1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x, y) and (a, b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

$$\mathrm{Succ}'(a,y) = \{(x,b)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$$

$$\mathrm{Succ}'(x,b) = \{(s,t) \in \mathrm{Succ}'(x,y) \cup \mathrm{Succ}'(a,b) | f(t) > f(b)\}$$

1.2 Lemma 3.2.

Suppose (a, b) and (x, y) are birth-death pairs of $f : X \to R$, y, b are consecutive in the ordering by f, and transposition of y, b is a switch. Then

$$\mathrm{Succ}''(x,b) = \{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s)\}$$

$$\mathrm{Succ}''(a,y) = \{(s,t) \in \mathrm{Succ}''(a,b) \cup \mathrm{Succ}''(x,y) | f(s) < f(a)\}$$

1.3 Lemma 3.3.

Suppose (a, b) and (x, y) are birth-death pairs of $f\colon X \to R$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then

$$\mathrm{Succ}(a,x) = \mathrm{Succ}(a,b) \text{ and } \mathrm{Succ}(b,y) = \mathrm{Succ}(x,y)$$

1.4 Hypothesis 1

Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), & \text{if } s \neq a, b \\ f(b), & \text{if } s = a \\ f(a), & \text{if } s = b \end{cases}$$

Let's denote $DP_f^{\min}$ the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f^{\min}(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f^{\min}(s) = DP_{f^*}^{\min}(s)$.

2 Model and Experiments

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0, 1]^d$. After this we calculate the Alpha Complex with these points, and then find its Depth Poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

As we know, an Alpha Complex is a Simplicial Complex, which can be represented as Lefschetz Complex. We also study the dual complexed transposing the border matrices over minor diagonal.

We can see the calculated cases in the table the given:

case	n	complex dim	alpha	dual
1	6	2	0	1
2	6	2	2	3
3	6	2	4	5
4	6	2	6	7
5	6	2	8	9
6	6	2	10	11
7	8	2	12	13
8	8	2	14	15
9	8	2	16	17
10	8	2	18	19
11	8	2	20	21
12	8	2	22	23
13	8	2	24	25
14	8	2	26	27
15	12	2	28	29
16	12	2	30	31
17	12	2	32	33
18	12	2	34	35
19	12	2	36	37
20	12	2	38	39
21	12	2	40	41
22	12	2	42	43
23	12	2	44	45
24	12	2	46	47
25	12	2	48	49
26	12	2	50	51
27	16	2	52	53
28	16	2	54	55
29	16	2	56	57
30	16	2	58	59
31	16	2	60	61
32	16	2	62	63
33	16	2	64	65
34	16	2	66	67
35	8	3	68	69
36	8	3	70	71
37	8	3	72	73
38	8	3	74	75
39	8	3	76	77
40	8	3	78	79
41	12	3	80	81
42	12	3	82	83
43	12	3	84	85
44	12	3	86	87
45	12	3	88	89
46	12	3	90	91
47	12	3	92	93
48	12	3	94	95
49	16	3	96	97
50	16	3	98	99
51	16	3	100	101
52	16	3	102	103
53	16	3	104	105
54	16	3	106	107
55	8	4	108	109
56	8	4	110	111
57	12	4	112	113
58	12	4	114	115
59	12	4	116	117
60	12	4	118	119
61	12	4	120	121
62	12	4	122	123
63	16	4	124	125
64	16	4	126	127

We can see the distribution of transposition types in each complex in the given table:

complex	switch forward birth-birth	switch forward death-death	switch forward birth-death	switch backward birth-birth	switch backward death-death	no switch birth-birth	no switch death-death	no switch birth-death
Total	95	95	1776	96	96	1533	1533	4470
0	0	0	2	1	0	3	4	5
1	0	0	2	0	1	4	3	5
2	0	1	0	1	0	3	3	6
3	1	0	0	0	1	3	3	6
4	0	0	2	1	0	3	4	5
5	0	0	2	0	1	4	3	5
6	1	1	2	0	1	4	3	3
7	1	1	2	1	0	3	4	3
8	1	0	2	0	0	3	4	3
9	0	1	2	0	0	4	3	3
10	1	0	3	0	0	3	4	5
11	0	1	3	0	0	4	3	5
12	0	1	2	1	1	6	5	6
13	1	0	2	1	1	5	6	6
14	0	1	0	2	1	5	5	9
15	1	0	0	1	2	5	5	9
16	0	1	4	1	1	6	5	3
17	1	0	4	1	1	5	6	3
18	1	0	2	0	0	5	6	9
19	0	1	2	0	0	6	5	9
20	1	1	1	0	1	6	5	9
21	1	1	1	1	0	5	6	9
22	0	0	0	0	1	6	5	8
23	0	0	0	1	0	5	6	8
24	0	0	5	1	1	6	6	4
25	0	0	5	1	1	6	6	4
26	0	1	0	0	0	7	6	8
27	1	0	0	0	0	6	7	8
28	1	4	1	1	0	11	9	12
29	4	1	1	0	1	9	11	12
30	1	1	2	1	0	9	10	14
31	1	1	2	0	1	10	9	14
32	2	0	0	2	1	7	10	14
33	0	2	0	1	2	10	7	14

complex	switch forward birth-birth	death-death	birth-death	switch backward birth-birth	death-death	no switch birth-birth	death-death	birth-death
34	1	1	3	0	1	10	9	10
35	1	1	3	1	0	9	10	10
36	1	0	0	1	0	8	10	19
37	0	1	0	0	1	10	8	19
38	1	2	2	0	0	10	9	16
39	2	1	2	0	0	9	10	16
40	1	0	2	2	1	7	9	14
41	0	1	2	1	2	9	7	14
42	0	0	1	2	1	9	10	17
43	0	0	1	1	2	10	9	17
44	0	3	4	0	0	12	9	9
45	3	0	4	0	0	9	12	9
46	0	1	2	1	2	10	8	13
47	1	0	2	2	1	8	10	13
48	1	2	1	1	1	11	10	14
49	2	1	1	1	1	10	11	14
50	0	0	2	0	0	10	10	14
51	0	0	2	0	0	10	10	14
52	0	0	2	3	2	14	15	21
53	0	0	2	2	3	15	14	21
54	1	0	2	0	1	15	15	21
55	0	1	2	1	0	15	15	21
56	0	3	1	2	0	13	12	23
57	3	0	1	0	2	12	13	23
58	3	2	0	0	1	14	14	23
59	2	3	0	1	0	14	14	23
60	2	1	2	2	1	15	17	17
61	1	2	2	1	2	17	15	17
62	0	1	0	1	0	15	15	24
63	1	0	0	0	1	15	15	24
64	0	2	2	1	1	16	14	18
65	2	0	2	1	1	14	16	18
66	0	0	1	1	0	14	15	24
67	0	0	1	0	1	15	14	24
68	0	0	7	1	0	6	7	17

complex	switch forward birth-birth	death-death	birth-death	switch backward birth-birth	death-death	no switch birth-birth	death-death	birth-death
69	0	0	7	0	1	7	6	17
70	0	0	8	0	0	7	7	19
71	0	0	8	0	0	7	7	19
72	1	1	9	1	0	7	8	13
73	1	1	9	0	1	8	7	13
74	1	1	7	0	0	6	6	17
75	1	1	7	0	0	6	6	17
76	1	3	6	2	1	8	7	15
77	3	1	6	1	2	7	8	15
78	0	0	9	0	1	8	7	18
79	0	0	9	1	0	7	8	18
80	0	1	18	0	0	15	14	41
81	1	0	18	0	0	14	15	41
82	1	3	13	3	2	14	13	40
83	3	1	13	2	3	13	14	40
84	2	1	13	1	1	10	11	39
85	1	2	13	1	1	11	10	39
86	0	0	16	1	1	14	14	44
87	0	0	16	1	1	14	14	44
88	0	0	22	1	0	12	13	30
89	0	0	22	0	1	13	12	30
90	1	1	20	1	0	10	11	42
91	1	1	20	0	1	11	10	42
92	0	0	18	1	2	13	12	44
93	0	0	18	2	1	12	13	44
94	1	0	16	0	1	10	10	47
95	0	1	16	1	0	10	10	47
96	1	1	20	2	0	21	23	64
97	1	1	20	0	2	23	21	64
98	1	1	17	0	0	19	19	72
99	1	1	17	0	0	19	19	72
100	0	1	22	0	0	19	18	67
101	1	0	22	0	0	18	19	67
102	0	0	22	2	1	22	23	75
103	0	0	22	1	2	23	22	75

complex	switch forward birth-birth	death-death	birth-death	switch backward birth-birth	death-death	no switch birth-birth	death-death	birth-death
104	0	0	18	2	0	22	24	65
105	0	0	18	0	2	24	22	65
106	1	3	17	0	0	27	25	61
107	3	1	17	0	0	25	27	61
108	2	0	28	1	0	6	9	34
109	0	2	28	0	1	9	6	34
110	0	1	21	0	1	9	7	32
111	1	0	21	1	0	7	9	32
112	1	0	43	0	0	18	19	98
113	0	1	43	0	0	19	18	98
114	0	1	47	1	1	21	20	90
115	1	0	47	1	1	20	21	90
116	1	1	41	1	1	23	23	103
117	1	1	41	1	1	23	23	103
118	0	0	44	1	0	20	21	101
119	0	0	44	0	1	21	20	101
120	0	1	62	2	2	17	16	93
121	1	0	62	2	2	16	17	93
122	2	0	60	2	2	14	16	103
123	0	2	60	2	2	16	14	103
124	1	0	106	1	2	29	29	174
125	0	1	106	2	1	29	29	174
126	4	2	83	1	0	40	43	157
127	2	4	83	0	1	43	40	157

3 Scores

jacard_nodes_filtration - The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_nodes_simplex - The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.

jacard_edges_filtration - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.

jacard_edges_simplex - The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.

jacard_l31a - Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(a,y)\}\cup\mathrm{Succ}'(a,b)\cup\{(s,t)\in \mathrm{Succ}'(x,y)|f(t)<f(b)\}$

jacard_l31b - Jacard Index of $\mathrm{Succ}'(x,b)$ and $\{(s,t)\in \mathrm{Succ}'(x,y)\cup\mathrm{Succ}'(a,b)|f(t)>f(b)\}$

jacard_l32a - Jacard Index of $\mathrm{Succ}''(x,b)$ and $\{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s) < f(x)\}$

jacard_l32b - Jacard Index of $\mathrm{Succ}''(a,y)$ and $\{(s,t) \in \mathrm{Succ}''(x,y) \cup \mathrm{Succ}''(a,b) | f(s) < f(a)\}$

jacard_l33a - Jacard Index of $\mathrm{Succ}(a,x)$ and $\mathrm{Succ}(a,b)$

jacard_l33b - Jacard Index of $\mathrm{Succ}(b,y)$ and $\mathrm{Succ}(x,y)$

jacard_nn_nodes - The Jacard index of subsets of nodes (s,t) from 2 depth posets, s.t. $s,t \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

jacard_nn_edges - The Jacard index of subsets of edges $((s_0,t_0),(s_1,t_1))$ from 2 depth posets, s.t. $s_0,t_0,s_1,t_1 \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Conclusions and Unexpected Cases

4.1 Lemma 3.1

Lemma 3.1 can be measured by 2 scores: **jacard_l31a** and **jacard_l31b**. And there are 0 of 95 switch-forward birth-birth transpositions found where these scores are not 1.

4.2 Lemma 3.2

Lemma 3.2 can be measured by 2 scores: **jacard_l32a** and **jacard_l32b**. And there are 0 of 95 switch-forward death-death transpositions found where these scores are not 1.

4.3 Lemma 3.3

Lemma 3.3 can be measured by 2 scores: **jacard_l33a** and **jacard_l33b**. And there are 0 of 1776 switch-forward birth-death transpositions found where these scores are not 1.

4.4 Hypothesis

The Hypothesis can be measured by 2 scores: **jacard_nn_nodes** and **jacard_nn_edges**. And there are 0 of 9822 transpositions found where these scores are not 1.