# Random Complexes and Persistent Homology

Zimin Fedor Vladimirovich

HSE

# Introduction

For percolation models on tori dimension $d \geq 3$

$$t_k^{perc} < t_k^{ec}$$

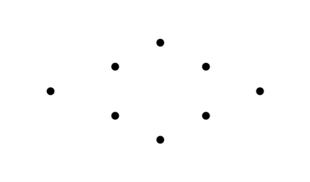
for $k < d/2$ and

$$t_k^{perc} > t_k^{ec}$$

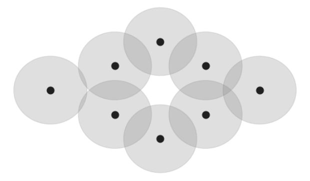
for $k > d/2$.

# Topological background

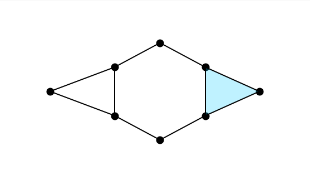
# Cloud; Čech and Rips complexes



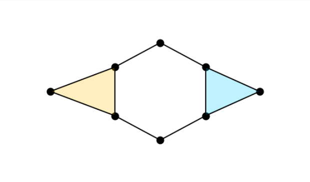
a. Cloud of points $P$



b. Union of balls $U(P)$



c. Čech complex $C(P)$



d. Rips complex $R(P)$

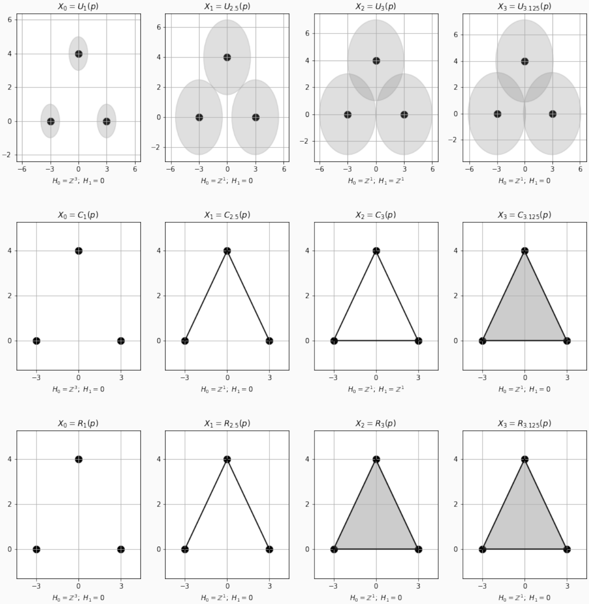
# Filtration

$$\emptyset = \mathbb{X}_0 \subset \mathbb{X}_1 \subset \cdots \subset \mathbb{X}_m = X$$

$$0 = H(\mathbb{X}_0) \to H(\mathbb{X}_1) \to \cdots \to H(\mathbb{X}_m) \to H(X)$$

$$0 \to \ldots \to 0 \to F \to \cdots \to F \to 0 \to \ldots \to 0$$

# Filtration


$X_0 = U_1(p)$
$X_1 = U_{2.5}(p)$
$X_2 = U_3(p)$
$X_3 = U_{3.125}(p)$
$H_0 = \mathbb{Z}^3;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = \mathbb{Z}^1$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$X_0 = C_1(p)$
$X_1 = C_{2.5}(p)$
$X_2 = C_3(p)$
$X_3 = C_{3.125}(p)$
$H_0 = \mathbb{Z}^3;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = \mathbb{Z}^1$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$X_0 = R_1(p)$
$X_1 = R_{2.5}(p)$
$X_2 = R_3(p)$
$X_3 = R_{3.125}(p)$
$H_0 = \mathbb{Z}^3;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$
$H_0 = \mathbb{Z}^1;\ H_1 = 0$

# Lattices

Root system:

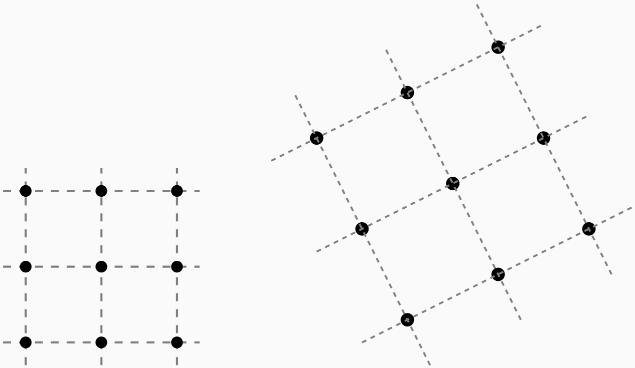
- $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \;\; \forall \alpha \in \Phi$
- $S_\alpha \Phi = \Phi \;\; \forall \alpha \in \Phi$

Reflection:

- $S_\alpha \alpha = -\alpha$
- $S_\alpha \beta = \beta \;\; \forall \beta \perp \alpha$

$$S_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

# Equivalence



# $Z^n$, $A_n$, $D_n$

$$\mathbb{Z}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : \ x_k \in \mathbb{Z} \ \forall k\} \subset \mathbb{R}^n$$

$$A_n = \{(x_0, ..., x_n) \in \mathbb{Z}^{n+1} : \ x_0 + \cdots + x_n = 0\} \subset \mathbb{Z}^{n+1}$$

$$D_n = \{(x_1, ..., x_n) \in \mathbb{Z}^n : \ x_1 + \cdots + x_n = 0 \pmod 2\} \subset \mathbb{Z}^n$$

# Dual Lattices

$\Gamma \subset \mathbb{R}^n$:

$$\Gamma^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \; \forall y \in \Gamma\}$$

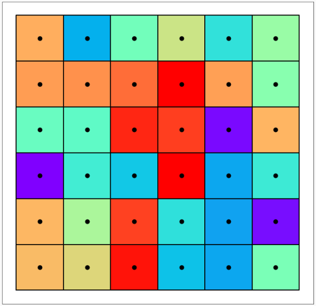
# About Voronoi Cells

$X$ - metric space

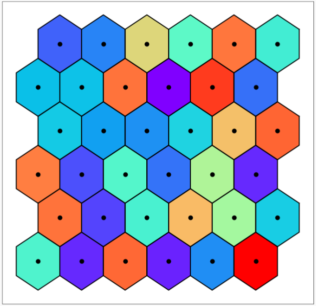
Cloud $P \subset X$

Voronoi cell of $p \in P$ is
$V_p = \{x \in X : \ \rho(x, p) \leq \rho(x, \hat{p}) \forall \hat{p} \in P/p\}$

# Voronoi cells of lattices



a. Voronoi cells of $\mathbb{Z}^2$

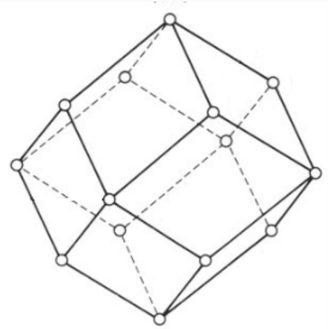


b. Voronoi Cells of $A_2$

# Voronoi cells of fcc and bcc-lattices

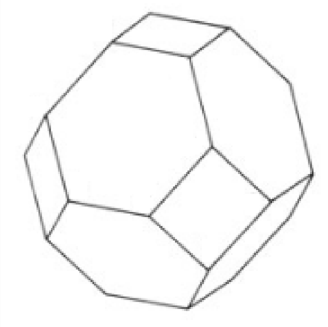
fcc-lattice

$A_3 \equiv D_3$



bcc-lattice

$A_3^* \equiv D_3^*$



# Extension of the Torus definition

# Torus definition and Notation

Torus definition: $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$

$$\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n \equiv (\mathbb{R}/v_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/v_1\mathbb{Z})$$

Notation:

$$\mathbb{T}_{(3,4)} = (\mathbb{R}/3\mathbb{Z}) \times (\mathbb{R}/4\mathbb{Z})$$

$$\mathbb{T}_5^4 = (\mathbb{R}/5\mathbb{Z})^4$$

# Lattice on Torus

$\Gamma \subset \mathbb{R}^n$

$\mathbb{T}_v = (\mathbb{R}/v_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/v_1\mathbb{Z})$

$a, b \in \mathbb{T}_v$. $a \equiv b \Leftrightarrow \exists k \in \mathbb{Z} : \quad a - b = kv$

$\Gamma$ on $\mathbb{T}_v$ means for $a, b \in \Gamma$ this equivalence relation is useble.

# Percolation on Cells

# Voronoi Cell Continuum Percolation Model

$P = \{p_1, ..., p_n\} \subset X$ - cloud.

$V = \{V_p\}_{p \in P}$

$T = \{t_p\}_{p \in P}$ iid randomly uniformly distributed in $[\mathbf{0}, \mathbf{1}]$.

$\{D_{t_p} = V_p\}_{p \in P}$

$$\bigcup_{p \in P} D_{t_p} = X$$

$$X_t = \bigcup_{t_p < t} D_{t_p}$$

# Notation

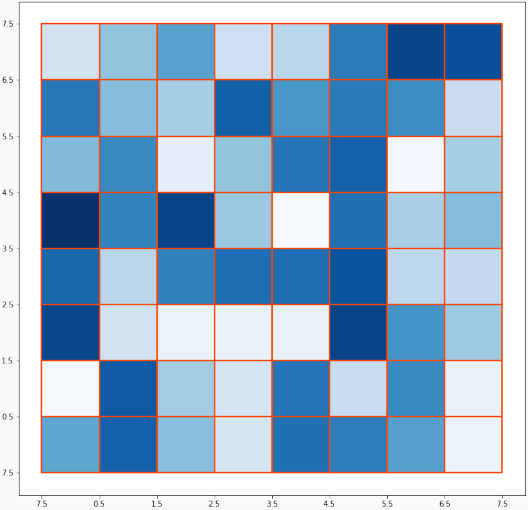
$$H_k(t) = H_k(X_t)$$

$$\beta_k(t) = \beta_k(X_t)$$

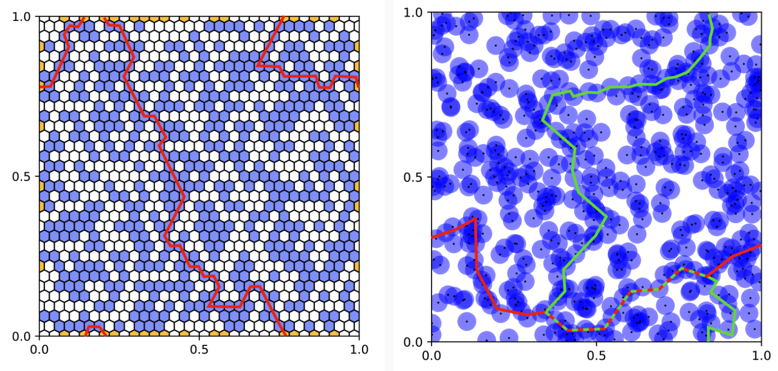
$$\chi(t) = \chi(X_t)$$

PVLM$(\Gamma, m)$ means the Voronoi cell continuum percolation model for lattice $\Gamma$ on $\mathbb{T}_m^{\dim \Gamma}$.

# PVLM$(Z^2, 8)$ EXAMPLE



# Review of the article of Boroski and Skraba




Examples of models, which Boborwski and Skraba researched

## The Bobrowski and Skraba's conjecture

For percolation models on tori dimension $d \geq 3$

$$t_k^{perc} < t_k^{ec}$$

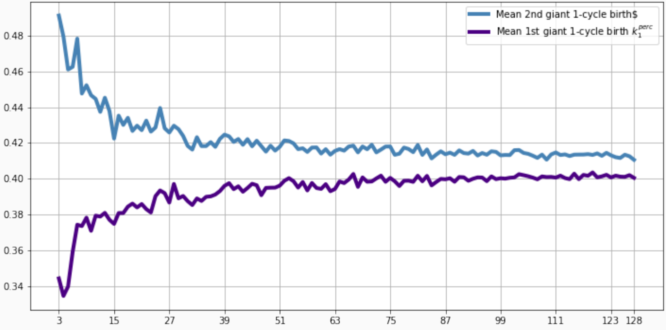
for $k < d/2$ and

$$t_k^{perc} > t_k^{ec}$$

for $k > d/2$.

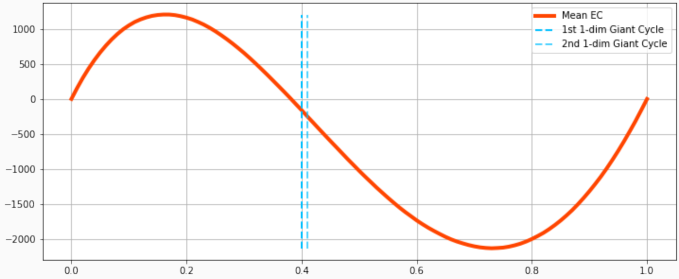
# Analyze PVLM models

# PVLM($Z^2, m$)



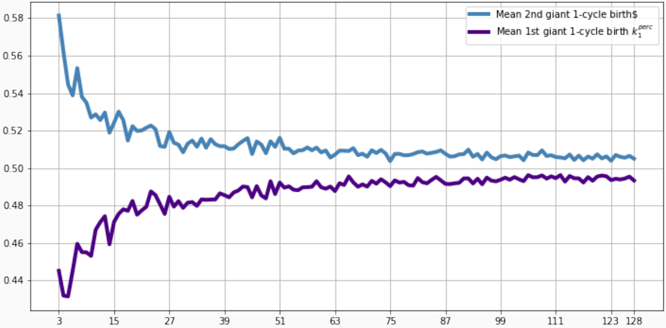

For $m = 128$ the mean 1st giant cycle birth is $t_1^{perc} \approx 0.400494$ and also the mean 2nd giant cycle birth $\approx 0.410593$.

# PVLM$(Z^2, m)$



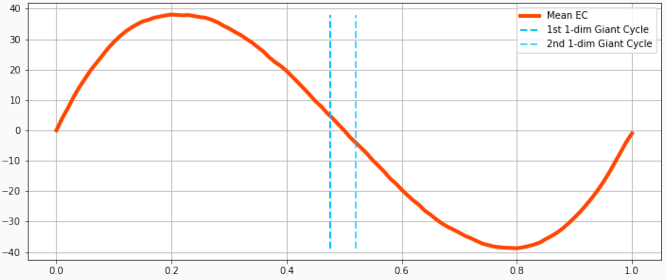

$$t_1^{perc} > t_1^{ec}$$

# PVLM($A_2, m$)



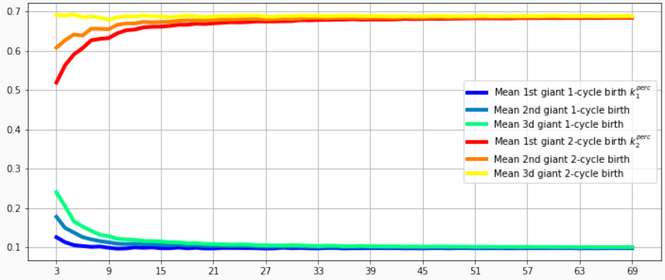

For $m = 128$ the mean 1st giant cycle birth is $t_1^{perc} \approx 0.475112$ and also the mean 2nd giant cycle birth $\approx 0.519803$.

# PVLM$(A_2, m)$



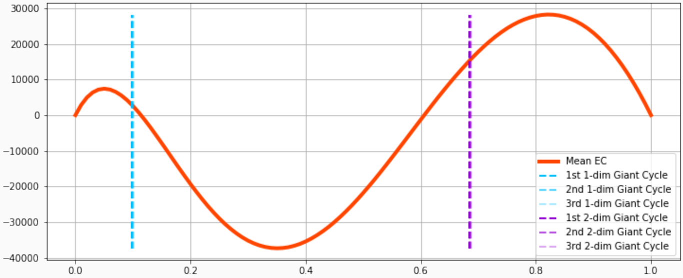

$$t_1^{perc} < t_1^{ec}$$

# PVLM$(\mathbb{Z}^3, m)$




| $k$ | | | | $(m = 69)$ |
|---|---|---|---|---|
| | Mean | 1st $k$-cycle birth | 2nd $k$-cycle birth | 3d $k$-cycle birth |
| 1 | | $t_1^{perc} = \mathbf{0.097873}$ | 0.099265 | 0.100511 |
| 2 | | $t_2^{perc} = \mathbf{0.683935}$ | 0.686005 | 0.688288 |

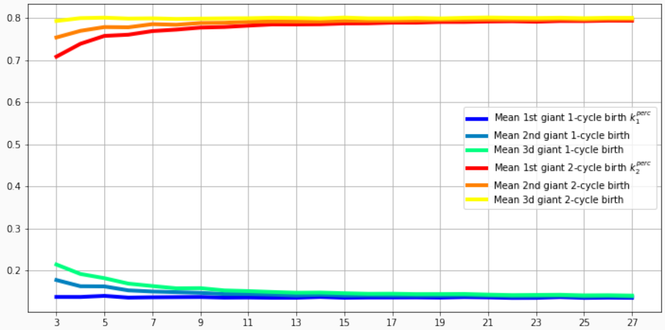
# PVLM$(\mathbb{Z}^3, m)$




$$t_1^{perc} < t_1^{EC}$$

$$t_2^{perc} > t_1^{EC}$$

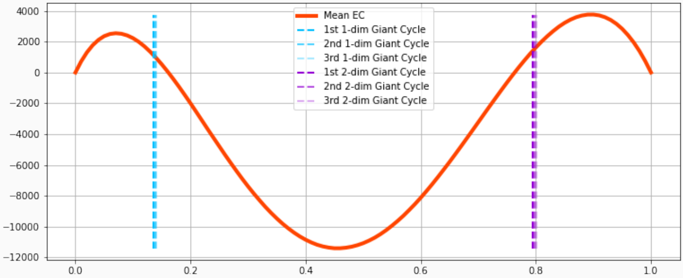
# PVLM(fcc, $2m$)




$k$ $(m = 27)$

| | | | | |
|---|---|---|---|---|
| | Mean | 1st $k$-cycle birth | 2nd $k$-cycle birth | 3d $k$-cycle birth |
| 1 | | $t_1^{perc} = \mathbf{0.135780}$ | 0.138391 | 0.141021 |
| 2 | | $t_2^{perc} = \mathbf{0.794358}$ | 0.797446 | 0.801073 |

# PVLM(fcc, $2m$)




$$t_1^{perc} < t_1^{EC}$$

$$t_2^{perc} > t_1^{EC}$$

Thank you for your attention