MODULAR BASICS
VIDEO 1

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In this video series I'll teach you, and myself, the basics of modular forms. Especially the geometric perspective. I do research in modular forms, but often find that I lack sufficient knowledge of the basics.

In this first video, I want to try and understand the structure of the quotient spaces $\Gamma\backslash\mathbb{H}^*$ for Γ a Fuchsian group of the first kind, as Riemann surfaces. This is an important ingredient in what will begin doing in the next video–namely using the Riemann-Roch theorem to compute the dimension of various spaces of modular forms, and especially spaces of vector-valued modular forms.

The whole series will mainly be based on Miyake for the elliptic modular forms stuff. When it comes to vector-valued modular forms, it will be based on my own notes, a paper by Borcherds, and every paper by Raum.

1 The Möbius action and the co-cycle j

Recall that we have the Möbius action of $\mathrm{GL}_2(\mathbb{C})$ on the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

for $z \neq \infty$. Since

$$\lim_{t\to\infty} \frac{ait+b}{cit+d} = \lim_{t\to\infty} \frac{ai+b/t}{ci+d/t} = a/c,$$

we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}.$$

Note that we use the convention $a/0 = \infty$.

We write for $\alpha = (a, b; c, d) \in \mathrm{GL}_2(\mathbb{C})$

$$a(\alpha) = a, b(\alpha) = b, c(\alpha) = c, \text{ and } d(\alpha) = d.$$

For $\alpha \in \mathrm{GL}_2(\mathbb{C})$ and $z \in \mathbb{P}$ we define

$$j(\alpha, z) = c(\alpha)z + d(\alpha).$$

It is a fact that j is a co-cycle, by which I mean that it satisfies

$$j(\alpha\beta, z) = j(\alpha, \beta z)j(\beta, z),$$

for "appropriate" α, β and z. What I mean by appropriate in this case is that

$$\alpha, \beta \in \mathrm{GL}_2(\mathbb{R}) \text{ and } z \in \mathbb{C} \setminus \mathbb{R}.$$

It probably works even for $\mathrm{GL}_2(\mathbb{C})$ and $z \in \mathbb{P}$, but I want to avoid gnarly infinities and whatnot.[1]

Anyway, the way to see this, and to simultaneously show that the Möbius action is indeed an action is that

$$\alpha \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = j(\alpha, z) \begin{pmatrix} \alpha z \\ 1 \end{pmatrix},$$

which yields that

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha\beta, z) \begin{pmatrix} (\alpha\beta) z \\ 1 \end{pmatrix},$$

but also

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha, \beta z) j(\beta, z) \begin{pmatrix} \alpha(\beta z) \\ 1 \end{pmatrix}.$$

The aforementioned conditions on α and β and z, ensure that

$$j(\alpha\beta, z), j(\alpha, \beta z), j(\beta, z) \neq 0,$$

so that we can conclude that j is a co-cycle, and that the Möbius action is an action.

Before continuing with more interesting things, we move to the more familiar setting of the upper half-plane. Since for $\alpha = (a, b; c, d) \in \mathrm{GL}_2(\mathbb{R})$ and $z \in \mathbb{P}$ we have that

$$\frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + bd + x(ad+bc) + iy(ad-bc)}{|cz+d|^2},$$

where $z = x + iy$, it holds that

$$\mathrm{Im}(\alpha z) = \frac{\det(\alpha)\mathrm{Im}(z)}{|cz+d|^2}.$$

Hence, for $\alpha \in \mathrm{GL}_2(\mathbb{R})$ with $\det(\alpha) > 0$ and $z \in \mathbb{H}$, we have that $\alpha z \in \mathbb{H}$.

We therefore define

$$\mathrm{GL}_2^+(\mathbb{R}) = \{\alpha \in \mathrm{GL}_2(\mathbb{R}) : \det(\alpha) > 0\},$$

and conclude that for $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ the map

$$\mathbb{H} \ni z \mapsto \alpha z \in \mathbb{H},$$

is a complex analytic automorphism, or in other words, a biholomorphic map (if you prefer the categorical point of view).

2 Elliptic, parabolic, and hyperbolic elements

In order to give a geometric definition of a "cusp", we first classify the elements in $\mathrm{GL}_2^+(\mathbb{R})$ depending on how the square of the trace relates to the determinant.

Definition 1. Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ be non-scalar. If

$$\mathrm{Tr}(\alpha)^2 < 4\det(\alpha),$$

then α is called **elliptic**, and if

$$\mathrm{Tr}(\alpha)^2 = 4\det(\alpha),$$

then α is called **parabolic**, and if

$$\mathrm{Tr}(\alpha)^2 > 4\det(\alpha),$$

then α is called **hyperbolic**.

[1] We want to let $0 \cdot \infty$ be undefined on the Riemann sphere, so $j(\alpha, \beta z)j(\beta, z)$ becomes problematic.

I frankly don't know where the terminology comes from, but I guess it will become clear as I mature.

If $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ is scalar, then $\alpha = aI$ and

$$\mathrm{Tr}(\alpha)^2 = (2a)^2 = 4a^2 = 4\det(\alpha),$$

so we could argue that scalar elements should be called parabolic. But then the terminology clashes with the following theorem, I think.[2]

Theorem 1. Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ be non-scalar. Then

- α is elliptic iff α has the fixed points z_0 and $\overline{z_0}$ for some $z_0 \in \mathbb{H}$
- α is parabolic iff α has a unique fixed point on $\mathbb{R} \cup \{\infty\}$
- α is hyperbolic iff α has two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

Proof. Miyake uses an in-line proof, but I don't like that, so here goes my reformulation.

Say that $\alpha = (a, b; c, d)$. If $c = 0$, then $\mathrm{Tr}(\alpha) = a + d$ and $\det(\alpha) = ad$, so

$$\mathrm{Tr}(\alpha)^2 - 4\det(\alpha) = (a+d)^2 - 4ad = (a-d)^2.$$

Hence α is parabolic iff $a = d$, and hyperbolic iff $a \neq d$.

If α is parabolic, we have that $\alpha z = z$ iff

$$z + \frac{b}{a} = z.$$

This is evidently only the case for $z = \infty$.

If α is hyperbolic, we have that $\alpha z = z$ iff

$$\frac{a}{d}z + \frac{b}{d} = z.$$

This equation has both finite and non-finite solutions. Clearly $z = \infty$ is a solution. The finite solution is

$$z = \frac{b}{d}\frac{d}{d-a} = \frac{b}{d-a}.$$

If $c \neq 0$, then $\alpha\infty = a/c \neq \infty$, so ∞ cannot be a fixed point. We have that $\alpha z = z$ iff

$$\frac{az+b}{cz+d} = z, \tag{1}$$

If $z = -d/c$, then the left-hand side is equal to ∞, so this cannot be a solution. Hence $cz + d \neq 0$ and we obtain that (1) is equivalent to

$$z^2 + \frac{d-a}{c}z - \frac{b}{c} = 0.$$

□

[2] I will be honest with my own knowledge gaps in this series. It will be important to be able to refer to misunderstandings further on.