Tangencies and the Conley index

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Abstract. We construct an algebraic method for detecting bifurcations of discrete dynamical systems using the Conley index theory. We define the transition matrix pair for discrete semidynamical systems, which detects bifurcations of connecting orbits. As an application, we give a sufficient condition for the occurrence of homoclinic and heteroclinic tangencies by showing that the occurrence of a heterodimensional cycle in the projectivization of a dynamical system implies the tangency of the original system.

1. *Introduction*

The Conley index, a generalization of the Morse index, is a topological tool designed for qualitative studies of dynamical systems (see Mischaikow-Mrozek [**8**] for general reference). It is defined for isolated invariant sets and takes the form of topological or algebraic object.

If an isolated invariant set has a Morse decomposition, that is, it is decomposed into the union of isolated invariant subsets and connecting orbits among these subsets, then there exist algebraic relations between the Conley index of the entire isolated invariant set and that of the invariant subsets. This can be taken as a generalization of the Morse inequality and stated in the form of the connection matrix (Franzosa [**3**]). Under some conditions, a non-zero entry of the connection matrix implies the existence of a connecting orbit.

Comparing connection matrices for two parameter values, we can also detect global bifurcations of connecting orbits in one-parameter families of flows. This is the purpose of the transition matrix. The simplest construction of transition matrix is due to Reineck [**11**] and referred to as the singular transition matrix. Similar to the case of connection matrix, a off diagonal non-zero entry of the transition matrix implies the occurrence of bifurcations. (see also Kokubu-Mischaikow-Oka [**4**] and McCord-Mischaikow [**6, 7**].)

The ideas described above were first defined for flows. The studies of the discrete Conley index, namely the Conley index for maps, appeared in late 80's (Mrozek [**9, 10**]) and rearranged by Szymczak [**14**] and Franks-Richeson [**2**] in categorical manners. Recently, Richeson [**12, 13**] gave the definition of the connection matrix pair for discrete semidynamical system, which takes the form of a pair of matrices, since the discrete Conley index is a pair of a space and its endomorphism, and hence is more complicated than that for flows.

The aim of this paper is to define the discrete transition matrix pair by reconstructing Reineck's method with Richeson's connection matrix pair, and to present some applications of that. For instance, we study homoclinic and heteroclinic tangencies using the transition matrix pair. Note that the Conley index itself does not carry the information on the differential since it is stable under perturbations in the C^0 topology. Therefore, we introduce the projectivization of a dynamical system, which involves only the directions of the tangent vectors, and apply the transition matrix pair to this system.

This paper consists of four sections. After the introduction, §2 is an overview of basic notions of the Conley index theory and the connection matrix pair according to Franks-Richeson [**2**] and Richeson [**12, 13**]. For simplicity, we define only the cohomological Conley index using relative cohomology groups. In §3, we discuss the change of connecting orbit structure when we let the drift go to zero, in parallel with Reineck [**11**]. In §4 we present the definition of the transition matrix pair for discrete semidynamical systems and examples in which the transition matrix pair actually finds the bifurcation of connecting orbits including the heterodimensional cycle. In §5, the final section, we discuss the method for detecting homoclinic and heteroclinic tangencies.

2. *Preliminary*

For simplicity, we use categorical notation. We denote by $\mathcal{V}$ the category of vector spaces and linear maps, and by $\mathcal{GV}$ the category of graded vector spaces and graded linear maps. Cochain complexes of vector spaces and cochain maps also form a category, which we denote by $\delta\mathcal{V}$. If C is a category, we define a new category $\mathtt{End}(C)$ as follows: the objects of $\mathtt{End}(C)$ are pairs (A, α) where A is an object of C and $\alpha : A \to A$ is a morphism of C; a morphism $\psi : (A, \alpha) \to (B, \beta)$ of $\mathtt{End}(C)$ is a morphism $\psi : A \to B$ of C such that $\psi \circ \alpha = \beta \circ \psi$. Define $\mathtt{Aut}(C)$ to be the full subcategory of $\mathtt{End}(C)$ that consists of pairs (A, α) where $\alpha : A \to A$ is an automorphism of C.

2.1. *The Conley Index*

Let X be a locally compact metric space and $f : X \to X$ a continuous map. We denote by $\operatorname{int} Y$, $\operatorname{cl} Y$ and ∂Y the interior, the closure and the boundary of Y, respectively.

A *complete orbit* of $x \in X$ with respect to f is a map $\sigma : \mathbb{Z} \to X$ such that $\sigma(0) = x$ and $f(\sigma(k)) = \sigma(k+1)$ for all $k \in \mathbb{Z}$. The *maximal invariant set* of $N \subset X$ is defined to be the set of $x \in N$ that has a complete orbit σ such that $\sigma(\mathbb{Z}) \subset N$ holds. We denote the maximal invariant set of N by $\operatorname{Inv}(N, f)$. Similarly, we define $\operatorname{Inv}^m(N, f) := \{f^m(y) \mid f^0(y), \ldots, f^{2m}(y) \in N\}$.

Proposition 2.1 ([**2**].) *If $N \subset X$ is compact, then* $\operatorname{Inv}(N, f) = \bigcap_{m=0}^{\infty} \operatorname{Inv}^m(N, f)$.

Definition. A compact set $N \subset X$ is an *isolating neighborhood* if $\operatorname{Inv}(N, f) \subset \operatorname{int} N$. If N is an isolating neighborhood, $S = \operatorname{Inv}(N, f)$ is called an *isolated invariant set*.

Definition. A *filtration pair* for an isolated invariant set S is a pair of compact sets (N, L) that satisfies the following:

(1) $\mathrm{cl}(N \setminus L)$ is an isolating neighborhood of S,

(2) L is a neighborhood of $N^- := \{x \in N \mid f(x) \notin N\}$, relative to N,

(3) $f(L) \cap \mathrm{cl}(N \setminus L) = \emptyset$.

There exists a filtration pair for any isolated invariant set (see [**2**], Theorem 3.6).

Let (N, L) be a filtration pair for S. Applying the Alexander-Spanier cohomology functor H^* to $f : (N, L) \to (N \cup f(L), L \cup f(L))$ and the inclusion map $i : (N, L) \to (N \cup f(L), L \cup f(L))$, we have

$$f^*, i^* : H^*(N \cup f(L), L \cup f(L)) \to H^*(N, L).$$

Since i^* is an isomorphism, we can define

$$F^*_{N,L} := f^* \circ (i^*)^{-1} : H^*(N, L) \to H^*(N, L).$$

Let us denote by $\varinjlim(A, \phi)$ the direct limit of the sequence $A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A \cdots$ where (A, ϕ) is an object of $\mathtt{End}(\mathcal{GV})$.

Definition. Let S be an isolated invariant set and (N, L) a filtration pair for S. The *cohomological Conley index* of S is defined by

$$\mathrm{Con}^*(S, f) = (CH^*(S, f), \chi^*(S, f)) := \varinjlim(H^*(N, L), F^*_{N,L}).$$

Note that $\chi^*(S, f)$ is an automorphism and $\mathrm{Con}^*(S, f)$ is therefore an object of $\mathtt{Aut}(\mathcal{GV})$. Theorem 4.3 of [**2**] shows that the cohomological Conley index is well-defined, that is, it does not depend on the choice of filtration pairs.

2.2. *The Morse Decomposition* As usual, we define the *ω-limit set* of $x \in X$ by $\omega(x) := \bigcap_{N>0} \mathrm{cl}\big(\bigcup_{n>N}\{f^n(x)\}\big)$. Since f may not be invertible, the same definition does not make sense for backward iterates. So, we define the *α-limit set* with respect to each complete orbit of x. Namely, for a complete orbit $\sigma : \mathbb{Z} \to X$, we define $\alpha(\sigma) := \bigcap_{N>0} \mathrm{cl}\big(\bigcup_{n>N}\{\sigma_f(-n)\}\big)$.

Let $A, B \subset X$. We denote the set

$$\{x \in X \setminus (A \cup B) \mid \exists \text{ a complete orbit } \sigma : \mathbb{Z} \to X \text{ s.t. } \alpha(\sigma) \subset A, \omega(x) \subset B\}.$$

by $C(A, B; X)$. Each complete orbit contained in $C(A, B; X)$ is called a *connecting orbit* from A to B.

If $A \subset X$ admits a compact neighborhood U such that $f(U) \subset \mathrm{int}\, U$ and $A = \bigcap_{n>0} f^n(U)$, then the set A is called an *attractor*. If S is an isolated invariant set, then an attractor with respect to the restricted map $f|_S : S \to S$ is called an attractor in S. We define the *dual repeller* of A to be the set $A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}$. If A is an attractor in S and A^* is the dual repeller, we call (A, A^*) an *attractor-repeller decomposition* of S.

We recall the definition of the ε-chain. Let $p, q \in \mathbb{Z}$ where $p < q$. A sequence $\{x_n\}_{n=p,p+1,\dots,q}$ is called an *ε-chain* from x_p to x_q if $d(f(x_n), x_{n+1}) < \varepsilon$ holds for $n = p, p+1, \dots, q-1$. Here d is the distance function on X.

Proposition 2.2 ([**2**].) *If (A, A^*) is an attractor-repeller decomposition of a compact invariant set S, then*

(1) $S = A \cup A^* \cup C(A^*, A; S)$,

(2) *there exists* $\varepsilon > 0$ *such that there is no* ε*-chain from A to* A^*,

(3) A^* *and A are disjoint isolated invariant sets.*

Let $\mathcal{P}$ be a finite set. A relation $<$ on $\mathcal{P}$ is a *partial order* if $p < q$ and $q < r$ implies $p < r$, and $p < p$ never holds. When we fix a partial order $<$ on $\mathcal{P}$, we may write $(\mathcal{P}, <)$.

A subset I of a partially ordered set $(\mathcal{P}, <)$ is called an *interval* if $p < r < q$ and $p, q \in I$ imply $r \in I$. An interval I is said to be *attracting* if $r < q$ and $q \in I$ imply $r \in I$. We denote by $\mathcal{I} = \mathcal{I}(\mathcal{P}, <)$ and $\mathcal{A} = \mathcal{A}(\mathcal{P}, <)$ the set of intervals and the set of attracting intervals, respectively.

An n-tuple of mutually disjoint intervals $(I_1, \ldots, I_n)$ such that $I_1 \cup \cdots \cup I_n \in \mathcal{I}$ is said to be *adjacent* if $i < j$ and $p \in I_i, q \in I_j$ imply $q \not< p$. We denote by $\mathcal{I}_n = \mathcal{I}_n(\mathcal{P}, <)$ the set of adjacent n-tuples. If $(I_1, \ldots, I_n) \in \mathcal{I}_n$, then the interval $I_1 \cup \cdots \cup I_n$ is denoted by $I_1 I_2 \cdots I_n$.

Now we are ready to define the Morse decomposition.

Definition. Let $\mathcal{P}$ be a finite set and S an isolated invariant set. A collection of mutually disjoint isolated invariant sets $\mathcal{M}(\mathcal{P}) = \{M(p) \mid p \in \mathcal{P}\}$ is called a *Morse decomposition* for S if there exists a partial order $<$ on $\mathcal{P}$ such that either of the following conditions holds for each complete orbit $\sigma : \mathbb{Z} \to S$:

(1) there exists $p \in \mathcal{P}$ such that $\sigma(\mathbb{Z}) \subset M(p)$,

(2) there exist $p, q \in \mathcal{P}$ such that $p < q$ and $\omega(\sigma(0)) \subset M(p)$, $\alpha(\sigma) \subset M(q)$.

If we consider a Morse decomposition with a partial order fixed, then we write $\mathcal{M}(\mathcal{P}, <)$. Moreover, we may write $\mathcal{M}(\mathcal{P}, <, S, f)$ to specify the setting. We call each $M(p) \in \mathcal{M}(\mathcal{P})$ a Morse set. Among partial orders that satisfy the definition of the Morse decomposition, the weakest one is called the map-defined order. Note that $p < q$ holds with respect to the map-defined order if and only if $C(M(q), M(p); S) \neq \emptyset$.

For each interval I, we define

$$M(I) := \bigcup_{p \in I} M(p) \cup \bigcup_{q, r \in I} C(M(q), M(r); S).$$

Proposition 2.3. *Let $\mathcal{M}(\mathcal{P}, <)$ be a Morse decomposition for S. Then,*

(1) *$M(I)$ is an isolated invariant set for each $I \in \mathcal{I}(\mathcal{P}, <)$.*

(2) *$M(I)$ is an attractor in S for each $I \in \mathcal{A}(\mathcal{P}, <)$.*

(3) *If $(I, J) \in \mathcal{I}_2$ then $(M(J), M(I))$ is an attractor-repeller decomposition for the isolated invariant set $M(IJ)$.*

Definition. Let $\mathcal{M}(\mathcal{P}, <)$ be a Morse decomposition. A collection of compact sets $\mathcal{N}(\mathcal{P}, <) = \{N(I) \subset X \mid I \in \mathcal{A}(\mathcal{P}, <)\}$ is called a *filtration* for $\mathcal{M}(\mathcal{P}, <)$ if for each pair $I, J \in \mathcal{A}(\mathcal{P}, <)$ we have the following conditions:

(1) $(N(I), N(\emptyset))$ is a filtration pair for $M(I)$,

(2) $N(I) \cap N(J) = N(I \cap J)$,

(3) $N(I) \cup N(J) = N(I \cup J)$.

THEOREM 2.4 ([**2, 13**].) *Every Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ for $S := \mathrm{Inv}(N, f)$ has a stable filtration $\mathcal{N}(\mathcal{P}, <)$. Namely, there exists a neighborhood $\mathcal{U}$ of f in C^0-topology such that if $g \in \mathcal{U}$ then $\mathcal{N}(\mathcal{P}, <)$ is a filtration for $\mathcal{M}(\mathcal{P}, <, \mathrm{Inv}(N, g), g)$.*

2.3. *The Connection Matrix Pair* To begin with, we recall the following theorem.

THEOREM 2.5 ([**12, 13**].) *If (A, R) is an attractor-repeller decompositions of an isolated invariant set S, then there exists the following long exact sequence in $\mathtt{Aut}(\mathcal{V})$:*

$$\cdots \xrightarrow{\delta^*} \mathrm{Con}^k(R) \longrightarrow \mathrm{Con}^k(S) \longrightarrow \mathrm{Con}^k(A) \xrightarrow{\delta^*} \mathrm{Con}^{k+1}(R) \longrightarrow \cdots .$$

If $A \cup R = S$, then the additivity of the Conley index implies $\mathrm{Con}^*(S) \cong \mathrm{Con}^*(A) \oplus \mathrm{Con}^*(R)$ in $\mathtt{Aut}(\mathcal{GV})$. Therefore the following holds.

COROLLARY 2.6. *If $\delta^* \neq 0$, $CH^*(S) \not\cong CH^*(A) \oplus CH^*(R)$ or $\chi^*(S) \not\cong \chi^*(A) \oplus \chi^*(R)$, then $C(R, A; S) \neq \emptyset$.*

Definition. A *cochain complex braid with endomorphism* is a collection $\mathcal{C} = \{(C(I), c(I)) \mid I \in \mathcal{I}\}$ of objects in $\mathtt{End}(\delta\mathcal{V})$ that satisfy the following:

(1) For any $(I, J) \in \mathcal{I}_2$ there exist cochain maps ρ, ι such that the following sequence is exact.
$$0 \longrightarrow (C(J), c(J)) \xrightarrow{\rho} (C(IJ), c(IJ)) \xrightarrow{\iota} (C(I), c(I)) \longrightarrow 0.$$

(2) For any $(I, J, K) \in \mathcal{I}_3$ the following diagram, which consists of the short exact sequence given in (1), commutes.

$$\begin{tikzcd}
0 & & 0 & & \\
& (C(I),c(I)) \arrow[ul] \arrow[ur] & & 0 & \\
& & (C(IJ),c(IJ)) \arrow[ul] \arrow[ur] & & 0 \\
& (C(IJK),c(IJK)) \arrow[uu, bend left] \arrow[ur] & & (C(J),c(J)) \arrow[ul] \arrow[ur] & \\
& & (C(JK),c(JK)) \arrow[ul] \arrow[ur] & & 0 \arrow[ul] \\
& (C(K),c(K)) \arrow[uu, bend left] \arrow[ur] & & 0 \arrow[ul] & \\
0 \arrow[ur] & & 0 \arrow[ul] & &
\end{tikzcd}$$

Definition. A *graded module braid with endomorphism* is a collection $\mathcal{H} = \{(H(I), a(I)) \mid I \in \mathcal{I}\}$ of objects in $\mathtt{End}(\mathcal{GV})$ that satisfy the following:

(1) For any $(I, J) \in \mathcal{I}_2$ there exist homomorphisms $\delta^*, \rho^*, \iota^*$ such that the following long sequence is exact.

$$\cdots \xrightarrow{\delta^{n-1}} (H^n(J), a^n(J)) \xrightarrow{\rho^n} (H^n(IJ), a^n(IJ)) \xrightarrow{\iota^n} (H^n(I), a^n(I)) \xrightarrow{\delta^n} \cdots .$$

(2) For any $(I, J, K) \in \mathcal{I}_3$ the following diagram, which consists of the long exact sequence given in (1), commutes.

$$\begin{tikzcd}
{} & & {} & & {} \\
(H^k(I), a^k(I)) \arrow[u, bend left] \arrow[urr] & & & & (H^{k+1}(K), a^{k+1}(K)) \arrow[u, bend right] \arrow[ull] \\
& & (H^k(IJ), a^k(IJ)) \arrow[ull] \arrow[urr] & & \\
(H^k(IJK), a^k(IJK)) \arrow[uu, bend left] \arrow[urr] & & & & (H^k(J), a^k(J)) \arrow[uu, bend right] \arrow[ull] \\
& & (H^k(JK), a^k(JK)) \arrow[ull] \arrow[urr] & & \\
(H^k(K), a^k(K)) \arrow[uu, bend left] \arrow[urr] & & & & (H^{k-1}(I), a^{k-1}(I)) \arrow[uu, bend right] \arrow[ull] \\
{} \arrow[u, bend left] & & {} \arrow[ull] \arrow[urr] & & {} \arrow[u, bend right]
\end{tikzcd}$$

The naturality of the connecting homomorphism implies the following.

PROPOSITION 2.7 ([**12, 13**].) *If we apply the cohomology functor to a cochain complex braid with endomorphism $\mathcal{C}$, we get a graded module braid with endomorphism, which we denote by $H\mathcal{C}$.*

Definition. Suppose we have two cochain complex braids with endomorphism $\mathcal{C} = \{(C(I), c(I)) \mid I \in \mathcal{I}\}$ and $\mathcal{C}' = \{(C'(I), c'(I)) \mid I \in \mathcal{I}\}$. A cochain complex braid endomorphism $\Psi : \mathcal{C} \to \mathcal{C}'$ is a collection $\{\psi(I) : (C(I), c(I)) \to (C'(I), c'(I)) \mid I \in \mathcal{I}\}$ of morphisms in $\mathtt{End}(\delta\mathcal{V})$ such that for each $(I, J) \in \mathcal{I}_2$ the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (C(J), c(J)) \arrow[r] \arrow[d, "\psi(J)"'] & (C(IJ), c(IJ)) \arrow[r] \arrow[d, "\psi(IJ)"'] & (C(I), c(I)) \arrow[r] \arrow[d, "\psi(I)"'] & 0 \\
0 \arrow[r] & (C(J), c(J)) \arrow[r] & (C(IJ), c(IJ)) \arrow[r] & (C(I), c(I)) \arrow[r] & 0.
\end{tikzcd}$$

If each $\psi(I)$ is an isomorphism, Ψ is called an isomorphism.

In a similar way, graded module braid endomorphisms and isomorphisms are defined. Passing to cohomology, a cochain complex braid isomorphism induces a graded module braid isomorphism.

Proposition 2.8 ([**12, 13**].) *Let* $\Psi : \mathcal{H} \to \mathcal{H}'$ *be a graded module braid endomorphism. If* $\psi^*(p)$ *is an isomorphism for each* $p \in \mathcal{P}$*, then* Ψ *is a graded module braid isomorphism.*

Let S be an isolated invariant set that has a filtration pair (N, L). Using the isomorphism $i^\sharp : C^\sharp(N \cup f(L), L \cup f(L)) \to C^\sharp(N, L)$, we define $F^\sharp_{N,L} := f^\sharp \circ {i^\sharp}^{-1}$ and $(LC(S), Lf(S)) := \varinjlim(C^\sharp(N, L), F^\sharp_{N,L})$. Here $C^\sharp$ denotes the Alexander-Spanier cochain. Suppose we have a Morse decomposition $\mathcal{M}(\mathcal{P}, <)$ and let $\mathcal{N}(\mathcal{P}, <)$ be a filtration for $\mathcal{M}(\mathcal{P}, <)$. We define

$$C(\mathcal{M}) := \{(LC(M(I)), Lf(M(I))) \mid I \in \mathcal{I}\},$$
$$\mathcal{H}(\mathcal{M}) := \{(CH^*(M(I)), \chi^*(M(I))) \mid I \in \mathcal{I}\}.$$

Since $\varinjlim$ commutes with H^* ([**5**], Appendix A.10), we have the following.

Proposition 2.9 ([**12, 13**].) *If* $\mathcal{M}(\mathcal{P}, <)$ *is a Morse decomposition,* $C(\mathcal{M})$ *is a cochain complex braid with endomorphism,* $\mathcal{H}(\mathcal{M})$ *is a graded module braid with endomorphism. Furthermore,* $\mathcal{H}(\mathcal{M}) \cong HC(\mathcal{M})$ *as graded module braids with endomorphism.*

Opposite to Proposition 2.7, sometimes a graded module braid with endomorphism yields a cochain complex braid with endomorphism. Let $\mathcal{H} = \{(H(I), a(I)) \mid I \in \mathcal{I}\}$ be a graded module braid with endomorphism and define

$$C\Delta(I) := \bigoplus_{p \in I} H(p),$$

where we regard $p \in \mathcal{P}$ as an interval $\{p\} \in \mathcal{I}$. Then, if $A : C\Delta(I) \to C\Delta(I)$ is a linear map, we can write A in the form of a $|\mathcal{P}| \times |\mathcal{P}|$ matrix $A = (A_{pq})_{p,q \in \mathcal{P}}$ where each $A_{pq} : H(q) \to H(p)$ is a linear map. For $I \in \mathcal{I}$, define $A(I) := (A_{pq})_{p,q \in I} : C\Delta(I) \to C\Delta(I)$.

Definition. Let $\mathcal{P} = (\mathcal{P}, <)$ be a partially ordered set. A matrix of linear maps $A = (A_{pq})_{p,q \in \mathcal{P}}$ is said to be *strictly lower triangular* if $A_{pq} \neq 0$ implies $p > q$ and *lower triangular* if $A_{pq} \neq 0$ implies $p \geq q$. If A is of degree $+1$ and $A^2 = 0$, then A is called a *coboundary operator*.

Now we suppose $\Delta = (\Delta_{pq})_{p,q \in \mathcal{P}} : C\Delta(\mathcal{P}) \to C\Delta(\mathcal{P})$ is a strictly lower triangular coboundary operator.

Proposition 2.10 ([**12, 13**].) *For each* $I \in \mathcal{I}$*, the submatrix* $\Delta(I)$ *is a strictly lower triangular coboundary operator, hence,* $(C\Delta(I), \Delta(I))$ *is a cochain complex.*

In addition, we suppose $a = (a_{pq})_{p,q \in \mathcal{P}} : C\Delta(\mathcal{P}) \to C\Delta(\mathcal{P})$ is a lower triangular cochain map of degree 0, that is, $\Delta \circ a = a \circ \Delta$ and each $a_{p,q}$ is of degree 0.

Proposition 2.11 ([**12, 13**].) *For each* $I \in \mathcal{I}$*, the submatrix* $a(I)$ *is a lower triangular cochain map of degree* 0 *with respect to the coboundary operator* $\Delta(I)$*.*

The preceding two propositions show that if we define

$$C\Delta := \{((C\Delta(I), \Delta(I)), a(I)) \mid I \in \mathcal{I}\},$$

then $C\Delta$ is a cochain complex braid with endomorphism. Indeed, you have ρ being an inclusion and ι being a projection since $C\Delta(IJ) = C\Delta(I) \oplus C\Delta(J)$. Applying Proposition 2.7 to $C\Delta$, we obtain a graded module braid with endomorphism $HC\Delta$, which we denote by $\mathcal{H}\Delta$.

Definition. A *connection matrix pair* for a graded module braid $\mathcal{H}$ is a pair (Δ, a) of a strictly lower triangular coboundary operator Δ and lower triangular cochain map a such that $\mathcal{H}\Delta \cong \mathcal{H}$ holds.

Definition. Let $\mathcal{M}(\mathcal{P}, <)$ be a Morse decomposition. We define a *connection matrix pair* for $\mathcal{M}(\mathcal{P}, <)$ to be a connection matrix pair for $\mathcal{H}\Delta(\mathcal{M})$.

The existence of a connection matrix pair is assured by the following:

THEOREM 2.12 ([**12, 13**].) *Suppose $\mathcal{H}$ is a graded module braid with endomorphism. If there exists a cochain complex braid C such that $\mathcal{H} \cong HC$, then there exists a connection matrix pair for $\mathcal{H}$.*

Since $\mathcal{H}(\mathcal{M}) \cong HC(\mathcal{M})$, we obtain the following theorem.

THEOREM 2.13. *For any isolated invariant set S and its Morse decomposition $\mathcal{M}(\mathcal{P}, <)$, there exists a connection matrix pair for $\mathcal{M}(\mathcal{P}, <)$.*

3. *Connections in One-parameter Families*

This section is devoted to proving Theorem 3.12, which will be used in later sections. We consider a one-parameter family $\{f_\lambda : X \to X\}_{\lambda\in\Lambda}$ with a continuing Morse decomposition. First, we put an artificial drift on the parameter space. It will be shown that there exists a corresponding Morse decomposition in $X \times \Lambda$. Then we investigate the change of connecting orbit structure of this Morse decomposition when we let the artificial drift go to zero. Finally, the occurrence of the bifurcation of the connecting orbits is derived from the argument above.

This section is similar to §3 of [**11**]. The main difference is that Lemma 3.6 of [**11**] is no longer true for the discrete case since connecting orbits are not connected. Hence we need to reprove Lemma 3.6 of this paper.

We begin by recalling the definition of the Hausdorff metric. Let X be a compact metric space and d the metric on X. The set of non-empty, closed subsets of X is denoted by $\mathcal{F}(X)$, and the ε-neighborhood of $Y \subset X$ is denoted by $N_\varepsilon(Y)$. Define $h, \rho : \mathcal{F}(X)\times\mathcal{F}(X) \to \mathbb{R}$ by

$$h(A, B) = \inf\{\varepsilon \mid A \subset N_\varepsilon(B)\}, \quad \rho(A, B) = \max\{h(A, B), h(B, A)\}.$$

Then $\mathcal{F}(X)$ is a compact metric space with ρ, which is called the Hausdorff metric. The following lemma is immediate.

LEMMA 3.1. *Let $\{A_n\}$ be a convergent sequence in $\mathcal{F}(X)$. Then*

$$\lim_{n\to\infty} A_n = \{x \in X \mid \exists\{x_n\} \subset A_n \text{ such that } \lim_{n\to\infty} x_n = x\}.$$

We suppose the parameter space Λ is a open interval containing $[0, 1]$ and fix a compact neighborhood $\bar{\Lambda}$ of $[0, 1]$ in Λ. Let $\pi_X : X\times\Lambda \to X$ and $\pi_\Lambda : X\times\Lambda \to \Lambda$ be the projections.

We introduce a map g_ϵ defined on Λ by $g_\epsilon(\lambda) := \lambda + \epsilon\lambda(\lambda - 1)$, which is called a slow-drift on the parameter space. When we talk of the slow-drift, it is always assumed that $\epsilon \geq 0$ is sufficiently small so that g_ϵ is strictly increasing on Λ. Define $F_\epsilon : X\times\Lambda \to X\times\Lambda$ by

$$F_\epsilon(x, \lambda) := (f_\lambda(x),\ g_\epsilon(\lambda)).$$

Note that F_0 coincides with $f \times \mathrm{id}$ and preserves λ.

We say that a family of isolated invariant sets $\{S_\lambda\}$ continues over Λ if there exists an isolated invariant set $\hat{S} \subset X \times \Lambda$ with respect to F_0 such that $S_\lambda = \hat{S} \cap (X \times \{\lambda\})$. Similarly, we define the continuation of Morse decompositions as follows:

Definition. A parameterized family $\mathcal{M}(\mathcal{P}, \lambda) = \{M_\lambda(p) \mid p \in \mathcal{P}\}$ of Morse decompositions *continues over* Λ if each $\{M_\lambda(p)\}$ continues over Λ and each $\mathcal{M}(\mathcal{P}, \lambda)$ is a Morse decomposition for S_λ where $\{S_\lambda\}$ is a family of isolated invariant sets that continues over Λ.

For the sake of simplicity, we suppose that N is an isolating neighborhood with respect to f_λ for every $\lambda \in \Lambda$ and define

$$S_\lambda := \mathrm{Inv}(N, f_\lambda).$$

Then $\{S_\lambda\}$ is obviously a family of isolated invariant sets continuing over Λ. Furthermore, we suppose the family S_λ has a family of Morse decompositions $\mathcal{M}(\mathcal{P}, \lambda)$ continuing over Λ. We denote by $<_\lambda$ the map-defined order on $\mathcal{P}$ with respect to f_λ.

Lemma 3.2. *If $\epsilon > 0$ is sufficiently small, $N \times \bar{\Lambda}$ is an isolating neighborhood with respect to F_ϵ.*

Proof. Suppose the lemma is false. Then there exists a sequence $\epsilon_n \to 0$ such that $N \times \bar{\Lambda}$ is not an isolating neighborhood with respect to F_{ϵ_n} for all n. By definition, $\partial(N \times \bar{\Lambda}) \cap \mathrm{Inv}(N \times \bar{\Lambda}, F_{\epsilon_n}) \neq \emptyset$. So, we can find a complete orbit $\sigma_n : \mathbb{Z} \to N \times \bar{\Lambda}$ such that $\sigma_n(0) \in \partial(N \times \bar{\Lambda})$ for each n. Since $\mathrm{Inv}(N \times \bar{\Lambda}, F_{\epsilon_n}) \subset N \times [0,1]$, we have $\sigma_n(0) \in \partial N \times [0,1]$. By compactness, we can choose a convergent subsequence of $\{\sigma_n(0)\}$. Change the label for $\sigma_n(0)$ so that $\{\sigma_n(0)\}$ itself is convergent, and denote $z_0 = (x_0, \lambda_0) := \lim_{n\to\infty} \sigma_n(0)$. Clearly $x_0 \in \partial N$. We will show that $z_0 \in \mathrm{Inv}(N \times [0,1], F_0|_{X\times[0,1]})$, which implies $x_0 \in \partial N \cap \mathrm{Inv}(N, f_{\lambda_0})$, a contradiction to the assumption that N is an isolating neighborhood for all λ. By Proposition 2.1, it will suffice to show $z_0 \in \mathrm{Inv}^m(N \times [0,1], F_0|_{X\times[0,1]})$ for all $m \geq 0$. Let z_{-1} be the limit of a convergent subsequence of $\{\sigma_n(-1)\}$. Change the label of σ_n so that $\sigma_n(-1) \to z_{-1}$. Repeating this process, we define z_k to be the limit of $\{\sigma_n(k)\}$ for $-m \leq k \leq m$. For each n and $k = -m, -m+1, \ldots, m-1$, we have

$$\begin{aligned} d(F_0(z_k), z_{k+1}) &\leq d(F_0(z_k), F_0(\sigma_n(k))) + d(F_0(\sigma_n(k)), z_{k+1}) \\ &\leq d(F_0(z_k), F_0(\sigma_n(k))) + d(F_0(\sigma_n(k)), F_{\epsilon_n}(\sigma_n(k))) + d(F_{\epsilon_n}(\sigma_n(k)), z_{k+1}). \end{aligned}$$

Here d is the distance function on $X \times \Lambda$. Choosing n large, the first term and the third term on the right hand side go to zero because

$$F_0(z_k) = F_0(\lim_{n\to\infty} \sigma_n(k)) = \lim_{n\to\infty} F_0(\sigma_n(k)), \text{ and}$$
$$z_{k+1} = \lim_{n\to\infty} \sigma_n(k+1) = \lim_{n\to\infty} F_{\epsilon_n}(\sigma_n(k)).$$

Also, the second term goes to zero because F_{ϵ_n} uniformly converges to F_0 on $X \times [0,1]$. It follows that $d(F_0(z_k), z_{k+1}) = 0$, hence $F_0(z_k) = z_{k+1}$ for $k = -m, -m+1, \ldots, m-1$. This implies $z_0 \in \mathrm{Inv}^m(N \times [0,1], F_0)$. □

In the rest of this section, we suppose that $\epsilon > 0$ is sufficiently small so that Lemma 3.2 holds and denote

$$\hat{S}_\epsilon := \mathrm{Inv}(N \times \Lambda, F_\epsilon).$$

We denote $A_\lambda := A \times \{\lambda\} \subset X \times \Lambda$ for $A \subset X$.

LEMMA 3.3. *If A is an isolated invariant set with respect to f_0, then A_0 is an isolated invariant set with respect to F_ϵ. Similarly, if A is an isolated invariant set with respect to f_1, then A_1 is an isolated invariant set with respect to F_ϵ.*

Proof. We prove only the statement for A_0. The proof for A_1 is the same. Let K be an isolating neighborhood such that $\mathrm{Inv}(K, f_0) = A$. Define $K' := K \times [-\gamma, \gamma] \subset X \times \Lambda$ for sufficiently small $\gamma > 0$. Let $(x, \lambda) \in K'$. We have $\mathrm{Inv}(K', F_\epsilon) \subset X_0$ because if $\lambda \neq 0$, then there exists no complete orbit of (x, λ) in K'. Since F_ϵ equals to f_0 on X_0, $\mathrm{Inv}(K', F_\epsilon) = \mathrm{Inv}(K_0, f_0) = A_0$. This proves the statement. □

LEMMA 3.4. *Suppose $\{M_0(p) \mid p \in (\mathcal{P}, <_0)\}$ is a Morse decomposition for S_0 and $\{M_1(r) \mid q \in (Q, <_1)\}$ a Morse decomposition for S_1. Then the collection $\{M(p) \mid p \in (\mathcal{P}, <_0)\} \cup \{M(q) \mid q \in (Q, <_1)\}$ is a Morse decomposition for $\hat{S}_\epsilon$ with respect to F_ϵ. In fact, we have an admissible partial order $\ll$ in $\mathcal{P} \cup Q$ given by*

$$\begin{aligned} p \ll q \quad & \text{if } p \in \mathcal{P} \text{ and } q \in Q, \\ p \ll p' \quad & \text{if } p, p' \in \mathcal{P} \text{ and } p <_0 p', \\ q \ll q' \quad & \text{if } q, q' \in Q \text{ and } q <_1 q'. \end{aligned}$$

Proof. It is evident that $\ll$ is a partial order on $\mathcal{P} \cup Q$. Lemma 3.3 shows that each $M_i(p)$ is an isolated invariant set with respect to F_ϵ. We claim that every complete orbit contained in $\hat{S}_\epsilon$ satisfies either (1) or (2) of the definition of the Morse decomposition. with respect to $(\mathcal{P} \cup Q, \ll)$.

If a complete orbit σ satisfies $\sigma(\mathbb{Z}) \cap S_0 \neq \emptyset$ then $\sigma(\mathbb{Z}) \subset S_0$ follows from the form of F_ϵ. Hence we have (1) or (2) since $\{M_0(p) \mid p \in (\mathcal{P}, <_0)\}$ is a Morse decomposition for S_0. Now our claim follows in this case since $p <_0 p'$ implies $p \ll p'$. The proof for the case $\sigma \cap S_1 \neq \emptyset$ is similar.

Next, take a complete orbit $\sigma(k) = (x(k), \lambda(k))$ such that $0 < \lambda(k) < 1$ holds for all k. Clearly, $\omega(\sigma) \subset S_0$ and $\alpha(\sigma) \subset S_1$.

We will show that there exist $p \in \mathcal{P}$ and $q \in Q$ such that $\omega(\sigma) \subset M_0(p)$ and $\alpha(\sigma) \subset M_1(q)$. This proves this lemma since $p \ll q$ by definition. We will prove $\alpha(\sigma) \subset M_1(q)$. The proof for $\omega(\sigma) \subset M_0(p)$ is similar. Note that each point in $\alpha(\sigma)$ has its complete orbit in $\alpha(\sigma)$ since $\alpha(\sigma)$ is f_1-invariant. In addition, it follows from the definition of Morse decomposition that the α-limit set of a point in $\alpha(\sigma)$ is contained in a Morse set. Hence, there exists $q \in Q$ such that $\alpha(\sigma) \cap M_1(q) \neq \emptyset$. We claim that $\alpha(\sigma) \subset M_1(q)$. If this does not hold, then there exists another $q' \in Q$ such that $\alpha(\sigma) \cap M_1(q') \neq \emptyset$. We may assume $q' \not<_1 q$. Define $I := \{r \in Q \mid r \leq_1 q\}$. Then I is an attracting interval and $q' \notin I$. Since $(I, Q \setminus I) \in \mathcal{I}_2(Q, <_1)$, Proposition 2.3 shows that $(M_1(I), M_1(Q \setminus I))$ is an attractor-repeller decomposition for S_1. We will show that for any $\varepsilon > 0$ there exists an ε-chain from $M_1(q) \subset M_1(I)$ to $M_1(q') \subset M_1(Q \setminus I)$ with respect to f_1. First, there exists $\delta > 0$

such that if $d(x, x') \leq \delta$ then $d(F_0(x), F_0(x')) < \varepsilon/3$ because F_0 is uniformly continuous on $N \times [0, 1]$. Next, there exists $0 < \lambda' < 1$ such that if $\lambda' \leq \lambda$ then $d(f_1(x), f_\lambda(x)) < \varepsilon/3$ holds for all $x \in N$ because F_ϵ converges uniformly to F_0. It follows from the form of g_ϵ that there exists a negative integer K' such that if $k \leq K'$ then $\lambda' \leq \lambda(k)$. Since $\alpha(\sigma) \subset S_1$, there exists a negative integer K'' such that if $k \leq K''$ then $d(\sigma(k), S_1) < \min(\varepsilon/3, \delta)$. Let $K = \min(K', K'')$. We can find $k \leq K$ such that $d(\sigma(k), M_1(q)) < \min(\varepsilon/3, \delta)$ since $\alpha(\sigma) \cap M_1(q)$. Similarly, we can find $k' < k$ such that $d(\sigma(k'), M_1(q')) < \min(\varepsilon/3, \delta)$. If $k' \leq i \leq k$, there exists a point in S_1 that is also in the $\min(\varepsilon/3, \delta)$-neighborhood of $\sigma(i)$ because $k \leq K$. Let x_i be such a point. Since F_0 coincides with f_1 on S_1 and $F_\epsilon(\sigma(i)) = \sigma(i + 1)$, we have

$$\begin{aligned} d(f_1(x_i), x_{i+1}) &\leq d(f_1(x_i), F_\epsilon(\sigma(i))) + d(F_\epsilon(\sigma(i)), x_{i+1}) \\ &\leq d(f_1(x_i), f_1(\sigma(i))) + d(F_0(\sigma(i)), F_\epsilon(\sigma(i))) + d(\sigma(i + 1), x_{i+1}) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which implies that $\{x_i\}_{i=k',k'+1,\dots,k}$ is an ε-chain from $M_1(q) \subset M_1(I)$ to $M_1(q') \subset M_1(Q \setminus I)$. This contradicts (2) of Proposition 2.2. □

We have supposed that there exists $\mathcal{M}(\mathcal{P}, \lambda)$, a family of Morse decompositions for $\{S_\lambda\}$. So, we obtain a Morse decomposition for $\hat{S}_\epsilon$ by the preceding lemma. In this case $\mathcal{P} = Q$, but $<_0 \neq <_1$ in general. We denote this Morse decomposition by $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$.

When $\epsilon = 0$, Lemma 3.2 is no longer true and we can not construct a Morse decomposition for $\epsilon = 0$. So, we consider what happens to $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$ if we let $\epsilon \to 0$, hoping that this consideration gives us some information about the dynamics with respect to F_0.

In particular, we are interested in the behavior of connecting orbits. Let $\epsilon_n \to 0$ be a convergent sequence and suppose $C(M_1(q), M_0(p); \hat{S}_{\epsilon_n}) \neq \emptyset$ for all n. Choose a point from $C(M_1(q), M_0(p); \hat{S}_{\epsilon_n})$. By definition it has at least one complete orbit contained in $\hat{S}_{\epsilon_n}$, which we denote by c_n. Let $\bar{c}_n := \mathrm{cl}(c_n) \in \mathcal{F}(N \times [0, 1])$. Since $\mathcal{F}(N \times [0, 1])$ is compact with the Hausdorff metric, there exists a convergent subsequence of $\bar{c}_n$. Relabeling $\bar{c}_n$, we assume that $\bar{c}_n$ is convergent and let $\bar{c} := \lim_{n\to\infty} \bar{c}_n$.

Lemma 3.5. *The set $\bar{c}$ is F_0-invariant.*

Proof. Since $\bar{c}, \bar{c}_n$ are compact, so are their images under F_0 and F_{ϵ_n}. By the triangle inequality for the Hausdorff metric,

$$d(F_0(\bar{c}), \bar{c}) \leq d(F_0(\bar{c}), F_0(\bar{c}_n)) + d(F_0(\bar{c}_n), F_{\epsilon_n}(\bar{c}_n)) + d(F_{\epsilon_n}(\bar{c}_n), \bar{c})$$

holds for each n. The third term on the right hand side equals $d(\bar{c}_n, \bar{c})$ because $\bar{c}_n$ is F_{ϵ_n}-invariant. The first term and the third term go to zero since $\bar{c}_n \to \bar{c}$ as $n \to \infty$. Also, the second term goes to zero since F_{ϵ_n} converges to F_0 uniformly on $N \times [0, 1]$. It follows that $d(F(\bar{c}), \bar{c}) = 0$, that means $F(\bar{c}) = \bar{c}$. □

Lemma 3.6. *The set $\pi_\Lambda(\bar{c})$ equals* $[0, 1]$.

Proof. We regard the projection π_Λ as $\pi_\Lambda : \mathcal{F}(N \times [0,1]) \ni A \mapsto \pi_\Lambda(A) \in \mathcal{F}[0,1]$. Then π_Λ is continuous with the Hausdorff metric because $\rho(\pi_\Lambda(A), \pi_\Lambda(B)) \leq \rho(A, B)$ holds for any $A, B \in \mathcal{F}(N \times [0,1])$. Therefore,

$$\pi_\Lambda(\bar{c}) = \pi_\Lambda(\lim_{n\to\infty} \bar{c}_n) = \lim_{n\to\infty} \pi_\Lambda(\bar{c_n}).$$

Clearly, $h(\pi_\Lambda(\bar{c_n}), [0,1]) = 0$. Fix $\varepsilon > 0$. Since g_{ϵ_n} converges uniformly to the identity on $[0,1]$, there exists $n(\varepsilon)$ such that if $n > n(\varepsilon)$ then $|g_{\epsilon_n}(\lambda) - \lambda| < \varepsilon$ holds for all $\lambda \in [0,1]$. It follows that an ε-neighborhood of $\pi_\Lambda(\bar{c}_n)$ is $[0,1]$ if $n > n(\varepsilon)$ and hence, $\rho([0,1], \pi_\Lambda(\bar{c}_n)) = h([0,1], \pi_\Lambda(\bar{c}_n)) \leq \varepsilon$ holds if $n > n_\varepsilon$. This implies $\rho(\pi_\Lambda(\bar{c}), [0,1]) = \rho(\lim_{n\to\infty} \pi_\Lambda(\bar{c}_n), [0,1]) = \lim_{n\to\infty} \rho(\pi_\Lambda(\bar{c_n}), [0,1]) = 0$. □

LEMMA 3.7. *For any $\lambda \in [0,1]$, the set $\bar{c}_\lambda := \bar{c} \cap N_\lambda$ is nonempty, compact and F_0-invariant.*

Proof. It follows from Lemma 3.6 that $\bar{c}_\lambda \neq \emptyset$ for every $0 \leq \lambda \leq 1$. By Lemma 3.5, $\bar{c}$ is F_0-invariant, hence so is $\bar{c}_\lambda = N_\lambda \cap \bar{c}$. Similarly, $\bar{c}_\lambda$ is compact because $\bar{c}$ and N_λ are compact. □

In the subsequent lemmas, we study the structure inside of $\bar{c}_\lambda$. Define

$$I_\lambda := \{p \in \mathcal{P} \mid \bar{c}_\lambda \cap M_\lambda(p) \neq \emptyset\}.$$

By the preceding lemmas, we can show that $I_\lambda \neq \emptyset$ for all $\lambda \in [0,1]$.

LEMMA 3.8. *Let a sequence $(x_n, \lambda_n) \in \bar{c}_n$ converge to $(x, \lambda) \in M_\lambda(p)$. If there exists a sequence $\{k_n\} \subset \mathbb{Z}_{\geq 0}$ such that $(y_n, \mu_n) := F_{\epsilon_n}^{k_n}(x_n, \lambda_n) \to (y, \lambda) \in M_\lambda(p')$, then $p' \leq_\lambda p$ holds with the map defined order $<_\lambda$. Here $F_{\epsilon_n}^{k_n}$ stands for $(F_{\epsilon_n})^{k_n}$.*

Proof. The proof is by contradiction. Thus, suppose $p' \not\leq_\lambda p$. Define $I := \{q \in \mathcal{P} \mid q \leq_\lambda p\}$. Then I is an attracting interval and $p' \notin I$. It follows from Lemma 2.3 that $(M_\lambda(I), M_\lambda(\mathcal{P} \setminus I))$ is an attractor-repeller decomposition for S_λ. We will show that for any $\varepsilon > 0$ there exist an ε-chain from $M_\lambda(p)$ to $M_\lambda(p')$ contained in S_λ, a contradiction with Proposition 2.2.

First we claim that for any $\varepsilon > 0$ there exists a positive integer K such that if $n \geq K$ then the orbit $\{(x_n, \lambda_n),\ F_{\epsilon_n}(x_n, \lambda_n), \ldots,\ F_{\epsilon_n}^{k_n}(x_n, \lambda_n)\}$ is contained in the $\varepsilon/3$-neighborhood of S_λ. Suppose the contrary, then we can find $\varepsilon > 0$ such that for each n there exists $0 \leq l_n \leq k_n$ satisfying $d(F_{\epsilon_n}^{l_n}(x_n, \lambda_n), S_\lambda) \geq \varepsilon/3$. Let $(z_n, \nu_n) := F_{\epsilon_n}^{l_n}(x_n, \lambda_n)$. From the form of g_{ϵ_n}, we have $\mu_n \leq \nu_n \leq \lambda_n$. Since we assumed $\mu_n, \lambda_n \to \lambda$, we have $\nu_n \to \lambda$. It follows from Lemma 3.1 that $(z_n, \nu_n) \to \bar{c}$, and therefore, $(z_n, \nu_n) \to \bar{c}_\lambda$. However, $\bar{c}_\lambda \subset S_\lambda$ since $\bar{c}_\lambda$ is f_λ-invariant and hence $(z_n, \nu_n) \to S_\lambda$, a contradiction with the fact $d(F_{\epsilon_n}^{l_n}(x_n, \lambda_n), S_\lambda) \geq \varepsilon/3$.

Now we have a positive integer K that satisfies the condition stated above. Enlarging K if needed, we assume that $n \geq K$ implies $d(F_0(w, \xi), F_{\epsilon_n}(w, \xi)) < \varepsilon/3$ for all $(w, \xi) \in N \times [0,1]$. Choose $n \geq K$ so that $d((x_n, \lambda_n), M_\lambda(p)) < \varepsilon/3$ and $d((y_n, \mu_n), M_\lambda(q)) < \varepsilon/3$ hold. By the definition of K, we can find $s_i \in S_\lambda$ such that $d(F_{\epsilon_n}^i(x_n, \lambda_n), s_i) < \varepsilon/3$ for $i \geq 0$. Let $s_{-1} \in M_\lambda$ be a point such that $d(f_\lambda(s_{-1}), (x_n, \lambda_n)) < \varepsilon$. By a triangle inequality similar to the one in the proof of Lemma 3.8, $\{s_{-1}, s_0, s_1, s_2, \ldots, s_{k_n}\}$ turns out to be an ε-chain from $M_\lambda(p)$ to $M_\lambda(p')$ contained in S_λ. □

Lemma 3.9. *For each $\lambda \in [0,1]$, the set $(I_\lambda, <_\lambda)$ is a totally ordered set.*

Proof. If $\lambda = 0$ or 1, this statement is trivial since $(I_\lambda, <_\lambda)$ has only one element. Suppose $\lambda \in (0,1)$ and $p, p' \in I_\lambda$. Let $(x,\lambda) \in M_\lambda(p) \cap \bar{c}_\lambda$ and $(y,\mu) \in M_\lambda(p') \cap \bar{c}_\lambda$. Lemma 3.1 shows that there exist convergent sequences $(x_n,\lambda_n), (y_n,\mu_n) \in \bar{c}_n$ such that $(x_n,\lambda_n) \to (x,\lambda)$ and $(y_n,\mu_n) \to (y,\mu)$. We may assume that $(x_n,\lambda_n), (y_n,\mu_n) \in c_n$ because all accumulation points of c_n are contained in $S_0 \cup S_1$. Since c_n is an orbit under F_{ϵ_n}, there exists $k_n \geq 0$ for each n such that either $(x_n,\lambda_n) = F_{\epsilon_n}^{k_n}(y_n,\mu_n)$ or $F_{\epsilon_n}^{k_n}(x_n,\lambda_n) = (y_n,\mu_n)$ holds. Taking a suitable subsequence n_i, we may assume one of the preceding two formulas holds for all n_i. By Lemma 3.8, in both cases, we have $p \leq_\lambda p'$ or $p' \leq_\lambda p$. □

Next, we consider how the structure inside of $\bar{c}_\lambda$ depends on λ. Define

$$A_p := \{\lambda \in [0,1] \mid \bar{c}_\lambda \subset M_\lambda(p)\}, \quad B_p := \{\lambda \in [0,1] \mid \bar{c}_\lambda \cap M_\lambda(p) \neq \emptyset\}$$

for each $p \in \mathcal{P}$. By definition, $A_p \subset B_p$.

Lemma 3.10. *For each $p \in \mathcal{P}$, A_p is open in $[0,1]$ and B_p is compact.*

Proof. Clearly

$$B_p = \pi_\Lambda\Big(\bar{c} \cap \bigcup_{\lambda\in[0,1]} M_\lambda(p)\Big).$$

Since $\bigcup_{\lambda\in[0,1]} M_\lambda(p)$ is an isolated invariant set with respect to $F_0|_{N\times[0,1]}$, it is compact, hence so is B_p. We will show

$$A_p = [0,1] \setminus (\bigcup_{p'\neq p} B_{p'}).$$

If this holds, A_p is open since $B_{p'}$ closed. First, suppose $\lambda \in A_p$, then $\bar{c}_\lambda \cap M_\lambda(p') = \emptyset$ for $p \neq p'$. It follows that $A_p \cap \bigcup_{p'\neq p} B_{p'} = \emptyset$, therefore, $\lambda \in [0,1] \setminus (\bigcup_{p'\neq p} B_{p'})$. Conversely, suppose $\lambda \in \Lambda \setminus (\bigcup_{p'\neq p} B_{p'})$. This implies $\bar{c}_\lambda \cap \bigcup_{p'\neq p} M_\lambda(p') = \emptyset$. By Lemma 3.7, $\bar{c}_\lambda$ is F_0-invariant, and therefore, each point of $\bar{c}_\lambda$ has its complete orbit and its α and ω-limit sets contained in $\bar{c}_\lambda$. Then if $\bar{c}_\lambda \not\subset M_\lambda(p)$, we have $\bar{c}_\lambda \cap M_\lambda(p') \neq \emptyset$ for some $p' \neq p$, a contradiction. Hence we have $\bar{c}_\lambda \subset M_\lambda(p)$, which implies $\lambda \in A_p$. □

Lemma 3.11. *Suppose $\lambda \in (0,1)$. If $p = \inf I_\lambda$ and $p' = \sup I_\lambda$, there exists an $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda) \subset A_p$ and $(\lambda, \lambda + \varepsilon) \subset A_{p'}$ hold.*

Proof. Suppose $(\lambda - \varepsilon, \lambda) \not\subset A_p$ for each $\varepsilon > 0$. Then there exists a monotone increasing sequence $\lambda_m \to \lambda$ such that for each m there exists $r_m \in \mathcal{P} \setminus p$ satisfying $\lambda_m \in B_{r_m}$. Choosing a suitable subsequence, we may assume $r_m = q$ for all m. We have $\lambda \in B_q$ since B_q is closed by Lemma 3.10. Let $(x,\lambda) \in M_\lambda(p) \cap \bar{c}_\lambda$, and $(y_m,\lambda_m) \in M_{\lambda_m}(q) \cap \bar{c}_{\lambda_m}$. Lemma 3.1 shows that there exist (x_i,μ_i), $(y_i^m,\nu_i^m) \in \bar{c}_i$ such that $\lim_{i\to\infty}(x_i,\mu_i) = (x,\lambda)$ and $\lim_{i\to\infty}(y_i^m,\nu_i^m) = (y_m,\lambda_m)$ hold. Since $\lim_{i\to\infty}(y_i^m,\nu_i^m) \in M_{\lambda_m}(q)$ and $\lambda_m < \lambda$, we can find a subsequence $\{i_m\}$ that satisfies the following: $i_m < i_{m+1}$; $d((y_{i_m}^m,\nu_{i_m}^m), M_{\lambda_m}(q)) < 1/m$; $\nu_{i_m}^m < \mu_{i_m}$. The sequence $\{(y_{i_m}^m,\nu_{i_m}^m)\}$ is contained in compact set $\bigcup_{\lambda\in[0,1]} M_\lambda(q)$, therefore, has a convergent subsequence. Relabeling the sequence, we assume $(y_{i_m}^m,\nu_{i_m}^m) \to (y,\nu)$. It follows from $\lim_{m\to\infty}\lambda_m = \lambda$ that $\nu = \lambda$, and therefore, we have $(y,\nu) \in M_\lambda(q)$. As in the proof of Lemma 3.9, it follows from $\lambda \in (0,1)$ that (x_{i_m},μ_{i_m}), $(y_{i_m}^m,\nu_{i_m}^m) \subset c_{i_m}$. Because c_{i_m}

is an orbit and $\nu^m_{i_m} < \mu_{i_m}$, there exists a sequence $k_m \geq 0$ such that $F^{k_m}_{\epsilon_{i_m}}(x_{i_m}, \mu_{i_m}) = (y^m_{i_m}, \nu^m_{i_m})$. Thus, Lemma 3.8 shows that $q \leq_\lambda p$, and it follows that $q <_\lambda p$ since we have assumed that $q \neq p$. Now we have $(y, \nu) = \lim_{m\to\infty}(y^m_{i_m}, \nu^m_{i_m}) \in \bar{c}_\lambda$ by Lemma 3.1, hence $q \in I_\lambda$. This contradicts the fact $p = \inf I_\lambda$. The proof for $A_{p'}$ is similar. □

As a conclusion of the preceding lemmas, we have the following:

THEOREM 3.12. *If $C(M_1(q), M_0(p)\ ; \hat{S}_{\epsilon_n}) \neq \emptyset$ for all n, then there exist $1 > \lambda_1 > \lambda_2 > \cdots \lambda_k > 0$ and $\{q = r_1, r_2, \ldots, r_{k+1} = p\} \subset \mathcal{P}$ such that $r_{i+1} <_i r_i$ where $<_i$ is the map-defined order with respect to f_{λ_i}.*

Proof. Define $K := [0,1] \setminus \bigcup_{p\in\mathcal{P}} A_p$. Then K is closed by Lemma 3.10, hence compact. Lemma 3.11 shows that for each $\lambda \in [0,1]$ there exist $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda) \cap K = \emptyset$ and $(\lambda, \lambda + \varepsilon) \cap K = \emptyset$. It follows that K is discrete in $[0,1]$ and hence finite. We have $K \subset (0,1)$ because $0 \in A_p$ and $1 \in A_q$. Define $\lambda_1 > \cdots > \lambda_k$ to be the elements of K and define $\lambda_0 = 1$. For each $i = 1, 2, \ldots, k$, we can find r_i such that $(\lambda_i, \lambda_{i-1}) \subset A_{r_i}$ because $\{A_p\}$ are mutually disjoint. By Lemma 3.11, $\sup I_{\lambda_i} = r_i$ and $\inf I_{\lambda_i} = r_{i+1}$. Hence $r_{i+1} <_i r_i$ by the definition of I_λ. Since $\{A_p\}$ are open and mutually disjoint, we have $(\lambda_1, 1) \subset A_{r_1}$ and $1 \in A_q$. This implies $A_q = A_{r_1}$, and therefore $q = r_1$. The same argument shows $p = r_{k+1}$. □

4. *Singular Transition Matrix Pair*

If we want to apply Theorem 3.12 to detect the bifurcations of connecting orbits, we must show the existence of the connecting orbit in $X \times \Lambda$ for all $n \geq 0$. The connection matrix pair discussed in § 2.3 can be used for this purpose.

The setting is same as in §3, so there exists a Morse decomposition $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$ for $\hat{S}_\epsilon$. For simplicity, we regards $\mathcal{P} \amalg \mathcal{P}$ as $\{p_0 \in \mathcal{P} \mid p \in \mathcal{P}\} \amalg \{p_1 \in \mathcal{P} \mid p \in \mathcal{P}\}$ with $M(p_1) := M_1(p)$ and $M(p_0) := M_0(p)$.

Theorem 2.13 assures that there exists a connection matrix pair $(\Delta_\epsilon, a_\epsilon)$ for the Morse decomposition $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$ with respect to F_ϵ. Since $p_0 \ll q_1$ holds for each pair $p, q \in \mathcal{P}$, the two lower triangular matrices

$$\Delta_\epsilon, a_\epsilon : \bigoplus_{p\in\mathcal{P}} CH^*(M(p_0), F_\epsilon) \bigoplus_{p\in\mathcal{P}} CH^*(M(p_1), F_\epsilon) \circlearrowleft$$

take the form

$$\Delta_\epsilon = \begin{pmatrix} \Delta_0 & 0 \\ T_\epsilon & \Delta_1 \end{pmatrix}, \quad a_\epsilon = \begin{pmatrix} a_0 & 0 \\ A_\epsilon & a_1 \end{pmatrix}.$$

Here the subscript ϵ for Δ_i and a_i are omitted. Now we have two submatrices

$$T_\epsilon, A_\epsilon : \bigoplus_{p\in\mathcal{P}} CH^*(M(p_0)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M(p_1)).$$

The discussion in §3 leads to the following definition.

Definition. A pair of matrices (T, A) is a *singular transition matrix pair* if there exists a sequence $\epsilon_n \to 0$ such that $T = T_{\epsilon_n}$ and $A = A_{\epsilon_n}$ for any $n \geq 0$. The set of singular transition matrix pairs is denoted by $\mathcal{T}_{0,1}$.

Remark. If the cohomology Conley index has coefficients in a finite field and the cohomology Conley index of each Morse set are of finite type, then the cardinality of possible connection matrix pairs for $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$ is finite. Hence, it immediately follows that $\mathcal{T}_{0,1} \neq \emptyset$ in this case.

For each $I \in \mathcal{I}(\mathcal{P} \amalg \mathcal{P}, \ll)$, denote by $a^*(I)$ the map induced on $\mathcal{H}\Delta(\mathcal{M})$ by the cochain map $a(I)$ and by $<_i$ the map-defined order on $\mathcal{P}$ with respect to f_{λ_i}.

THEOREM 4.1. *Suppose* $(T, A) \in \mathcal{T}_{0,1}$ *and* $(p_0, q_1) \in \mathcal{I}_2(\mathcal{P} \amalg \mathcal{P}, \ll)$. *If either* $T_{q_1 p_0} \neq 0$ *or* $a^*(p_0) \oplus a^*(q_1) \not\cong a^*(p_0 \cup q_1)$, *then there exists* $1 > \lambda_1 > \lambda_2 > \cdots \lambda_k > 0$ *and* $\{q = r_1, r_2, \ldots, r_{k+1} = p\} \subset \mathcal{P}$ *such that* $r_{i+1} <_i r_i$ *holds.*

Proof. By definition, there exists a sequence $\epsilon_n \to 0$ such that $T = T_{\epsilon_n}$ and $A = A_{\epsilon_n}$ for all n. Thus, (T, A) is a submatrix pair of the connection matrix pair for the Morse decomposition $\mathcal{M}(\mathcal{P} \amalg \mathcal{P}, \ll)$ of $\hat{S}_{\epsilon_n}$. Since $(M(p_0), M(q_1))$ is an attractor-repeller decomposition of $M(q_1 \cup p_0)$, Theorem 2.5 shows that there exists a long exact sequence

$$\cdots \xrightarrow{\delta^*} \mathrm{Con}^k(M(q_1), F_\epsilon) \longrightarrow \mathrm{Con}^k(M(q_1 \cup p_0), F_\epsilon) \longrightarrow \mathrm{Con}^k(M(p_0), F_\epsilon) \xrightarrow{\delta^*} \cdots .$$

It follows that

$$\left(\begin{pmatrix} 0 & 0 \\ T_{q_1 p_0} & 0 \end{pmatrix}, \begin{pmatrix} A_{p_0 p_0} & 0 \\ A_{q_1 p_0} & A_{q_1 q_1} \end{pmatrix} \right)$$

is a connection matrix pair for this attractor-repeller decomposition. Thus, by definition we have

$$\begin{aligned} \delta^* &\cong T_{q_1 p_0}, & \chi^*(M(q_1)) &\cong a^*(q_1), \\ \chi^*(M(p_0)) &\cong a^*(p_0), & \chi^*(M(q_1 \cup p_0)) &\cong a^*(q_1 \cup p_0). \end{aligned}$$

Therefore, $T_{q_1 p_0} \neq 0$ implies $\delta^* \neq 0$, and $a^*(p_0) \oplus a^*(q_1) \not\cong a^*(p_0 \cup q_1)$ implies $\chi^*(M(q_1 \cup p_0)) \not\cong \chi^*(M(q_1)) \oplus \chi^*(M(p_0))$. By Corollary 2.6, it follows that $C(M(q_1), M(p_0); \hat{S}_{\epsilon_n}) \neq \emptyset$ for all n. To finish the proof, apply Lemma 3.12. □

This theorem shows that the transition matrix pair has some information about bifurcations of connecting orbits. However, the theorem says nothing about the computation of the transition matrix pairs. So, we hope to know the algebraic relations between the transition matrix pairs and the connection matrix pairs for S_0 and S_1, which can be computed from the filtrations.

Suppose

$$\Delta_\epsilon = \begin{pmatrix} \Delta_0 & 0 \\ T_\epsilon & \Delta_1 \end{pmatrix}, \quad a_\epsilon = \begin{pmatrix} a_0 & 0 \\ A_\epsilon & a_1 \end{pmatrix}$$

is a connection matrix pair for $\hat{S}_\epsilon$. Identifying $\hat{S}_\epsilon \cap N_i$ with S_i, it follows from the definition of the connection matrix pairs that (Δ_0, a_0) and (Δ_1, a_1) are connection matrix pairs for $\mathcal{M}(\mathcal{P}, \ll, S_0, F_\epsilon)$ and $\mathcal{M}(\mathcal{P}, \ll, S_1, F_\epsilon)$, respectively. The following proposition relates $\mathcal{CM}(\mathcal{M}(\mathcal{P}, \ll, S_i, F_\epsilon))$ and $\mathcal{CM}(\mathcal{M}(\mathcal{P}, <_i, S_i, f_i))$.

PROPOSITION 4.2. *Let* i *be* 0 *or* 1. *Suppose* $(\Delta_i, a_i) \in \mathcal{CM}(\mathcal{M}(\mathcal{P}, \ll, S_i, F_\epsilon))$. *Then there exist* $(\tilde{\Delta}_i, \tilde{a}_i) \in \mathcal{CM}(\mathcal{M}(\mathcal{P}, <_i, S_i, f_i))$ *and graded module braid isomorphism* θ_i *of degree* $-i$

such that (Δ_1, a_1) *is conjugate to* $(\tilde{\Delta}_1, \tilde{a}_1)$ *by* θ_i*, namely, the following diagram commutes:*

$$\begin{array}{ccc}
\bigoplus_{p\in\mathcal{P}} CH^*(M(p_i), F_\epsilon) & \xrightarrow{\Delta_i\,(or\ a_i)} & \bigoplus_{p\in\mathcal{P}} CH^*(M(p_i), F_\epsilon) \\
\theta_i \Big\downarrow & & \theta_i \Big\downarrow \\
\bigoplus_{p\in\mathcal{P}} CH^*(M_i(p), f_i) & \xrightarrow{\tilde{\Delta}_i\,(or\ \tilde{a}_i)} & \bigoplus_{p\in\mathcal{P}} CH^*(M_i(p), f_i).
\end{array}$$

To prove this proposition, we need the following lemma.

Lemma 4.3. *Let* $\mathcal{H}$ *and* $\tilde{\mathcal{H}}$ *be graded module braids with endomorphisms, and* $\Psi = \{\phi^* : H(I) \to \tilde{H}(I) \mid I \in \mathcal{I}\}$ *a graded module braid isomorphism. If* (Δ, A) *is a connection matrix pair for* $\mathcal{H}$ *then*

$$\tilde{\Delta}_{pq} := \phi^*(p) \circ \Delta_{pq} \circ \phi^*(q)^{-1}, \quad \tilde{a}_{pq} := \phi^*(p) \circ a_{pq} \circ \phi^*(q)^{-1}$$

form a connection matrix pair $(\tilde{\Delta}, \tilde{a})$ *for* $\tilde{\mathcal{H}}$.

Proof. It is evident that $\tilde{\Delta}$ is a strictly upper triangular coboundary operator and $\tilde{a}$ is a cochain map with respect to $\tilde{\Delta}$. By the definition of the connection matrix pair, we have $\mathcal{H} \cong \mathcal{H}\Delta$, and by our assumption $\mathcal{H} \cong \tilde{\mathcal{H}}$. Therefore, if we can show $\mathcal{H}\Delta \cong \tilde{\mathcal{H}}\tilde{\Delta}$, it will follow that $\tilde{\mathcal{H}} \cong \tilde{\mathcal{H}}\tilde{\Delta}$, which implies this lemma. Thus, we will make a graded module braid isomorphism from $\mathcal{H}\Delta$ to $\tilde{\mathcal{H}}\tilde{\Delta}$. By definition, for each $I \in \mathcal{I}$ we have the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{p\in I} H(p) & \xrightarrow{\Delta(I)\ (\text{or } a(I))} & \bigoplus_{p\in I} H(p) \\
\oplus\phi^*(p) \Big\downarrow & & \oplus\phi^*(p) \Big\downarrow \\
\bigoplus_{p\in I} \tilde{H}(p) & \xrightarrow{\tilde{\Delta}(I)\ (\text{or } \tilde{a}(I))} & \bigoplus_{p\in I} \tilde{H}(p).
\end{array}$$

Therefore, for each $(I, J) \in \mathcal{I}_2$, the following diagram of $\mathtt{End}(\delta\mathcal{V})$ commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{p\in J} H(p) & \longrightarrow & \bigoplus_{p\in IJ} H(p) & \longrightarrow & \bigoplus_{p\in I} H(p) & \longrightarrow & 0 \\
& & \oplus\phi^*(p) \Big\downarrow & & \oplus\phi^*(p) \Big\downarrow & & \oplus\phi^*(p) \Big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{p\in J} \tilde{H}(p) & \longrightarrow & \bigoplus_{p\in IJ} \tilde{H}(p) & \longrightarrow & \bigoplus_{p\in I} \tilde{H}(p) & \longrightarrow & 0
\end{array}$$

Here we denote $((\bigoplus_{p\in I} H(p), \Delta(I)), a(I))$ simply by $\bigoplus_{p\in I} H(p)$, *etc*. Thus the family of maps $\Psi = \{\phi^*(I) : C\Delta \to \tilde{C}\tilde{\Delta} \mid I \in \mathcal{I}\}$ is a cochain complex braid endomorphism. Moreover, Ψ is an isomorphism since each $\phi^*(p)$ is an isomorphism. Taking cohomology, it induces the isomorphism from $\mathcal{H}\Delta$ to $\tilde{\mathcal{H}}\tilde{\Delta}$. □

Proof of Proposition4.2. First, we consider the case $i = 0$. Let $\mathcal{N}(\mathcal{P}, <_0, S_0, f_0) = \{N(I) \mid I \in \mathcal{A}\}$ be a stable filtration for $\mathcal{M}(\mathcal{P}, <_0, S_0, f_0)$. Denote by $\mathcal{H}(\mathcal{M}_0)$ the graded module braid with endomorphism induced by $\mathcal{N}(\mathcal{P}, <_0, S_0, f_0)$. Because $\mathcal{N}(\mathcal{P}, <_0, S_0, f_0)$ is stable, it can be shown that $\{\hat{N}(I) := N(I) \times [-\varepsilon, \varepsilon] \mid I \in \mathcal{A}\}$ will serve as a filtration for

$\mathcal{M}(\mathcal{P}, \ll, S_0, F_\epsilon)$ for sufficiently small ε. We denote by $\mathcal{H}(\hat{\mathcal{M}}_0)$ the graded module braid with endomorphism induced by $\{\hat{N}(I) \mid I \in \mathcal{A}\}$. By Lemma 4.3, it will suffice to find a graded module braid isomorphism $\{\theta_0^*(I) \mid I \in \mathcal{I}\} : \mathcal{H}(\hat{\mathcal{M}}_0) \to \mathcal{H}(\mathcal{M}_0)$. Let $(I, J) \in \mathcal{I}_2$ and define

$$Q_I := \{p \in \mathcal{P} \setminus I \mid \exists q \in I,\ p < q\}.$$

Then there exists a commutative diagram

$$\begin{array}{ccccc}
(N(Q_I I), N(Q_I)) & \longrightarrow & (N(Q_{IJ} IJ), N(Q_{IJ})) & \longrightarrow & (N(Q_{IJ} IJ), N(Q_{IJ} I)) \\
\downarrow & & \downarrow & & \downarrow \\
(\hat{N}(Q_I I), \hat{N}(Q_I)) & \longrightarrow & (\hat{N}(Q_{IJ} IJ), \hat{N}(Q_{IJ})) & \longrightarrow & (\hat{N}(Q_{IJ} IJ), \hat{N}(Q_{IJ} I))
\end{array}$$

where all arrows are the inclusion maps. Applying the cohomology functor, we obtain the isomorphism $\{\theta_0^*(I) \mid I \in \mathcal{I}\}$ since all vertical arrows are homotopy equivalence.

Next, we prove the case $i = 1$. As in the case $i = 0$, let $\mathcal{N}(\mathcal{P}, <_1, S_1, f_1) = \{N(I) \mid I \in \mathcal{A}\}$ be a stable filtration for $\mathcal{M}(\mathcal{P}, <_1, S_1, f_1)$ and denote by $\mathcal{H}(\mathcal{M}_1)$ the graded module braid with endomorphism induced from $\mathcal{N}(\mathcal{P}, <_1)$. Let

$$\hat{N}(I) := (N(I) \times [1-\varepsilon, 1+\varepsilon]) \cup (N(\mathcal{P}) \times ([1-\varepsilon, 1-\varepsilon'] \cup [1+\varepsilon', 1+\varepsilon])).$$

It can be shown that if we choose $0 < \varepsilon' < \varepsilon$ suitable, then $\{\hat{N}(I) \mid I \in \mathcal{A}\}$ serves as a filtration for $\mathcal{M}(\mathcal{P}, \ll, S_0, F_\epsilon)$. Denote by $\hat{\mathcal{H}}(\mathcal{M}_1)$ the graded module braid with endomorphism induced from $\{\hat{N}(I) \mid I \in \mathcal{A}\}$. Suppose $I \in \mathcal{I}$ and let $(K, L) := (N(Q_I I), N(Q_I))$ and $(\hat{K}, \hat{L}) := (\hat{N}(Q_I I), \hat{N}(Q_I))$. Define

$$\begin{aligned}
K_- &:= \{\, K \times [1-\varepsilon, 1+\varepsilon') \,\} \cup \{\, N(\mathcal{P}) \times ([1-\varepsilon, 1-\varepsilon'] \,\}, \\
K_+ &:= \{\, K \times (1-\varepsilon', 1+\varepsilon] \,\} \cup \{\, N(\mathcal{P}) \times ([1+\varepsilon', 1+\varepsilon] \,\}, \\
L_- &:= \{\, L \times [1-\varepsilon, 1+\varepsilon') \,\} \cup \{\, N(\mathcal{P}) \times ([1-\varepsilon, 1-\varepsilon'] \,\}, \\
L_+ &:= \{\, L \times (1-\varepsilon', 1+\varepsilon] \,\} \cup \{\, N(\mathcal{P}) \times ([1+\varepsilon', 1+\varepsilon] \,\}.
\end{aligned}$$

For simplicity, we denote

$$\begin{aligned}
K' &:= K \cup f_1(L), \quad & L' &:= L \cup f_1(L), \\
\hat{K}' &:= \hat{K} \cup F_\epsilon(\hat{L}), \quad & \hat{L}' &:= \hat{L} \cup F_\epsilon(\hat{L}),
\end{aligned}$$

and

$$K'_\pm := K_\pm \cup F_\epsilon(K_\pm), \quad L'_\pm := L_\pm \cup F_\epsilon(L_\pm).$$

Note that the definition of $K'_\pm$ is different from that of K' and $\hat{K}'$. Now we consider the Meyer-Vietoris sequence. First, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\sharp(\hat{K}, \hat{L}) & \longrightarrow & C^\sharp(K_-, L_-) \oplus C^\sharp(K_+, L_+) & \longrightarrow & C^\sharp(K_+ \cap K_-, L_+ \cap L_-) & \longrightarrow & 0 \\
 & & \big\uparrow{\scriptstyle F_\epsilon^\sharp} & & \big\uparrow{\scriptstyle F_\epsilon^\sharp \oplus F_\epsilon^\sharp} & & \big\uparrow{\scriptstyle F_\epsilon^\sharp} & & \\
0 & \longrightarrow & C^\sharp(\hat{K}', \hat{L}') & \longrightarrow & C^\sharp(K'_-, L'_-) \oplus C^\sharp(K'_+, L'_+) & \longrightarrow & C^\sharp(K'_+ \cap K'_-, L'_+ \cap L'_-) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle j^\sharp} & & \big\downarrow{\scriptstyle i^\sharp \oplus i^\sharp} & & \big\downarrow{\scriptstyle i^\sharp} & & \\
0 & \longrightarrow & C^\sharp(\hat{K}, \hat{L}) & \longrightarrow & C^\sharp(K_-, L_-) \oplus C^\sharp(K_+, L_+) & \longrightarrow & C^\sharp(K_+ \cap K_-, L_+ \cap L_-) & \longrightarrow & 0,
\end{array}$$

where i and j are inclusions. It can be shown that $K_\pm$ is homotopic to $L_\pm$ in $K_\pm$, hence, $H^*(K_\pm, L_\pm) = 0$. In addition, it can be shown that

$$(H^*(K'_+ \cap K'_-, L'_+ \cap L'_-, F_\epsilon^* \circ (i^*)^{-1}) \cong (H^*(K, L), f_1^* \circ (i^*)^{-1})$$

as objects of $\mathtt{Aut}(\mathcal{GV})$. Therefore, the following long exact sequence with the connecting homomorphism δ^* commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H^k(K,L) \arrow[r, "\delta^k"] \arrow[d, "f_1^*\circ(i^*)^{-1}"'] & H^{k+1}(\hat{K},\hat{L}) \arrow[r] \arrow[d, "F_\epsilon^*\circ(j^*)^{-1}"'] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H^k(K,L) \arrow[r, "\delta^k"] & H^{k+1}(\hat{K},\hat{L}) \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Since $\varinjlim$ commutes with H^*, we have an isomorphism

$$\theta(I) : CH^*(M_1(I), F_\epsilon) \to CH^*(M_1(I), f_1).$$

We claim that the family $\{\theta(I) \mid I \in \mathcal{I}\}$ is a graded module braid isomorphism. Indeed, for any $(I, J) \in \mathcal{I}_2$ and inclusions

$$(N(Q_I I), N(Q_I)) \xrightarrow{\iota} (N(Q_{IJ} IJ), N(Q_{IJ})) \xrightarrow{\rho} (N(Q_{IJ} IJ), N(Q_{IJ} I)),$$
$$(\hat{N}(Q_I I), \hat{N}(Q_I)) \xrightarrow{\hat{\iota}} (\hat{N}(Q_{IJ} IJ), \hat{N}(Q_{IJ})) \xrightarrow{\hat{\rho}} (\hat{N}(Q_{IJ} IJ), \hat{N}(Q_{IJ} I)),$$

we can show that $\theta(I) \circ \iota^* = \hat{\iota}^* \circ \theta(IJ)$ and $\theta(IJ) \circ \rho^* = \hat{\rho}^* \circ \theta(I)$. Therefore, we have

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^{k-1}"] & \mathrm{Con}^k(M(J)) \arrow[r] \arrow[d, "\theta^k(J)"'] & \mathrm{Con}^k(M(IJ)) \arrow[r] \arrow[d, "\theta^k(IJ)"'] & \mathrm{Con}^k(M(I)) \arrow[r, "\delta^k"] \arrow[d, "\theta^k(I)"'] & \cdots \\
\cdots \arrow[r, "\delta^{k-1}"] & \mathrm{Con}^k(\hat{M}(J)) \arrow[r] & \mathrm{Con}^k(\hat{M}(IJ)) \arrow[r] & \mathrm{Con}^k(\hat{M}(I)) \arrow[r, "\delta^k"] & \cdots
\end{tikzcd}$$

and the claim follows. Finally, $\theta_1(I) := \theta(I)^{-1}$ is the map we wanted. □

We will prove some lemmas which are useful to compute the transition matrix.

Lemma 4.4. *If $(T, A) \in \mathcal{T}_{0,1}$, then $Ta_0 + \Delta_1 A = A\Delta_0 + a_1 T$.*

Proof. By definition,

$$\begin{pmatrix} \Delta_0 & 0 \\ T & \Delta_1 \end{pmatrix}\begin{pmatrix} a_0 & 0 \\ A & a_1 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ A & a_1 \end{pmatrix}\begin{pmatrix} \Delta_0 & 0 \\ T & \Delta_1 \end{pmatrix}.$$

The bottom left entry of this formula is this lemma. □

Lemma 4.5. *Let (A, R) be an attractor-repeller decomposition for an isolated invariant set S. If $\mathrm{Con}^*(S) = 0$ then $\delta^* : \mathrm{Con}^*(A) \to \mathrm{Con}^*(R)$ defined in Theorem 2.5 is an isomorphism.*

Proof. Exactness of the long sequence in Theorem 2.5 implies the assertion. □

Lemma 4.6. *Let $K \subset N$ be an isolating neighborhood. If $\mathrm{Inv}(K, f_\lambda) = M_\lambda(p)$ and $K \cap f_\lambda(N \setminus K) = \emptyset$ holds for all $\lambda \in \Lambda$, then $T_{p_1 p_0}$ is an isomorphism.*

Proof. By Lemma 3.2, we may assume that $K \times \bar{\Lambda}$ is an isolating neighborhood. Consider the homotopy $G_t(x, \lambda) := (f_{t\lambda}, g_\epsilon(\lambda))$ from $G_0 = f_0 \times g_\epsilon$ to $G_1 = F_\epsilon$. Since $K \times \bar{\Lambda}$ remains to be an isolating neighborhood through this homotopy, we have $\mathrm{Con}^*(\mathrm{Inv}(K \times \bar{\Lambda}, F_\epsilon)) = \mathrm{Con}^*(\mathrm{Inv}(K \times \bar{\Lambda}, f_0 \times g_\epsilon))$ from the continuity of the Conley index. Similarly, consider an obvious homotopy from g_ϵ to g', where $g'(\lambda)$ is defined to be $\lambda - \gamma$ for a constant $\gamma > 0$. Again from the continuity of the Conley index, $\mathrm{Con}^*(\mathrm{Inv}(K \times \bar{\Lambda}, f_0 \times g_\epsilon)) = \mathrm{Con}^*(\mathrm{Inv}(K \times \bar{\Lambda}, f_0 \times g'))$. Since $\mathrm{Inv}(K \times \bar{\Lambda}, f_0 \times g') = \emptyset$, we have $\mathrm{Con}^*(\mathrm{Inv}(K, f_0 \times g')) = 0$, and hence, it follows that $\mathrm{Con}^*(\mathrm{Inv}(K, F_\epsilon)) = 0$. Next we will see $\mathrm{Inv}(K, F_\epsilon) = M(p_0 \cup p_1) = M(p_0) \cup M(p_1) \cup C(M(p_1), M(p_0); \hat{S}_\epsilon)$. By the definition of F_ϵ, it follows that $\mathrm{Inv}(K, F_\epsilon) \subset M(p_0 \cup p_1)$. By our assumption it follows that a point outside of $K \times [0, 1]$ will never enter $K \times [0, 1]$, hence we have $C(M(p_1), M(p_0); \hat{S}_{\epsilon_n}) \subset \mathrm{Inv}(K, F_\epsilon)$, therefore, $M(p_0 \cup p_1) \subset \mathrm{Inv}(K, F_\epsilon)$. Finally, applying Lemma 4.5, it follows that $\delta^* \cong T_{p_1 p_0}$ is an isomorphism. □

Example A. We consider a one-parameter family $\{f_\lambda\}$ of 2-dimensional maps. Figures 1 and 2 illustrate the behavior of the maps f_1 and f_0 where N_p, N_q, N_r are the three rectangles from right to left.

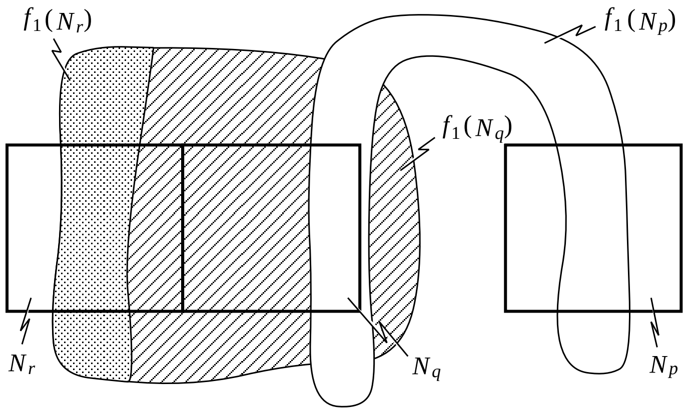

FIGURE 1. f_1

Clearly, $\{\mathrm{Inv}(N_p, f_i), \mathrm{Inv}(N_q, f_i), \mathrm{Inv}(N_r, f_i)\}$ is a Morse decomposition of the isolated invariant set $\mathrm{Inv}(N_p \cup N_q \cup N_r, f_i)$, for $i = 0, 1$. In fact, we can adapt the order $<_1$ defined by $r <_1 p$, and the order $<_0$ defined by $r <_0 p$, $r <_0 q$ as the partial orders for f_1 and f_0, respectively.

Suppose that $\{\mathrm{Inv}(N_p, f_\lambda), \mathrm{Inv}(N_q, f_\lambda), \mathrm{Inv}(N_r, f_\lambda)\}$ is a Morse decomposition continuing over Λ. Then we claim that for some $\lambda \in (0, 1)$, there exists a connection from $\mathrm{Inv}(N_p, f_\lambda)$ to $\mathrm{Inv}(N_q, f_\lambda)$. To see this, we will compute the transition matrix pair. Let $M_\lambda(\eta) := \mathrm{Inv}(N_\eta, f_\lambda)$ where $\eta = p, q, r$.

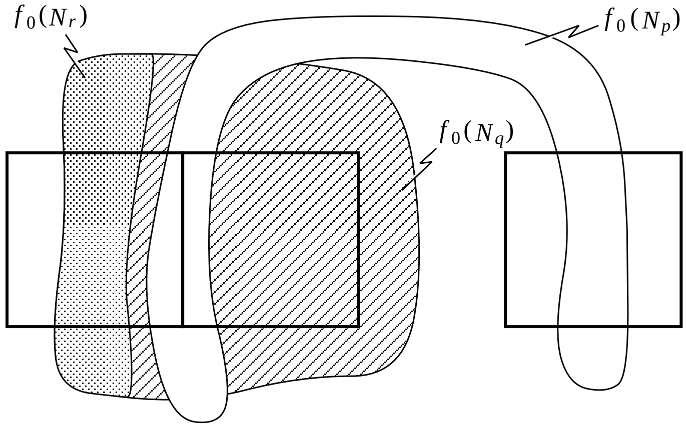

FIGURE 2. f_0

First, we give filtrations with respect to f_1 and f_0. According to Figure 3, define

$$\begin{aligned} N(p \cup q \cup r) &= N_p \cup N_q \cup N_r, & N(p) &= N_p \cup N(\emptyset), \\ N(p \cup r) &= N(p) \cup N(r), & N(r) &= \text{striped region} \cup N(\emptyset), \\ N(q \cup r) &= N_q \cup N_r \cup N(r), & N(\emptyset) &= \text{dotted region}. \end{aligned}$$

This family serves as a filtration for f_1. Similarly, according to Figure 4, take

$$\begin{aligned} N(p \cup q \cup r) &= N_p \cup N_q \cup N_r, & N(r) &= \text{striped region} \cup N(\emptyset), \\ N(p \cup r) &= N_p \cup N(r), & N(\emptyset) &= \text{dotted region}, \\ N(q \cup r) &= N_q \cup N_r \cup N(r) \end{aligned}$$

for a filtration for f_0.

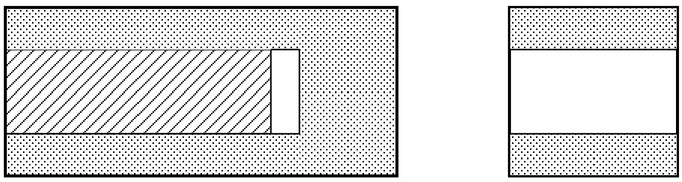

FIGURE 3. filtration for f_1

For $i = 0, 1$,

$$\mathrm{Con}^k(M_i(p)) \cong \mathrm{Con}^k(M_i(r)) \cong \begin{cases} (\mathbb{F}, 1) & (k = 1) \\ (0, 0) & (k \neq 1) \end{cases},$$

$$\mathrm{Con}^k(M_i(q)) \cong \begin{cases} (\mathbb{F}, 1) & (k = 2) \\ (0, 0) & (k \neq 2) \end{cases}$$

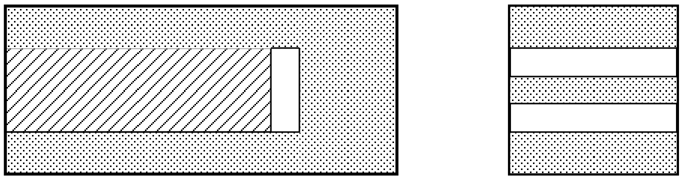

FIGURE 4. filtration for f_0

is computed from the filtrations above. Let (Δ_i, a_i) be a connection matrix pair for $\lambda = i = 0, 1$. Taking $\mathbb{F} = \mathbb{Z}_2$ we have

$$\Delta_0 = \Delta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

because all entries are zero except the (r, q)-entry since Δ_i is of degree $+1$ and strictly lower triangular. And Lemma 4.5 shows that the (r, q)-entry is an isomorphism of $\mathbb{Z}_2$. Next, we have

$$a_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where we compute the (p, r)-entries directly from the filtrations and all other entries from a consideration similar to the case of Δ_0 and Δ_1. Let $(T, A) \in \mathcal{T}_{0,1}$. These matrices are in the form of

$$T = \begin{pmatrix} t_{r_1 r_0} & 0 & t_{r_1 p_0} \\ 0 & t_{q_1 q_0} & 0 \\ t_{p_1 r_0} & 0 & t_{p_1 p_0} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_{r_1 q_0} & 0 \\ 0 & 0 & 0 \\ 0 & a_{p_1 q_0} & 0 \end{pmatrix},$$

where 0 entries are determined by considering degrees. Applying Lemma 4.4, we have $t_{p_1 r_0} + t_{p_1 p_0} = a_{p_1 q_0} + t_{p_1 r_0}$. It follows from Lemma 4.6 that $a_{p_1 q_0} = 1 \neq 0$. On the other hand,

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_{p_1 q_0} & 1 \end{pmatrix} \right)$$

is a connection matrix pair for the attractor-repeller decomposition $(M(q_0), M(p_1))$. Since the coboundary operator is 0, we have

$$a^*(p_1 \cup q_0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \not\cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a^*(p_1) \oplus a^*(q_0),$$

and hence, our claim follows from Theorem 4.1.

Example B. Here we consider the 3-dimensional analogue of Example 4, namely the one-parameter family $\{g_\lambda\}$ illustrated in Figures 5 and 6, where N_p, N_q, N_r are the three cubes from right to left.

We can find that the product of a closed interval and the filtrations given in *Example A* serve as filtrations. Therefore, the Conley indices and connection matrix pairs are

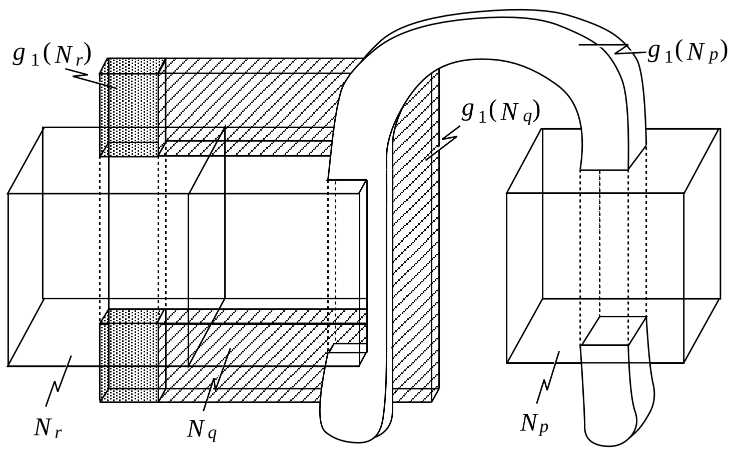

FIGURE 5. g_1

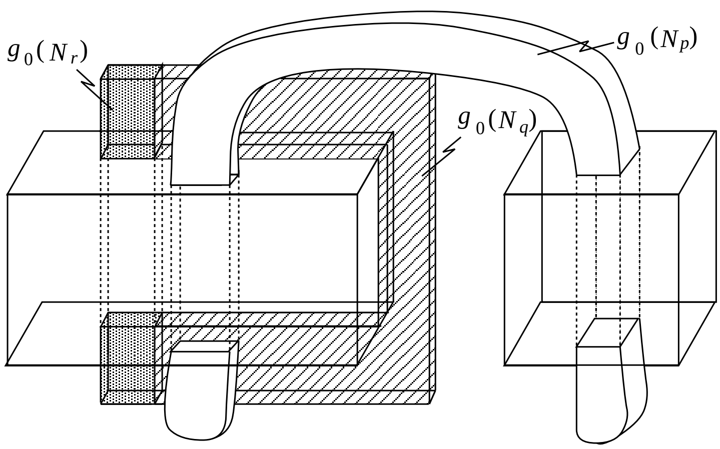

FIGURE 6. g_0

exactly the same as those of *Example A*. So, the argument in *Example A* also shows that if $\{\mathrm{Inv}(N_p, g_\lambda), \mathrm{Inv}(N_q, g_\lambda), \mathrm{Inv}(N_r, g_\lambda)\}$ is a Morse decomposition continuing over Λ, then for some $\lambda \in (0, 1)$ there exists a connecting orbit from $\mathrm{Inv}(N_p, g_\lambda)$ to $\mathrm{Inv}(N_q, g_\lambda)$.

Remark. If we make the isolated invariant sets in *Example B* hyperbolic fixed points, then the argument above would imply the occurrence of a heterodimensional cycle (see Diaz-Rocha [**1**]).

5. *Homoclinic and Heteroclinic Tangencies*

Let M be an m-dimensional manifold and f a diffeomorphism on M. We denote by TM the tangent bundle of M and df the differential of f.

Let us construct the projectivization $Pf : PM \to PM$ of $f : M \to M$. The space PM is defined by

$$PM = \coprod_{x \in M} P_x M := \coprod_{x \in M} \{1\text{-dimensional subspace } V \subset T_x M\}.$$

We denote by Pf the induced map on PM. Namely, $Pf([v]) := [df(v)]$, where $v \neq 0$ is an element of TM and $[v]$ is the subspace spanned by v. If we identify M with the image of zero section from M to TM, then the following diagram with the projections p and p' commutes:

$$\begin{array}{ccc} TM \setminus M & \xrightarrow{df|_{TM\setminus M}} & TM \setminus M \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ PM & \xrightarrow{Pf} & PM \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ M & \xrightarrow{f} & M. \end{array}$$

The topology of PM is induced by p. Then, the following proposition is evident.

Proposition 5.1. *If M is of class C^r and f is of class C^s where $s \leq r$, then PM is a fiber bundle of class C^{r-1} and Pf is a diffeomorphism of class C^{s-1}.*

This proposition shows that PM becomes a locally compact metric space and hence we can apply our transition matrix pair for $Pf : PM \to PM$. Indeed, if $r > 1$, obviously PM is metric space with the Riemannian metric; in the case $r = 1$, it follows from the fact that a locally compact Hausdorff space satisfying the second countable axiom is metrizable.

Let $x \in M$ be a hyperbolic fixed point of f and $T_x M = \tilde{E}^s_x \oplus \tilde{E}^u_x$ the corresponding splitting. Define $E^s_x := p(\tilde{E}^s_x \setminus \{0\})$ and $E^u_x := p(\tilde{E}^u_x \setminus \{0\})$.

Proposition 5.2. *The spaces E^s_x, E^u_x are isolated invariant sets with respect to $Pf : PM \to PM$.*

Proof. Let $L \subset M$ be an isolating neighborhood of $\{x\}$ with respect to $f : M \to M$ such that $PU \cong U \times \mathbb{R}P^{m-1}$ if $p'(U) \subset L$. Since E^s_x and E^u_x are disjoint compact sets in PM, there exist disjoint compact neighborhoods U and U' of E^s_x and E^u_x, respectively, such that $p'(U), p'(U') \subset L$. We will show that $\mathrm{Inv}(U, Pf) = E^s_x$ and $\mathrm{Inv}(U', Pf) = E^u_x$. Choose $z = (y, p(v)) \in U$, where $0 \neq v \in T_y M$. If $y \neq x$, then clearly $z \notin \mathrm{Inv}(U, Pf)$. Assume $y = x$ and let $v = (v^s, v^u)$ where $v^s \in \tilde{E}^s_x$ and $v^u \in \tilde{E}^u_x$. Then we have

$$Pf^n(z) = (f^n(x), p(df^n(v))) = (x, p(df^n(v^s) + df^n(v^u))).$$

If $z \notin E^s_x$, then $v^u \neq 0$ and therefore, $p(df^n(v))$ will be arbitrary close to E^u_x when $n \to \infty$. It follows that $Pf^n(z)$ exits U for some positive n. Thus, U is an isolating neighborhood and $\mathrm{Inv}(U, Pf) = E^s_x$. The proof for E^u_x is the same. □

Let us denote the stable and unstable manifold of a hyperbolic fixed point p by $W^s(p)$ and $W^u(p)$, respectively.

Proposition 5.3. *Let p, q be hyperbolic fixed points of f, and assume that* $\dim W^u(p) + \dim W^s(q) \leq n$. *If there exists a compact set $S \subset PM$ such that $C(E^u_p, E^s_q; S) \neq \emptyset$, then $W^u(p)$ and $W^s(q)$ have a non-transverse intersection.*

Proof. Suppose $z \in C(E^u_p, E^s_q; S)$ and $x = p'(z)$. It follows from $(Pf)^n(z) \in P_{f^n(x)}M$ that $\omega(x) = q$ and $\alpha(x) = p$. Furthermore, $\lim_{n\to\infty} f^n(x) = q$ and $\lim_{n\to-\infty} f^n(x) = p$, since each complete orbit of x is bounded in the compact set S. Thus, $x \in W^u(p) \cap W^s(q)$.

If $\dim W^u(p) + \dim W^s(q) < n$, then the assertion is immediate, since $W^u(p) \cap W^s(q) \neq \emptyset$ implies non-transversality. Next, we consider the case $\dim W^u(p) + \dim W^s(q) = n$. First we claim that if $W^u(p)$ is transverse to $W^s(q)$, then for each n, a tangent vector of $W^u(p)$ at $f^n(x)$ spans the one-dimensional space $(Pf)^n(z) \subset T_{f^n(x)}M$. If this is false, then $(Pf)^n(z)$ is transverse to $W^u(p)$, thus the λ-lemma implies that $\lim_{n\to-\infty}(Pf)^n(z) = E^s_p$. However, since E^u_p and E^s_p are disjoint closed sets, this contradicts $\alpha(z) \subset E^u_p$. Therefore, a tangent vector of $W^u(p)$ spans $(Pf)^n(z)$. Since we assumed that $W^u(p)$ is transverse to $W^s(q)$, we can apply the λ-lemma again, and we obtain $\lim_{n\to\infty}(Pf)^n = E^u_q$. This contradicts $\omega(z) \subset E^s_q$. □

If $\dim W^u(p) + \dim W^s(q) = n$, then the converse of the preceding proposition is partially true, namely:

Proposition 5.4. *If $W^u(p)$ and $W^s(q)$ have a non-transverse intersection and* $\dim W^u(p) + \dim W^s(q) \geq n$, *then $C(E^u_p, E^s_q; PM) \neq \emptyset$.*

Proof. By assumption, there exists $x \in W^u(p) \cap W^s(q)$ such that $T_x(W^u(p)) \oplus T_x(W^s(q)) \neq T_xM$. If $T_x(W^u(p)) \cap T_x(W^s(q)) = \emptyset$, then $\dim W^u(p) + \dim W^s(q) \geq n$ implies $T_x(W^u(p)) + T_x(W^s(q))) = T_xM$ This contradicts the assumption. Thus, we can find a $v \in T_x(W^u(p)) \cap T_x(W^s(q))$. It follows that $\lim_{n\to\infty} Pf^n([v]) = E^s_q$ and $\lim_{n\to-\infty} Pf^n([v]) = E^u_p$, which imply $C(E^u_p, E^s_q; PM) \neq \emptyset$. □

The following theorem is immediate from Theorem 4.1 and Proposition 5.3.

Theorem 5.5. *Let $(T, A) \in \mathcal{T}_{0,1}$. Suppose $(p_0, q_1) \in \mathcal{I}_2(\mathcal{P} \amalg \mathcal{P}, \ll)$ and $(q, p) \in \mathcal{I}_2(\mathcal{P}, <_\lambda)$ for any $\lambda \in [0, 1]$. If either $T_{q_1p_0} \neq 0$ or $a^*(p_0) \oplus a^*(q_1) \not\cong a^*(p_0 \cup q_1)$ holds, then there exists $\lambda \in (0, 1)$ such that $W^u(p(\lambda))$ and $W^s(q(\lambda))$ have a non-transverse intersection.*

Example C. We consider a family $\{f_\lambda\}$ of 2-dimensional diffeomorphism illustrated in Figure 7. The map f_0 is a broken horseshoe shown in [**12**]. Suppose that $p = p(\lambda) := \mathrm{Inv}(K, f_\lambda)$ and $q = q(\lambda) := \mathrm{Inv}(L, f_\lambda)$ are hyperbolic saddles for all $\lambda \in \Lambda$. Then we have a Morse decomposition in PM illustrated in Figure 8, where $\mathrm{Inv}(N^u_p, Pf_\lambda) = E^u_p$, $\mathrm{Inv}(N^s_q, Pf_\lambda) = E^s_q$, $\mathrm{Inv}(N^u_q, Pf_\lambda) = E^u_q$. The images of N_1 under Pf_1 and Pf_0 are shown in Figure 9. We claim that if the Morse decomposition $\{E^u_p, E^s_q, E^s_q\}$ continues over Λ, then there exists $\lambda \in (0, 1)$ such that $W^u(p(\lambda))$ and $W^s(q(\lambda))$ have a non-transverse intersection with respect to f_λ. Namely, f_λ has a heteroclinic tangency. Since this family of Morse decomposition is topologically same as that of *Example B*, we apply Theorem 5.5 and the claim follows.

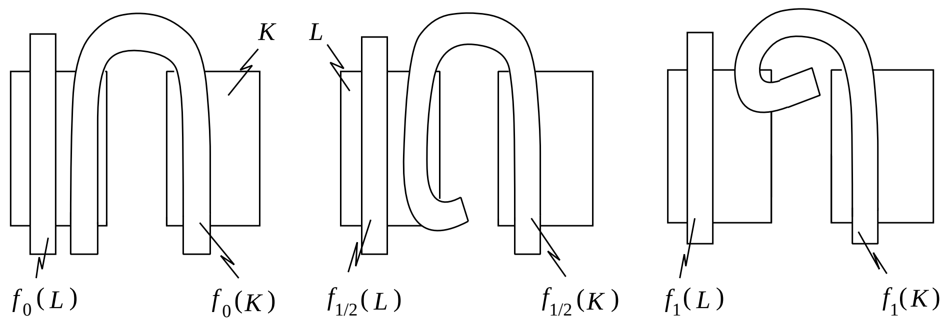

FIGURE 7. f_λ

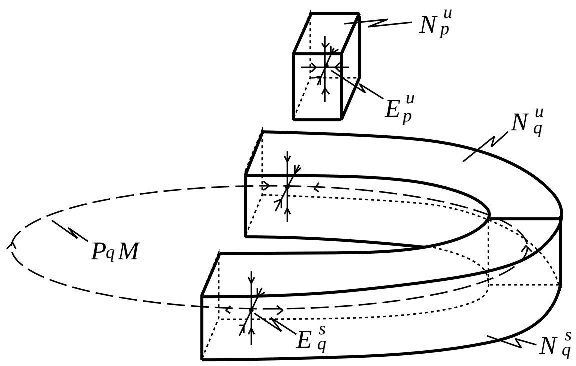

FIGURE 8. isolating neighborhoods for Pf_λ

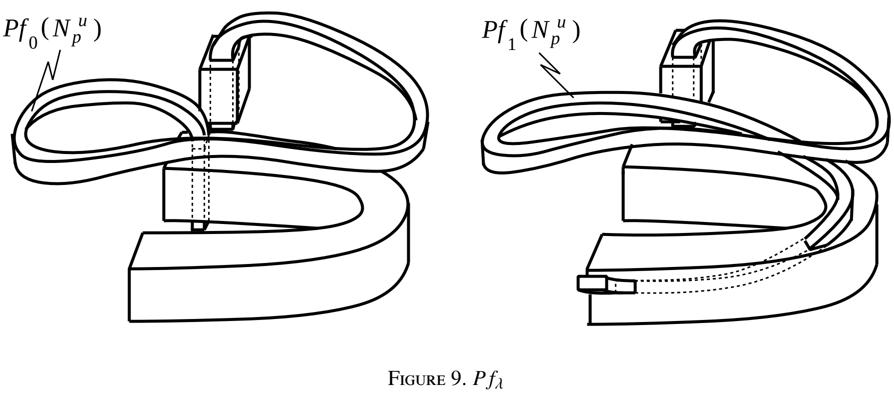

FIGURE 9. Pf_λ

Example D. We consider a family $\{g_\lambda\}$ of 2-dimensional diffeomorphisms illustrated in Figure 10. As in the case of *Example C*, suppose p, q are hyperbolic fixed points for all

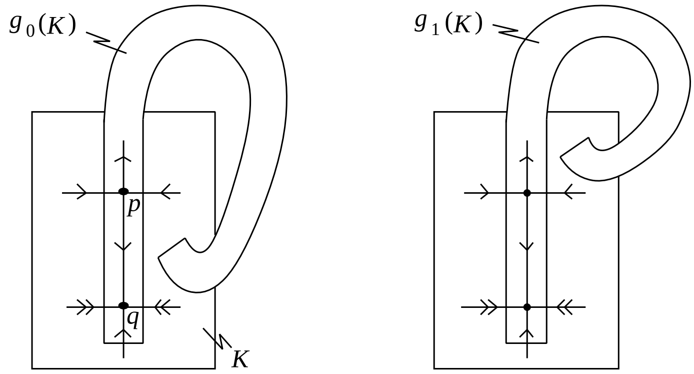

FIGURE 10. g_λ

$\lambda \in \Lambda$ and consider the Morse decomposition $\{E^u_p, E^s_p, E^{ss}_q\}$ in PM, where E^{ss}_q stands for the strong stable direction of q. In Figure 11, $\mathrm{Inv}(N^u_p, Pg_\lambda) = E^u_p$, $\mathrm{Inv}(N^s_p, Pg_\lambda) = E^s_p$ and $\mathrm{Inv}(N^{ss}_q, Pg_\lambda) = E^{ss}_q$ are shown. The images of N^u_p under Pg_0 and Pg_1 are shown in

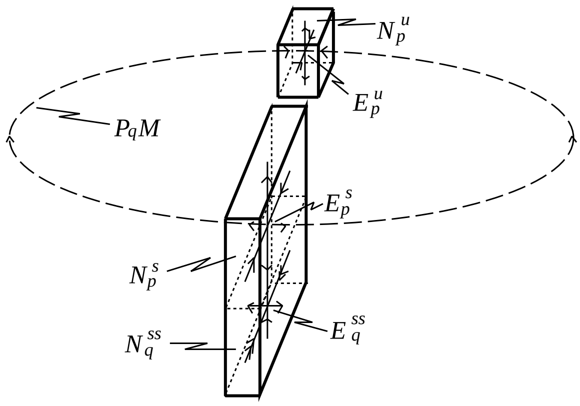

FIGURE 11. isolating neighborhoods for Pg_λ

Figure 12. We claim that if the Morse decomposition $\{E^u_p, E^s_p, E^{ss}_q\}$ continues over Λ, then there exists $\lambda \in (0, 1)$ such that p has a non-transverse homoclinic point with respect to g_λ. Namely, g_λ has a homoclinic tangency. Since this family of Morse decomposition is also topologically the same as that of *Example B*, we apply Theorem 5.5 and the claim follows.

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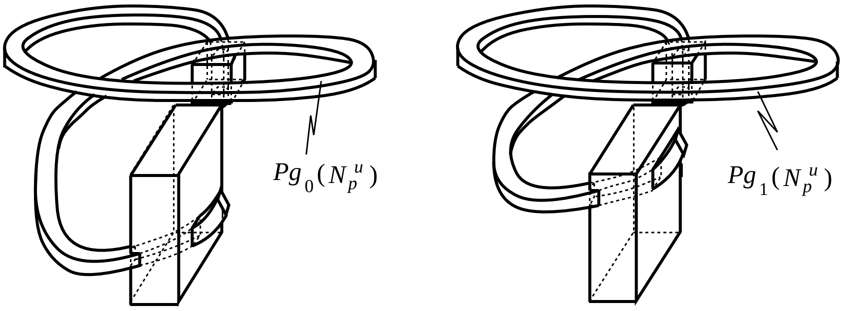

FIGURE 12. Pg_λ

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