1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x,y) and (a,b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

a $W'_{\text{after}}[y,t] = \begin{cases} W'_{\text{before}}[y,t] + W'_{\text{before}}[b,t], & \text{if } t \neq b \\ W'_{\text{before}}[y,b] + 1, & \text{otherwise} \end{cases}$

b $W'_{\text{after}}[b,t] = W'_{\text{before}}[b,t]$

These 2 formulas can be descripted in archaic way:

a $(y,t) \in B'_{\text{after}}$ iff one of the sentences is true:

- $t \neq b$ and $(y,t) \in B'_{\text{before}}$ and $(b,t) \notin B'_{\text{before}}$
- $t \neq b$ and $(y,t) \notin B'_{\text{before}}$ and $(b,t) \in B'_{\text{before}}$
- $t = b$ and $(y,b) \notin B'_{\text{before}}$

b $(b,t) \in B'_{\text{aftr}}$ iff $(b,t) \in B'_{\text{before}}$

And they have implications:

c $\mathrm{Succ}'_{\text{after}}(a,y) = \begin{cases} \mathrm{Succ}'_{\text{before}}(x,y) \cup \mathrm{Succ}'_{\text{before}}(a,b) \cup \{(x,b)\}, & \text{if } (a,b) \notin \mathrm{Succ}'_{\text{before}}(x,y) \\ \mathrm{Succ}'_{\text{before}}(x,y) / (\{a,b\} \cup \{(s,t) \in \mathrm{Succ}'_{\text{before}}(a,b) : \ \kappa(t,a) = 0\}) \end{cases}$

d $\mathrm{Succ}'_{\text{after}}(x,b) = \mathrm{Succ}'_{\text{before}}(a,b)$

1.2 Lemma 3.3

Suppose (a,b) and (x,y) are birth-death pairs of $f : X \to \mathbb{R}$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then the successors of the two pairs are preserved, while the predecessors may change:

a $\mathrm{Succ}_{\text{after}}(a,x) = \mathrm{Succ}_{\text{before}}(a,b)$

b $\mathrm{Succ}_{\text{after}}(b,y) = \mathrm{Succ}_{\text{before}}(x,y)$

c $\mathrm{Pred}_{\text{after}}(a,x) = \{(s,t) \mid t \in \mathrm{Col}_{\text{before}}(a,x)\} \cup \mathrm{Pred}''_{\text{before}}(a,b)$

d $\mathrm{Pred}_{\text{after}}(b,y) = \{(s,t) \mid s \in \mathrm{Row}_{\text{before}}(b,y)\} \cup \mathrm{Pred}'_{\text{before}}(x,y)$

in which the left- and right-hand sides of the equations correspond to the states after and before the transposition, respectively.

1.3 Hypothesis 1

Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), \text{ if } s \neq a, b \\ f(b), \text{ if } s = a \\ f(a), \text{ if } s = b \end{cases}$$

Let's denote $DP_f^{\min}$ the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f^{\min}(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$

2 Model and Experiments

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0, 1]^d$. After this we calculate the Alpha Complex with these points, and then find its Depth Poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

As we know, an Alpha Complex is a Simplicial Complex, which can be represented as Lefschetz Complex. We also study the dual complexed transposing the border matrices over minor diagonal.

We can see the calculated cases in the table the given:

case	n	complex dim	alpha	dual	case	n	complex dim	alpha	dual
1	6	2	2	3	11	8	2	22	23
2	6	2	4	5	12	8	2	24	25
3	6	2	6	7	13	8	2	26	27
4	6	2	8	9	14	8	2	28	29
5	6	2	10	11	15	8	3	30	31
6	6	2	12	13	16	8	3	32	33
7	8	2	14	15	17	8	3	34	35
8	8	2	16	17	18	8	3	36	37
9	8	2	18	19	19	8	3	38	39
10	8	2	20	21	20	8	3	40	41

We also check few specific transpositions for cases, with given border matrices:

complex	dim	matrix	transpositions
0	2	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	1
1	1	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	1

We can see the distribution of transposition types in each complex in the given table:

	switch forward			switch backward		no switch		
	birth-birth	death-death	birth-death	birth-birth	death-death	birth-birth	death-death	birth-death
complex								
Total	28	26	150	19	19	257	217	354
0	1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	2	0	0	0	0	3	4	7
3	0	2	0	0	0	5	2	7
4	0	1	0	0	1	5	2	5
5	1	0	0	1	0	3	4	5
6	1	0	2	1	0	3	4	3
7	0	1	2	0	1	5	2	3
8	0	0	1	0	0	5	4	5
9	0	0	1	0	0	5	4	5
10	1	0	1	0	0	4	4	5
11	0	1	1	0	0	5	3	5
12	1	1	2	0	1	5	3	3
13	1	1	2	1	0	4	4	3
14	0	0	3	1	1	7	6	6
15	0	0	3	1	1	7	6	6
16	0	0	2	1	1	6	5	9
17	0	0	2	1	1	6	5	9
18	0	0	1	0	0	7	6	9
19	0	0	1	0	0	7	6	9
20	0	1	1	0	0	7	5	8
21	1	0	1	0	0	6	6	8
22	0	1	1	0	0	9	7	5
23	1	0	1	0	0	8	8	5
24	2	2	2	1	0	6	6	5
25	2	2	2	0	1	7	5	5
26	1	2	0	1	0	6	5	10
27	2	1	0	0	1	6	5	10
28	1	0	1	2	1	4	5	9
29	0	1	1	1	2	6	3	9
30	0	0	7	1	1	10	9	17
31	0	0	7	1	1	10	9	17
32	1	2	10	0	0	8	6	16
33	2	1	10	0	0	7	7	16

complex	switch forward birth-birth	death-death	birth-death	switch backward birth-birth	death-death	no switch birth-birth	death-death	birth-death
34	1	0	12	0	2	7	5	13
35	0	1	12	2	0	6	6	13
36	2	1	11	1	0	9	10	14
37	1	2	11	0	1	11	8	14
38	1	0	10	0	0	7	7	13
39	0	1	10	0	0	8	6	13
40	0	1	8	0	2	10	6	15
41	1	0	8	2	0	7	9	15

3 Scores

jacard_nodes_filtration - The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_nodes_simplex - The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.

jacard_edges_filtration - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_edges_simplex - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.

l31a - Returns the portion of simplices t which are births or deaths in both depth posets such that $W'_{\text{after}}[y,t] = \begin{cases} W'_{\text{before}}[y,t] + W'_{\text{before}}[b,t], & \text{if } t \neq b \\ W'_{\text{before}}[y,b] + 1, & \text{otherwise} \end{cases}$

l31b - Returns the portion of simplices t which are births or deaths in both depth posets such that $W'_{\text{after}}[b,t] = W'_{\text{before}}[b,t]$

l31c - If $(a,b) \notin \text{Desc}'(x,y)$ will return the Jacard index of sets: $\text{Desc}'_{\text{after}}(a,y)$ and $\text{Desc}'_{\text{before}}(x,y) \cup \text{Desc}'_{\text{before}}(a,b) \cup \{(x,b)\}$. And will return the Jacard index of sets $\text{Desc}'_{\text{after}}(a,y)$ and $\text{Desc}'_{\text{before}}(x,y)/(\{a,b\} \cup \{(s,t) \in \text{Desc}'_{\text{before}}(a,b) : \kappa(t,a) = 0\})$ otherwise.

l31d - The Jacard index of sets $\text{Desc}'_{\text{after}}(x,b)$ and $\text{Desc}'_{\text{before}}(a,b)$

l33a - The Jacard index of sets $\text{Succ}_{\text{after}}(a,x)$ and $\text{Succ}_{\text{before}}(a,b)$

l33b - The Jacard index of sets $\text{Succ}_{\text{after}}(b,y)$ and $\text{Succ}_{\text{before}}(x,y)$

jacard_nn_nodes - The Jacard index of subsets of nodes (s,t) from 2 depth posets, s.t. $s,t \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

jacard_nn_edges - The Jacard index of subsets of edges $((s_0,t_0),(s_1,t_1))$ from 2 depth posets, s.t. $s_0,t_0,s_1,t_1 \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Conclusions and Unexpected Cases

4.1 Lemma 3.1

Lemma 3.1 can be measured by 2 scores: **l31a**, **l31b**, **l31c**, **l31d**. And there are 0 of 28 switch forward birth-birth transpositions found where these scores are not 1.

4.2 Lemma 3.3

Lemma 3.3 can be measured by 2 scores: **l33a**, **l33b**. And there are 0 of 150 switch forward birth-death transpositions found where these scores are not 1.

4.3 Hypothesis

Hypothesis can be measured by 2 scores: **jacard_nn_nodes**, **jacard_nn_edges**. And there are 2 of 1070 transpositions found where these scores are not 1.

We can see these situations in the table:

case	complex	simplex 0	simplex 1	jacard_nn_nodes	jacard_nn_edges	Figure
	0	1	2	NaN	NaN	Figure 1
	1	1	2	NaN	NaN	Figure 2

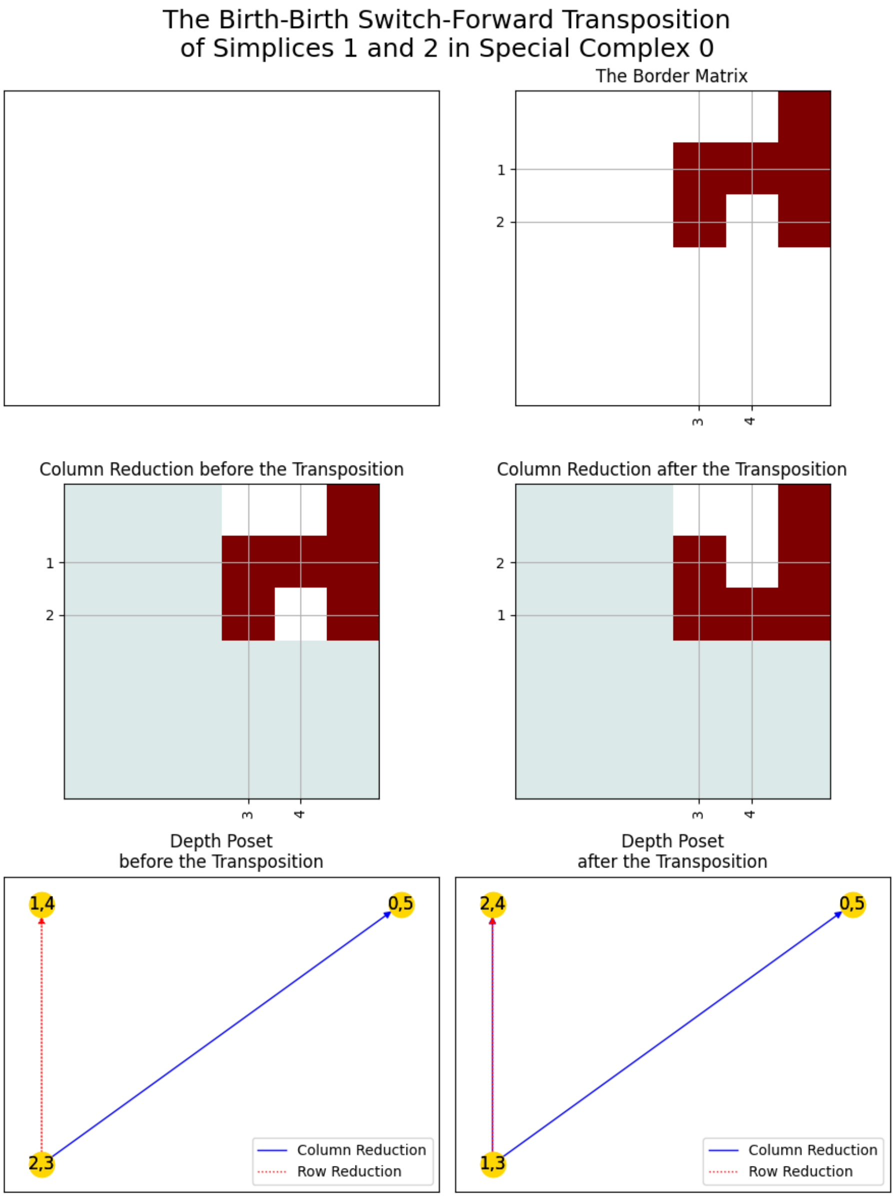

Figure 1: The birth-birth Transposition of simplices 1 and 2 in complex 0

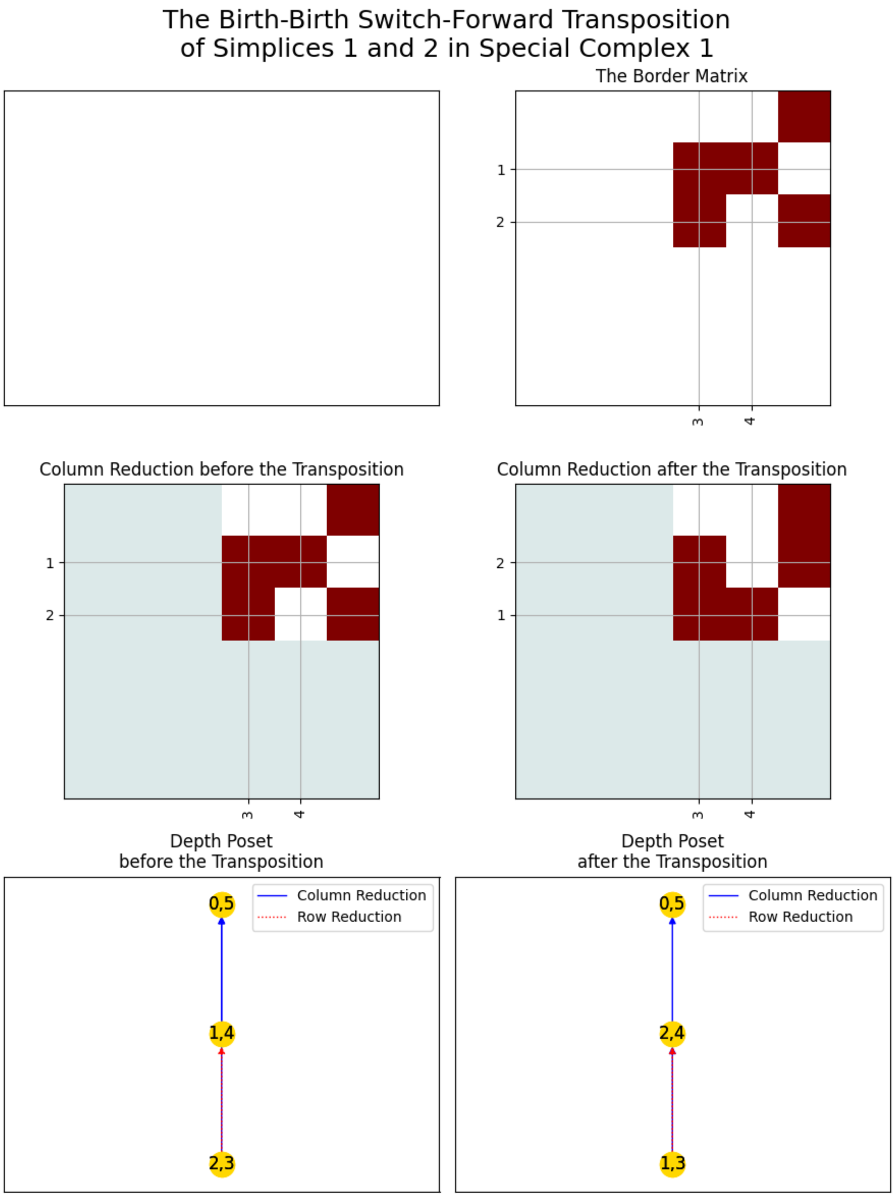

Figure 2: The birth-birth Transposition of simplices 1 and 2 in complex 1