MONODROMY AND BIFURCATIONS OF THE HÉNON MAP

ZIN ARAI
DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY

Abstract. We discuss the structure of the parameter space of the Hénon family. Our main tool is the monodromy representation that assigns an automorphism of the full shift to each loop in the hyperbolic parameter locus of the complex Hénon family. We show that the monodromy carries the information of the bifurcations taking place inside the loop, and this enables us to construct pruning fronts, a generalization of kneading theory to the real Hénon family. Furthermore, assuming that there exist infinitely many non-Wieferich prime numbers (it suffices to assume "abc conjecture"), we show that monodromy automorphisms must satisfy a certain algebraic condition, which imposes geometric restrictions on the structure of the parameter space.

1. Monodromy of the *complex* Hénon Map

We discuss the structure of the parameter space of the *complex* Hénon map

$$H_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2 : (x, y) \mapsto (x^2 + c - ay, x), \quad (a, c) \in \mathbb{C}^2$$

and the pruning front of the *real* Hénon map $H_{a,c}|_{\mathbb{R}^2}$ for $(a, c) \in \mathbb{R}^2$.

Let us define the filled Julia set by

$$K^{\mathbb{C}}_{a,c} := \{p \in \mathbb{C}^2 : \{H^n_{a,c}(p)\}_{n\in\mathbb{Z}} \text{ is bounded}\}$$

The set $K^{\mathbb{C}}_{a,c}$ is compact and invariant with respect to $H_{a,c}$. We denote its real slice $K^{\mathbb{C}}_{a,c} \cap \mathbb{R}^2$ by $K^{\mathbb{R}}_{a,c}$. Note that when the parameters are real, that is, $(a, c) \in \mathbb{R}^2$, $K^{\mathbb{R}}_{a,c}$ is invariant with respect to $H_{a,c}$.

More precisely, we study the following sets:

$$\mathcal{H}^{\mathbb{C}} := \{(a, c) \in \mathbb{C}^2 : H_{a,c}|K^{\mathbb{C}}_{a,c} \text{ is a hyperbolic full horseshoe}\},$$
$$\mathcal{H}^{\mathbb{R}} := \{(a, c) \in \mathbb{R}^2 : H_{a,c}|K^{\mathbb{R}}_{a,c} \text{ is a hyperbolic full horseshoe}\}.$$

Here we mean by a hyperbolic full horseshoe a uniformly hyperbolic invariant set which is topologically conjugate to the full shift map σ defined on $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$, the space of bi-infinite sequences of two symbols. From the classical theorem of Devaney and Nitecki (and also by Hubbard and Overste-Vorth), we know that for every $a \neq 0$, if $|c|$ is large enough, then $(a, c) \in \mathcal{H}^{\mathbb{C}}$.

Let us fix a basepoint $(a_0, c_0) \in \mathcal{H}^{\mathbb{R}}$ and a topological conjugacy $h_0 : K^{\mathbb{C}}_{a_0,c_0} \to \Sigma_2$. Given a loop $\gamma : [0, 1] \to \mathcal{H}$ based at (a_0, c_0), we construct a continuous family of conjugacies $h_t : K^{\mathbb{C}}_{\gamma(t)} \to \Sigma_2$ along γ. Then we define $\rho(\gamma) := h_1 \circ (h_0)^{-1} : \Sigma_2 \to \Sigma_2$. It is easy to see that ρ defines a group homomorphism $\rho : \pi_1(\mathcal{H}, (a_0, c_0)) \to \mathrm{Aut}(\Sigma_2)$ where $\mathrm{Aut}(\Sigma_2)$ is the group of the automorphisms of Σ_2. We call ρ the monodromy homomorphism. Let us denote the image of ρ by Γ.

In analogy with the one dimensional complex dynamics, John Hubbard raised the following conjecture, which implies that the topological structure of $\mathcal{H}$ should be extremely rich.

Hubbard's Conjecture. $\Gamma \cup \{\sigma\}$ *generates* $\mathrm{Aut}(\Sigma_2)$.

In a previous paper [1], the author has shown that Γ contains non-trivial elements and in particular, it contains elements of infinite order. Roughly saying, this implies "Γ is large". The main purpose of this paper is to say the contrary, "Γ is not too large".

To give the precise statement of our result, we need to recall a number-theoretic notion.

Definition 1. A prime number p is a Wieferich prime if p^2 divides $2^{p-1} - 1$.

Note that Fermat's little theorem says that every odd prime p divides $2^{p-1} - 1$. The only known Wieferich primes are 1093 and 3511 and it is proven that there is no other Wieferich prime smaller than 6.7×10^{15}. It has been conjectured that there exists infinitely many Wieferich primes, but it has not been proved. Moreover, it is still not known if there exist infinitely many *non*-Wieferich primes.

Theorem 2. *Assume that there exists infinitely many non-Wieferich primes. Then the image* Γ *is contained in* $\mathrm{Inert}(\Sigma_2)$*, the subgroup of inert automorphisms. In particular,* Γ *does not contain any iteration of* σ.

Here we want to remark that it is proven by J. H. Silverman that if the *abc conjecture* holds, then there exist infinitely many many non-Wieferich primes [7]. Furthermore, S. Mochizuki recently announced a proof of the *abc conjecture* [6].

2. Proof of Theorem 2

To prove our theorem, we make use of the following algebraic condition on the automorphism of Σ_2. Let $\phi \in \mathrm{Aut}(\Sigma_2)$ be an automorphism. We begin by defining two numbers associated to ϕ.

The first one, the n-th sign number $s_n(\phi)$ of ϕ, is defined to be 0 or 1 according to the action of ϕ on periodic orbits: $s_n(\phi) = 0$ if the permutation induced by ϕ on the set of periodic orbits of least period n is an even permutation, and $s_n(\phi) = 1$ if it is an odd permutation.

While the sign number computes how the orbits are interchanged, the n-th gyration number $g_n(\phi)$, the second number, is related to which extent each orbit is "gyrated" by ϕ. To define it, we choose an arbitrary element $x_U \in U$ for each periodic orbit U of least period n. Since $\phi(x_U)$ and $x_{\phi(U)}$ are in the same periodic orbit, we can find an integer $k(U)$ such that $\phi(x_U) = \sigma^{k(U)}(x_{\phi(U)})$. Then we define the n-th gyration number $g_n(\phi) \in \mathbb{Z}_n$ by

$$g_n(\phi) := \sum_U k(U) \mod n$$

where the sum is taken over all the periodic orbit U of least period n.

Definition 3. The k-th sign-gyration homomorphism ($k \geq 2$) is defined by

$$\mathrm{SGCC}_k : \mathrm{Aut}(\Sigma_2) \to \mathbb{Z}_k : \phi \mapsto g_k(\phi) + \sum_{i>0} s_{k/2^i}(\phi)$$

where we take $s_{k/2^i} = 0$ whenever $k/2^i$ is not an integer, and for even k we consider $\mathbb{Z}_2$ as a subgroup $\{0, k/2\}$ of $\mathbb{Z}_k$. We say that ϕ satisfies the sign-gyration compatibility condition (SGCC) if $\mathrm{SGCC}_k(\phi) = 0$ for all $k \geq 2$.

In the context of the monodromy homomorphism, this algebraic property has a geometric and dynamical interpretation. That is, the monodromy action can be described by the bifurcation of the periodic points.

For $n \geq 1$, define polynomials F_n and G_n by

$$H^n_{a,c}(x, y) = (F_n(x, y, a, c), G_n(x, y, a, c)).$$

Then for a periodic sequence $s \in \Sigma_2$ of length n, denote by $\tilde{X}_s$ the component of the affine algebraic variety ($\subset \mathbb{C}^4 = \{(x, y, a, c)\}$) defined by

$$F_n(x, y, a, c) - x = 0$$
$$G_n(x, y, a, c) - y = 0$$

containing $h_{(a_0,c_0)}(s)$ and by X_s the intersection of $\tilde{X}_s$ and the algebraic variety defined by

$$F_n(x, y, a, c) - x = 0$$
$$G_n(x, y, a, c) - y = 0$$
$$\det\begin{pmatrix} \frac{dF_n}{dx} - 1 & \frac{dF_n}{dy} \\ \frac{dG_n}{dx} & \frac{dG_n}{dy} - 1 \end{pmatrix} = 0.$$

The variety $\tilde{X}_s$ describes the continuation of the periodic points having the code s over the parameter space. Note that it may happen that $\tilde{X}_s = \tilde{X}_t$ for different periodic codes s and t of length n. On the other hand, the variety X_s corresponds to the parameter and the location of non-transversal periodic orbits which appears as the continuation of the periodic point having the code s.

Let V_s be the projection of X_s to (a, c) space. Since the continuation of transversal periodic orbit is unique, it is easy to prove the following theorem.

Theorem 4. *Let γ be a loop in $\mathcal{H}$ and $s \in \Sigma_2$ a periodic sequence. If*

$$\rho(\gamma)(s) \neq s$$

then γ is non-trivial in $\pi_1(\mathbb{C}^2 \setminus V_s)$.

Now we consider the relation between SGCC and the bifurcation of periodic points. We begin by the simplest case: the fixed points and periodic orbits of period two. Recall that in $\mathrm{Aut}(\Sigma_2)$, there are two fixed points $\overline{0}$ and $\overline{1}$, and only one periodic orbit of period two, $\overline{01}$. Then $SGCC_2(\phi) = 0$ implies that ϕ interchanges the two fixed points if and only if it rotates the period 2 orbit. We claim that this property holds for every $\phi \in \Gamma$.

Lemma 5. *Let $\phi \in \Gamma$. Then ϕ satisfies* $\mathrm{SGCC}_2(\phi) = 0$.

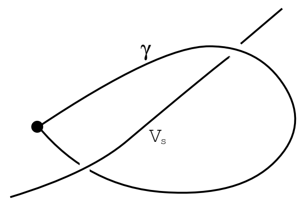

FIGURE 1. non-trivial loop γ

Proof. Let $\phi = \rho(\gamma)$ for a loop γ in $\mathcal{H}$. It is easy to see that

$$V_{\overline{0}} = V_{\overline{1}} = \{(a, c) \mid c = \frac{1}{4}(b+1)^2\}$$

and

$$V_{\overline{01}} = \{(a, c) \mid c = -\frac{3}{4}(b+1)^2\}.$$

These varieties are homotopic in $\mathbb{C}^2\mathcal{H}$ via the family $V_\lambda = \{(a, c) \mid c = \lambda(b+1)^2\}$ and therefore the loop γ determines the same element in

$$\pi_1(\mathbb{C}^2 \setminus V_{\overline{0}}) \cong \pi_1(\mathbb{C}^2 \setminus V_{\overline{01}}) \cong \mathbb{Z}.$$

Then by direct computation, we can show that ϕ interchanges $\overline{0}$ and $\overline{1}$ if and only if γ determines an odd element in $\mathbb{Z}$. On the other hand, ϕ gyrates $\overline{01}$ when the same condition holds. This prove our claim. □

What we want to say is that the same property holds for all periodic orbits of higher periods. However, it is quite difficult to prove this in a similar manner since we need to compute the topology of V_s for long sequences s, which are algebraic varieties defined by polynomials with extremely high degrees.

Therefore, we first consider the periodic orbits of prime periods, for which we can control the topology of V_s and then extends the result to all natural numbers using an algebraic consideration.

Lemma 6. *Let $\phi \in \Gamma$. Then ϕ satisfies* $\mathrm{SGCC}_p(\phi) = 0$ *for all prime number p.*

Proof. Since p is a prime number, a bifurcation curve β involving p-periodic orbit lies on the boundary of "main cardioid" defined by

$$\mathcal{M} = \{(a, c) \mid \text{ there exists an attracting fixed point of } H_{a,c}\}.$$

By a simple calculation, we can construct a homotopy equivalence between $\mathcal{M}$ and the saddle-node bifurcation curve of fixed points. It follows that if a loop $\gamma \in \mathcal{H}$ turns around β, it should also turns around $\bar{\beta}$, the complex conjugate of β, for exactly the same times. Thus the rotation around β and $\bar{\beta}$ cancel each other out. □

Lemma 7. *For an odd prime p,* $\mathrm{SGCC}_p(\sigma) = 0$ *if and only if p is a Wieferich prime.*

Proof. Since the number of fixed points of σ^p is 2^p and this number counts all points in periodic orbits of least period p and two fixed points $\overline{0}$ and $\overline{1}$ of σ, the number of periodic *orbits* of of least period p is exactly

$$\frac{2^p - 2}{p} = 2 \cdot \frac{2^{p-1} - 1}{p}.$$

Since each periodic orbit is gyrated one step ahead by the shift map, the gyration number is

$$2 \cdot \frac{2^{p-1} - 1}{p} \pmod{p},$$

which is 0 when p is a Wieferich prime, by definition. □

Proof of Theorem 2. First we recall that there is a decomposition

$$\mathrm{Aut}(\Sigma_2) = \mathbb{Z} \oplus \mathrm{Inert}(\Sigma_2)$$

where $\mathbb{Z}$ is generated by σ [4]. Here we denote this decomposition as a direct sum since σ commutes with $\mathrm{Inert}(\Sigma_2)$ although the group $\mathrm{Aut}(\Sigma_2)$ itself is non-commutative.

Let $\phi \in \Gamma$. By the decomposition above, we can write it as $\phi = (\sigma^k, \phi')$ where k is an integer and $\phi' \in \mathrm{Inert}(\Sigma_2)$. Then

$$\sigma^{-k} \circ \phi = (\sigma^{-k} \circ \sigma^k, \phi') = (\mathrm{id}, \phi') \in \mathrm{Inert}(\Sigma_2)$$

and therefore we have $\mathrm{SGCC}_m(\sigma^{-k} \circ \phi) = 0$ for all $m \geq 2$. We also have $\mathrm{SGCC}_p(\phi) = 0$ for all prime p. Since

$$g_p(\sigma^{-k}) = (-k) \cdot g_p(\sigma) = (-k) \cdot \frac{2^{p-1} - 1}{p} \pmod{p},$$

by choosing a sufficiently large non-Wieferich prime p which is relatively prime to k (this is possible because of Lemma 7), we can assume $g_p(\sigma^{-k}) \neq 0$. Since σ does not permute periodic orbits, this implies $\mathrm{SGCC}_p(\sigma^{-k}) \neq 0$. Finally, we have

$$0 = \mathrm{SGCC}_p(\sigma^{-k} \circ \phi) = \mathrm{SGCC}_p(\sigma^{-k}) + \mathrm{SGCC}_p(\phi) = \mathrm{SGCC}_p(\sigma^{-k}) \neq 0.$$

This is a contradiction. □

3. Application to pruning fronts of the *real* Hénon Map

The key to relate the monodromy of the complex Hénon map to the pruning front of the real Hénon map is the following theorem.

Theorem 8 (ZA [1])**.** *For $(a, c) \in \mathcal{H} \cap \mathbb{R}^2$ and a path α connecting (a_0, b_0) to (a, c), define $\gamma := \alpha \cdot (\bar{\alpha})^{-1}$. Then $\rho(\gamma)$ is an involution and $H_{a,c} : K^{\mathbb{R}}_{a,c} \to K^{\mathbb{R}}_{a,c}$ is topologically conjugate to $\sigma|_{\mathrm{Fix}(\rho(\gamma))} : \mathrm{Fix}(\rho(\gamma)) \to \mathrm{Fix}(\rho(\gamma))$.*

By virtue of the theorem, we can define the pruning front for these real Hénon map using the monodromy action.

Definition 9. For a hyperbolic parameter value $(a, c) \in \mathcal{H} \cap \mathbb{R}^2$, we choose a path α connecting (a_0, b_0) to (a, c) in $\mathcal{H}$. Define the pruning front of $H_{a,c}$ to be the set of symbolic sequences

$$P := ([0] \cap (\rho(\gamma))^{-1}[1]) \cup ([1] \cap (\rho(\gamma))^{-1}[0]).$$

where $\gamma = \alpha \cdot (\bar{\alpha})^{-1}$.

By the previous theorem, the pruning front P completely determines the dynamics of the *r*eal Hénon map acting on $K^{\mathbb{R}}_{a,c}$.

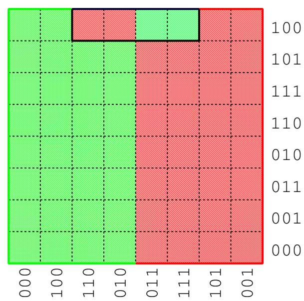

FIGURE 2. The pruning front: two block 10100 and 11100 are interchanged.

Although the definition of the pruning front depends on the choice of α, we can precisely describe how it depends thorough the following proposition.

Proposition 10. *Let α and β be paths connecting to (a_0, b_0) to (a, c) in $\mathcal{H}$. Let $\gamma_\alpha := \alpha \cdot (\bar{\alpha})^{-1}$ and $\gamma_\beta := \beta \cdot (\bar{\beta})^{-1}$. Then*

$$\rho(\gamma_\beta) = \rho(\delta)^{-1} \circ \rho(\gamma_\alpha) \circ \rho(\delta)$$

where $\delta := \beta \cdot \alpha^{-1}$.

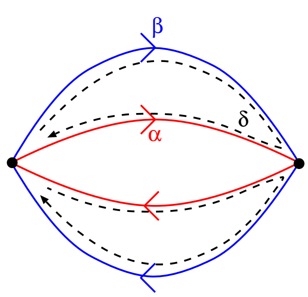

FIGURE 3. the loop $\delta = \beta \cdot \alpha^{-1}$

It is known that SGCC holds for any automorphisms of a full shift which is a composition of finite-order automorphisms. It follows that

Proposition 11. *If γ is symmetric (i.e. $\bar{\gamma} = \gamma$) then $\rho(\gamma)$ must satisfy SGCC.*

This implies there is a restriction on the shape of pruning fronts. For example, although $0{}^{0}_{1}10$ is possible (and in fact, is the pruning front for $a = -1, c = -5$), $0{}^{0}_{1}100$ is not allowed.

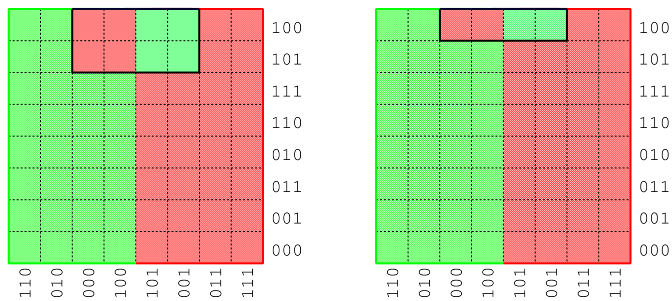

FIGURE 4. Orientation reversing case: $0{}^{0}_{1}10$ (left, allowed) and $0{}^{0}_{1}100$ (right, prohibited).

This can be proven by computing the sign and gyration number for periodic orbits of lower periods. Figure 3 illustrates these possible and prohibited pruning fronts using the identification of Σ_2 with the orientation reversing horseshoe. It is interesting to see that in the orientation preserving case, the realizability of these geometric configuration are interchanged. In fact, Figure 3 shows a possible pruning front $1{}^{0}_{1}100$ and prohibited one $1{}^{0}_{1}10$ using the identification of Σ_2 with the orientation preserving horseshoe.

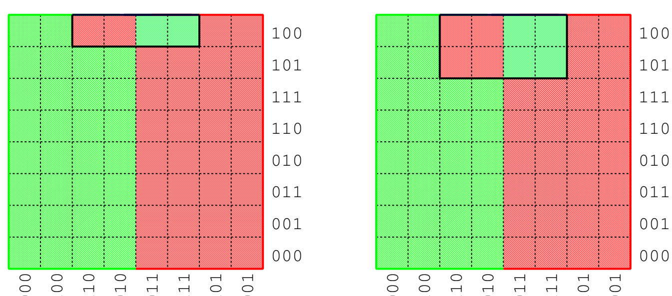

FIGURE 5. Orientation preserving case: $1{}^{0}_{1}100$ (left, allowed) and $1{}^{0}_{1}10$ (right, prohibited)

Recently, Nicholas Long proved the following related result posing an algebraic restriction on subshifts that can be the fixed point set of an involution.

Theorem 12 (Long [5])**.** *If a SFT Y is the fixed point set of an inert involution of a mixing shift of finite type X, then* $\mathrm{Per}(X)\backslash\mathrm{Per}(Y)$ *is the disjoint union of 2-cascades.*

Combined with our results, this theorem suggests that in a hyperbolic SFT that appears via pruning, if a periodic orbit P is missing, then all periodic orbits on the period-doubling cascade beginning at P should also be missing. The detailed analysis of this phenomena will appear elsewhere.

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