Strong convergence of propagation of chaos for McKean-Vlasov SDEs with singular interactions

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N-particle systems

- Consider the following N-particle systems on $\mathbb{R}^d$:

$$\mathrm{d}X_t^{N,i} = F\Big(t, X_t^{N,i}, \frac{1}{N}\sum_{j=1}^{N}\varphi(X_t^{N,i} - X_t^{N,j})\Big)\mathrm{d}t + \sigma(t, X_t^{N,i})\mathrm{d}B_t^i, \quad (\mathcal{N})$$

  where $i = 1, 2, .., N$,

  - F: Some nonlinear force;
  - φ: interaction kernel;
  - B^i: independent Brownian motions on $\mathbb{R}^d$ (random phenomena).

McKean-Vlasov equations

- When F, φ and σ are smooth, the solution of the N-particle systems $X^{N,i}$ convergences to the solution of the following McKean-Vlasov SDE, which is also called Distribution Dependent SDE:

$$\mathrm{d}X_t = F\Big(t, X_t, \int_{\mathbb{R}^d} \varphi(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t\Big) + \sigma(t, X_t)\mathrm{d}B_t, \tag{1}$$

where μ_t is the distribution of X_t and B_t is a standard BM.

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  where μ_t is the distribution of X_t and B_t is a standard BM.

- Well-known results for well-posedness

  i) Linear growth F and φ:
  Mishura-Veretennikov (arXiv-16), Lacker (arXiv-21).

  ii) $L^q_T L^p$ interaction:
  Röckner-Zhang (Bernoulli-21), Zhao (arXiv-20)
  $\|f\|_{L^q_T L^p} := \left(\int_0^T \|f(t)\|_p^q \mathrm{d}t \right)^{1/q}$

Propagation of chaos

- Denote by $P_t^{N,k}$ and P_t the distribution of $(X_t^{N,1},...,X_t^{N,k})$ for $k = 1, 2, ., N$ and X_t respectively. It is natural to ask whether we have

$$P_t^{N,k} \to P_t^{\otimes k}, \quad \text{if} \quad P_0^{N,k} \to P_0^{\otimes k} \text{ (which is called } P_0\text{-chaotic).}$$

This is called propagation of chaos, which originally goes as far back as Maxwell and Boltzmann, and was formalized by Kac in 1950s.

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  - Jabin-Wang (JFA-16 & Invent-18): Assume φ is bounded (kinetic case) and $\varphi, \mathrm{div}\varphi \in W^{-1,\infty}$(in $\mathbb{T}^d$),

$$\mathcal{H}(P_t^{N,N}, P_t^{\otimes N}) \le C.$$

$$\Rightarrow \ \|P_t^{N,k} - P_t^{\otimes k}\|_{var} \le C\sqrt{\frac{k}{N}}.$$

  - Lacker (EJP-18 & arXiv-21) Assume φ is bounded,

$$\|P_t^{N,k} - P_t^{\otimes k}\|_{var} \le C\frac{k}{N}.$$

Strong propagation of chaos

- Apart from the above convergence results for $P^{N,k}$, assuming that $F(t,x,r) = r$ and φ is Lipschitz, Sznitman in his famous lecture in 1991 showed us the following results:

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |X_t^{N,i} - X_t^i| \Big) \leq C\Big(\sqrt{\frac{1}{N}} + \mathbb{E}|X_0^{N,i} - X_0^i| \Big),$$

  where X_t^i is the solution to (1) driven by BM B_t^i.
- We call this type of convergence the strong convergence of propagation of chaos.

Moderate case

- When $\varphi(\cdot) = \varphi_N(\cdot) = \varepsilon_N^{-d}\phi(\varepsilon_N^{-1}\cdot)$, which is called moderately interacting kernel, and $\varepsilon_N \to 0$ as $N \to \infty$, we rewrite our N-particle systems as follow:

$$\mathrm{d}X_t^{N,i} = F\Big(t, X_t^{N,i}, \frac{1}{N}\sum_{j=1}^{N}\varphi_N(X_t^{N,i} - X_t^{N,j})\Big)\mathrm{d}t + \sigma(t, X_t^{N,i})\mathrm{d}B_t^i. \quad (\mathcal{M})$$

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- At this time, $\varphi_N \to \delta$. We expect that the limit equation is the following Density Dependent SDE (is also called McKean-Vlasov SDE of Nemytskii-type):

$$\mathrm{d}X_t = F(t, X_t, \rho_t(X_t))\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \tag{2}$$

where ρ_t stands for the density of X_t.

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  where ρ_t stands for the density of X_t.
- Here ρ_t solves the following nonlinear Fokker-Planck equation:

$$\partial_t \rho_t = \partial_i \partial_j (a_{ij}\rho_t) + \mathrm{div}(F(\rho_t)\rho_t).$$

  Barbu-Röckner (AoP-2020), (JFA-2021)......

Well-known results and question

- (Oelschläger, PTRF-85)
  $F(t, \cdot, \cdot)$ and $r \to rF(r)$ are Lipschitz, $\varphi = W * W$ with some $W \in H^\alpha$
  $\Rightarrow$ Weak convergence when $\varepsilon_N = N^{\beta/d}$ with $\beta \in (0, 1)$.
- (Jourdain-Méléard, AIHP-98)
  F, ϕ and σ are smooth $\Rightarrow$ strong convergence rate of propagation of chaos when $\varepsilon_N \asymp (\ln N)^\delta$ with some $\delta > 0$.

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- (Jourdain-Méléard, AIHP-98)
  F, ϕ and σ are smooth $\Rightarrow$ strong convergence rate of propagation of chaos when $\varepsilon_N \asymp (\ln N)^\delta$ with some $\delta > 0$.
- However, Lipschitz assumptions on F, ϕ and σ are too strong in practice. In fact, many of the interesting physical models have bounded measurable or even singular interaction kernels.
- (Question:) Can we establish the strong convergence of propagation of chaos with singular (L^∞ and $L^q_T L^p$) interaction both for classical one ($\mathcal{N}$) and moderate one ($\mathcal{M}$)?

Well-posedness for N-particle systems

- Before consider the strong convergence of propagation of chaos, it is nature to ask whether there is a strong well-posedness for both N-particle systems and limit McKean-Vlasov SDE.

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- Let $F(t,x,r) = r$ and $\varphi \in L^p$. Consider the following N-particle systems:

$$\mathrm{d}X_t^{N,i} = \frac{1}{N}\sum_{j=1}^{N}\varphi(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \mathrm{d}B_t^i.$$

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- Cépa-Lépingle (PTRF-1997) : $d = 1$; $\varphi(x) = \frac{1}{x}$.
- Krylov-Röckner (PTRF-2005) : $\varphi = \nabla V$ with continuously differentiable potential V in $\mathbb{R}^d \setminus \{0\}$.
- Fontbona-Martinez (JSP-2007): $d = 2$; Biot-Savart law.

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  How to obtain strong well-posedness of N-particle systems with general L^p-interaction kernels with $p > d$.
- By strong well-posedness results of SDE with general L^p drift in Krylov-Röckner (PTRF-05), we need

$$\varphi^i(x) := \frac{1}{N}\sum_{j=1}^{N}\varphi(x_i - x_j) \in L^p \text{ with } p > Nd.$$

Property of interaction kernel φ^i

- Notice that for $\mathbf{p_0} = (p, \infty, \infty, ..., \infty)$, $\varphi^1 \in L^{\mathbf{p_0}}$, where

$$\|f\|_{L^{\mathbf{p}}} := \Big(\int_{\mathbb{R}^d} \Big(\cdot \cdot \big(\int_{\mathbb{R}^d} |f(x)|^{p_1} \mathrm{d}x_1 \big)^{p_1/p_2} \cdot \cdot \Big)^{p_N/p_{N-1}} \mathrm{d}x_N \Big)^{1/p_N}.$$

- Ling-Xie (POTA 2021) Strong well-posedness for the following SDE

$$\mathrm{d}X_t = F(X_t)\mathrm{d}t + \mathrm{d}W_t$$

where $F \in L^{\mathbf{p}}$ with $\frac{d}{p_1} + \frac{d}{p_2} + \cdot \cdot + \frac{d}{p_N} < 1$.

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where $F \in L^{\mathbf{p}}$ with $\frac{d}{p_1} + \frac{d}{p_2} + \cdots + \frac{d}{p_N} < 1$.

- However, notice that $(\varphi^1, ..., \varphi^N) \notin L^{\mathbf{p_0}}$ because of permutation. Actually, we only have

$$\sup_{x_j, j \neq i} \Big(\int_{\mathbb{R}^d} |\varphi^i(..., x_i, ...)|^p \mathrm{d}x_i \Big)^{1/p} < \infty.$$

And $L^{p_1}L^{p_2} \notin L^{p_2}L^{p_1}$.

Mixed L^p space with permutation

- For multi-index $\mathbf{p} = (p_1, \cdots, p_d) \in [1, \infty]^N$ and any permutation $\mathbf{x} = (x_{i_1}, ..., x_{i_N})$, the mixed $L^{\mathbf{p}}_{\mathbf{x}}$-space is defined by

$$\|f\|_{L^{\mathbf{p}}_{\mathbf{x}}} := \left(\left(\int_{\mathbb{R}^d} \cdots \left(\int_{\mathbb{R}^d} |f(x_1, \cdots, x_d)|^{p_d} \mathrm{d}x_{i_1} \right)^{\frac{p_{d-1}}{p_d}} \cdots \right)^{\frac{p_1}{p_2}} \mathrm{d}x_{i_N} \right)^{\frac{1}{p_1}}.$$

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- Note that for general $\mathbf{x} \neq \mathbf{x}'$ and $\mathbf{p} \neq \mathbf{p}'$,

$$L^{\mathbf{p}'}_{\mathbf{x}} \neq L^{\mathbf{p}}_{\mathbf{x}} \neq L^{\mathbf{p}}_{\mathbf{x}'}.$$

  For multi-indices $\mathbf{p} \in [1, \infty]^N$, we shall use the following notations:

$$\left| \frac{d}{\mathbf{p}} \right| = \sum_{i=1}^{d} \frac{d}{p_i}.$$

- $\varphi^i \in L^{\mathbf{p_0}}_{\mathbf{x}_i}$ with $\mathbf{x}_i = (x_i, x_1, ..., x_N)$.

Main result

Assumptions

- Let indices $(q_i, \mathbf{p}_i)$, $i = 0, 1, .., N$ satisfy $\frac{2}{q_i} + \left|\frac{d}{\mathbf{p}_i}\right| < 1.$
- Suppose that $\nabla\sigma \in L_T^{q_0}(L_{\mathbf{x}_0}^{\mathbf{p}_0})$,

$$\kappa_0^{-1}|\xi| \leqslant |\sigma(t,x)\xi| \leqslant \kappa_0|\xi|, \quad \forall x, \xi \in \mathbb{R}^d$$

- For any $T > 0$ and $i = 1, \cdots, N$, there are permutations $\mathbf{x}_i$ such that

$$\sup_{\mu \in C([0,T];\mathcal{P}(\mathbb{R}^d))} \| \sup_{r \geqslant 0} |b^i(\cdot, \cdot, r, \mu.)| \|_{L_T^{q_i}(L_{\mathbf{x}_i}^{\mathbf{p}_i})} \leqslant \kappa_1,$$

and for some $h_i \in L_T^{q_i}(L_{\mathbf{x}_i}^{\mathbf{p}_i})$ and for all $t, x \in [0,T] \times \mathbb{R}^d$, $r, r' \geqslant 0$ and $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$|b^i(t,x,r,\mu) - b^i(t,x,r',\nu)| \leqslant h_i(t,x)\big(|r - r'| + \|\mu - \nu\|_{\mathrm{var}}\big).$$

Theorem 1 (H., Röckner and Zhang)

Under the assumptions, for any probability measure $\mu_0(\mathrm{d}x) = \rho_0(x)\mathrm{d}x$ with $\rho_0 \in L^\infty$, there is a unique strong solution to the following dDDSDE with initial distribution μ_0

$$\mathrm{d}X_t = b(t, X_t, \rho_t(X_t), \mu_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,$$

where $\rho_t(x)$ and μ_t are the density and dietribution of X_t respectively and $b(t, x, r, \mu) = (b^1, ..., b^N) : \mathbb{R}_+ \times \mathbb{R}^{Nd} \times \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{Nd}$.

If b does not depend on the density variable, then we can drop the assumption $\mu_0(\mathrm{d}x) = \rho_0(x)\mathrm{d}x$.

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If b does not depend on the density variable, then we can drop the assumption $\mu_0(\mathrm{d}x) = \rho_0(x)\mathrm{d}x$.

- Strong well-posedness for N-particle systems for some $L_T^q L^p$ interaction kernels.
- Strong well-posedness for limit equation (both distribution dependent case and density dependent case) by taking $N = 1$. This extends the results in H.-Röckner-Zhang (JDE 2021) and Wang (arXiv-2021).

Sketch of the proof

- (SDE) For $b(t,x,r,\mu) = b(t,x)$, establish the Krylov's type estimate. Consider the following PDE and construct Zvonkin's transformation

$$\partial_t u = a_{ij}\partial_i\partial_j u - \lambda u + b \cdot \nabla u + f, \quad u(0) = 0. \tag{3}$$

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     We use Zvonkin's transform and heat kernel estimates here.
- (dDDSDE) Girsaonv' Theorem and Picard-iteration.

Assumptions

- Let $\frac{2}{q} + \frac{d}{p} < 2$.

($\mathbf{H}^\sigma$) There are $\kappa_0 \geqslant 1$ and $\gamma_0 \in (0, 1]$ such that,

$$\kappa_0^{-1}|\xi| \leqslant |\sigma(t,x)\xi| \leqslant \kappa_0|\xi|, \; \|\sigma(t,x) - \sigma(t,x')\|_{HS} \leqslant \kappa_0|x - x'|^{\gamma_0},$$

where $\|\cdot\|_{HS}$ is the usual Hilbert-Schmidt norm of a matrix. Moreover, $\nabla\sigma \in L_T^q L^p$.

($\mathbf{H}^b$) Suppose that $\varphi(0) = 0$ and for some measurable $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ and $\kappa_1 > 0$,

$$|F(t,x,r)| \leqslant h(t,x) + \kappa_1|r|, \;\; |F(t,x,r) - F(t,x,r')| \leqslant \kappa_1|r - r'|,$$

and for some $p_0 > d$

$$\|h\|_{L_T^q L^p} + \|\varphi\|_{p_0} \leqslant \kappa_1.$$

Main results

Theorem 2 (Strong convergence)

Let $T > 0$. Under above assumptions, suppose that $P_0^{N,N}$ is symmetric and P_0-chaotic, and

$$\lim_{N\to\infty} \mathbb{E}|X_0^{N,1} - X_0|^2 = 0,$$

then for any $\gamma \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) = 0.$$

Main results

Theorem 3 (Strong convergence rate)

Let $T > 0$. Assume the same assumptions as the above theorem. Let

$$\kappa_2 := \sup_N \mathcal{H}\left(P_0^{N,N} | P_0^{\otimes N}\right) < \infty. \tag{4}$$

Also assume that h and φ are bounded measurable. Then for any $\gamma \in (0,1)$, there is a constant $C > 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) \leqslant C\left(\mathbb{E}|X_0^{N,1} - X_0|^2 + \frac{\kappa_2 + 1}{N}\right)^{\gamma}.$$

Strong convergence for moderately case

Theorem 4
Let $T > 0$. Suppose that $(\mathbf{H}^\sigma)$ holds, and

$$|F(t,x,r)| \leqslant \kappa_1, \ \ |F(t,x,r) - F(t,x,r')| \leqslant \kappa_1 |r - r'|,$$

and

$$\varphi(x) = \varphi_{\varepsilon_N}(x) = \varepsilon_N^{-d} \phi(x/\varepsilon_N),$$

where ϕ is a bounded probability density function in $\mathbb{R}^d$ with support in the unit ball. Under (4), for any $T > 0$, $\beta \in (0, \gamma_0)$ and $\gamma \in (0,1)$, there is a constant $C > 0$ such that for all N,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) \leqslant C\mathrm{e}^{C\varepsilon_N^{-2d}} \left(\mathbb{E}|X_0^{N,1} - X_0|^2 + \frac{\kappa_2 + 1}{N}\right)^\gamma + C\varepsilon_N^{2\beta\gamma}.$$

Remarks

- Suppose that for some $C > 0$,

$$\mathbb{E}|X_0^{N,1} - X_0|^2 \leqslant C/N.$$

If one chooses $\varepsilon_N = (\ln N)^{-1/(2d)}$, then for some $C > 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) \leqslant \frac{C}{(\ln N)^{(\beta\gamma)/d}}.$$

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- Although we assume F is bounded, once we can establish the existence of bounded solutions to Fokker-Planck equation under linear growth assumption of F on r, then the boundedness of F is no longer a restriction. We illustrate this by the following example.

Example

- Consider the following special case:

$$\partial_t \rho = \Delta \rho + \mathrm{div}(F(\rho)\rho), \tag{5}$$

where $F : \mathbb{R}_+ \to \mathbb{R}^d$ satisfies $\sum_{i=1}^d |F_i'(r)| \leqslant \kappa_1$.

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- Notice that this equation can be written as the following transport form:

$$\partial_t \rho = \Delta \rho + (F(\rho) + F'(\rho)\rho) \cdot \nabla \rho. \tag{6}$$

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- Assume that there is a unique strong solution ρ and

$$n_0 := \|\rho_0\|_\infty < \infty.$$

- Let $G(x) := F(\eta(x))$ where $\eta \in C_b^\infty$ and

$$\eta(x) = x, \quad x \in [-2n_0, 2n_0].$$

Then, $\sup_{x\in\mathbb{R}} |G(x)| \leq C\|\eta\|_\infty < \infty$.

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- By the maximum principle, we have

$$\|\tilde{\rho}_t\|_\infty \leqslant \|\rho_0\|_\infty = n_0,$$

which implies that $G(\tilde{\rho}) = F(\tilde{\rho})$.

- By uniqueness, we know that

$$\rho = \tilde{\rho} \quad \Rightarrow \|\rho_t\|_\infty \leq n_0.$$

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$$\partial_t \rho = \Delta \rho + \operatorname{div}(G(\rho)\rho),$$

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- (Numerical experiment of Burgers equation)
Consider $d = 1$ and $F(r) = r$ and take $\phi(x) = 1_{[-1,1]}(x)/2$:

$$\frac{1}{N}\sum_{j=1}^{N} \varphi_N(X_t^{N,i} - X_t^{N,j}) = \frac{1}{2N\varepsilon_N}\sum_{j=1}^{N} 1_{|X_t^{N,i}-X_t^{N,j}|\leqslant\varepsilon_N}.$$

This form is useful for numerical experiments.

Recall

$$\mathrm{d}X_t^{N,i} = \frac{1}{N}\sum_{j=1}^{N} \varphi_N(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \mathrm{d}B_t^i.$$

$$\mathrm{d}X_t^i = \rho_t(X_t^i)\mathrm{d}t + \mathrm{d}B_t^i.$$

Thank you!

Danke!