 8
25	49	(6,)	(5,)	1.00	0.91	
26	50	(1, 3)	(5, 13)	1.00	0.67	Figure 9
26	51	(5, 13)	(1, 3)	1.00	0.67	

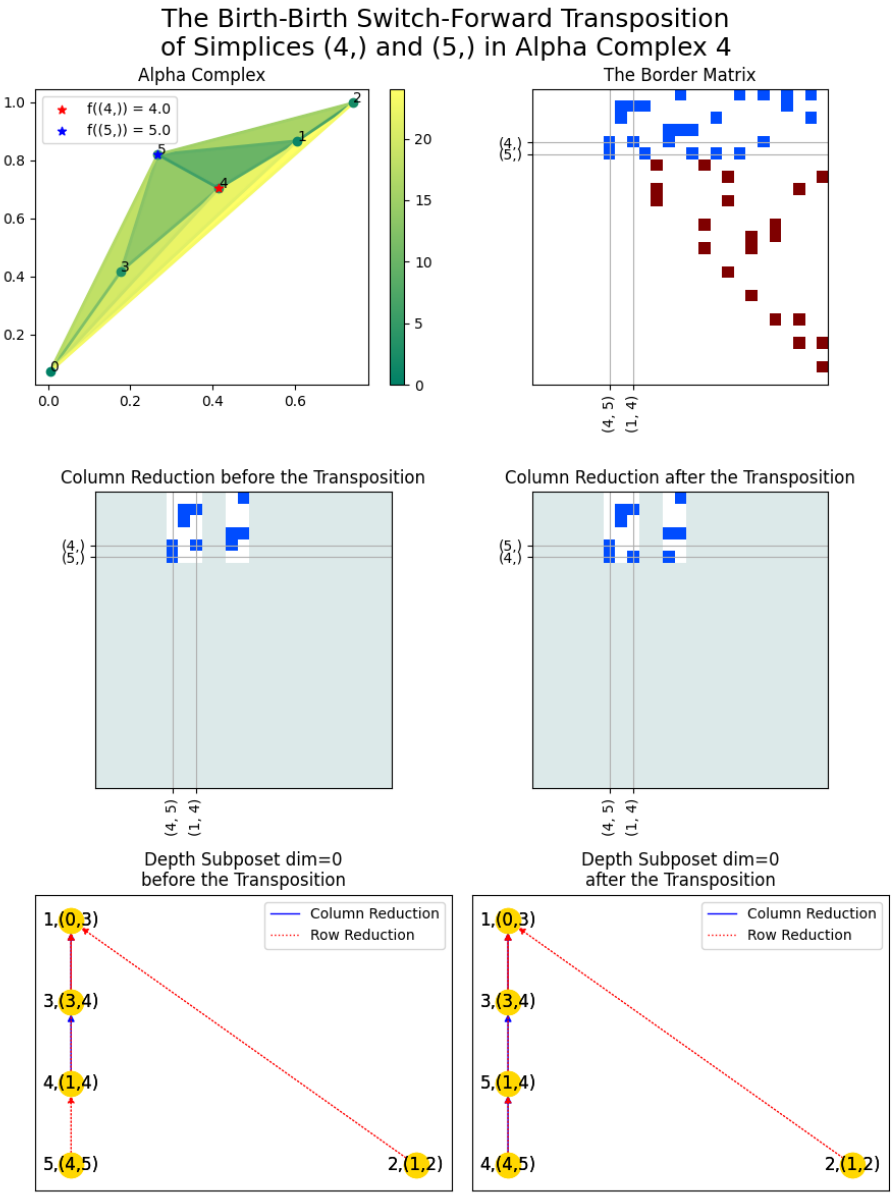

Figure 1: The birth-birth Transposition of simplices (4,) and (5,) in complex 4

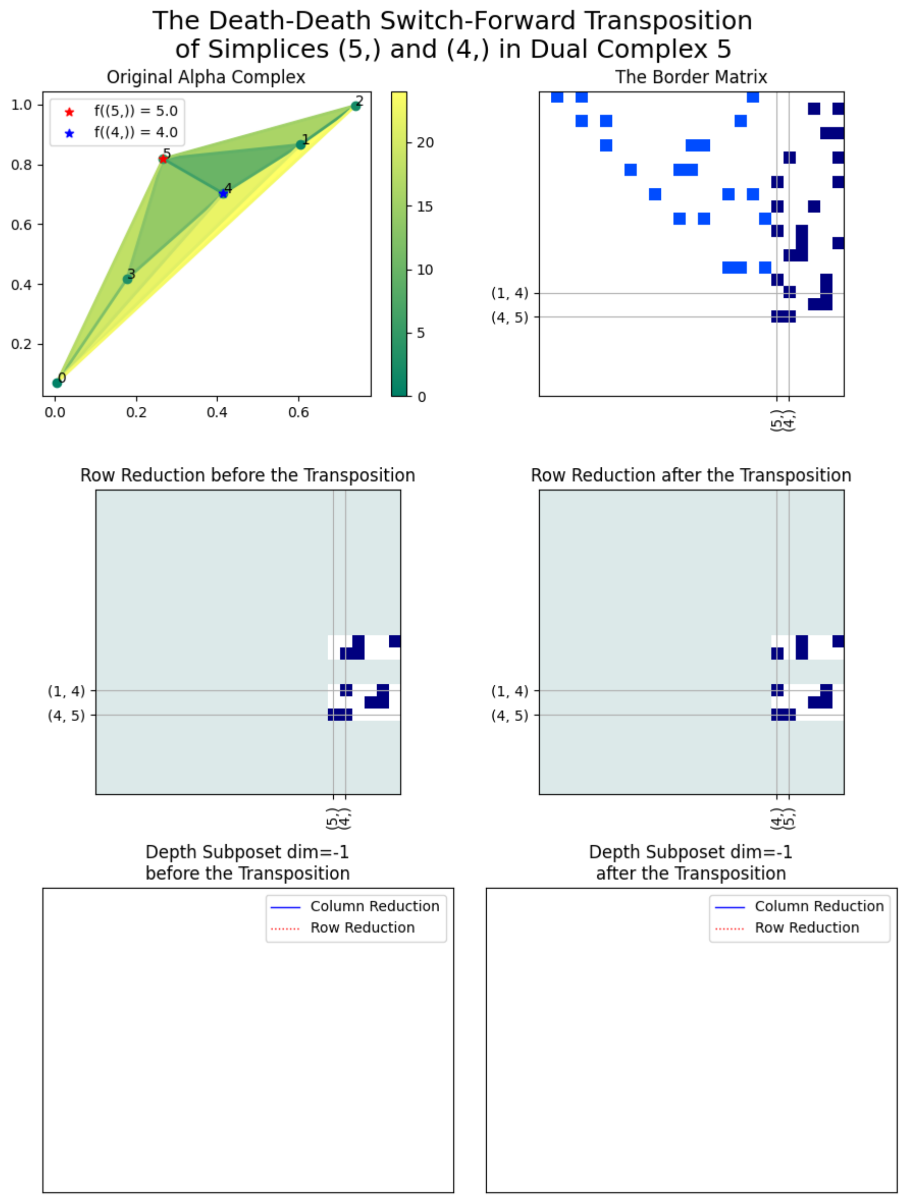

Figure 2: The death-death Transposition of simplices (5,) and (4,) in complex 5

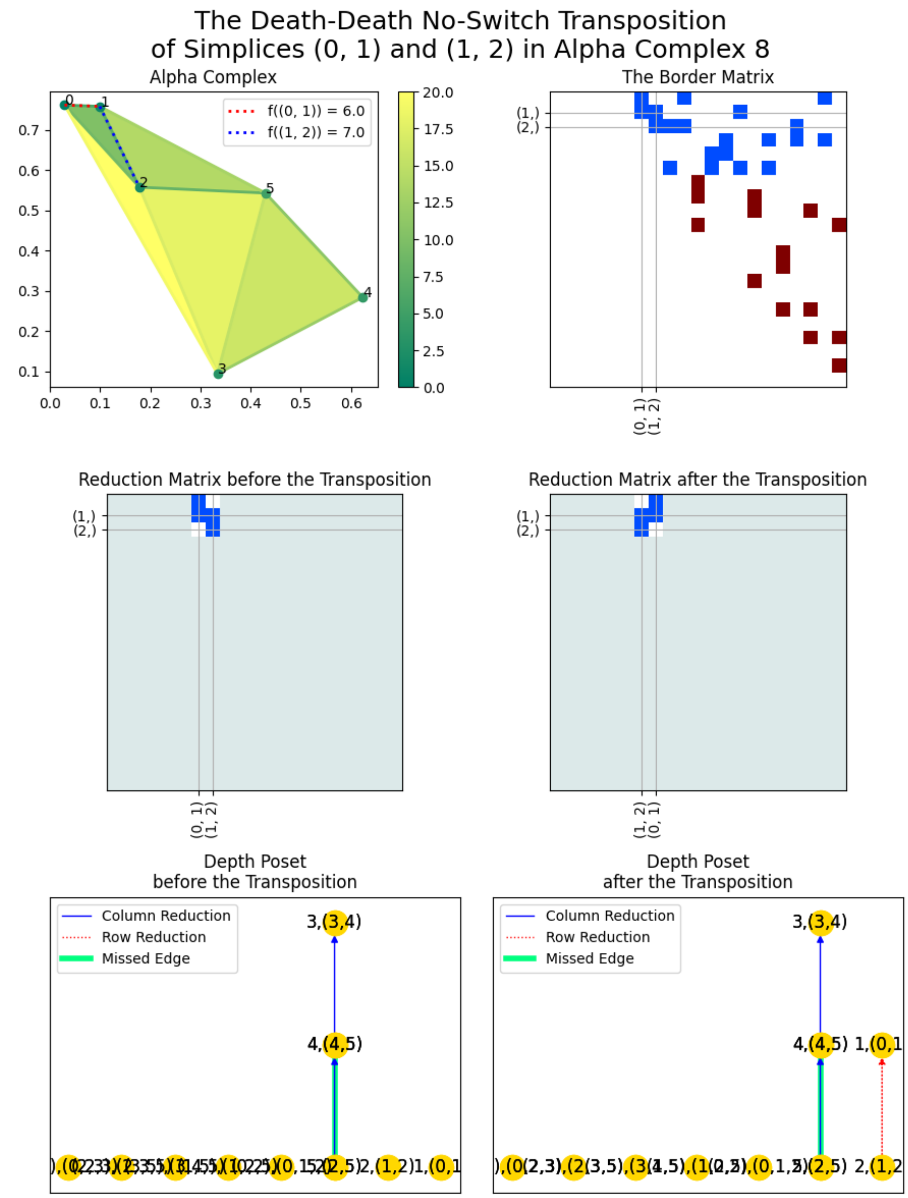

Figure 3: The death-death Transposition of simplices (0, 1) and (1, 2) in complex 8

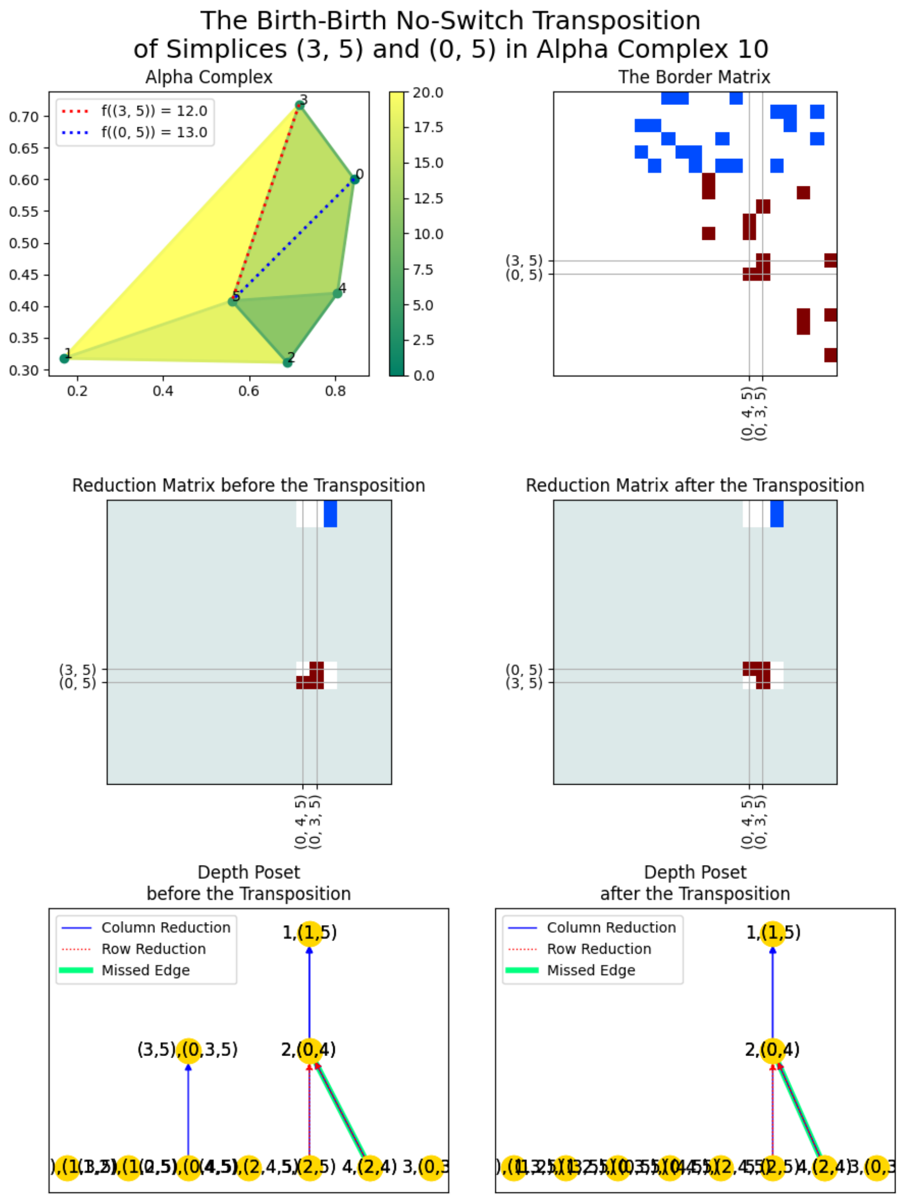

Figure 4: The birth-birth Transposition of simplices (3, 5) and (0, 5) in complex 10

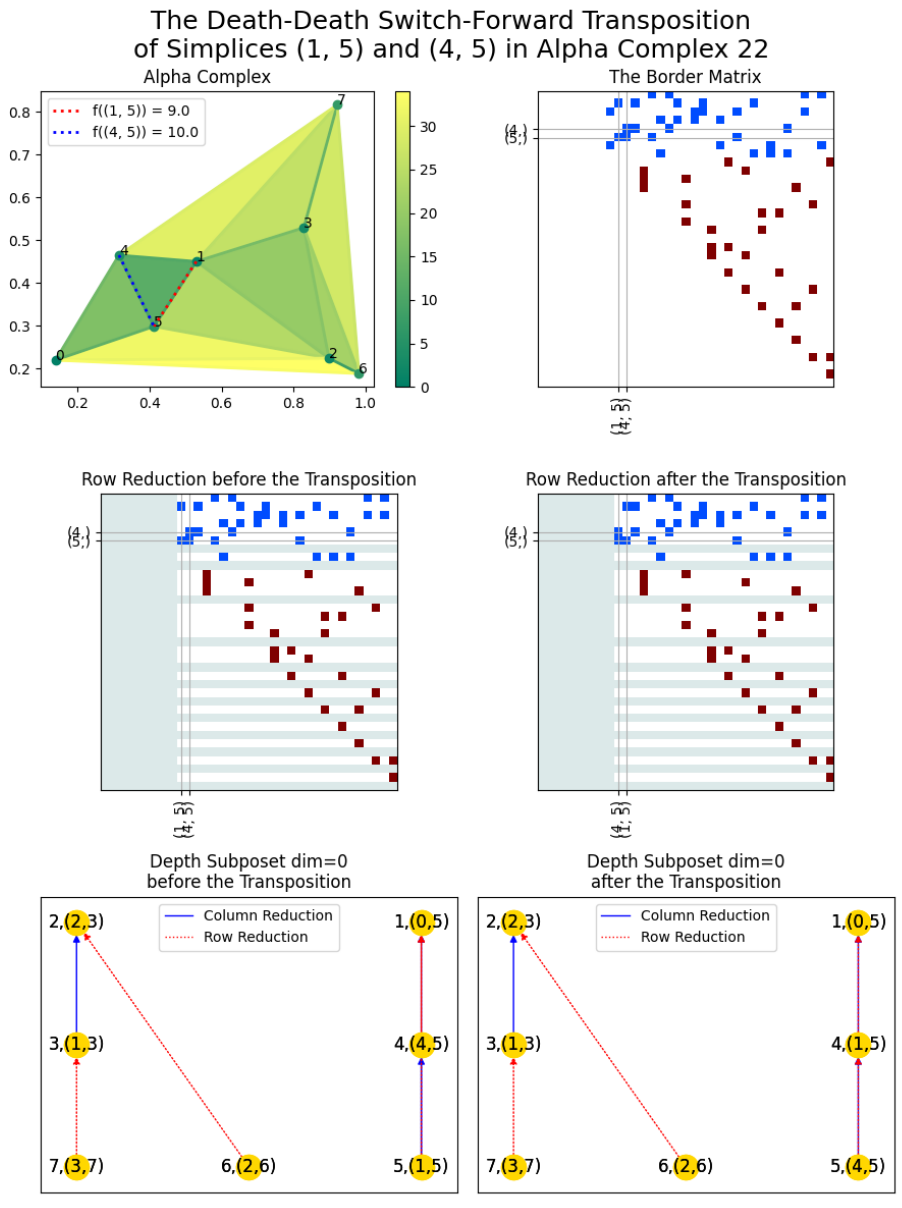

Figure 5: The death-death Transposition of simplices (1, 5) and (4, 5) in complex 22

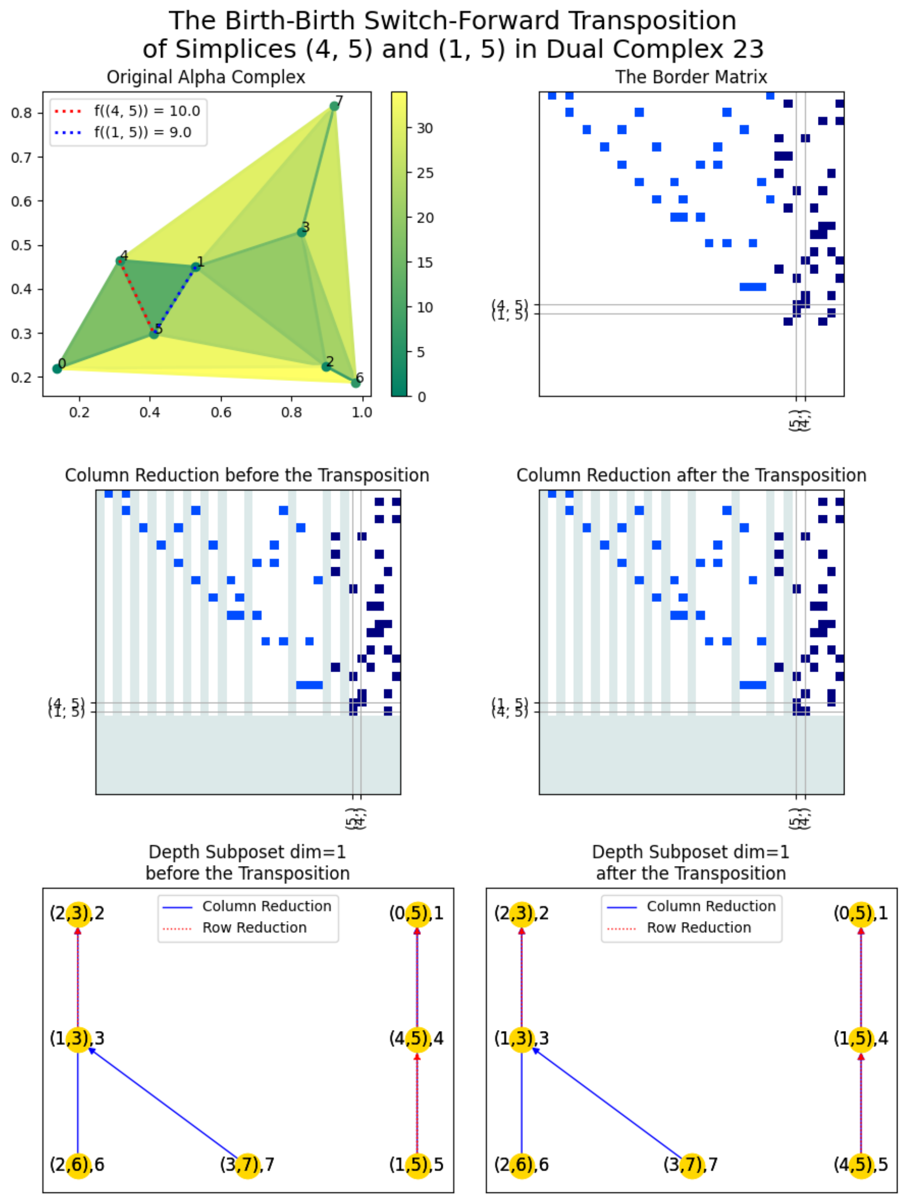

Figure 6: The birth-birth Transposition of simplices (4, 5) and (1, 5) in complex 23

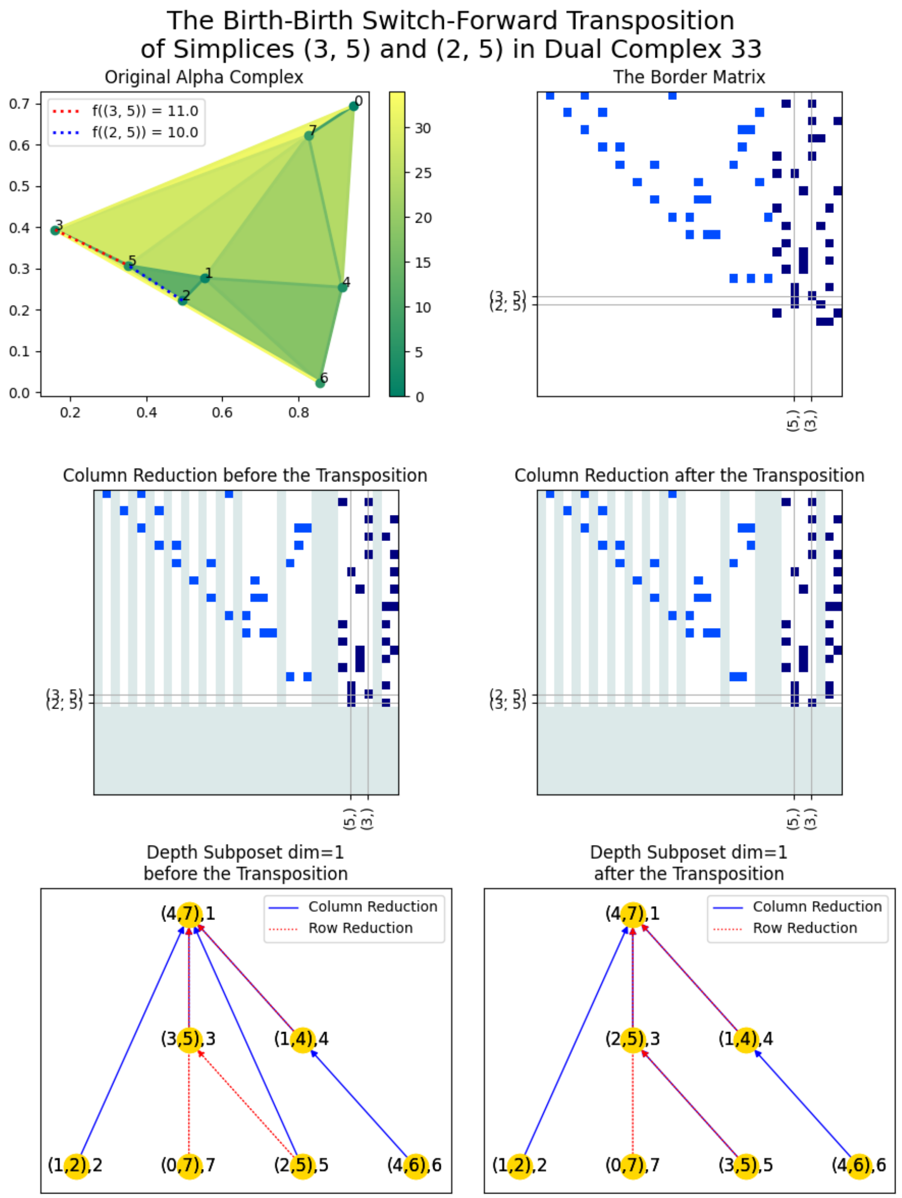

Figure 7: The birth-birth Transposition of simplices (3, 5) and (2, 5) in complex 33

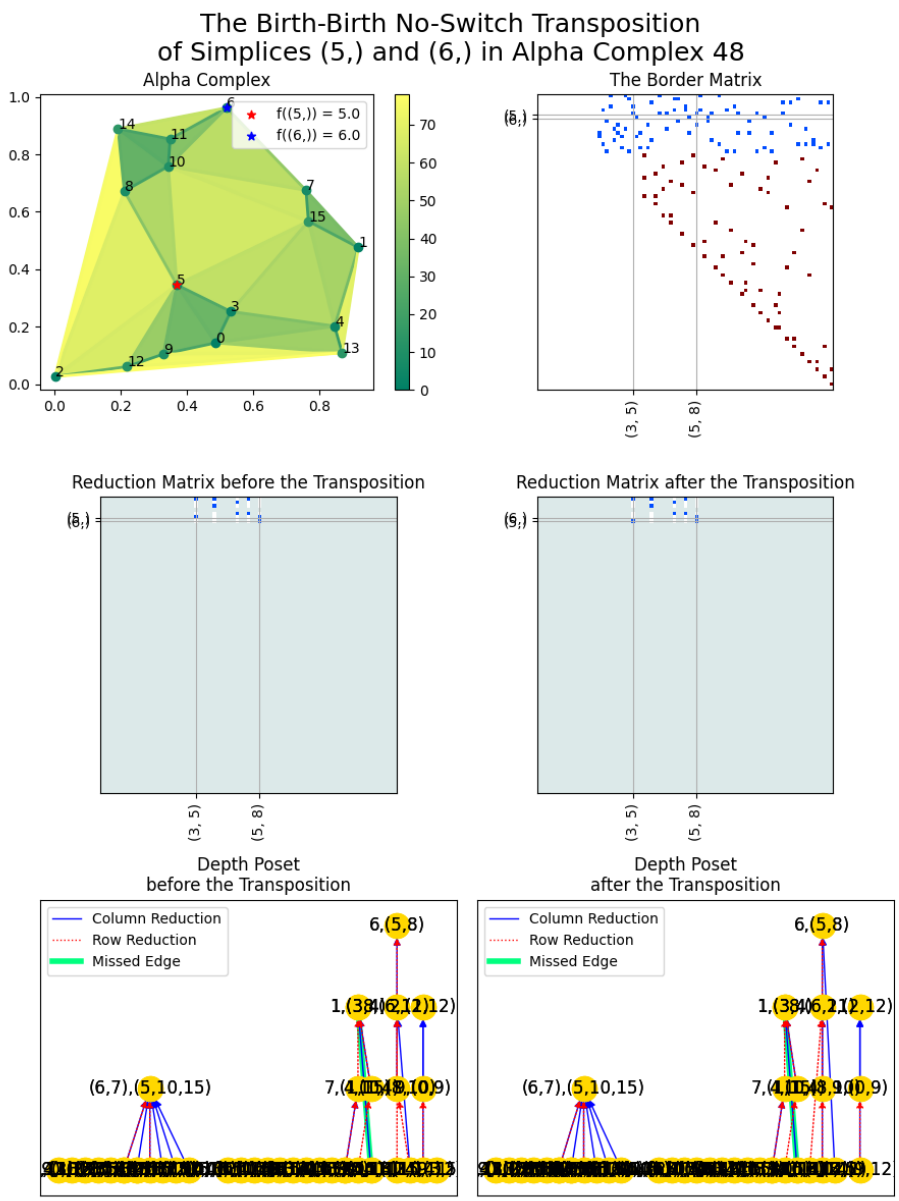

Figure 8: The birth-birth Transposition of simplices (5,) and (6,) in complex 48

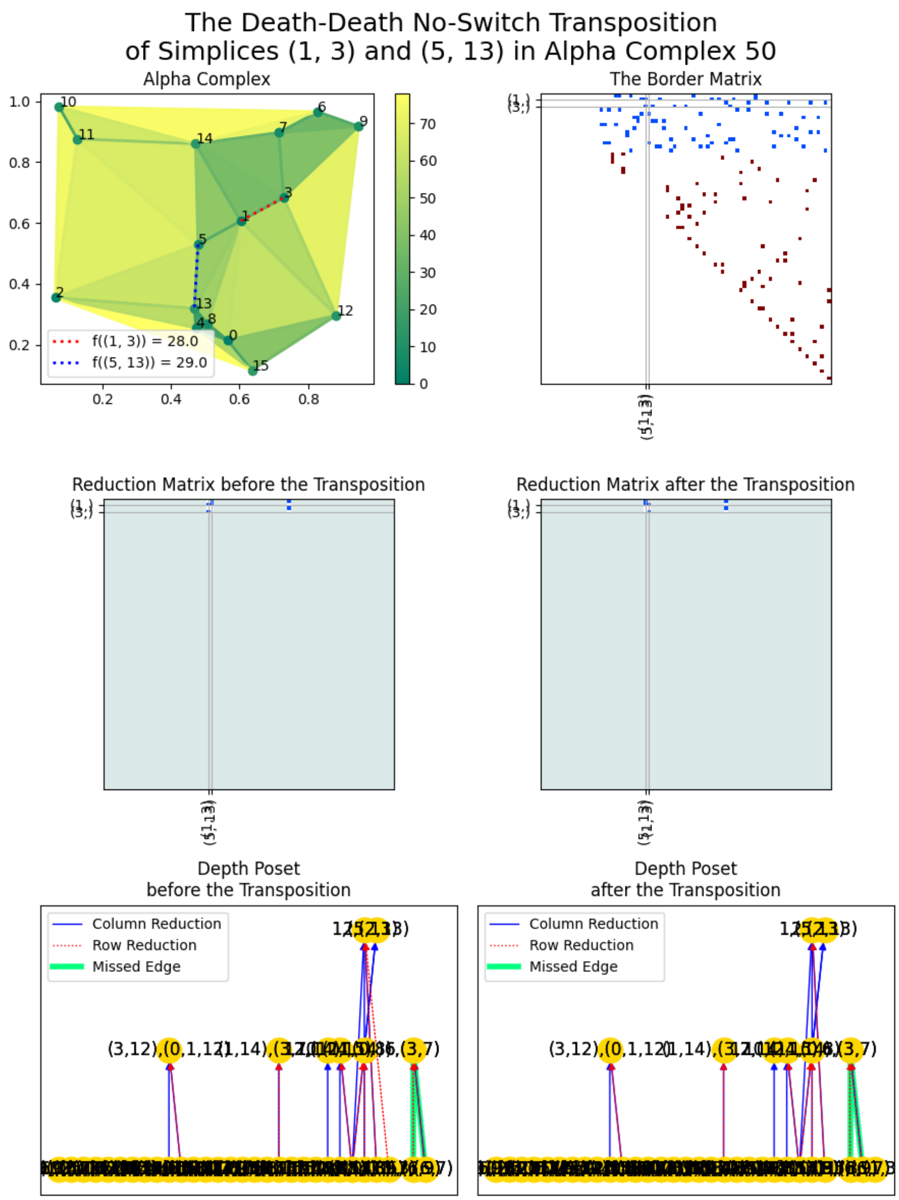

Figure 9: The death-death Transposition of simplices (1, 3) and (5, 13) in complex 50