VE281
Data Structures and Algorithms

Minimum Spanning Tree

Learning Objectives:

- Know what a minimum spanning tree (MST) is
- Know the Prim’s algorithm for finding the MST
- Know how the various choices of the supporting data structures affect the runtime of the Prim’s algorithm

Outline

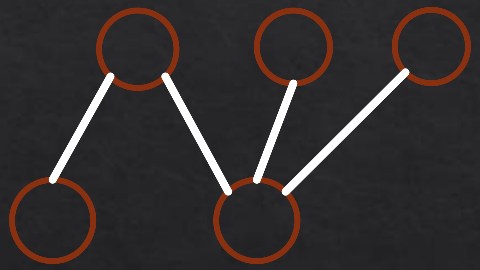
- Minimum Spanning Tree
  - Problem
  - Prim's Algorithm

Tree and Graph

◈A **tree** is an **acyclic, connected undirected** graph.

The tree we see before

However, this is also a tree



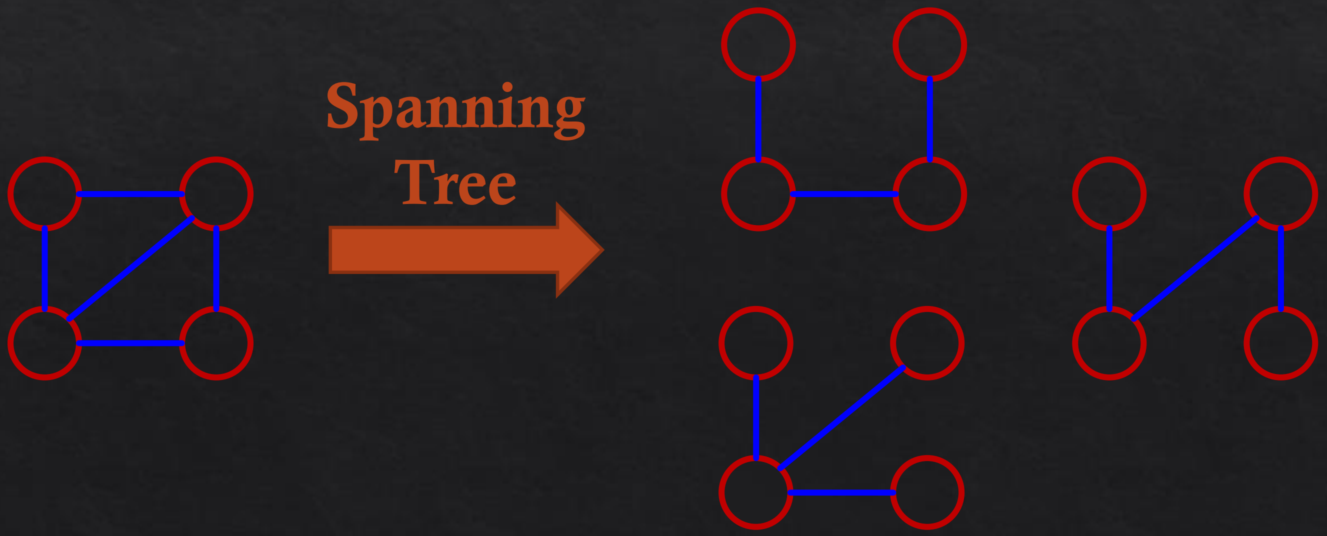
Any node can be the root of the tree.

◈For a tree, $|E| = |V| - 1$.

◈Claim: Any **connected** graph with N nodes and $N - 1$ edges is a tree.

Subgraph and Spanning Tree

- $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$.
- A **spanning tree** of a **connected undirected** graph G is a subgraph of G that
  1. contains all the nodes of G;
  2. is a tree, i.e., connected and acyclic.

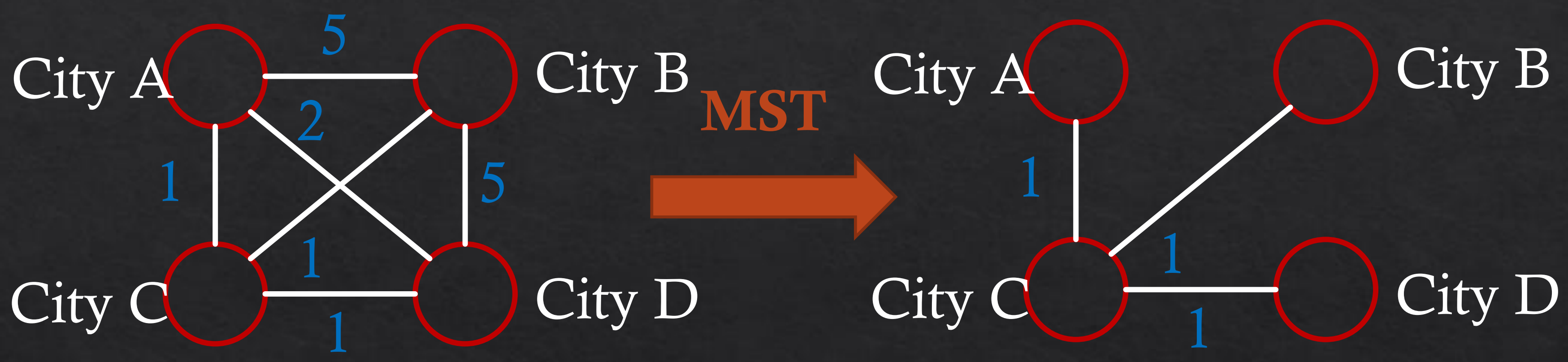


Minimum Spanning Tree (MST)

- Given a weighted, connected, undirected graph $G = (V, E)$, a **minimum spanning tree** T of G is a spanning tree of G whose sum of all edge weights is the minimal.

Application of MST

- A government planning a freeway system to connect all the cities.



- A power company planning where to lay down high-voltage power lines.

Minimum Spanning Tree Algorithms

- Main idea: greedily select edges one by one and add to a growing sub-graph.

- Two standard algorithms:
  - Prim’s algorithm
  - Kruskal’s algorithm

Outline

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  - Problem
  - Prim’s Algorithm

Prim's Algorithm

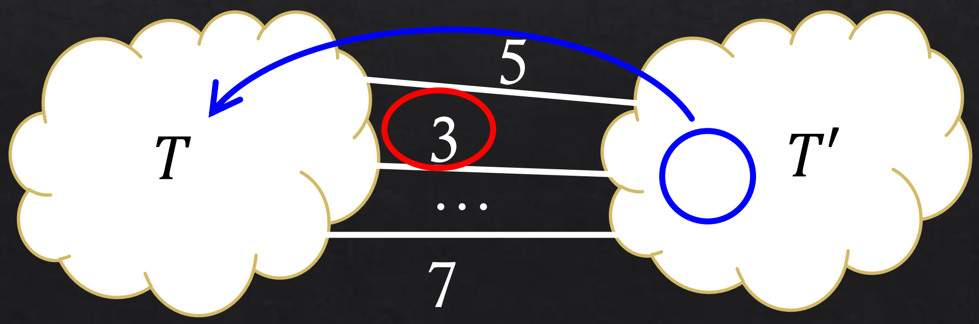
- Separate V into two sets:
  - T: the set of nodes that have been added to the MST.
  - T': those nodes that have not been added to the MST, i.e., $T' = V - T$.

- Prim's algorithm initially sets $T = \{s\}$, where s is an arbitrarily picked **node**, and $T' = V - \{s\}$. The algorithm moves one node from T' to T in each iteration. After the last iteration, $T = V$ and we have constructed the MST.

Prim's Algorithm
Basic Version

1. Arbitrarily pick one node s; set $T = \{s\}$ and $T' = V - \{s\}$.
2. While $T' \neq \emptyset$

◈Select an edge with the **smallest weight** that connects between a node in T and a node in T'. Suppose the edge connects with node v in T'. Move v from T' to T.



Selecting the Smallest Edge and Node

- For each node $v \in T'$, keep a measure $D(v)$, storing the "**current**" **smallest weight** over all edges that connect v to a node in T.
  - Will be updated later.

- To choose the edge with the smallest weight that connects between a node in T and a node in T', we pick the node $v \in T'$ with **the smallest** $D(v)$.
  - If edge (u, v) gives **the smallest** $D(v)$, then (u, v) is the edge with the smallest weight **across** set T and T'.

Updating *v*'s Neighbor

- If we move a node v from T' to T, then for each of v's neighbor u that is **still** in T', we update its $D(u)$ as follows:
  - If $D(u) > w(v,u)$, then let $D(u) = w(v,u)$.
  - I.e., update $D(u)$ if the weight of edge (v,u) is smaller than the weight of any other edge that connects a node in T to u.

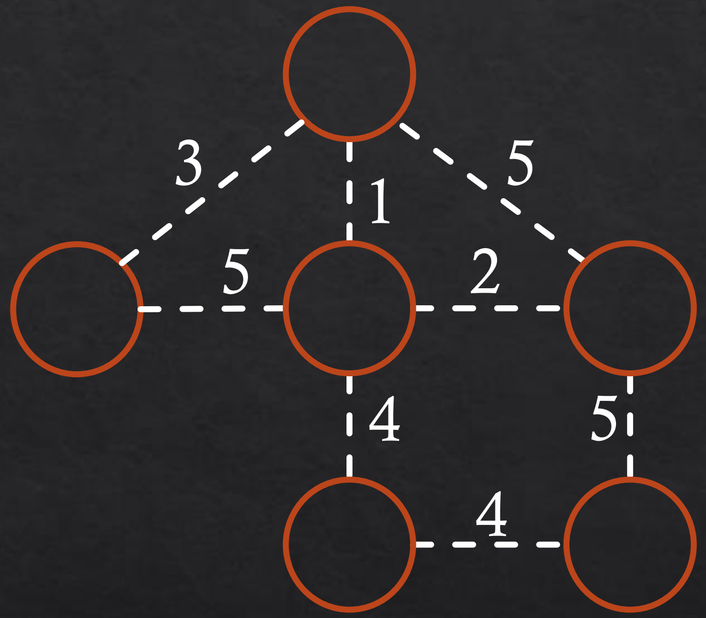
Prim's Algorithm
Full Version

- ⟐We keep $P(v)$ for each node v: $(P(v), v)$ is the edge chosen in the MST.

1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
    1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
    2. For each of v's **neighbors** u that is **still** in T',
if $D(u) > w(v, u)$, then update $D(u)$ as $w(v, u)$ and $P(u)$ as v.

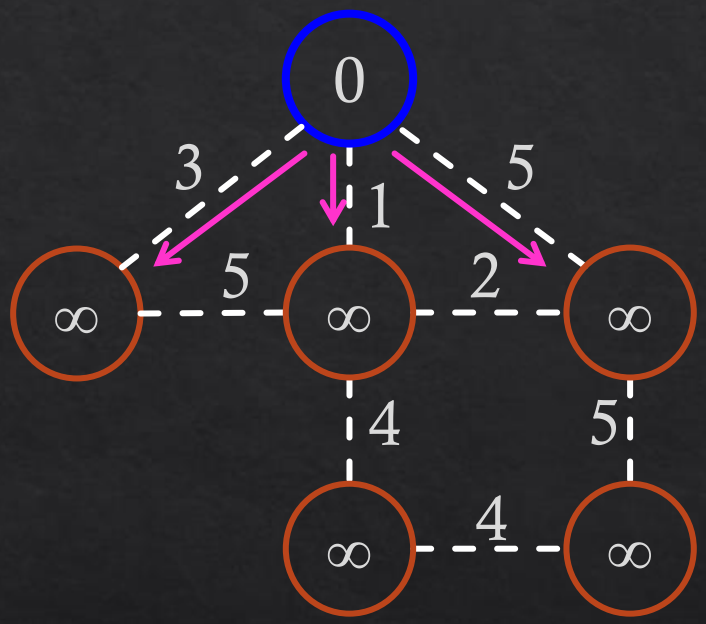
Prim's Algorithm

Example

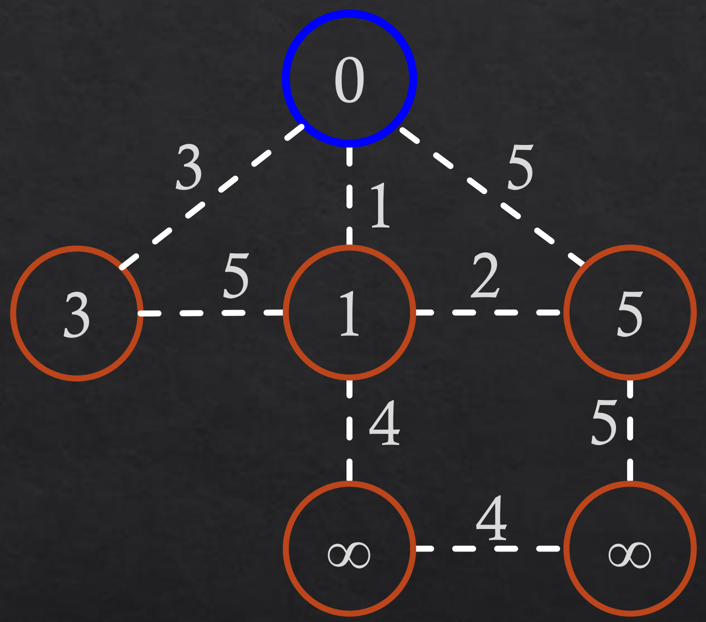


Prim's Algorithm

Example

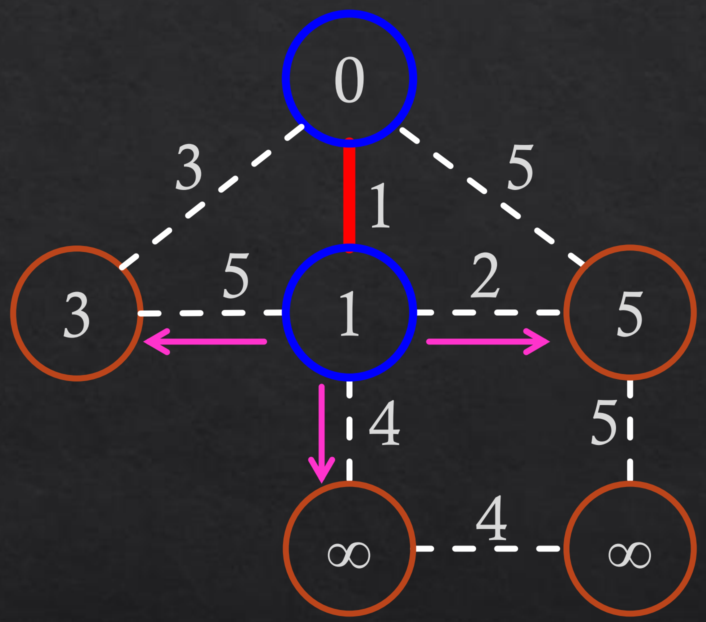


Prim's Algorithm
Example



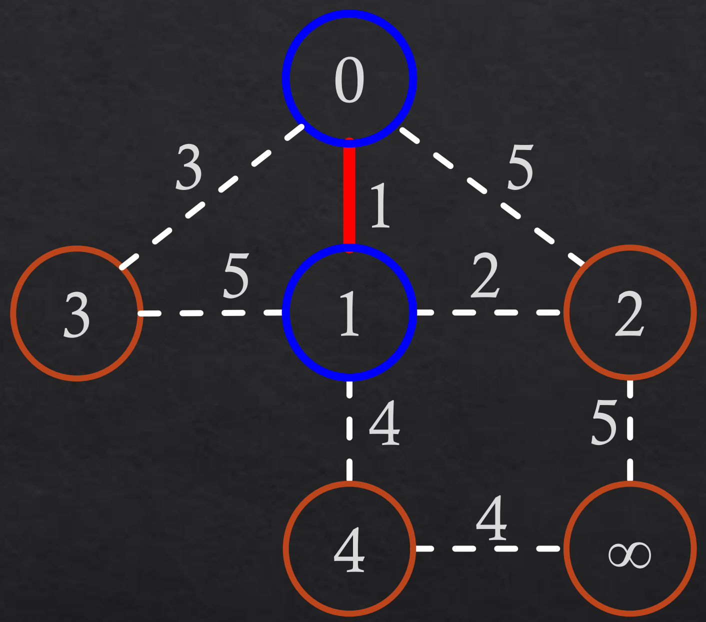
Prim’s Algorithm

Example

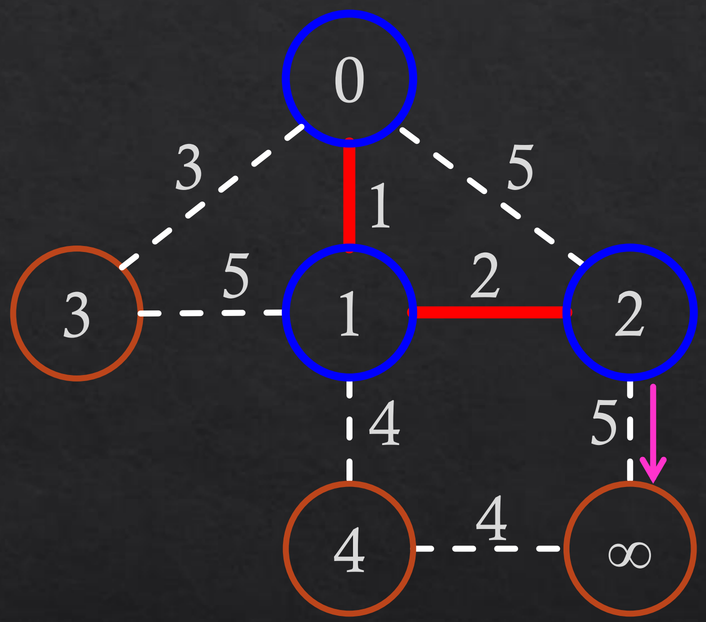


Prim's Algorithm

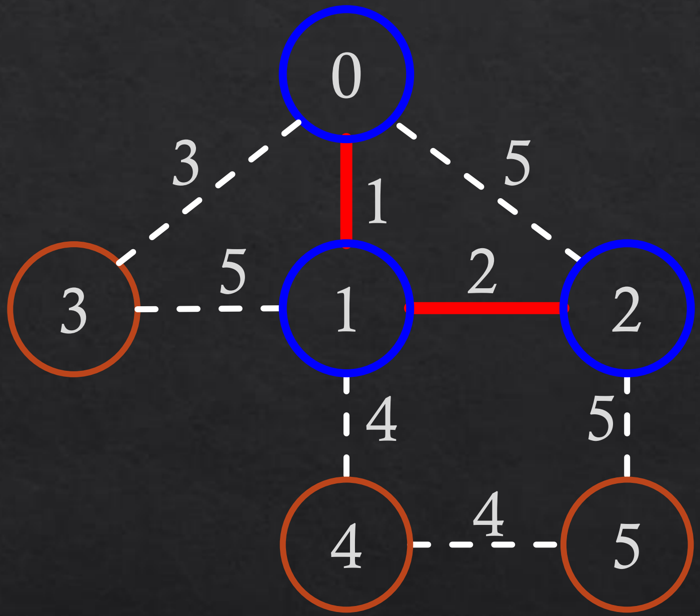
Example



Prim's Algorithm
Example

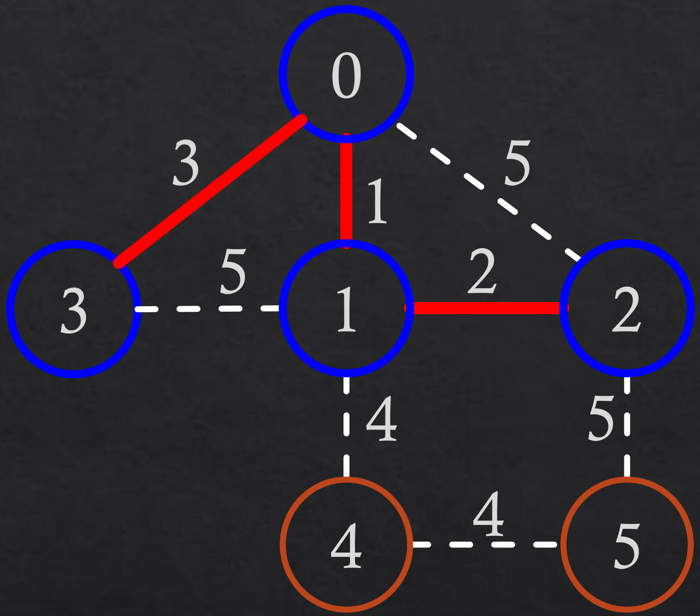


Prim's Algorithm
Example

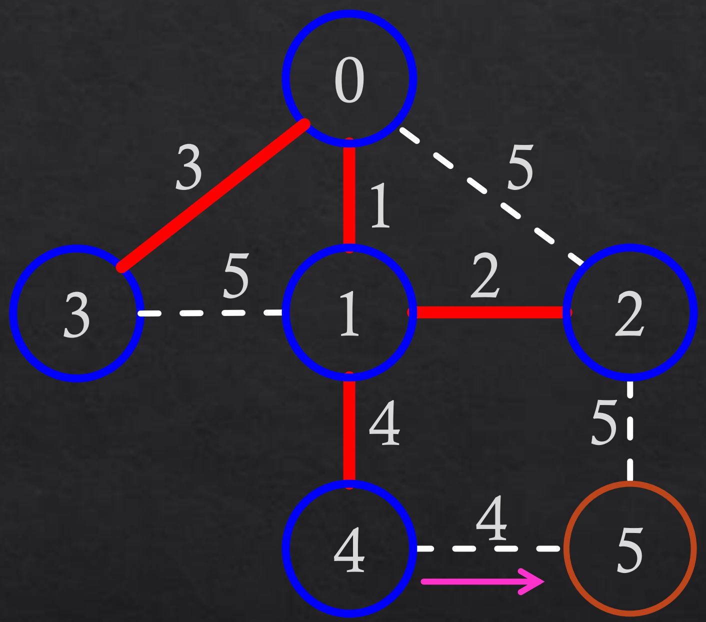


Prim's Algorithm

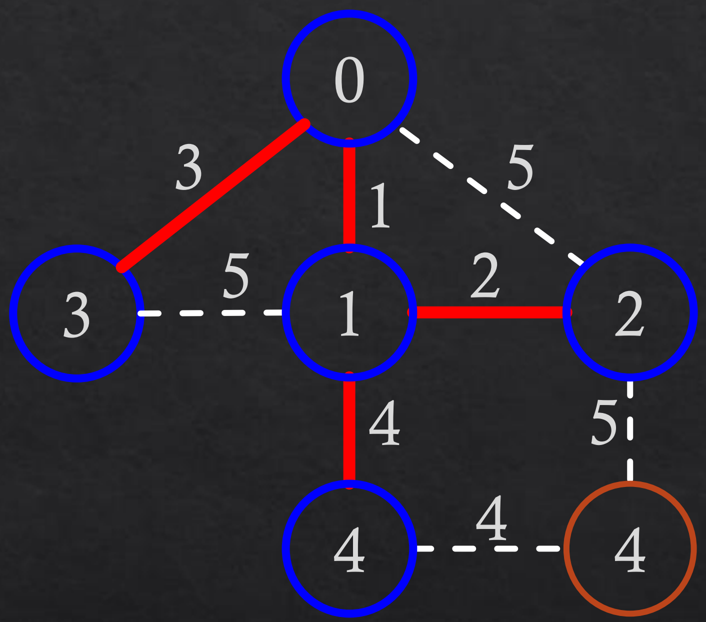
Example



Prim's Algorithm
Example

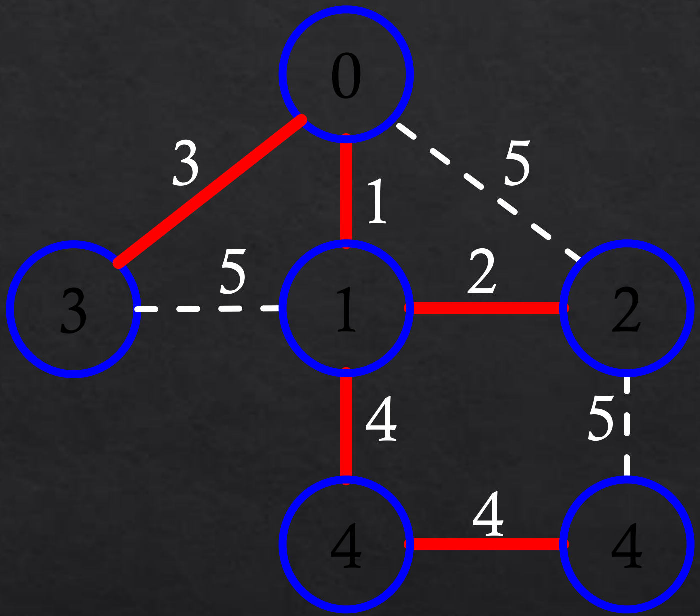


Prim's Algorithm
Example



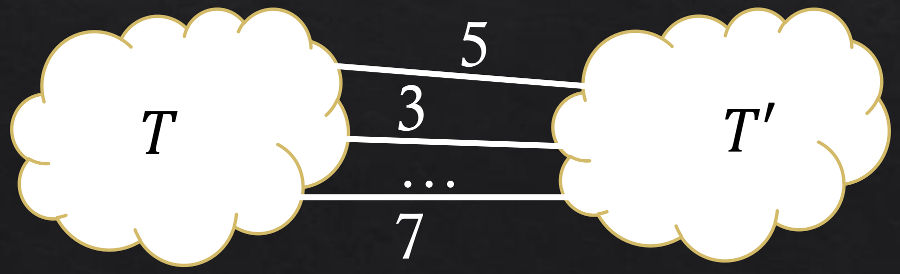
Prim's Algorithm

Example



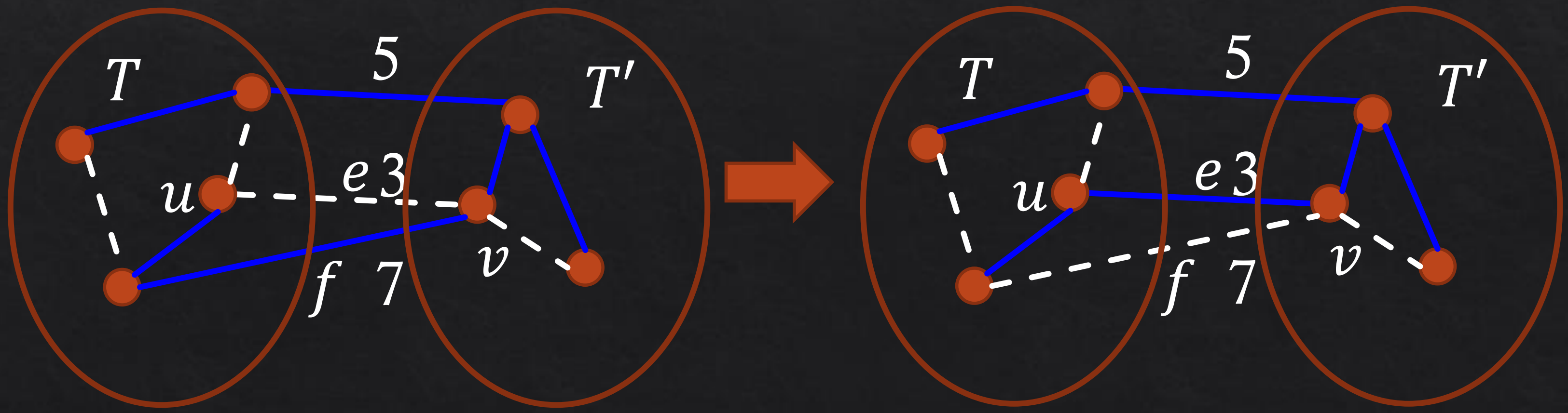
Prim's Algorithm
Justification

- Claim: the obtained subgraph is a tree
- Proof:
  - The nodes in set T are connected (can be shown by induction)
  - Furthermore, $|V| = |E| + 1$
  - Claim: Any **connected** graph with N nodes and $N - 1$ edges is a tree



Prim's Algorithm
Justification

- Claim: the obtained subgraph is an MST
- Proof by contradiction:
  - Assume the MST does not contain the cheapest edge e between T and T'
  - Assume $e = (u, v)$. Its weight is w
  - In the MST, there exists a unique path between u and v. On this path, there is an edge f across T and T'. Its weight $> w$
  - We replace f by e in original MST.
  - The new graph is a tree (Why?) with smaller sum of edge weights



Prim's Algorithm
Time Complexity

1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
    1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
    2. For each of v's **neighbors** u that is **still** in T',
       if $D(u) > w(v, u)$, then update $D(u)$ as $w(v, u)$ and $P(u)$ as v.

What is the time complexity of Prim's algorithm?

Prim's Algorithm
Time Complexity

- Method 1: linear scan the set T' to find the smallest $D(v)$.
- Number of times to find the smallest $D(v)$: $|V|$.
  - Each cost: $O(|V|)$.

- **Maximal** number of times to update the neighbors: $|E|$.
  - Since each neighbor of each node could be **potentially** updated.
  - Each cost: $O(1)$.

- Total running time is $O\left(|E| + |V|^2\right) = O(|V|^2)$.

Prim's Algorithm
Time Complexity

- Method 2: use a binary heap to store $D(v)$'s.
- Number of times to extract the smallest $D(v)$: $|V|$.
  - Each cost: $O(\log|V|)$.

- <u>**Maximal**</u> number of times to update the neighbors: $|E|$.
  - Each cost is $O(\log|V|)$, since after updating $D(v)$, we should percolate up new $D(v)$ into right location of binary heap.

- Total running time is $O(|V|\log|V| + |E|\log|V|)$
  $= O\big((|V| + |E|)\log|V|\big)$.

Prim's Algorithm
Time Complexity

- Method 3: use a Fibonacci heap to store $D(v)$'s.
- Number of times to extract the smallest $D(v)$: $|V|$.
  - Each cost: $O(\log|V|)$.

- **Maximal** number of times to update the neighbors: $|E|$.
  - Each cost is $O(1)$ (decreaseKey operation; amortized time).

- Total running time is $O(|V|\log|V| + |E|)$.

Prim’s Algorithm
Time Complexity

- Method 1: linear scan the set T' to find the smallest $D(v)$
  - Total runtime: $O(|V|^2)$
- Method 2: use a binary heap to store $D(v)$’s
  - Total runtime: $O((|V| + |E|) \log |V|)$
- Method 3: use a Fibonacci heap to store $D(v)$’s
  - Total runtime: $O(|V| \log|V| + |E|)$
- Which one is the best?
  - Answer: Fibonacci heap.
  - For sparse graphs, i.e., $|E| \approx \Theta(|V|)$, using binary heap has same runtime complexity as Fibonacci heap. The runtime complexity is $O(|V| \log|V|)$