VE281

Data Structures and Algorithms

Comparison Sort

Learning Objectives:

- Know the difference between comparison sort and non-comparison sort
- Know the procedures of merge sort and quick sort
- Know the master theorem
- Know different characteristics of sorting algorithms, such as time complexity, stability, etc.

Outline

- Sorting Basics
- Merge Sort
- Quick Sort
- Comparison Sort Summary

Sorting

- Given array A of size N, reorder A so that its elements are in order.
  - "In order" with respect to a consistent comparison function, such as "≤" or "≥".

- Sorting order
  - Ascending order
  - Descending order
- Unless otherwise specified, we consider sorting in ascending order.

Characteristics of Sorting Algorithms

- Average-case time complexity
- Worst-case time complexity
- Space usage: **in place** or not?
  - **In place**: requires $O(1)$ additional memory
  - Don't forget the stack space used in recursive calls
  - **In place is better**
    - Why? The data can fit into cache, not main memory
  - Real example: quick sort versus merge sort. Both have average-case time complexity of $O(n \log n)$. Quick sort runs faster in practice, due to in place

Characteristics of Sorting Algorithms

- **Stability**: whether the algorithm maintains the relative order of records with equal keys

(4, b), (3, e), (3, b), (5, b) → (3, e), (3, b), (4, b), (5, b) **Stable!**

Sort on the first number

→ (3, b), (3, e), (4, b), (5, b) **Not Stable!**

- Usually there is a secondary key whose ordering you want to keep. Stable sort is thus useful for sorting over multiple keys
- Example: sort complex numbers a+bi
  - Ordering rule: first compare a; when there is a tie, compare b
  - One sorting method: first sort b, then sort a

3+5i, 2+6i, 3+4i, 5+2i → Sort on b → 5+2i, 3+4i, 3+5i, 2+6i

→ … sort on a → 2+6i, 3+4i, 3+5i, 5+2i

Stability is important!

Types of Sorting Algorithms

- Sorting algorithms can be classified as **comparison sort** and **non-comparison sort**.
- **Comparison sort**: each item is compared against others to determine its order.
- **Non-comparison sort**: each item is put into predefined “bins” independent of the other items presented.
  - No comparison with other items needed.
  - It is also known as **distribution-based sort**.

Types of Sorting Algorithms

- General types of comparison sort
  - Insertion-based: insertion sort
  - Selection-based: selection sort, heap sort
  - Exchange-based: bubble sort, quick sort
  - Merging-based: merge sort

- Non-comparison sort:
  counting sort, bucket sort, radix sort

Insertion Sort

- `A[0]` alone is a sorted array.
- For `i=1` to `N-1`
  - **Insert** `A[i]` into the appropriate location in the sorted array `A[0], …, A[i-1]`, so that `A[0], …, A[i]` is sorted.
  - To do so, save `A[i]` in **a temporary variable** `t`, **shift** sorted elements greater than `t` right, and then **insert** `t` in the gap.
- Time complexity? $O(N^2)$
- In place? Yes. O(1) additional memory.
- Stable?
  - Yes, because elements are visited in order and equal elements are inserted after its equals.

Insertion Sort
Best Case Time Complexity

- For `i=1` to `N-1`
  - **Insert** `A[i]` into the appropriate location in the sorted array `A[0], …, A[i-1]`, so that `A[0], …, A[i]` is sorted.

- The **best case** time complexity is $O(N)$.
  - It happens when the array is already sorted.
  - For other sorting algorithms we will talk, their best case time complexity is $\Omega(N \log N)$.

Selection Sort

- For `i=0` to `N-2`
  - Find the smallest item in the array `A[i], …, A[N-1]`. Then, swap that item with `A[i]`.

- Finding the smallest item requires **linear scan**.

Which Statements Are Correct for Selection Sort?

For **`i=0`** to **`N-2`**

Find the smallest item in the array **`A[i], …, A[N-1]`**. Then, swap that item with **`A[i]`**.

- **A.** Its worse-case time complexity is $O(N^2)$
- **B.** Its best-case time complexity is $\Omega(N^2)$
- **C.** It is not in-place
- **D.** It is stable

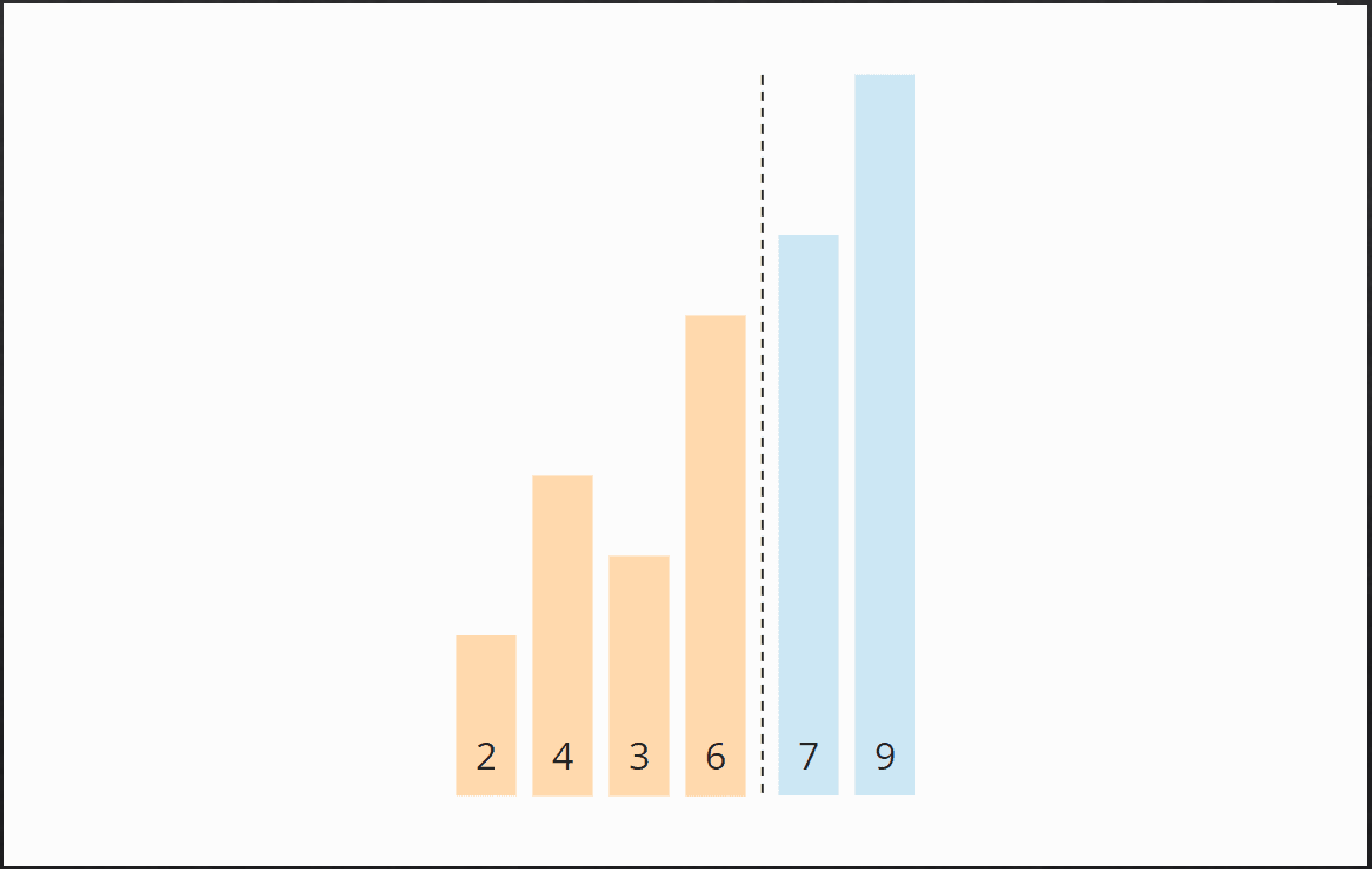
Bubble Sort

```
For i=N-2 downto 0
  For j=0 to i
    If A[j]>A[j+1] swap A[j] and A[j+1]
```

- Compares two adjacent items and swap them to keep them in ascending order.
  - From the beginning to the end. The last item will be the largest.

- Time complexity? $O(N^2)$
- In place? Yes.
- Stable?
  - Yes, because equal elements will not be swapped.

Animation



Worst-Case Complexity

Average Time Complexity

Unfortunately, the average time complexity of Bubble Sort cannot – in contrast to most other sorting algorithms – be explained in an illustrative way.

Without proving this mathematically (this would go beyond the scope of this article), one can roughly say that in the average case, one has about half as many exchange operations as in the worst case since about half of the elements are in the correct position compared to the neighboring element. So the number of exchange operations is:

$$\tfrac{1}{4}\,(n^2 - n)$$

It becomes even more complicated with the number of comparison operations, which amounts to (source: this German Wikipedia article; the English version doesn't cover this):

$$\tfrac{1}{2}\,(n^2 - n \times \ln(n) - (\gamma + \ln(2) - 1) \times n) + O(\tilde{A}n)$$

Two Problems with Simple Sorts

- They learn only one piece of information per comparison and hence might compare every pair of elements.
  - Contrast with binary search: learns N/2 pieces of information with first comparison.
- They often move elements one place at a time (bubble sort and insertion sort), even if the element is “far” from its **final place**.
  - Contrast with selection sort, which moves each element exactly to its final place.

- Fast sorts attack these two problems.
  - Two famous ones: **merge sort** and **quick sort**.

Short Break

- 5 min
- Time for questions!
- Try to keep your volume low so others can hear the question clearly!

Outline

- Sorting Basics
- Merge Sort
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Merge Sort Algorithm

- Spilt array into two (roughly) equal subarrays.
- Merge sort each subarray recursively.
  - The two subarrays will be sorted.
- Merge the two sorted subarrays into a sorted array.

[8, 3, 13, 6, 2, 14, 5, 9, 10]

[8, 3, 13, 6, 2]   [14, 5, 9, 10]

⋮   ⋮

[2, 3, 6, 8, 13]   [5, 9, 10, 14]

[2, 3, 5, 6, 8, 9, 10, 13, 14]

Merge Sort
Pseudo-code

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

Merge Two Sorted Arrays

- For example, merge A = (2, 5, 6) and B = (1, 3, 8, 9, 10).

- Compare the smallest element in the two arrays A and B and move the smaller one to an additional array C.
- Repeat until one of the arrays becomes empty.
- Then append the other array at the end of array C.

Merge Two Sorted Arrays
Implementation

- We actually do not "remove" element from arrays A and B.
  - We just keep a pointer indicating the smallest element in each array.
  - We "remove" element by incrementing that pointer.

```
i = j = k = 0;
while(i < sizeA && j < sizeB) {
  if(A[i]<=B[j]) C[k++]=A[i++];
  else C[k++]=B[j++];
}
if(i == sizeA) append(C, B);
else append(C, A);
```

Time complexity?

$O(sizeA + sizeB)$

Merge Sort

Time Complexity

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

$T(N/2)$

$T(N/2)$

$O(N)$

- Let $T(N)$ be the time required to merge sort N elements.
- Merge two sorted arrays with total size N takes $O(N)$.

Recursive relation: $T(N) = 2T(N/2) + O(N)$

How to solve the recurrence?

Solve Recurrence: Master Method

- A "black box" for solving recurrence.
- However, there is an important assumption: all sub-problems have roughly **equal** sizes.
  - E.g., merge sort
  - Not apply to unbalanced division.

Solve Recurrence: Master Method

- Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$
  - Base case: $T(n) \leq \boldsymbol{constant}$ for all sufficiently small n.
  - a = number of recursive calls (integer ≥ 1)
  - b = input size shrinkage factor (integer > 1)
  - $O(n^d)$: the runtime of merging solutions. d is a real value ≥ 0.
  - a, b, d are independent of n.
- Claim:

base doesn't matter

$$T(n) = \begin{cases} O\left(n^d \log n\right) & if\ a = b^d \\ O\left(n^d\right) & if\ a < b^d \\ O\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$$

base matters!

Idea of the Proof

Example of Merge Sort

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O\left(n^d \log n\right) & if\ a = b^d \\ O\left(n^d\right) & if\ a < b^d \\ O\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$

$a = 2, b = 2, d = 1$
$T(n) = O(n \log n) \Rightarrow b^d = a$

What are a, b, d for Binary Search?

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O\left(n^d \log n\right) & if\ a = b^d \\ O\left(n^d\right) & if\ a < b^d \\ O\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$

A. a =2, b = 2, d = 0 **B.** a =1, b = 2, d =0
C. a =2, b = 2, d = 1 **D.** a =1, b = 2, d =1

Merge Sort
Characteristics

- Not in-place (Why?)
  - For efficient merging two sorted arrays, we **need an auxiliary $\boldsymbol{O(N)}$ space**.
  - Recursion needs up to $O(\log N)$ **stack space**.

- Stable if **`merge()`** maintains the relative order of equal keys. (How?)

```
merge(a, left, mid, right);
```

Divide-and-Conquer Approach

- Merge sort uses the **divide-and-conquer** approach.

- Recursively **breaking** down a problem into two or more sub-problems of the same (or related) type, until these become simple enough to be solved directly.
  - For merge sort, split an array into two and sort them respectively.

- The solutions to the sub-problems are then **combined** to give a solution to the original problem.
  - For merge sort, merge two sorted arrays.

Short Break

- 5 min
- Time for questions!
- Remember to keep your discussion among yourselves

Outline

- Sorting Basics
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- Quick Sort
- Comparison Sort Summary

Quick Sort Algorithm

- Choose an array element as **pivot**.
- Put all elements < pivot to the left of pivot.
- Put all elements ≥ pivot to the right of pivot.
- Move pivot to its correct place in the array.
- Sort left and right subarrays recursively (not including pivot).

`partition()`

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

Another divide-and-conquer approach to sort

Choice of Pivot

- If your input is random, you can choose the **first** element.
  - But this is very bad for presorted input.

- A better strategy: **randomly** pick an element from the array as pivot.
  - **Claim**: **for any input**, the average running time is $O(n \log n)$.
    - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.

Partitioning the Array

- Once pivot is chosen, swap pivot to the beginning of the array.
- When another array B is available, scan original array A from left to right.
  - Put elements < pivot at the left end of B.
  - Put elements ≥ pivot at the right end of B.
  - The pivot is put at the remaining position of B.
  - Copy B back to A.

A	6	2	8	5	11	10	4	1	9	7	3
B	2	5	4	1	3	6	7	9	10	11	8

In-Place Partitioning the Array

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
    - **`A[i]`** is the leftmost item ≥ pivot.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
    - **`A[j]`** is the rightmost item < pivot.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

In-Place Partitioning the Array
Example

`i` ■ `j` ■

A	6	2	8	5	11	10	4	1	9	7	3
A	6	2	3	5	11	10	4	1	9	7	8
A	6	2	3	5	1	10	4	11	9	7	8
A	6	2	3	5	1	4	10	11	9	7	8

- Now, `j < i`, swap the first element (pivot) with **`A[j]`**.

A	4	2	3	5	1	6	10	11	9	7	8

In-Place Partitioning the Array
Time Complexity

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

- Scan the entire array no more than twice.
- Time complexity is $O(N)$, where N is the size of the array.

Quick Sort
Time Complexity

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

$O(N)$

$T(LeftSz)$

$T(RightSz)$

- Let $T(N)$ be the time needed to sort N elements.
  - $T(0) = c$, where c is a constant.
- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

  - $LeftSz + RightSz = N - 1$

Quick Sort

Worst Case Time Complexity

- Recursive relation: $T(N) = T(LeftSz) + T(RightSz) + O(N)$

- Worst case happens when each time the pivot is the smallest item or the largest item
  - $T(N) = T(N-1) + T(0) + O(N)$

$$\leq T(N-1) + T(0) + dN$$

$$\leq T(N-2) + 2T(0) + d(N-1) + dN$$

$$\dots$$

$$\leq T(0) + NT(0) + d + 2d + \cdots + d(N-1) + dN$$

$$= O(N^2)$$

Quick Sort
Best Case Time Complexity

- Recursive realtaion: $T(N) = T(LeftSz) + T(RightSz) + O(N)$

- Best case happens when each time the pivot divides the array into two equal-sized ones.
  - $T(N) = T((N-1)/2) + T((N-1)/2) + O(N)$
  - The recursive relation is similar to that of merge sort.
  - $T(N) = O(N \log N)$

Quick Sort
Average Time Complexity

- Average time complexity of quick sort can be proved to be $O(N \log N)$.
  - Assume **randomly** pick an element from the array as pivot.
  - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.
  - The claim holds for any input.

Quick Sort
Other Characteristics

- In-place?
  - In-place partitioning.
  - Worst case needs $O(N)$ stack space. (Why?)
  - Average case needs $O(\log N)$ stack space.
    - "Weakly" in-place.

- Not stable. (Why?)
- Remedy?

(4, b), (3, e), (3, b), (5, b) $\Rightarrow$ (4, b,1), (3, e,2), (3, b,3), (5, b,4)

Quick Sort
Summary

- Like merge sort, quick sort is a divide-and-conquer algorithm.

- Merge sort: easy division, complex combination.
- Quick sort: complex division (partition with pivot step), easy combination.

- Insertion sort runs faster than quick sort for small arrays.
    - Terminate quick sort when array size is below a threshold. Do insertion sort on subarrays.

Outline

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Comparison Sorts
Summary

	Worst Case Time	Average Case Time	In Place	Stable
Insertion	$O(N^2)$	$O(N^2)$	Yes	Yes
Selection	$O(N^2)$	$O(N^2)$	Yes	No
Bubble	$O(N^2)$	$O(N^2)$	Yes	Yes
Merge Sort	$O(N \log N)$	$O(N \log N)$	No	Yes
Quick Sort	$O(N^2)$	$O(N \log N)$	Weakly	No

For comparison sort, is $O(N \log N)$ the best we can do in the **worst case**?

Comparison Sorts
Worst Case Time Complexity

- Theorem: A sorting algorithm that is based on pairwise comparisons must use $\Omega(N \log N)$ operations to sort in the worst case.

That is All for today!

- Questions?