1 Problems

Write $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) = \mathrm{Succ}''(s,t)$ for the succesors and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) = \mathrm{Pred}''(s,t)$ for the predecessors (ancessors) of the pair in D, D', D'', respectively.

Since $D = D' \cup D''$, we also have $\mathrm{Succ}(s,t) = \mathrm{Succ}'(s,t) \cup \mathrm{Succ}''(s,t)$ and $\mathrm{Pred}(s,t) = \mathrm{Pred}'(s,t) \cup \mathrm{Pred}''(s,t)$.

1.1 Lemma 3.1.

Suppose (x,y) and (a,b) are birth-death pairs of $f : X \to R$, a, x are consecutive in the ordering of the cellls by f, and the transposition a, x is a switch. Then

$$\mathrm{Succ}'(a,y) = \{(x,b)\} \cup \mathrm{Succ}'(a,b) \cup \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) < f(b)\}$$

$$\mathrm{Succ}'(x,b) = \{(s,t) \in \mathrm{Succ}'(x,y) | f(t) > f(b)\}$$

1.2 Lemma 3.2.

Suppose (a,b) and (x,y) are birth-death pairs of $f : X \to R$, y, b are consecutive in the ordering by f, and transposition of y, b is a switch. Then

$$\mathrm{Succ}''(x,b) = \{(a,y)\} \cup \mathrm{Succ}''(a,b) \cup \{(s,t) \in \mathrm{Succ}''(x,y) | f(a) < f(s) < f(x)\}$$

$$\mathrm{Succ}''(a,y) = \{(s,t) \in \mathrm{Succ}''(x,y) | f(s) < f(a)\}$$

1.3 Lemma 3.3.

Suppose (a,b) and (x,y) are birth-death pairs of $f\colon X \to R$, b, x are consecutive in the ordering by f, and the transposition of b, x is a switch. Then

$$\mathrm{Succ}(a,x) = \mathrm{Succ}(a,b) \text{ and } \mathrm{Succ}(b,y) = \mathrm{Succ}(x,y)$$

1.4 Hypothesis 1

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Suppose a and b are 2-simplices consecutive in the ordering by f. And there is another Morse function f^*:

$$f^*(s) = \begin{cases} f(s), & \text{if } s \neq a, b \\ f(b), & \text{if } s = a \\ f(a), & \text{if } s = b \end{cases}$$

Let's denote DP_f the transitive reduction of the Depth Poset defined by the filtration f. And let's deonote $DP_f(s)$ the set of nodes in $DP_f(s)$ which are pairs containing cell s and the set of edges with these nodes.

Hypothesis: if the cell s has no faces and cofaces with a and b, then $DP_f(s) = DP_{f^*}(s)$.

2 Model

The probabilistic model is simple. The first we just generate the cloud of n points uniformly distributed in $[0, 1]^d$. After this we calculate the Alpha complex with these points, and then find its depth poset. Then we itarate all neighbour pairs of simplices and check if their transposition will be possible filtration, calculating the scores for the switch-forward transpositions.

We can see the generated cloud and its simplicial complex in the Figure 2.

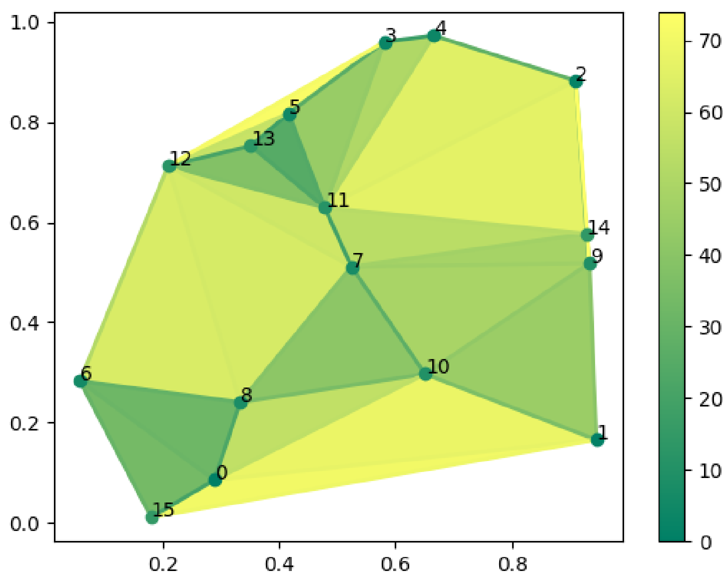

Figure 1: The generated cloud and coresponding simplicial complex

3 Scores description

- **jacard_nodes_filtration**: The Jacard index of node sets from 2 depth posets. The birth-death pairs are equal in terms of filtration values.
- **jacard_nodes_simplex**: The Jacard index of nodes from 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_edges_filtration**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of filtration values.

- **jacard_edges_simplex**: The Jacard index of edge sets from transitive reductions of 2 depth posets. The birth-death pairs are equal in terms of simplices.
- **jacard_l31a**: Jacard Index of $\mathrm{Succ}'(a, y)$ and $(\{(x, b)\} \cup \mathrm{Succ}'(a, b) \cup \{(s, t) \in \mathrm{Succ}'(x, y) | f(t) < f(b)\})$
- **jacard_l31b**: Jacard Index of $\mathrm{Succ}'(x, b)$ and $\{(s, t) \in \mathrm{Succ}'(x, y) | f(t) > f(b)\}$
- **jacard_l32a**: Jacard Index of $\mathrm{Succ}''(x, b)$ and $\{(a, y)\} \cup \mathrm{Succ}''(a, b) \cup \{(s, t) \in \mathrm{Succ}''(x, y) | f(a) < f(s) < f(x)\}$
- **jacard_l32b**: Jacard Index of $\mathrm{Succ}''(a, y)$ and $\{(s, t) \in \mathrm{Succ}''(x, y) | f(s) < f(a)\}$
- **jacard_l33a**: Jacard Index of $\mathrm{Succ}(a, x)$ and $\mathrm{Succ}(a, b)$
- **jacard_l33b**: Jacard Index of $\mathrm{Succ}(b, y)$ and $\mathrm{Succ}(x, y)$
- **jacard_nn_nodes**: The Jacard index of subsets of nodes (s, t) from 2 depth posets, s.t. $s, t \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.
- **jacard_nn_edges**: The Jacard index of subsets of edges $((s_0, t_0), (s_1, t_1))$ from 2 depth posets, s.t. $s_0, t_0, s_1, t_1 \notin \nabla\partial\sigma_0 \cup \partial\nabla\sigma_0 \cup \nabla\partial\sigma_1 \cup \partial\nabla\sigma_1$, where σ_0 and σ_1 are transposing simplices.

4 Results

Here are the tables, containing the scores for the switch transpositions in the experiment:

index 0	index 1	simplex 0	simplex 1	type	jacard_nodes_filtration	jacard_nodes_simplex
23	24	[11, 13]	[5, 11]	birth-death	1.00	0.90
29	30	[6, 8]	[6, 15]	birth-death	1.00	0.90
35	36	[8, 10]	[1, 10]	death-death	1.00	0.90
46	47	[7, 9]	[7, 14]	birth-birth	1.00	0.90
62	63	[7, 8, 12]	[6, 8, 12]	death-death	1.00	0.90
65	66	[2, 4, 11]	[2, 11, 14]	death-death	1.00	0.90

index 0	index 1	simplex 0	simplex 1	type	jacard_edges_filtration	jacard_edges_simplex
23	24	[11, 13]	[5, 11]	birth-death	0.96	0.81
29	30	[6, 8]	[6, 15]	birth-death	0.92	0.85
35	36	[8, 10]	[1, 10]	death-death	0.92	0.71
46	47	[7, 9]	[7, 14]	birth-birth	1.00	0.85
62	63	[7, 8, 12]	[6, 8, 12]	death-death	1.00	0.85
65	66	[2, 4, 11]	[2, 11, 14]	death-death	1.00	0.60

index 0	index 1	simplex 0	simplex 1	type	jacard_l31a	jacard_l31b
23	24	[11, 13]	[5, 11]	birth-death	NaN	NaN
29	30	[6, 8]	[6, 15]	birth-death	NaN	NaN
35	36	[8, 10]	[1, 10]	death-death	0.80	0.00
46	47	[7, 9]	[7, 14]	birth-birth	1.00	1.00
62	63	[7, 8, 12]	[6, 8, 12]	death-death	0.67	1.00
65	66	[2, 4, 11]	[2, 11, 14]	death-death	0.86	1.00

index 0	index 1	simplex 0	simplex 1	type	jacard_l32a	jacard_l32b
23	24	[11, 13]	[5, 11]	birth-death	NaN	NaN
29	30	[6, 8]	[6, 15]	birth-death	NaN	NaN
35	36	[8, 10]	[1, 10]	death-death	0.00	0.00
46	47	[7, 9]	[7, 14]	birth-birth	0.00	0.00
62	63	[7, 8, 12]	[6, 8, 12]	death-death	0.00	0.00
65	66	[2, 4, 11]	[2, 11, 14]	death-death	0.00	0.00

index 0	index 1	simplex 0	simplex 1	type	jacard_l33a	jacard_l33b
23	24	[11, 13]	[5, 11]	birth-death	0.50	0.50
29	30	[6, 8]	[6, 15]	birth-death	0.00	0.00
35	36	[8, 10]	[1, 10]	death-death	NaN	NaN
46	47	[7, 9]	[7, 14]	birth-birth	NaN	NaN
62	63	[7, 8, 12]	[6, 8, 12]	death-death	NaN	NaN
65	66	[2, 4, 11]	[2, 11, 14]	death-death	NaN	NaN

index 0	index 1	simplex 0	simplex 1	type	jacard_nn_nodes	jacard_nn_edges
23	24	[11, 13]	[5, 11]	birth-death	1.00	1.00
29	30	[6, 8]	[6, 15]	birth-death	1.00	1.00
35	36	[8, 10]	[1, 10]	death-death	1.00	1.00
46	47	[7, 9]	[7, 14]	birth-birth	1.00	1.00
62	63	[7, 8, 12]	[6, 8, 12]	death-death	1.00	1.00
65	66	[2, 4, 11]	[2, 11, 14]	death-death	1.00	1.00

5 Unexpected results

The Hypothesis problem can be measured by 2 scores with expected values: **jacard_nn_edges**, **jacard_nn_nodes** There are 1 cases, which does not require the expectations:

index 0	index 1	simplex 0	simplex 1	type	jacard_nn_edges	jacard_nn_nodes	Figure
30	31	[6, 15]	[0, 6]	birth-birth	0.95	1.00	Figure 5

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The Lemma 3.1 problem can be measured by 2 scores with expected values: **jacard_l31a**, **jacard_l31b** There are 0 cases, which does not require the expectations.

The Lemma 3.2 problem can be measured by 2 scores with expected values: **jacard_l32a**, **jacard_l32b** There are 3 cases, which does not require the expectations:

index 0	index 1	simplex 0	simplex 1	type	jacard_l32a	jacard_l32b	Figure
35	36	[8, 10]	[1, 10]	death-death	0.00	0.00	Figure 5
62	63	[7, 8, 12]	[6, 8, 12]	death-death	0.00	0.00	Figure 5
65	66	[2, 4, 11]	[2, 11, 14]	death-death	0.00	0.00	Figure 5

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The Lemma 3.3 problem can be measured by 2 scores with expected values: **jacard_l33a**, **jacard_l33b** There are 2 cases, which does not require the expectations:

index 0	index 1	simplex 0	simplex 1	type	jacard_l33a	jacard_l33b	Figure
23	24	[11, 13]	[5, 11]	birth-death	0.50	0.50	Figure 5
29	30	[6, 8]	[6, 15]	birth-death	0.00	0.00	Figure 5

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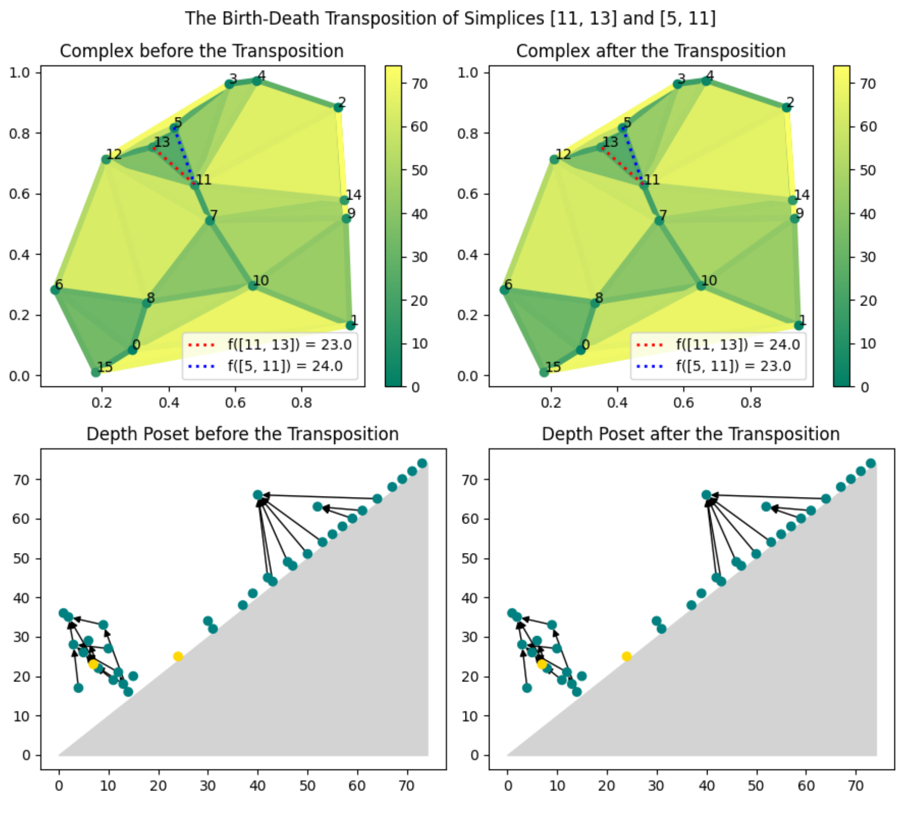

Figure 2: The Birth-Death Transposition of simplices [11, 13] and [5, 11]

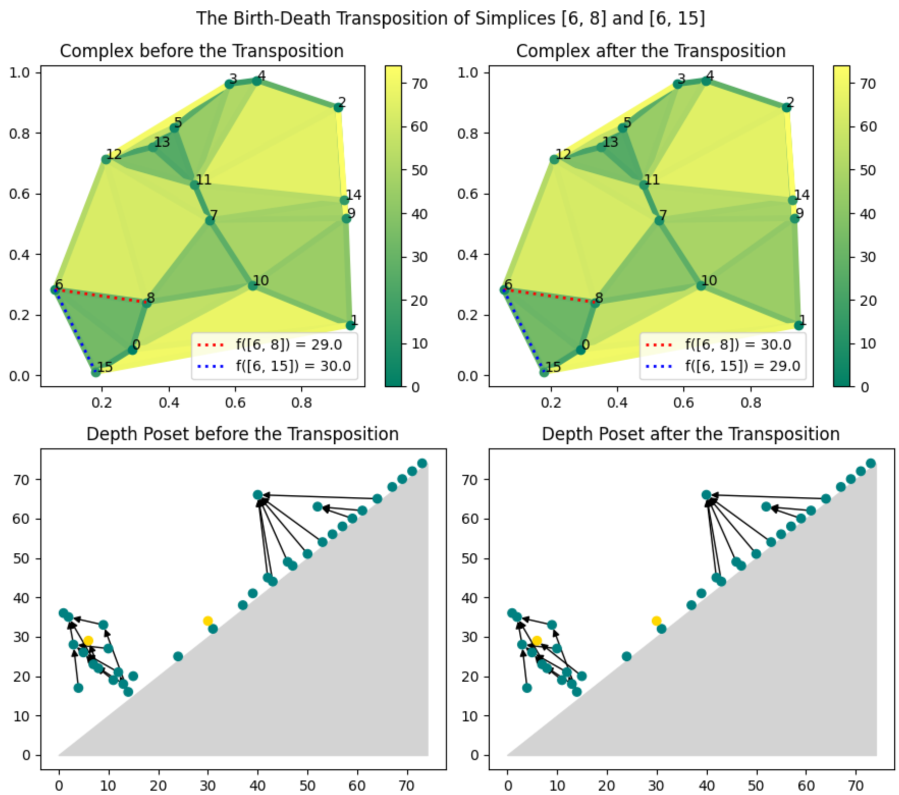

Figure 3: The Birth-Death Transposition of simplices [6, 8] and [6, 15]

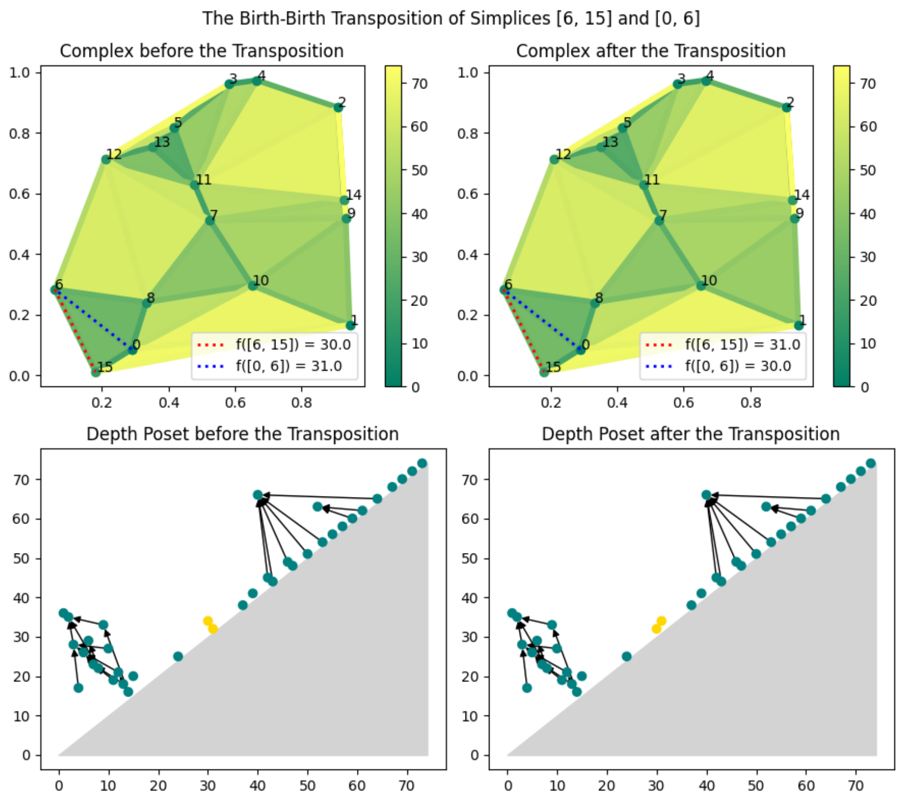

Figure 4: The Birth-Birth Transposition of simplices [6, 15] and [0, 6]

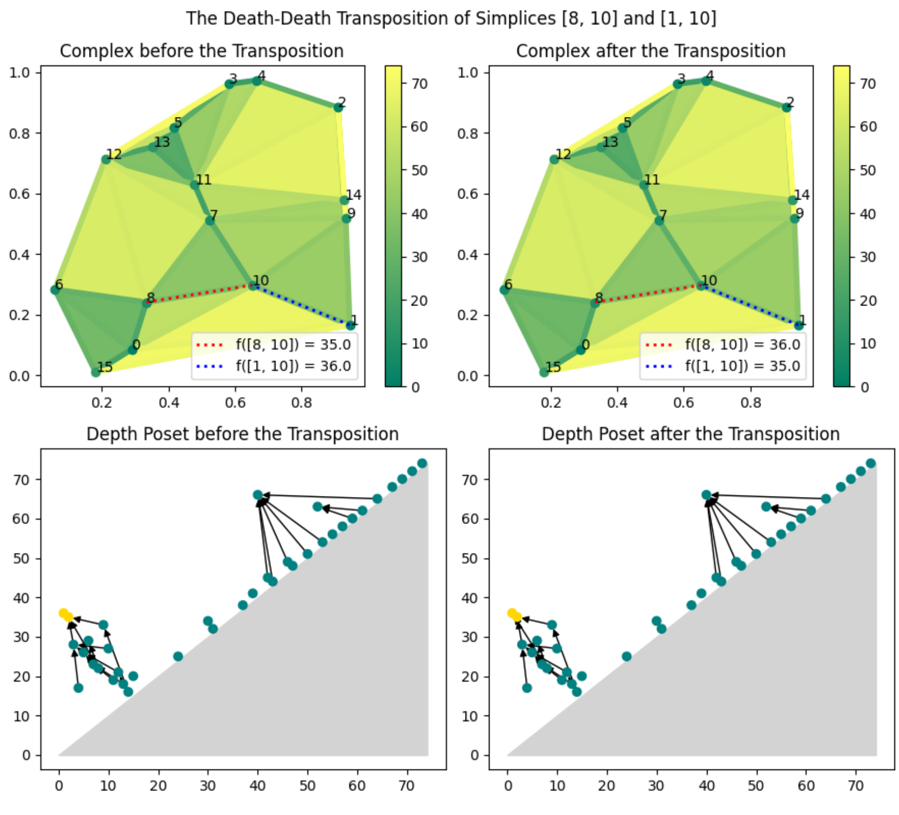

Figure 5: The Death-Death Transposition of simplices [8, 10] and [1, 10]

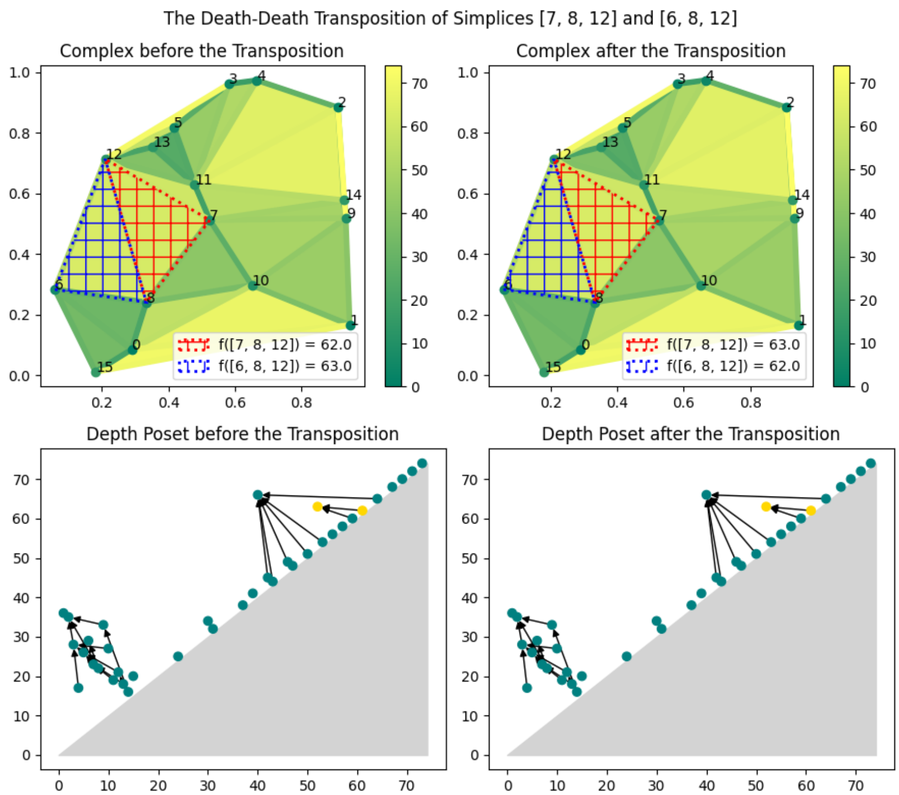

Figure 6: The Death-Death Transposition of simplices [7, 8, 12] and [6, 8, 12]

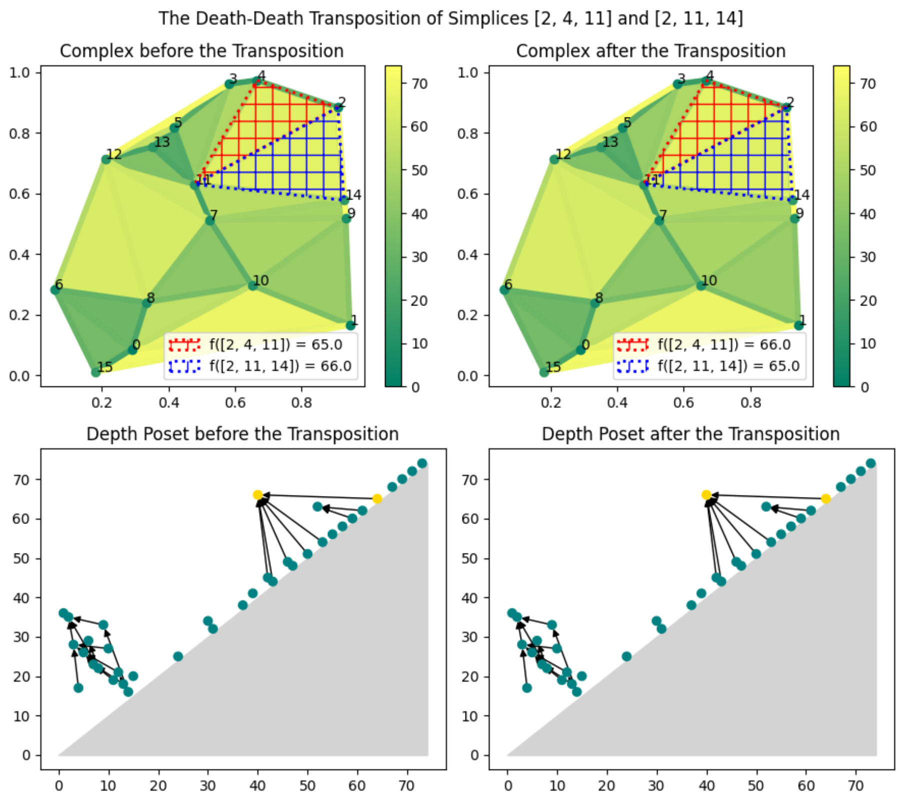

Figure 7: The Death-Death Transposition of simplices [2, 4, 11] and [2, 11, 14]