Flow-distribution dependent SDEs and Navier-Stokes equations with fractional Brownian motion

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Probability representation of the NSE

- Consider the following Navier-Stokes equation on $\mathbb{R}^d$ with $d = 2, 3$:

$$\begin{cases} \partial_t u = \Delta u + u \cdot \nabla u + \nabla p, \\ \mathrm{div} u = 0, \quad u_0 = \varphi, \end{cases}$$

- (Constantin-Iyer 2008, CPAM)

$$\begin{cases} X_t^x = x + \int_0^t u(s, X_s^x) \mathrm{d}s + \sqrt{2} W_t, \quad t \geq 0, \\ u(t, x) = \mathbf{P}\mathbb{E}[\nabla_x^{\mathrm{t}} Y_t^x \cdot \varphi(Y_t^x)], \end{cases}$$

where Y_t^x is the inverse of the flow mapping $x \to X_t^x$, ∇^{t} denotes the transpose of the Jacobi matrix $(\nabla X)_{ij} := \partial_{x_j} X^i$, and $\mathbf{P} := \mathbb{I} - \nabla \Delta^{-1} \mathrm{div}$ is the Leray projection.

Probability representation of the NSE

- Velocity & Vorticity:

$$w = \mathrm{curl} u = \begin{cases} \partial_2 u_1 - \partial_1 u_2, & d = 2; \\ \nabla \times u, & d = 3; \end{cases}$$

  and

$$u = K_d * w, \ \ d = 2, 3,$$

  with the Biot-Savart law:

$$K_2(x) := \frac{(x_2, -x_1)}{2\pi |x|^2}, \ K_3(x)h = \frac{x \times h}{4\pi |x|^3}.$$

Probability representation of the NSE

- (Zhang 2016, AoAP)

$$w(t,x)=\begin{cases}\mathbb{E}\left((\mathrm{curl}\varphi)(Y_t^x)\det(\nabla Y_t^x)\right), & d=2,\\ \mathbb{E}\left(\nabla_x^{\mathrm{t}} Y_t^x\cdot(\mathrm{curl}\varphi)(Y_t^x)\right), & d=3.\end{cases}$$

and

$$u(t,x)=\begin{cases}\mathbb{E}\left(\int_{\mathbb{R}^2} K_2(x-X_t^y)\cdot(\mathrm{curl}\varphi)(y)\mathrm{d}y\right), & d=2,\\ \mathbb{E}\left(\int_{\mathbb{R}^3} K_3(x-X_t^y)\cdot\nabla X_t^y\cdot(\mathrm{curl}\varphi)(y)\mathrm{d}y\right), & d=3.\end{cases}$$

Flow-distribution dependent SDEs

- $d = 2$, we define

$$B(x, \mu^{\cdot}) := \int_{\mathbb{R}^2} (K_2 * \mu^y)(x)\mathrm{curl}\varphi(y)\mathrm{d}y.$$

Then X_t^x solves the following closed SDE:

$$X_t^x = x + \int_0^t B(X_s^x, \mu_s^{\cdot})\mathrm{d}s + \sqrt{2}W_t. \tag{1}$$

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- When $\mathrm{curl}\varphi \in \mathscr{P}(\mathbb{R}^2)$, flow-distribution dependent SDE (FDSDE) can induce the following distributional dependent SDE (DDSDE):

$$X_t = X_0 + \int_0^t (K_2 * \mu_s)(X_s)\mathrm{d}s + \sqrt{2}W_t, \quad X_0 \overset{(d)}{=} \mathrm{curl}\varphi(y)\mathrm{d}y, \tag{2}$$

by letting

$$\mathbb{P} \circ (X_{\cdot})^{-1} := \int_{\mathbb{R}^2} \mathbb{P} \circ (X_{\cdot}^{y})^{-1}\mathrm{curl}\varphi(y)\mathrm{d}y.$$

Flow-distribution dependent SDEs

- The FDSDE (1) was introduced by [Chorin 1973, JFM] as the random vortex method to simulate viscous incompressible fluid flows for smooth kernels.
- [Beale-Majda 1981, MoC], [Marchioror-Pulvirenti 1982, CMP], [Goodman 1987, CPAM], [Long 1988, JAMS].
- Propagation of chaos for interaction particle system: [Jabin-Wang 2018, Invent], [Feng-Wang 2023], [Wang 2024]; [Wang-Zhao-Zhu 2024, ARMA]......
- Moderately interacting particle systems: [Flandoli-Olivera-Simon 2020, SIAM J. MATH. ANAL], [Olivera-Richard-Tomašević 2021]......

Flow-distribution dependent SDEs

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- Moderately interacting particle systems: [Flandoli-Olivera-Simon 2020, SIAM J. MATH. ANAL], [Olivera-Richard-Tomašević 2021]......
- Well-posedness of DDSDE (2): [Zhang 2023, CMS], [Chaudru de Raynal-Jabir-Menozzi, 2023], [Barbu-Röckner-Zhang, 2023], [H.-Röckner-Zhang 2024, AoP]......
- (Question:) Well-posedness of FDSDE (1)?

FDSDEs related to the 3D NSE

- When $d = 3$, we introduce a matrix-valued process $U_t^x := \nabla X_t^x$. Then U solves the following linear ODE:

$$U_t^x = \mathbb{I}_{3\times 3} + \int_0^t \bar{\mathbb{E}}\left(\int_{\mathbb{R}^3} \nabla K_3(X_s^y - \bar{X}_s^y)\cdot \bar{U}_s^y \cdot (\mathrm{curl}\varphi)(y)\mathrm{d}y\right)\mathrm{d}s,$$

  where $\bar{U}$ is an independent copy.

- Let $(\mu^x)_{x\in\mathbb{R}^3}$ be a family of probability measures over $\mathbb{R}^3 \times \mathcal{M}^3$, where $\mathcal{M}^3$ stands for the space of all 3×3-matrices. Now let us introduce

$$B(x,\mu) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathcal{M}^3} K_3(x-z)\cdot M\mu^y(\mathrm{d}z\times\mathrm{d}M)\cdot(\mathrm{curl}\varphi)(y)\mathrm{d}y.$$

- Then we obtain the following closed FDSDE

$$\begin{cases} X_t^x = x + \int_0^t B(X_r^x, \mu_r^\cdot)\mathrm{d}r + \sqrt{2}W_t, \\ U_t^x = \mathbb{I}_{3\times 3} + \int_0^t \nabla B(\cdot, \mu_r^\cdot)(X_r^x)U_r^x\mathrm{d}r, \end{cases}$$

  where $\mu_t^x := \mathbb{P}\circ(X_t^x, U_t^x)^{-1} \in \mathscr{P}(\mathbb{R}^3\times\mathcal{M}^3)$ for $x\in\mathbb{R}^3$.

Probability representation of the NSE-backward form

- On the other hand, setting $\tilde{u}(t,x) := u(T-t,x)$ and $\tilde{p}(t,x) := p(T-t,x)$, then $\tilde{u}$ solves the following backward Navier-Stokes equation:

$$\begin{cases} \partial_t \tilde{u} + \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = 0, \\ \mathrm{div} \tilde{u} = 0, \quad \tilde{u}_T = \varphi. \end{cases}$$

- (Zhang 2010, PTRF)

$$\begin{cases} \tilde{X}^x_{s,t} = x + \int_s^t \tilde{u}(r, \tilde{X}^x_{s,r}) \mathrm{d}r + \sqrt{2}(W_t - W_s), \quad (s,t) \in \mathbb{D}_T, \\ \tilde{u}(t,x) = \mathbf{P}\mathbb{E}[\nabla^{\mathrm{t}} \tilde{X}^x_{t,T} \cdot \varphi(\tilde{X}^x_{t,T})]. \end{cases} \tag{3}$$

Backward flow-distribution dependent SDEs

- Similarly, (3) can be transformed into the following backward FDSDE:

$$\tilde{X}^x_{s,t} = x + \int_s^t \tilde{B}(\tilde{X}^x_{s,r}, \tilde{\mu}^{\cdot}_{r,T})\mathrm{d}r + \sqrt{2}(W_t - W_s),$$

where $\mu^x_{s,t}$ is the law of $X^x_{s,t}$, and

$$\tilde{B}(x, \mu^{\cdot}) = K_2 * \left(\int_{\mathbb{R}^2} \mathrm{curl}\varphi(y)\mu^{\cdot}(\mathrm{d}y) \right)(x).$$

Backward flow-distribution dependent SDEs

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- Recall the previous drift B in forward FDSDE (1):

$$B(x, \mu^{\cdot}) := \int_{\mathbb{R}^2} (K_2 * \mu^y)(x)\mathrm{curl}\varphi(y)\mathrm{d}y.$$

Forward & backward FDSDEs

(i) Forward FDSDE:

$$X_t^x = x + \int_0^t B(r, X_r^x, \mu_r^{\cdot})\mathrm{d}r + \int_0^t \Sigma(r, X_r^x, \mu_r^{\cdot})\mathrm{d}W_r, \quad t \in [0, T].$$

(ii) Backward FDSDE: $0 \le s \le t \le T$,

$$\tilde{X}_{s,t}^x = x + \int_s^t \tilde{B}(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^{\cdot})\mathrm{d}r + \int_s^t \Sigma(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^{\cdot})\mathrm{d}W_r.$$

Forward FDSDEs and non-linear PDEs

(i) Forward FDSDE:

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▷ For some $\nu_0 \in \mathscr{P}(\mathbb{R}^d)$, we let

$$B(r, x, \mu^{\cdot}) = b(t, x, \nu_0 \odot \mu^{\cdot}), \quad \Sigma(r, x, \mu^{\cdot}) = \sigma(t, x, \nu_0 \odot \mu^{\cdot}),$$

where $(\nu_0 \odot \mu^{\cdot})(\mathrm{d}y) := \int_{\mathbb{R}^d} \mu^x(\mathrm{d}y)\nu_0(\mathrm{d}x)$.

▷ $\mu_t := \nu_0 \odot \mu_t^{\cdot}$ solves the following nonlinear Fokker-Planck equation

$$\partial_t \mu_t = \partial_{y_i}\partial_{y_j}(a_{ij}(t, y, \mu_t)\mu_t) - \mathrm{div}_y(b(t, y, \mu_t)\mu_t), \quad \mu_0(\mathrm{d}y) = \nu_0.$$

Forward FDSDEs and non-linear PDEs

(i) Forward FDSDE:

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$$\partial_t \mu_t = \partial_{y_i}\partial_{y_j}(a_{ij}(t, y, \mu_t)\mu_t) - \mathrm{div}_y(b(t, y, \mu_t)\mu_t), \quad \mu_0(\mathrm{d}y) = \nu_0.$$

▷ If we choose a random variable $\xi \overset{(d)}{=} \nu_0$, which is independent of X_t^x, then

$$Y_t := X_t^x|_{x=\xi}$$

satisfies the classical DDSDE.

Backward FDSDEs and non-linear PDEs

(ii) Backward FDSDE: $0 \leq s \leq t \leq T$,

$$\tilde{X}^x_{s,t} = x + \int_s^t \tilde{B}(r, \tilde{X}^x_{s,r}, \tilde{\mu}^{\cdot}_{r,T})\mathrm{d}r + \int_s^t \Sigma(r, \tilde{X}^x_{s,r}, \tilde{\mu}^{\cdot}_{r,T})\mathrm{d}W_r.$$

▷ For some function ϕ, we let

$$B(r, x, \mu^{\cdot}) = \tilde{b}(t, x, \mu^{\cdot}(\phi)), \quad \Sigma(r, x, \mu^{\cdot}) = \tilde{\sigma}(t, x, \mu^{\cdot}(\phi)),$$

where $\mu^{\cdot}(\phi) := \int_{\mathbb{R}^d} \phi(y)\mu^{\cdot}(\mathrm{d}y)$.

▷ Then $u(t, x) = \tilde{\mu}^x_{t,T}(\phi)$ solves the non-divergence and nonlinear PDE:

$$\partial_t u(t,x) + \tilde{a}_{ij}(t, x, u(t,\cdot))\partial_i\partial_j u(t,x) + \tilde{b}(t, x, u(t,\cdot)) \cdot \nabla u(t,x) = 0, \quad u(T) = \phi.$$

Backward FDSDEs and non-linear PDEs

(ii) Backward FDSDE: $0 \le s \le t \le T$,

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- ▷ If

$$(\tilde{a}_{ij}, \tilde{b})(t, x, u(t,\cdot)) = (\tilde{a}_{ij}, \tilde{b})(t, x, u(t,x)),$$

then we get a typical second order quasi-linear PDE.

Forward & backward FDSDEs

(i) Forward FDSDE:

$$X_t^x = x + \int_0^t B(r, X_r^x, \mu_r^\cdot)\mathrm{d}r + \int_0^t \Sigma(r, X_r^x, \mu_r^\cdot)\mathrm{d}W_r, \quad t \in [0, T].$$

▷ $\mu_t = \int_{\mathbb{R}^d} \mu_t^x(\mathrm{d}y)\nu_0(\mathrm{d}x)$ solves

$$\partial_t \mu_t = \partial_i \partial_j (a_{ij}(\mu_t)\mu_t) - \mathrm{div}_y(b(\mu_t)\mu_t), \quad \mu_0 = \nu_0.$$

(ii) Backward FDSDE: $0 \le s \le t \le T$,

$$\tilde{X}_{s,t}^x = x + \int_s^t \tilde{B}(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^\cdot)\mathrm{d}r + \int_s^t \Sigma(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^\cdot)\mathrm{d}W_r.$$

▷ $u(t,x) = \tilde{\mu}_{t,T}^x(\phi) = \mathbb{E}\phi(\tilde{X}_{t,T}^x)$ solves

$$\partial_t u + \tilde{a}_{ij}(u)\partial_i \partial_j u + \tilde{b}(u) \cdot \nabla u = 0, \quad u(T) = \phi.$$

General cases

- Consider

$$X_{s,t}^x = x + \int_s^t B(r, X_{s,r}^x, \mu_{r,T}^{\cdot}, \mu_{s,r}^{\cdot})\mathrm{d}r + \int_s^t \Sigma(r, X_{s,r}^x, \mu_{r,T}^{\cdot}, \mu_{s,r}^{\cdot})\mathrm{d}W_r. \qquad (4)$$

- Let $\mathfrak{F} := (\Omega, \mathscr{F}, (\mathscr{F}_s)_{s \geqslant 0}, \mathbb{P})$. We call a pair of stochastic processes

$$\left((X_{s,t}^x)_{(s,t,x) \in \mathbb{D}_T \times \mathbb{R}^d}, (W_t)_{t \in [0,T]}\right), \quad \mathbb{D}_T := \{0 \le s \le t \le T\},$$

  defined on $\mathfrak{F}$ a weak solution of FDSDE (4), if
  - W_t is a standard d-dimensional $\mathscr{F}_t$-Brownian motion;
  - For each $(s,t) \in \mathbb{D}_T$, $\mathbb{R}^d \ni x \to \mu_{s,t}^x := \mathbb{P} \circ (X_{s,t}^x)^{-1} \in \mathscr{P}(\mathbb{R}^d)$ is weakly continuous;
  - For each $(s,x) \in [0,T] \times \mathbb{R}^d$, equation (4) holds a.e. for all $t \in [s,T]$ provided that

$$\int_s^T |B(r, X_{s,r}^x, \mu_{r,T}^{\cdot}, \mu_{s,r}^{\cdot})|\mathrm{d}r + \int_s^T |\Sigma(r, X_{s,r}^x, \mu_{r,T}^{\cdot}, \mu_{s,r}^{\cdot})|^2\mathrm{d}r < \infty, \quad \mathbb{P} - \text{a.s.}$$

- If in addition that $X_{s,t}^x$ is adapted to the filtration generated by the Brownian motion, then it is called a strong solution of FDSDE (4).

Flow Wasserstein-1 distance

- Let $\mathscr{P}_1 := \mathscr{P}_1(\mathbb{R}^d)$ be the space of all probability measures with finite first order moment w.r.t. the Wasserstein-1 distance $\mathscr{W}_1$ defined by

$$\mathscr{W}_1(\mu,\nu) := \inf_{\mathbb{P}\circ X^{-1}=\mu, \mathbb{P}\circ Y^{-1}=\nu} \mathbb{E}|X-Y|.$$

- The duality of Monge-Kantorovich:

$$\mathscr{W}_1(\mu,\nu) = \sup_{\|g\|_{\mathrm{Lip}}\leqslant 1} |\mu(g)-\nu(g)|.$$

- Denote $\mathcal{C}\mathscr{P}_1 := C(\mathbb{R}^d;\mathscr{P}_1)$ by the space of $\mathscr{P}_1$-valued continuous functions on $\mathbb{R}^d$. For two $\mu^\cdot, \nu^\cdot \in \mathcal{C}\mathscr{P}_1$, we introduce a distance between $\mu^\cdot$ and $\nu^\cdot$ by

$$\mathrm{d}_{\mathcal{C}\mathscr{P}_1}(\mu^\cdot,\nu^\cdot) := \sup_{x\in\mathbb{R}^d} \frac{\mathscr{W}_1(\mu^x,\nu^x)}{1+|x|}.$$

- For simplicity, we write

$$\|\mu^\cdot\|_{\mathcal{C}\mathscr{P}_1} := \mathrm{d}_{\mathcal{C}\mathscr{P}_1}(\mu^\cdot,\delta_0) = \sup_{x\in\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} |y|\mu^x(\mathrm{d}y)}{1+|x|}.$$

Well-posedness

(Assumptions):
There are constants $\kappa_0, \kappa_2, \kappa_3, \kappa_4 > 0$ and $\kappa_1 \in \mathbb{R}$ such that for any $(t, x, \mu, \nu) \in [0, T] \times \mathbb{R}^d \times \mathcal{C}\mathscr{P}_1 \times \mathcal{C}\mathscr{P}_1$,

$$\langle x, B(t, x, \mu^{\cdot}, \nu^{\cdot})\rangle + 2\|\Sigma(t, x, \mu^{\cdot}, \nu^{\cdot})\|_{\mathrm{HS}}^2 \leq \kappa_0 + \kappa_1 |x|^2 + \kappa_2(\|\mu^{\cdot}\|_{\mathcal{C}\mathscr{P}_1}^2 + \|\nu^{\cdot}\|_{\mathcal{C}\mathscr{P}_1}^2),$$

and for any $(t, x_i, \mu_i, \nu_i) \in [0, T] \times \mathbb{R}^d \times \mathcal{C}\mathscr{P}_1 \times \mathcal{C}\mathscr{P}_1$, $i = 1, 2$,

$$\begin{aligned}&\langle x_1 - x_2, B(t, x_1, \mu_1^{\cdot}, \nu_1^{\cdot}) - B(t, x_2, \mu_2^{\cdot}, \nu_2^{\cdot})\rangle + 2\|\Sigma(t, x_1, \mu_1^{\cdot}, \nu_1^{\cdot}) - \Sigma(t, x_2, \mu_2^{\cdot}, \nu_2^{\cdot})\|_{\mathrm{HS}}^2 \\ &\quad \leq \kappa_3 |x_1 - x_2|^2 + \kappa_4 (1 + |x_1|^2 + |x_2|^2) \left(\mathrm{d}_{\mathcal{C}\mathscr{P}_1}^2 (\mu_1^{\cdot}, \mu_2^{\cdot}) + \mathrm{d}_{\mathcal{C}\mathscr{P}_1}^2 (\nu_1^{\cdot}, \nu_2^{\cdot}) \right).\end{aligned}$$

Theorem 1

Under the assumptions, there is a unique strong solution to FDSDE (4). *Moreover, if* $\kappa_1 < 0$ *and* $\kappa_1 + 2\kappa_2 < 0$, *then*

$$\mathbb{E}|X_{s,t}^x|^2 \leqslant \mathrm{e}^{\kappa_1 (t-s)} |x|^2 + (\kappa_0 + \kappa_5)(\mathrm{e}^{\kappa_1 (t-s)} - 1)/\kappa_1,$$

where $\kappa_5 := 2\kappa_2(|\kappa_1| + \kappa_0)/(|\kappa_1| - 2\kappa_2)$.

Examples

- Let $K_1, K_2 : \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$(1 + |x|)\, K_1, \nabla K_1 \in L^1 \ \text{ and } \nabla K_2 \in L^\infty.$$

Let $\varphi_1, \varphi_2 : \mathbb{R}^d \to \mathbb{R}$ be two Borel measurable functions with

$$\varphi_1, \nabla\varphi_1 \in L^\infty \text{ and } (1 + |x|)\, \varphi_2 \in L^1.$$

For $\mu, \nu \in \mathcal{C}\mathcal{P}_1$,

$$B(x, \mu^\cdot, \nu^\cdot) := \int_{\mathbb{R}^d} K_1(x-y)\mu^y(\varphi_1)\mathrm{d}y + \int_{\mathbb{R}^d} (K_2 * \nu^z)(x)\varphi_2(z)\mathrm{d}z.$$

Examples

- Let $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfy
$$|\sigma(t,x,r) - \sigma(t,x',r')| \leqslant C(|x-x'| + |r-r'|).$$
Let ϕ_ε be a family of mollifiers. For $\mu \in \mathcal{C}\mathcal{P}_1$, we introduce
$$\Sigma_\varepsilon(t,x,\mu^\cdot) := \sigma\left(t, x, \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\langle \mu^y, \varphi\rangle \mathrm{d}y\right).$$
- Then the following SDE admits a unique solution
$$X^{x,\varepsilon}_{s,t} = x + \int_s^t \Sigma_\varepsilon(r, X^{x,\varepsilon}_{s,r}, \mu^{\cdot,\varepsilon}_{r,T})\mathrm{d}W_r.$$
- (Open question): Whether we can take limits $\varepsilon \to 0$ so that we can give a probability representation $u(s,x) = \mathbb{E}\varphi(X^{x,0}_{s,T})$ for local quasi-linear PDE:
$$\partial_s u + \frac{1}{2}\sum_{i,j,k}(\sigma_{ik}\sigma_{jk})(s,x,u)\partial_i\partial_j u = 0, \quad u(T) = \varphi,$$
where $X^{x,0}_{s,T}$ solves the following nonlinear-SDE:
$$X^{x,0}_{s,t} = x + \int_s^t \sigma(r, X^{x,0}_{s,r}, \mu^{x,0}_{r,T}(\varphi))\mathrm{d}W_r.$$

FDSDEs from NSE

- Aim: Well-posedness for
  - (forward)

$$X_t^x = x + \int_0^t B(X_r^x, \mu_r^\cdot)\mathrm{d}r + \sqrt{2}W_t,$$

where

$$B(x, \mu^\cdot) := \int_{\mathbb{R}^2} (K_2 * \mu^y)(x)\mathrm{curl}\varphi(y)\mathrm{d}y;$$

  - (backward)

$$\tilde{X}_{s,t}^x = x + \int_s^t \tilde{B}(\tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^\cdot)\mathrm{d}r + \sqrt{2}(W_t - W_s),$$

where

$$\tilde{B}(x, \mu^\cdot) = K_2 * \left(\int_{\mathbb{R}^2} \mathrm{curl}\varphi(y)\mu^\cdot(\mathrm{d}y)\right)(x).$$

- $K_2 \in L^2_{loc}(\mathbb{R}^2)$.

Forward FDSDEs with singular kernel

- Consider the DDSDE related to NSE:

$$X_t = X_0 + \int_0^t (K_2 * \mu_r)(X_r)\mathrm{d}r + \sqrt{2}W_t.$$

- (H.-Röckner-Zhang 2024, AoP)
  Strong well-posedness holds when X_0 admits an $L^{1+}(\mathbb{R}^2)$ density.
- (Question): well-posedness for any $X_0 = x$?
- Consider the D(F)DSDE driven by fractional Brownian motion (fBM):

$$X_t = X_0 + \int_0^t (K_2 * \mu_r)(X_r^x)\mathrm{d}r + W_t^H.$$

DDSDEs driven by fBM

- Consider the DDSDE driven by fBM:

$$X_t = x + \int_0^t (K * \mu_r)(r, X_r)\mathrm{d}r + W_t^H.$$

- (Galeati-Harang-Mayorcas 2023, PTRF), (Galeati-Gerencsér 2022)
  Strong well-posedness holds when $K \in L_t^q L_x^p$ with

$$q \in (1,2], \quad \frac{Hd}{p} + \frac{1}{q} < 1 - H.$$

DDSDEs driven by fBM

- Consider the DDSDE driven by fBM:

$$X_t = x + \int_0^t (K * \mu_r)(r, X_r)\mathrm{d}r + W_t^H.$$

- (Galeati-Harang-Mayorcas 2023, PTRF), (Galeati-Gerencsér 2022)
  Strong well-posedness holds when $K \in L_t^q L_x^p$ with

$$q \in (1, 2], \quad \frac{Hd}{p} + \frac{1}{q} < 1 - H.$$

- (Biot-Sarvat law case): $K_2 \in L^{2-} \Rightarrow H < \frac{1}{4}$.
- (Question): well-posedness for $H \in (\frac{1}{4}, \frac{1}{2})$?

SDE driven by fBM

- Consider the following SDE driven fBM:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}W_t^H.$$

- Well-posedness: (Nualart-Ouknine 2002, SPA), (Lê 2020, EJP), (Galeati- Gubinelli 2021, RMI), (Butkovsky-Lê-Mytnik 2023), (Butkovsky-Gallay 2023), ...
- $b \in L_t^q L_x^p$ (strong well-posedness):

$$q \in (1, 2], \quad \frac{Hd}{p} + \frac{1}{q} < 1 - H.$$

- (Butkovsky-Mytnik 2024)
  Weak well-posedness: $b \in \mathbf{C}^\alpha$ with $\alpha > \frac{1}{2} - \frac{1}{2H}$.
- Biot-Savart law:

$$b = K_d \in L_x^{d-}.$$

SDE driven by fBM

- Consider the following SDE driven fBM:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}W_t^H.$$

- Our results: weak well-posedness when $b \in L_t^q L_x^p$ with
  - (Butkovsky-Gallay 2023)+ related entropy:

$$\frac{Hd}{p} + \frac{1-H}{q} < (1-H)^2;$$

  - A new type of Krylov estimate + Girsanov theorem:

$$p, q \geq \frac{1}{1-H}, \quad \frac{Hd}{p} + \frac{1-H}{q} < \frac{1}{2}.$$

- Open question: (weak or strong) well-posedness for the following scaling sub-critical condition:

$$\frac{Hd}{p} + \frac{1}{q} < (1-H).$$

FDSDEs driven by fBM

- Consider the FDSDE driven by fBM:

$$X_{s,t}^x = x + \int_s^t B(r, X_{s,r}^x, \mu_{r,T}^{\cdot}, \mu_{s,r}^{\cdot})\mathrm{d}r + W_t^H - W_s^H, \quad (s,t) \in \mathbb{D}_T. \tag{5}$$

FDSDEs driven by fBM

- Consider the FDSDE driven by fBM:

$$X^x_{s,t} = x + \int_s^t B(r, X^x_{s,r}, \mu^{\cdot}_{r,T}, \mu^{\cdot}_{s,r})\mathrm{d}r + W^H_t - W^H_s, \ \ (s,t) \in \mathbb{D}_T. \qquad (5)$$

- Localized L^p space: For $p \in [1, \infty]$, let $\mathbb{L}^p = L^p(\mathbb{R}^d)$ be the usual L^p-space.

$$\tilde{\mathbb{L}}^p := \left\{ f \in L^p_{loc}(\mathbb{R}^d) : |||f|||_p := \sup_{z \in \mathbb{R}^d} \|1_{|z-\cdot| \le 1} f\|_p < \infty \right\}.$$

- **Flow total variation distance**: the total variation distance $\|\cdot\|_{\mathrm{var}}$ is defined by

$$\|\mu - \nu\|_{\mathrm{var}} := \sup_{A \in \mathscr{B}(\mathbb{R}^d)} |\mu(A) - \nu(A)|.$$

Let $\mathscr{C}\mathscr{P}_0$ be the space of all continuous probability kernels $x \mapsto \mu^x$ w.r.t $\|\cdot\|_{\mathrm{var}}$ with the distance

$$\|\mu^{\cdot} - \nu^{\cdot}\|_{\mathscr{C}_{\mathrm{var}}} := \sup_{x \in \mathbb{R}^d} \|\mu^x - \nu^x\|_{\mathrm{var}}.$$

FDSDEs driven by fBM

($\mathbf{H}_1^s$) Let $(p_1, q_1) \in [\frac{1}{1-H}, \infty]^2$ satisfy $\frac{1}{q_1} + \frac{Hd}{p_1} < \frac{1}{2}$. There is a $\kappa_1 > 0$,

$$\|B(\cdot, \mu^\cdot, \nu^\cdot)\|_{\mathbb{L}_T^{q_1}\tilde{\mathbb{L}}^{p_1}} \leq \kappa_1, \ \ \forall \mu^\cdot, \nu^\cdot \in \mathcal{C}\mathscr{P}_0,$$

and there is a function $\ell \in \mathbb{L}_T^{q_1}$ such that for all $\mu_i^\cdot, \nu_i^\cdot \in \mathcal{C}\mathscr{P}_0$, $i = 1, 2$,

$$\|B(t, \cdot, \mu_1^\cdot, \nu_1^\cdot) - B(t, \cdot, \mu_2^\cdot, \nu_2^\cdot)\|_{p_1} \leq \ell(t)(\|\mu_1^\cdot - \mu_2^\cdot\|_{\mathcal{C}\mathscr{P}_0} + \|\nu_1^\cdot - \nu_2^\cdot\|_{\mathcal{C}\mathscr{P}_0}).$$

Theorem 2

Under ($\mathbf{H}_1^s$)*, there is a unique weak solution to FDSDE* (5).

NSE related to fBM

- Let ν_0 be a finite signed measure. Consider

$$X_t^x = x + \int_0^t B_{\nu_0}(X_s^x, \mu_s^\cdot)\mathrm{d}s + W_t^H, \tag{NSE1}$$

where

$$B_{\nu_0}(x, \mu^\cdot) = \int_{\mathbb{R}^2} (K_2 * \mu^y)(x)\nu_0(\mathrm{d}y), \quad K_2(x) = (x_2, -x_1)/(2\pi|x|^2).$$

Theorem 3

Let $H \in (0, \frac{1}{2})$. For any initial vorticity ν_0 being a finite singed measure, there is a unique strong solution $X_t^\cdot$ to FDSDE (NSE1)*. Moreover, if we let*

$$u^H(t, x) := \int_{\mathbb{R}^2} \mathbb{E}K_2(x - X_t^y)\nu_0(\mathrm{d}y) = B_{\nu_0}(x, \mu_t^\cdot),$$

then for any $p > 1$ and $j \in \mathbb{N}$, there is a constant $C > 0$ such that for all $t \in (0, T]$,

$$\|\nabla^j u^H(t)\|_p \lesssim_C t^{-2H(p-1)/p-(j-1)H}\|\nu_0\|_{\mathrm{var}}^j.$$

Moreover, for any $p \in (1, 2)$ and $\varepsilon > 0$ and $0 < s < t \leq T$,

$$\|u^H(t) - u^H(0)\|_p \lesssim_{p,\varepsilon} t^{[H(\frac{2}{p}-1)]\wedge[\frac{1-2H}{1-H}]-\varepsilon},$$

and

$$\|u^H(t) - u^H(s)\|_\infty \lesssim_{p,\varepsilon} s^{-2H}|t-s|^{\frac{H}{3}-\varepsilon}.$$

Open questions

- When $H \to 1/2$, whether $u^H \to u$, where u is the solution to the real NSE?
- How to understand u^H?
- Does u^H solve any PDE?

Backward FDSDEs with L^p kernels

- Consider the backward FDSDE:

$$X_{s,t}^x = x + \int_s^t B(r, X_{s,r}^x, \mu_{r,T}^{\cdot})\mathrm{d}r + \sqrt{2}(W_t - W_s). \tag{6}$$

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- Flow kernel space:
  - Let $\mathcal{KP}$ be the set of all probability kernels from $\mathbb{R}^d$ to $\mathcal{P}$.
  - For given $p \in [1, \infty]$, we introduce two subclasses

$$\mathcal{L}^p\mathcal{P} := \left\{ \mu^{\cdot} \in \mathcal{KP} : \|\mu^{\cdot}\|_p := \sup_{\|\phi\|_p \leqslant 1} \|\mu^{\cdot}(\phi)\|_p < \infty \right\}$$

  and

$$\tilde{\mathcal{L}}^p\mathcal{P} := \left\{ \mu^{\cdot} \in \mathcal{KP} : |\!|\!|\mu^{\cdot}|\!|\!|_p := \sup_{|\!|\!|\phi|\!|\!|_p \leqslant 1} |\!|\!|\mu^{\cdot}(\phi)|\!|\!|_p < \infty \right\}.$$

  - $\tilde{\mathcal{L}}^p\mathcal{P}$ is complete, but $\mathcal{L}^p\mathcal{P}$ is not complete.

Backward FDSDEs with L^p kernels

($\mathbf{H}_2^s$) For some $(p_1, q_1) \in (2, \infty)$ with $\frac{2}{q_1} + \frac{d}{p_1} < 1$ and $p_0 \in (1, p_1]$, there is a function $\ell \in \mathbb{L}_T^{q_1}$ such that for some $\beta \geqslant 0$ and all $t \in [0, T]$ and $\mu^{\boldsymbol{\cdot}} \in \tilde{\mathcal{L}}^{p_0}\mathcal{P}$,

$$\|B(t, \cdot, \mu^{\boldsymbol{\cdot}})\|_{p_1} \leq \ell(t)(1 + \|\mu^{\boldsymbol{\cdot}}\|_{p_0}^{\beta}),$$

and for all $t \in [0, T]$ and $\mu^{\boldsymbol{\cdot}}, \nu^{\boldsymbol{\cdot}} \in \tilde{\mathcal{L}}^{p_0}\mathcal{P}$,

$$\|B(t, \cdot, \mu^{\boldsymbol{\cdot}}) - B(t, \cdot, \nu^{\boldsymbol{\cdot}})\|_{p_1} \leq \ell(t)\|\mu^{\boldsymbol{\cdot}} - \nu^{\boldsymbol{\cdot}}\|_{p_0}.$$

Theorem 4

Under ($\mathbf{H}_2^s$), there is a time $T \in (0, 1)$ such that for each $x \in \mathbb{R}^d$, there is a unique strong solution to FDSDE (6) on $[0, T]$. When $\beta = 0$, the time can be taken arbitrarily large. Moreover, if we replace all the norms in ($\mathbf{H}_2^s$) by $\| \cdot \|_p$, then the conclusion still holds.

Backward FDSDEs related to NSE

- Recall the backward version of Navier-Stokes equation:

$$X^x_{s,t} = x + \int_s^t B_g(X^x_{s,r}, \mu^{\cdot}_{r,T})\mathrm{d}r + \sqrt{2}(W_t - W_s), \tag{7}$$

where

$$B_g(x, \mu^{\cdot}) = (K_2 * \mu^{\cdot}(g))(x).$$

Theorem 5

Let $g \in \mathbb{L}^{1+} = \cup_{p>1}\mathbb{L}^p$. There is a unique strong solution

$$(X^x_{s,t})_{0\le s\le t\le T, x\in\mathbb{R}^d}$$

to FDSDE (7).

Moreover, $u(s,x) := B_g(x, \mu^{\cdot}_{s,T}) \in C([0,T]; C^\infty_b(\mathbb{R}^2))$ solves the following backward Navier-Stokes equation:

$$\begin{cases} \partial_s u + \Delta u + u\cdot\nabla u + \nabla p = 0, \\ \mathrm{div} u = 0, \quad u(T) = K_2 * g. \end{cases}$$

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- Recall the backward version of Navier-Stokes equation:

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- (Open question): What about the case driven by fBM?

Thank you!