DK Book Proofs: Chapter 4 Lemma 1

Zimo Li

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Definition

x and y are jointly normally distributed random vectors with

$$E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$$
$$Var\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xx}^{'} & \Sigma_{yy} \end{bmatrix}$$

where Σ_{yy} is assumed to be a nonsingular matrix.

The conditional distribution of x given y is normal with mean vector $E[x|y]$ and variance matrix $Var[x|y]$ where:

$$E[x|y] = \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$$
$$Var[x|y] = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^{'}$$

Proof

Let $x = x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y) + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$.
Let $z = x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$.

Since z is a linear transformation of x and y, it is also normally distributed. We have:

$$
\begin{aligned}
E[z] &= E[x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)] \\
&= E[x] - \Sigma_{xy}\Sigma_{yy}^{-1}E[y - \mu_y] \\
&= \mu_x - \Sigma_{xy}\Sigma_{yy}^{-1}(\mu_y - \mu_y) \\
&= \mu_x \\
Var[z] &= Var[x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)] \\
&= Var[x] - Var[\Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)] \\
&= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yy}(\Sigma_{yy}^{-1})'\Sigma_{xy}' \\
&= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}'
\end{aligned}
$$

By calculating the covariance of z and y, we found that z and y are uncorrelated.

$$
\begin{aligned}
Cov[y, z] &= E[yz] - E[y]E[z] \\
&= E[xy - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)y] - \mu_y\mu_x \\
&= E[xy] - \Sigma_{xy}\Sigma_{yy}^{-1}E[y^2] + \Sigma_{xy}\Sigma_{yy}^{-1}\mu_y E[y] - \mu_x\mu_y \\
&= E[xy] - \Sigma_{xy}\Sigma_{yy}^{-1}(Var[y] + \mu_y^2) + \Sigma_{xy}\Sigma_{yy}^{-1}\mu_y E[y] - \mu_x\mu_y \\
&= E[xy] - \Sigma_{xy} - \mu_x\mu_y \\
&= (E[xy] - E[x]E[y]) - Var[xy] \\
&= 0
\end{aligned}
$$

Therefore z is distributed independently of y, and the conditional distribution of z given y is the same as its unconditional distribution.

We can then calculate the mean and covariance of the distribution of x given y:

$$
\begin{aligned}
E[x|y] &= E[z + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)|y] \\
&= E[z|y] + \Sigma_{xy}\Sigma_{yy}^{-1}(E[y|y] - E[\mu_y|y]) \\
&= E[z] + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y) \\
&= \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y) \\
Var[x|y] &= Var[z + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)|y] \\
&= Var[z|y] + \Sigma_{xy}\Sigma_{yy}^{-1}Var[y|y]\Sigma_{xy}'(\Sigma_{yy}^{-1})' \\
&= Var[z] \\
&= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}'
\end{aligned}
$$