1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x,y) and (a,b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

$$\mathrm{Succ}'(a,y) = \{(x,b)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$$

$$\mathrm{Succ}'(x,b) = \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) > f(b)\}$$

1.2 Lemma 3.2.

Suppose (a,b) and (x,y) are birth-death pairs of $f : X \to R$, y, b are consecutive in the ordering by f, and transposition of y, b is a switch. Then

$$\mathrm{Succ}''(x,b) = \{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s) < f(x)\}$$

$$\mathrm{Succ}''(a,y) = \{(s,t) \in \mathrm{Succ}''(x,y) | f(s) < f(a)\}$$

1.3 Lemma 3.3.

Suppose (a,b) and (x,y) are birth-death pairs of $f \colon X \to R$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then

$$\mathrm{Succ}(a,x) = \mathrm{Succ}(a,b) \text{ and } \mathrm{Succ}(b,y) = \mathrm{Succ}(x,y)$$

1.4 Hypothesis 1

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Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), \text{ if } s \neq a, b \\ f(b), \text{ if } s = a \\ f(a), \text{ if } s = b \end{cases}$$

Let's denote DP_f the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f(s) = DP_{f^*}(s)$.

2 Model

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0,1]^d$. After this we calculate the Alpha complex with these points, and then find its depth poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

3 Scores

- **jacard_nodes_filtration**: The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.
- **jacard_nodes_simplex**: The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_edges_filtration**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.
- **jacard_edges_simplex**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_l31a**: Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(a,y)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$
- **jacard_l31b**: Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(s,t) \in \mathrm{Succ}'(x,y) | f(t) > f(b)\}$
- **jacard_l32a**: Jacard Index of $\mathrm{Succ}''(x,b)$ and $\{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s) < f(x)\}$
- **jacard_l32b**: Jacard Index of $\mathrm{Succ}''(a,y)$ and $\{(s,t) \in \mathrm{Succ}''(x,y) | f(s) < f(a)\}$
- **jacard_l33a**: Jacard Index of $\mathrm{Succ}(a,x)$ and $\mathrm{Succ}(a,b)$
- **jacard_l33b**: Jacard Index of $\mathrm{Succ}(b,y)$ and $\mathrm{Succ}(x,y)$
- **jacard_nn_nodes**: The Jacard index of subsets of nodes (s,t) from 2 depth posets, s.t. $s,t \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.
- **jacard_nn_edges**: The Jacard index of subsets of edges $((s_0,t_0),(s_1,t_1))$ from 2 depth posets, s.t. $s_0,t_0,s_1,t_1 \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Results

Here are the tables, containing the scores for the switch transpositions in the experiment:

index 0	index 1	simplex 0	simplex 1	type	jacard_nodes_filtration	jacard_nodes_simplex
47	48	[13, 15]	[8, 11]	birth-death	1.00	0.92
53	54	[1, 8]	[1, 3]	birth-death	1.00	0.92
57	58	[11, 12]	[12, 15]	birth-death	1.00	0.92
65	66	[0, 10]	[5, 10]	birth-death	1.00	0.92
87	88	[2, 9, 14]	[9, 14, 17]	death-death	1.00	0.92

index 0	index 1	simplex 0	simplex 1	type	jacard_edges_filtration	jacard_edges_simplex
47	48	[13, 15]	[8, 11]	birth-death	0.93	0.68
53	54	[1, 8]	[1, 3]	birth-death	1.00	0.73
57	58	[11, 12]	[12, 15]	birth-death	1.00	0.58
65	66	[0, 10]	[5, 10]	birth-death	0.96	0.83
87	88	[2, 9, 14]	[9, 14, 17]	death-death	1.00	0.79

index 0	index 1	simplex 0	simplex 1	type	jacard_l31a	jacard_l31b
47	48	[13, 15]	[8, 11]	birth-death	NaN	NaN
53	54	[1, 8]	[1, 3]	birth-death	NaN	NaN
57	58	[11, 12]	[12, 15]	birth-death	NaN	NaN
65	66	[0, 10]	[5, 10]	birth-death	NaN	NaN
87	88	[2, 9, 14]	[9, 14, 17]	death-death	0.00	0.00

index 0	index 1	simplex 0	simplex 1	type	jacard_l32a	jacard_l32b
47	48	[13, 15]	[8, 11]	birth-death	NaN	NaN
53	54	[1, 8]	[1, 3]	birth-death	NaN	NaN
57	58	[11, 12]	[12, 15]	birth-death	NaN	NaN
65	66	[0, 10]	[5, 10]	birth-death	NaN	NaN
87	88	[2, 9, 14]	[9, 14, 17]	death-death	1.00	1.00

index 0	index 1	simplex 0	simplex 1	type	jacard_l33a	jacard_l33b
47	48	[13, 15]	[8, 11]	birth-death	1.00	1.00
53	54	[1, 8]	[1, 3]	birth-death	1.00	1.00
57	58	[11, 12]	[12, 15]	birth-death	1.00	1.00
65	66	[0, 10]	[5, 10]	birth-death	1.00	1.00
87	88	[2, 9, 14]	[9, 14, 17]	death-death	NaN	NaN

index 0	index 1	simplex 0	simplex 1	type	jacard_nn_nodes	jacard_nn_edges
47	48	[13, 15]	[8, 11]	birth-death	1.00	1.00
53	54	[1, 8]	[1, 3]	birth-death	1.00	1.00
57	58	[11, 12]	[12, 15]	birth-death	1.00	1.00
65	66	[0, 10]	[5, 10]	birth-death	1.00	1.00
87	88	[2, 9, 14]	[9, 14, 17]	death-death	1.00	1.00

4.1 Unexpected results

The Hypothesis problem can be measured by 2 scores with expected values: **jacard_nn_edges**, **jacard_nn_nodes** There are 0 cases, which does not corespond the expectations.

The Lemma 3.1 problem can be measured by 2 scores with expected values: **jacard_l31a**, **jacard_l31b** There are 0 cases, which does not corespond the expectations.

The Lemma 3.2 problem can be measured by 2 scores with expected values: **jacard_l32a**, **jacard_l32b** There are 0 cases, which does not corespond the expectations.

The Lemma 3.3 problem can be measured by 2 scores with expected values: **jacard_l33a**, **jacard_l33b** There are 0 cases, which does not corespond the expectations.