1 Theory

1.1 Persistent Homology and Giant Cycles

The first we should understand, which objects topological data analysis research.

People call cloud a collection of points $\{x_\alpha\} \subset X$, where X is a metric space. That's interesting to convert that data to some structure, so points can be representated as vertices of some combinatorial graph. And that graph can become scaffold of a simplicial complex. That's a good way to research data, ignoring high dimension of the space [4].

A simplicial complex X (I mean abstract simplicial complex) is a set of vertices $\{v_\alpha\}$ and a collection of its subsets, called simplices (simplex a which is set of elements k have dimension $k-1$ and can be called k-simplex), X such that: for all $a \in X$ for all $b \subset a$ $b \in S$ [3]. The dimension of a simplicial complex is the maximal dimension of it's simplices ($\dim X = \max\limits_{a \in S} \dim a = \max\limits_{a \in S} |a| - 1$).

When simplicial complex is defined, let's continue way to define simplicial homology. Let $\Delta_n(X)$ be the free abelian group with basis on n-simplices e_α^n of X. That groups elements can be rewritten as $\sum_\alpha n_\alpha e_\alpha^n$ and called n-chains. We also can represent them as $\sum_\alpha n_\alpha \sigma_\alpha$ where the $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of .

The boundary of n-simplex $(v_0, ..., v_n)$ is $(n-1)$-simplices $[v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$. So let the boundary be $\sum_i (-1)^i F_i$. The signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented. Using that geometry we can define a boundary homomorphism $\delta : \Delta_n(X) \to \Delta_{n-1}(X)$:

$$\delta_n(\sigma_\alpha) = \sum_i (-1^i)\sigma_\alpha \ : \ [v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$$

So there is the lemma, which said that the composition $\Delta_n(X) \xrightarrow{\delta_n} \Delta_{n-1}(X) \xrightarrow{\delta_{n-1}}\xrightarrow{\delta_{n-2}} \Delta_{n-2}(X)$ is zero. That's not hard to prove. We have

$$\delta_n(\sigma) = \sum_i (-1)^i \sigma \ : \ [v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$$

and hence

$$\begin{aligned} \delta_{n-1}\delta_n = &\sum_{j<i} (-1)^i(-1)^j \sigma : [v_0, ..., v_{j-1}, v_{j+1}, ..., v_{i-1}, v_{i+1}, ..., v_n] \\ + &\sum_{j>i} (-1)^i(-1)^{j-1} \sigma : [v_0, ..., v_{i-1}, v_{i+1}, ..., v_{j-1}, v_{j+1}, ..., v_n] = 0 \end{aligned}$$

So we have a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \to \cdots \to C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$

such that $\delta_n\delta_{n+1} = 0$ for each n. Sequences like that are called chain complexes. Cause the equation $\delta_n\delta_{n+1} = 0$ is equivalent to the inclusion $\text{Im } \delta_{n+1} \subset \text{Ker } \delta_n$, we can defind the n-th homology group of the chain complex as the quotient group $H_n = \text{Ker } \delta_n / \text{Im } \delta_{n+1}$. In the case of simplicial complex $C_n = \Delta_n(X)$, so the homology group $\text{Ker } \delta_n / \text{Im } \delta_{n+1}$ be called the n-th homology group of X and can be noted $H_n^\Delta(X)$. People call the elements of $\text{Ker } \delta_n$ cycles and the elements of $\text{Im } \delta_{n+1}$ boundaries. [1]

One of the natural methods to represent a cloud as a simplicial complex is the Cech complex. For a given cloud $\{x_\alpha\} \subset \mathbb{E}^n$ the Cech complex C_ϵ is the simplicial complex whose k-simplices (the simplices dimension k: $a : |a|-1 = k$) are determined by unordered $(k+1)$-tuples of points $\{x_\alpha\}_0^k$ whose closed $\epsilon/2$-ball neighbourhoods have a point of common intersection [4].

Another one natural method to represent a cloud as a simplicial complex is the Rips complex. For a given cloud $\{x_\alpha\} \subset \mathbb{E}^n$ the Rips complex R_ϵ is determined by unordered $(k+1)$-tuples of points whose for each pair of points $\{x_\alpha\}_0^k$ the distance between that pairs points less or equal ϵ.

... (the Cech (Nerve) theorem)

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1.2 Lattices Voronoi Cells and their Interpretations on Thorus

In this chapter we will show default definitions about lattices in $\mathbb{R}^n$, talk about Voronoi cells and then extropolate their definitions to the thorus case.

A lattice in $\mathbb{R}^n$ is a subset $\Gamma \subset \mathbb{R}^n$ with the property that there exxists a basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ s.t. $\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$. [2]

Let's throw few examples of lattices, which will be interesting in this work:

The Lattices Z_n: ...

The Lattices A_n: ...

The Lattices D_n: ...

Let's define a Γ^* dual to Γ as $\{x \in \mathbb{R}^\ltimes : \; x \cdot y \in \mathbb{Z} \; \forall y \in \Gamma\}$.

...

Let's define d-dimensional thorus as $\mathbb{R}^n/\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$. Not hard to see, that $\mathbb{R}/a_1\mathbb{Z} \times \cdots \times \mathbb{R}/a_n\mathbb{Z}$ $(a_1, ..., a_n \in \mathbb{R}_{>0})$ will be the homeomorphically-same object.

Let's redefine lattice thinking, that' lattices lie not just on $\mathbb{R}^n$, but on some thorus with defined equivalence relation. So...

1.3 Random Filtration on Cells

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References

[1] Hatcher, A. (2001). Algebraic topology. Proceedings of The Edinburgh Mathematical Society - PROC EDINBURGH MATH SOC. 46. 511-512. 10.1017/S0013091503214620.

[2] Ebeling, Wolfgang. (2002). Lattices and Codes. 10.1007/978-3-322-90014-2.

[3] Prasolov, V. V. (2006), Elements of combinatorial and differential topology, American Mathematical Society, ISBN 0-8218-3809-1, MR 2233951

[4] Ghrist, Robert. (2008). Barcodes: The persistent topology of data. BULLETIN (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY. 45. 10.1090/S0273-0979-07-01191-3.