Mayer-Vietoris sequences and coverage problems in sensor networks

(Running title: Mayer-Vietoris sequences in sensor networks)

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Abstract

A coverage problem of sensor networks is studied in this paper. Following the recent work by Ghrist et al., in which a computational topological method is applied for the coverage problem, this paper presents an algorithm to compute the first homology group of a planar Rips complex in a distributed manner. The key idea is to decompose the Rips complex into small pieces of subcomplexes, and to make use of Mayer-Vietoris sequences in order to sum up local homology groups. Since the coverage condition can be described by the first homology group, the proposed algorithm enables a distributed computation for the condition and turns out to be an efficient method for the purpose.

Key words. Mayer-Vietoris sequences, sensor networks, coverage problems

1 Introduction

Wireless sensor network, which consists of a number of sensor nodes with signal processing and communication capabilities, has been drawing a lot of research interest since its origins in the early nineties. What make the sensor network different from conventional broadband wireless communications systems are crucial requirements for the sensor nodes of low cost and high energy efficiency, since a large number of battery-operated sensor nodes have to be utilized in many applications. Radio communication and signal processing are often major power-intensive operations in the sensor nodes [13], therefore the communications range is strictly restricted and reducing the amount of signal processing in the sensor nodes is of great importance in the design of the wireless sensor network.

One of the fundamental problems in the wireless sensor network is a coverage problem. This is because the coverage problem is closely related to both the network deployment problem, which is directly related to the cost of the network, and the scheduling problem of sleep and wake-up modes for low power consumption. Moreover, the coverage is also regarded as one of the important measures for the quality of service provided by the wireless sensor network.

For the coverage problems in sensor networks, Ghrist et al. have recently presented computational topological methods [3][4][5]. Their idea is to capture global information about the covering from homology groups of a certain geometrical object which can be constructed from local connection information between sensors. One of the advantages of the topological methods is that we do not assume information of absolute locations or orientations for the sensors, i.e., the methods are coordinate-free. Moreover, there are also no probabilistic assumptions such as a uniform distributions of sensors in a given domain. These assumptions, which are sometimes regarded as unrealistic settings, are necessary for conventional techniques based on computational geometry (e.g.[7][10][11][17])

or probabilistic approaches (e.g.[9][14][16]), and so the topological methods have potentials for various practical applications of sensor networks.

Following the paper [3], let us here describe the settings of a coverage problem we treat.

Assumption

A1 Suppose P is a finite set of sensors(or also called nodes) in a compact connected domain $D \subset \mathbb{R}^2$.

A2 Two sensors $v, w \in P$ can communicate with each other when their distance is less than broadcast radius r_b.

A3 Each sensor has radially symmetric covering domains of cover radius $r_c \geq r_b/\sqrt{3}$.

A4 The boundary ∂D is assumed to be connected and piecewise-linear with nodes marked fence nodes. The distances between any adjacent two fence nodes are less than r_b.

Let $B(v; r_c)$ be a disk with radius r_c centered at $v \in P$ and $U = \bigcup_{v \in P} B(v; r_c)$. Then the coverage problem is formulated to studying $D \subset U$. In this setting, the essential step of the topological methods is to build up a Rips complex [6][15], which can be constructed by local connection information determined by r_b and possesses reasonable covering information U. Here, the Rips complex $\mathcal{R}_{r_b}$ of P is an abstract simplicial complex and is defined as follows. All nodes in P are assigned to be 0-simplexes, and a $(k+1)$-tuple of nodes $v_0, \cdots, v_k$ determines a k-simplex $|v_0 \cdots v_k|$ in $\mathcal{R}_{r_b}$ if and only if any two nodes of them are within the distance r_b. In this paper, we abbreviate $\mathcal{R}_{r_b}$ to $\mathcal{R}$, and treat their homology groups $H_*(\mathcal{R})$ as integer coefficients (see e.g.[12] for standard homology theory).

Let us note that, from **A4**, the boundary ∂D induces a one dimensional simplicial subcomplex $\mathcal{F}$ in $\mathcal{R}$. Then, in the above setting, de Silva and Ghrist have presented the following sufficient condition for the coverage $D \subset U$:

Theorem 1 [3] *The sensor covering U contains D if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\delta_2[\alpha] \neq 0$ in $H_1(\mathcal{F})$, where δ_2 is the connecting homomorphism arising in the long exact sequence of the pair $(\mathcal{R}, \mathcal{F})$:*

$$\longrightarrow H_2(\mathcal{F}) \longrightarrow H_2(\mathcal{R}) \longrightarrow H_2(\mathcal{R}, \mathcal{F}) \xrightarrow{\delta_2} H_1(\mathcal{F}) \longrightarrow .$$

The most important point of this theorem is, as we already mentioned, that it does not require absolute coordinates of the sensors or probabilistic assumptions. The only necessary information is a graph structure for the set of nodes P, which is equivalent to the 1-skelton of $\mathcal{R}$ and is available from the conditions of communications between sensors.

Theorem 1 itself is a well worthy result in the sense of providing a novel method which can deal with difficulties for conventional methods, however, it includes several problems, especially for implementations, as mentioned in [3]. The most crucial problem among them is that, although the construction of the Rips complex only requires the local connection information, a naive computation of homology groups needs centralized processes. It should be avoided in the following sense.

First, it is not desirable for all sensors to transmit local connection information to a central base station, because this procedure requires huge amount of communications or long distance communications, and is, after all, equivalent to assuming the global information, like computational geometrical approaches. Due to the restriction of batteries for sensors, such communications should not be adopted. Second, even if one can obtain the integrated connection information at one place, the present computations of homology groups is in the range of orders from quadratic to quintic with respect to the number of simplices ([8]), and we can not ignore this cost for practical implementations.

One possibility to overcome this problem is to adopt a distributed way of homology computations. That is to say, we would like to somehow sum up local homology groups of small pieces of subregions in parallel in order to obtain the global homology group. This is the central subject to discuss in this paper.

Our main result in this paper is to present a parallelized algorithm for checking the sufficient condition for the coverage under the assumptions **A1**-**A5** ("non-pinching condition" **A5** and its verifiable substitution $\widetilde{\mathbf{A5}}$ are explained in section 3). The technique is based on distributed homology computations for $H_1(\mathcal{R})$, and deeply relies on Mayer-Vietoris sequences (e.g.[12]), which are classical tools in algebraic topology to calculate a homology group of a union of two appropriate topological objects. Our algorithm consists of two parts. One is to derive a decomposition of a given Rips complex $\mathcal{R} = \cup_{k=1}^{K} \mathcal{R}_k$ (Algorithm 5), and the other is to sum up local homology groups $H_1(\mathcal{R}_k)$ to $H_1(\mathcal{R})$ (Algorithm 7).

We here note that it is not trivial whether we can always calculate $H_1(\mathcal{R}_1 \cup \mathcal{R}_2)$ by the corresponding Mayer-Vietoris sequence, since this exact sequence only gives relationships among $H_*(\mathcal{R}_1), H_*(\mathcal{R}_2), H_*(\mathcal{R}_1 \cap \mathcal{R}_2)$, and $H_*(\mathcal{R}_1 \cup \mathcal{R}_2)$. The key for the computability of $H_1(\mathcal{R}_1 \cup \mathcal{R}_2)$, which is shown in Proposition 3, results from the following geometrical good property of a planer Rips complex.

Suppose $p : \mathcal{R} \to \mathbb{R}^2$ is a projection map which maps each simplex in $\mathcal{R}$ affinely onto the convex hull of its nodes in $\mathbb{R}^2$. The projection of $\mathcal{R}$ is called the shadow and denoted by S. Let $\pi_1(\mathcal{R})$ and $\pi_1(S)$ be the fundamental groups whose base points are naturally identified by p. The projection map p induces a homomorphism $\pi(p) : \pi_1(\mathcal{R}) \to \pi_1(S)$ on these fundamental groups. Then, the following theorem holds:

Theorem 2 [1] *Let $\mathcal{R}$ be a Rips complex generated by a finite number of points in $\mathbb{R}^2$. Then $\pi(p) : \pi_1(\mathcal{R}) \to \pi_1(S)$ is an isomorphism. Especially, the first homology group $H_1(\mathcal{R})$ is free.*

Let us also note the relationship between the sufficient condition for the coverage in Theorem 1 and the first homology group. Due to the help of Theorem 2 again and the non-pinching assumption **A5** or $\widetilde{\mathbf{A5}}$ with slight modifications of fence nodes, Proposition 4 states that studying $H_1(\mathcal{R})$ is enough to check the sufficient condition in Theorem 1.

Therefore Algorithm 7 enables a distributed computations for the coverage condition. On the other hand, $H_1(\mathcal{R})$ itself is important to repair holes in the covering as discussed in [3].

This paper is organized as follows. In section 2, we prove a proposition, which is necessary for the construction of our algorithm via Mayer-Vietoris sequences. We show several equivalent sufficient conditions for the coverage of a non-pinching domain D in section 3. Section 4 is the main part of this paper and is devoted to present the distributed $H_1(\mathcal{R})$ computations, which enable us to parallelize the computations for checking a sufficient condition of the coverage. In section 5, the estimates for computational costs and further extensions of the algorithm are discussed.

2 Mathematical preliminary

The purpose of this section is to prove the following proposition.

Proposition 3 *Let $\mathcal{R}_1, \mathcal{R}_2$ be Rips complexes satisfying*

$$H_0(\mathcal{R}_1) \cong H_0(\mathcal{R}_2) \cong \mathbb{Z}, \quad H_0(\mathcal{R}_1 \cap \mathcal{R}_2) \cong \mathbb{Z}^r, \quad H_1(\mathcal{R}_1) \cong \mathbb{Z}^n, \quad H_1(\mathcal{R}_2) \cong \mathbb{Z}^m.$$

Then the first homology group of $\mathcal{R}_1 \cup \mathcal{R}_2$ is given by

$$H_1(\mathcal{R}_1 \cup \mathcal{R}_2) \cong \begin{cases} \mathbb{Z}^{n+m-L} & (r = 0) \\ \mathbb{Z}^{n+m-L+r-1} & (r \geq 1) \end{cases}$$

where L denotes the rank of the homomorphism i_1 that appears in the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_1(\mathcal{R}_1 \cap \mathcal{R}_2) \xrightarrow{i_1} H_1(\mathcal{R}_1) \oplus H_1(\mathcal{R}_2) \longrightarrow H_1(\mathcal{R}_1 \cup \mathcal{R}_2) \longrightarrow \cdots.$$

Proof. We consider the Mayer-Vietoris sequence for $\mathcal{R}_1$ and $\mathcal{R}_2$

$$\begin{tikzcd} \cdots \arrow[r, "\delta_2"] & H_1(\mathcal{R}_1 \cap \mathcal{R}_2) \arrow[r, "i_1"] & H_1(\mathcal{R}_1) \oplus H_1(\mathcal{R}_2) \arrow[r, "j_1"] & H_1(\mathcal{R}_1 \cup \mathcal{R}_2) \arrow[dll, "\delta_1"] & \\ & H_0(\mathcal{R}_1 \cap \mathcal{R}_2) \arrow[r, "i_0"] & H_0(\mathcal{R}_1) \oplus H_0(\mathcal{R}_2) \arrow[r, "j_0"] & H_0(\mathcal{R}_1 \cup \mathcal{R}_2) \arrow[r] & 0. \end{tikzcd}$$

The short exact sequence

$$0 \longrightarrow \text{Im } j_1 \longrightarrow H_1(\mathcal{R}_1 \cup \mathcal{R}_2) \xrightarrow{\delta_1} \text{Im } \delta_1 \longrightarrow 0$$

splits since Im δ_1, being a subgroup of $H_0(\mathcal{R}_1 \cap \mathcal{R}_2) \cong \mathbb{Z}^r$, is free. Therefore we have

$$H_1(\mathcal{R}_1 \cup \mathcal{R}_2) \cong \text{Im } j_1 \oplus \text{Im } \delta_1.$$

By virtue of Theorem 2, $H_1(\mathcal{R}_1 \cup \mathcal{R}_2)$ is free and hence so is Im j_1. Thus, for computing Im j_1, it suffices to know the rank of i_1. That is, we have

$$\text{Im } j_1 \cong (H_1(\mathcal{R}_1) \oplus H_1(\mathcal{R}_2))/\text{Ker } j_1 \cong \mathbb{Z}^{n+m}/\text{Im } i_1 \cong \mathbb{Z}^{n+m-L}.$$

Next, we compute Im δ_1, which depends on r. If $r = 0$ then Im δ_1 is trivial and nothing is to be proved. Assume $r \geq 1$. Since $H_0(\mathcal{R}_1 \cap \mathcal{R}_2)$ is non-trivial, $\mathcal{R}_1$ and $\mathcal{R}_2$ must have at least one 0-simplex in common, and therefore it follows from $H_0(\mathcal{R}_1) \cong H_0(\mathcal{R}_2) \cong \mathbb{Z}$ that $H_0(\mathcal{R}_1 \cup \mathcal{R}_2) \cong \mathbb{Z}$. Therefore, the bottom row of the Mayer-Vietoris sequence induces

$$0 \longrightarrow \text{Im } \delta_1 \longrightarrow \mathbb{Z}^r \xrightarrow{i_0} \mathbb{Z}^2 \xrightarrow{j_0} \mathbb{Z} \longrightarrow 0$$

which implies Im $\delta_1 \cong \mathbb{Z}^{r-1}$. ∎

We note that in view of Theorem 2, the assumptions on the first homology groups H_1 in Proposition 3 is not restrictive at all.

3 Sufficient conditions for the coverage

First of all, we add one assumption on the locations of nodes. This assumption guarantees "non-pinching" via the Rips shadow around the boundary as shown in Figure 1. From the mathematical viewpoint, the condition is precisely described as follows:

==== Figure 1 ====

Assumption

A5 The Rips shadow S is included in D.

As you immediately observe, this condition is difficult to check from the local information of each sensor. However, the following verifiable mild condition may be sufficient in practice from the engineering viewpoint.

For each fence node v, let us denote its two neighboring fence nodes on both sides as $v_{l_1}, v_{l_2}, v_{r_1}, v_{r_2}$. A non-pinching condition for v is introduced in such a way that all fence nodes except for $v_{l_1}, v_{l_2}, v_{r_1}, v_{r_2}$ do not have edges to v (See Figure 1). Then **A5** can be replaced by a verifiable condition:

$\widetilde{\textbf{A5}}$ All fence nodes satisfy the non-pinching condition.

==== Figure 2 ====

Rigorously speaking, it is possible to show exceptional examples like in Figure 2 such that $\widetilde{\mathbf{A5}}$ does not induce **A5**. The reason for the existence of such examples is related to the definition of the fence nodes, but we can remove these singular situations by $\widetilde{\mathbf{A5}}$ with slight modifications of fence nodes, e.g., dealing with v as a fence node in Figure 2. Another example of the exception can be considered as depicted in Figure 3. However, we can ignore the exception as we will see in the end of this section. Therefore, for the practical purposes, $\widetilde{\mathbf{A5}}$ is enough to guarantee **A5**.

==== Figure 3 ====

Then, the sufficient condition for coverage in Theorem 1 can be described in the following equivalent forms.

Proposition 4 *Under the assumption* **A1**-**A5**, *the following four conditions are equivalent:*

1. *There exists* $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ *such that* $\delta_2[\alpha] \neq 0$ *in* $H_1(\mathcal{F})$.
2. $j_1 : H_1(\mathcal{R}) \to H_1(\mathcal{R}, \mathcal{F})$ *is an isomorphism.*
3. *The inclusion map* $i_1 : H_1(\mathcal{F}) \to H_1(\mathcal{R})$ *is the trivial map* $i_1 = 0$.
4. $H_1(\mathcal{R}) = 0$.

Here j_1 *and* i_1 *are homomorphisms in the long exact sequence of the pair* $(\mathcal{R}, \mathcal{F})$

$$\begin{aligned} &\longrightarrow H_2(\mathcal{F}) \xrightarrow{i_2} H_2(\mathcal{R}) \xrightarrow{j_2} H_2(\mathcal{R}, \mathcal{F}) \xrightarrow{\delta_2} \\ &\xrightarrow{\delta_2} H_1(\mathcal{F}) \xrightarrow{i_1} H_1(\mathcal{R}) \xrightarrow{j_1} H_1(\mathcal{R}, \mathcal{F}) \xrightarrow{\delta_1} H_0(\mathcal{F}) \xrightarrow{i_0} H_0(\mathcal{R}) \longrightarrow . \end{aligned}$$

Proof. The proofs for $2 \Rightarrow 3$ and $4 \Rightarrow 1$ are trivial from the exactness.
$(1 \Rightarrow 2)$: Since Im δ_2 is a subgroup of $H_1(\mathcal{F}) \cong \mathbb{Z}$, it takes a form Im $\delta_2 \cong c\mathbb{Z}$. From the assumption, c is a nonzero integer. Since

$$H_1(\mathcal{F})/\mathrm{Ker}\ i_1 \cong H_1(\mathcal{F})/\mathrm{Im}\ \delta_2 \cong \mathbb{Z}/c\mathbb{Z} \cong \mathbb{Z}_c$$

and $H_1(\mathcal{R})$ is free, the integer c must be one. This leads to $i_1 = 0$, and so j_1 is injective. In addition, Ker $i_0 = 0$ results in $\delta_1 = 0$, so implies the surjectivity of j_1. This leads to the statement 2.
$(3 \Rightarrow 4)$: First, note that $S \subset D$ holds from **A5**. To prove $H_1(\mathcal{R}) = 0$, it suffices to show that $S = D$. Because, this implies $\pi_1(S)$ is trivial and then it follows from Theorem 2 that so is $\pi_1(R)$.

By contradiction, assume that $S \neq D$. Then $H_1(S)$ is a non-trivial group generated by loops around the components of $D \setminus S$. In particular, the shadow of the fence cycle is non-zero in $H_1(S)$ since $p(\mathcal{F}) = \partial D$.

Now, consider the following diagram:

$$\begin{array}{ccccc}
\pi_1(\mathcal{F}) & \xrightarrow{i_*} & \pi_1(\mathcal{R}) & \xrightarrow{p_*} & \pi_1(S) \\
\psi\Big\downarrow & & \psi\Big\downarrow & & \psi\Big\downarrow \\
H_1(\mathcal{F}) & \xrightarrow{i_1} & H_1(\mathcal{R}) & \xrightarrow{p_1} & H_1(S)
\end{array}$$

where ψ is the Hurewicz homomorphism. The functorial property of the Hurewicz homomorphisms implies that this diagram commutes. Regard the fence cycle $\mathcal{F}$ as an element of $\pi_1(\mathcal{F})$. Then

$$\psi \circ p_* \circ i_*(\mathcal{F}) = p_1 \circ \psi \circ i_*(\mathcal{F}) \neq 0$$

since it corresponds to the shadow of the fence cycle in $H_1(S)$. It follows that

$$\psi \circ i_*(\mathcal{F}) = i_1 \circ \psi(\mathcal{F}) \neq 0.$$

This contradicts our assumption $i_1 = 0$. ∎

It should be noted that the condition **A5** is necessary only for the proof ($3 \Rightarrow 4$). This proof shows that another exceptional case that $\widetilde{\mathbf{A5}}$ does not induce **A5** like Figure 3 is allowed for the validity of Proposition 4. This is because a shadow edge outside of D can be now homotopically deformable to some portions of ∂D. In this sense, the condition $\widetilde{\mathbf{A5}}$, with slight modifications concerning fence nodes if necessary, is suitable for practical implementations. The reason to add **A5** or $\widetilde{\mathbf{A5}}$ is related to distributed computations which will be proposed in the next section. We will explain this subject in section 5 in more detail.

4 Distributed computation

In this section, we present two algorithms for parallelized computations of the coverage condition under **A1**-**A5**(or $\widetilde{\mathbf{A5}}$). For this purpose, let us distinguish the first K sensors as core sensors and, to each core sensor v_k ($k = 1, \cdots, K$), assign a primary ID k. The essence of the following algorithm is divided into two processes. (1) Spreading primary IDs to adjacent nodes which have not yet obtained primary IDs based on the breadth first search. The adjacent nodes with different primary IDs mutually share their ID information as sub IDs. (2) End sensors of corresponding tree structure for each primary ID k make so called "k-connection sheet" and return it to the upstream sensor with adding necessary connection information. Here a k-connection sheet consists of the following two lists:

(1) List of edges composed by two sensors both of which have a primary or
sub ID number k.
(2) List of the primary ID and all sub ID numbers for each sensor appearing
in the list (1).

Algorithm 5 (Decomposition of Rips Complexes)

1. *Each sensor (say v_k) which already holds own primary ID (say k) performs the following (a) and (b), and waits until proper connection sheets will be sent by the process 2 or 3. However, sensors which can execute neither (a) nor (b) proceed to 2.*

   (a) *The sensor v_k assigns ID number k as the primary ID to its adjacent sensors $v_{k_1}, \cdots v_{k_l}$ whose primary IDs have not yet been determined. The sensors which obtained their primary IDs here repeat the same process.*

   (b) *The sensor v_k assigns ID number k as sub IDs to its adjacent sensors $v_{k_{l+1}}, \cdots, v_{k_L}$ which already have their primary IDs other than k. The sensors which obtained sub IDs here proceed to 3.*

2. *Each sensor v_k makes its k-connection sheet and returns it to the parent sensor for ID k. Then, it proceeds to 4.*

3. *A sensor who obtained a sub ID k makes the k-connection sheet after all its adjacent sensors' primary IDs are assigned, and returns it to the parent sensors for ID k. Then, it proceeds to 4.*

4. *The parent sensor obtains all the k-connection sheets from its child sensors, sums up to one k-connection sheet with adding connection information for itself, and returns the list to its own parent sensor. Repeat this process until all core sensors complete their connection sheets.*

Let us remark that sensors may have multiple sub IDs and the process of 3 and 4 in the algorithm above are executed for each sub ID in parallel. All sensors appearing in the k-connection sheet of the core sensor v_k must possess the primary or sub ID number k, and vice versa.

We note that the k-connection sheet defines a graph G_k by the list of edges. Moreover, let us define a subgraph G_{kl} by the list of all edges in the k-connection sheet whose nodes on both sides have the primary or sub ID of l. It is obvious that $G_{kl} = G_{lk}$. Let us denote by $\mathcal{R}_k$ and $\mathcal{R}_{kl}$ the Rips complexes determined by the graph G_k and G_{kl}, respectively. Then, the following proposition about Algorithm 5 holds.

Proposition 6

1. *Algorithm 5 finishes in finite steps.*

2. *If the Rips complex $\mathcal{R}$ is connected, all nodes obtain their unique primary IDs.*

3. *If the Rips complex $\mathcal{R}$ is connected, $\mathcal{R} = \cup_{k=1}^{K} \mathcal{R}_k$.*

4. $\mathcal{R}_k \cap \mathcal{R}_l = \mathcal{R}_{kl}$.

Proof. The statements 1, 2, 4 and $\mathcal{R} \supset \cup_{k=1}^{K} \mathcal{R}_k$ are clear. Suppose $\sigma \in \mathcal{R}$. Let us assume there exist two nodes v_a and v_b in σ holding different primary IDs k_a and k_b, respectively. Since v_b is an adjacent node of v_a, it has a sub ID k_a. Therefore, all nodes in σ are included in $\mathcal{R}_a$, and $\sigma \in \mathcal{R}_a$ holds. ∎

==== Figure 4 ====

We remark that sub IDs possess information about intersections of the Rips subcomplexes $\mathcal{R}_k, k = 1, \cdots, K$. It is obvious that, if the Rips complex is not connected, then allocating one core sensor at least to each connected component guarantees the same results of the statements 2 and 3 in Proposition 6. In Figure 4, we describe an example of assignments of primary and sub IDs in the case of $K = 2$, where x_1 and x_2 are core sensors. The corresponding 1 and 2-connection sheets are shown in Table 1.

==== Table 1 ====

Now, we are in the position to present an algorithm for distributed $H_1(\mathcal{R})$ computations by Proposition 3 and Algorithm 5.

Algorithm 7 (Distributed $H_1(\mathcal{R})$ computations)

1. *Decompose the Rips complex* $\mathcal{R}$ *into Rips subcomplexes* $\{\mathcal{R}_k | k = 1, \cdots, K\}$ *by Algorithm 5.*
2. *Calculate* $H_1(\mathcal{R}_k), k = 1, \cdots, K,$ *in parallel by each core sensor.*
3. *Calculate* $H_1(\mathcal{R}_1 \cup \mathcal{R}_2)$ *by means of Proposition 3.*
4. *Change primary and sub ID numbers* 2 *to* 1 *in all connection sheets.*
5. *Repeat the processes 3 and 4 for* $H_1(\mathcal{R}_1 \cup \mathcal{R}_k)$ *until* $k = K$*.*

By this algorithm, the homology group $H_1(\mathcal{R})$ is calculated by summing up the local homology groups $H_1(\mathcal{R}_k)$. From Proposition 4 the sufficient condition for the coverage under the assumption **A1**-**A5**(or $\widetilde{\mathbf{A5}}$) can be expressed as $H_1(\mathcal{R}) = 0$. Therefore this algorithm enables parallelized computations of the coverage condition, which will be indispensable for practical implementations.

Let us note that the above procedure for summing up the local homology $H_1(\mathcal{R}_k)$ are straightforward. Although the explanation becomes clear, it causes computational unbalance focusing on the core sensor v_1. In order to avoid this problem, a simple modification should be taken into account. For example, we may adopt distributed computations in Algorithm 7 in such a way that some preassigned core sensors execute processes 3 and 4 in parallel to their nearby Rips complexes.

5 Discussions

In this paper, we have discussed a coverage problem and presented an algorithm to check the sufficient condition for the coverage expressed by means of homology groups in a distributed manner. We here discuss its computational costs and further extensions.

First of all, let us consider the total computational cost of Algorithm 7. Obviously, homology computations determine the total computational complexity. Most homology computations are based on Smith normal forms and, in general, recent algorithms need polynomial orders with respect to the number of simplices. The order p of polynomials depends on geometrical settings of problems in the range $2 \leq p \leq 5$.

In the homology computations in Algorithm 7, we need to calculate $H_1(\mathcal{R}_k)$, $H_1(\mathcal{R}_1 \cap \mathcal{R}_k), k = 1, \cdots, K$, and the rank L defined in Proposition 3 for each summing process. It should be noted that we can apply the method discussed in the paper [2] to the computations of rank L. This method enables us to detect whether a cycle with k edges is contractible in our settings and its computational cost is $O(k)$ with $O(m \log n)$ preprocess, where m and n are the numbers of edges and points in the Rips complex. That is, the computations for the rank L is easier than homology computations. Another approach from the engineering viewpoint can even exclude the computation of rank L by scheduling Algorithm 5 and Algorithm 7 to have $H_1(\mathcal{R}_1 \cap \mathcal{R}_k) = 0$, i.e., rank $L = 0$. In these sense, Algorithm 7 succeeds in the parallelization and reduces the total computational time roughly to $1/K^p$. The more detailed estimates with respect to computation-communication tradeoffs are future problems.

Next, let us discuss the extensions of the presented distributed computations. One extension of the algorithm is to construct distributed computation algorithms also for homology groups except for $H_1(\mathcal{R})$. It should be noted that the equivalence of the sufficient conditions 1 and 2 under **A1**-**A4** in Proposition 4 is easily checked. However, the reason that we have to put the assumption **A5** or $\widetilde{\mathbf{A5}}$ for our settings is to avoid computations of $H_1(\mathcal{R}, \mathcal{F})$ and the homomorphism j_1. Therefore, further extensions of distributed computations to other homology groups may remove the non-pinching condition **A5** or $\widetilde{\mathbf{A5}}$.

Another direction may be to apply into a three dimensional coverage problem as studied in [3]. In this case, a corresponding Rips complex is derived from a set of finite points in $\mathbb{R}^3$. For both cases, we need to clarify the topological properties of Rips complexes further, like [1], and these are challenging problems from both engineering and mathematical viewpoints.

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1-connection sheet	
Edge list	ID list
(1,3)	x_1: (1)
(1,4)	x_3: (1)
(3,4)	x_4: (1,2)
(3,5)	x_5: (1,2)
(4,6)	x_6: (2,1)
(5,6)	x_7: (2,1)
(5,7)	

2-connection sheet	
Edge list	ID list
(2,7)	x_2: (2)
(2,8)	x_4: (1,2)
(4,6)	x_5: (1,2)
(5,6)	x_6: (2,1)
(5,7)	x_7: (2,1)
(6,8)	x_8: (2)
(7,8)	

Table 1: 1 and 2-connection sheets of $\mathcal{R}_1$ and $\mathcal{R}_2$ in Figure 4, respectively. In the edge list, each pair of numbers expresses the edge with the corresponding vertices. In the ID list, the first number shows the primary ID and the next shows the sub ID.

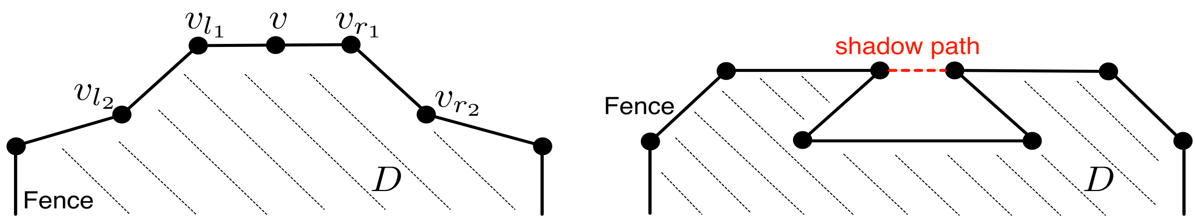

Figure 1: The left figure shows a non-pinching situation and the right figure shows a pinching situation, respectively.

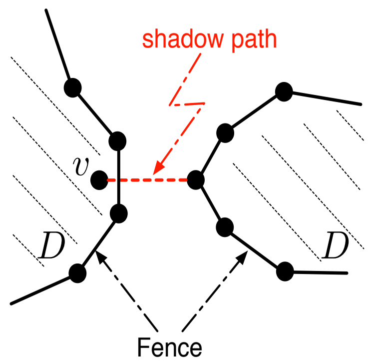

Figure 2: An exceptional example satisfying $\widetilde{\mathbf{A5}}$, but not $\mathbf{A5}$.

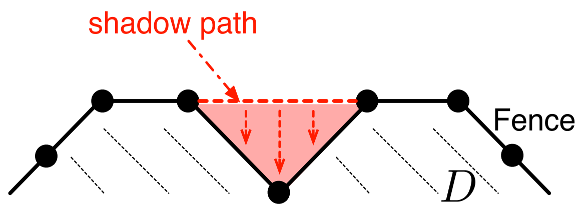

Figure 3: Another exceptional example. But this shadow path can be homotopically deformable to a portion of ∂D via a 2-simplex. Hence, Proposition 4 is still valid in this case.

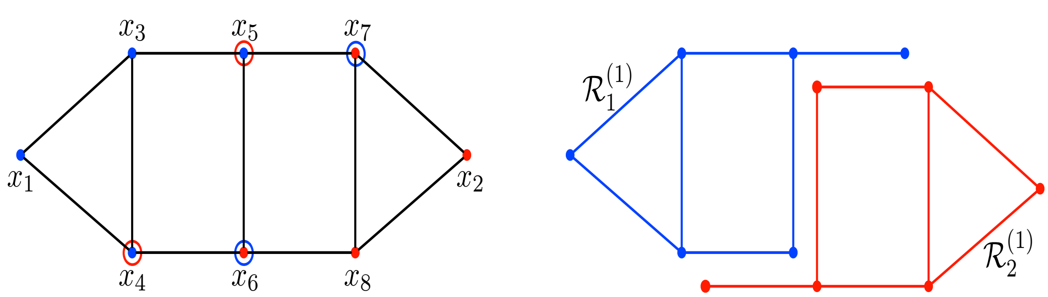

Figure 4: The left figure shows assignments of primary and sub IDs to the communication network, where nodes x_1 and x_2 are core sensors. Blue(red) solid and open circles express primary and sub ID 1(2), respectively. The Right figure shows the derived 1-skeltons of the Rips subcomplexes $\mathcal{R}_1$ and $\mathcal{R}_2$.