STEPAN MAXIMOV

Part I.
Symmetry of bilinear forms.

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

Hermitian

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal Hermitian Symplectic

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

$$\left(\mathbb{R}^n, \langle -, - \rangle \right)$$

$$\langle x, y \rangle = \langle y, x \rangle$$

Hermitian

Symplectic

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

$$(\mathbb{R}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \langle y, x \rangle$$

Hermitian

$$(\mathbb{C}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

Symplectic

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

$$(\mathbb{R}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \langle y, x \rangle$$

Hermitian

$$(\mathbb{C}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

Symplectic

$$(\mathbb{R}^{2n}, \omega)$$

$$\omega(x, y) = -\omega(y, x)$$

$$[\omega] = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

Linear geometry is the geometry of a vector space equipped with a bilinear form.

Orthogonal

$$(\mathbb{R}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \langle y, x \rangle$$

Hermitian

$$(\mathbb{C}^n, \langle -, - \rangle)$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

Symplectic

$$(\mathbb{R}^{2n}, \omega)$$

$$\omega(x, y) = -\omega(y, x)$$

$$[\omega] = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

Why!?

Answer:
Reasonable notion of orthogonality.

Answer:
Reasonable notion of orthogonality.

$$B(x, y) = 0 \Rightarrow B(y, x) = 0$$

Answer:
Reasonable notion of orthogonality.

$$B(x,y)=0 \Rightarrow B(y,x)=0$$

Bilinear forms satisfying this condition are called reflexive.

Theorem 1.

Let $B\colon V \times V \longrightarrow F$ be a reflexive bilinear form on a vector space V over F with $\mathrm{char}(F) \neq 2$. Then B is either symmetric or skew-symmetric.

Lemma 1.

Any reflexive bilinear form $B\colon V \times V \longrightarrow F$ **satisfies the relation**

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Lemma 1.

Any reflexive bilinear form $B\colon V \times V \longrightarrow F$ **satisfies the relation**

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Proof:

For all $u, v, w \in V$ **we have**

$$B(v,u)B(v,w) - B(v,u)B(v,w) = 0$$

Lemma 1.

Any reflexive bilinear form $B\colon V \times V \longrightarrow F$ satisfies the relation

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Proof:

For all $u, v, w \in V$ **we have**

$$B(v,u)B(v,w) - B(v,u)B(v,w) = 0$$

$$B(v, uB(v,w)) - B(v, wB(v,u)) = 0$$

Lemma 1.

Any reflexive bilinear form $B\colon V \times V \longrightarrow F$ satisfies the relation

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Proof:

For all $u, v, w \in V$ we have

$$B(v,u)B(v,w) - B(v,u)B(v,w) = 0$$

$$B(v,uB(v,w)) - B(v,wB(v,u)) = 0$$

$$B(v,uB(v,w) - wB(v,u)) = 0$$

Lemma 1.

Any reflexive bilinear form $B \colon V \times V \longrightarrow F$ **satisfies the relation**

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Proof:

For all $u, v, w \in V$ **we have**

$$B(v,u)B(v,w) - B(v,u)B(v,w) = 0$$

$$B(v, uB(v,w)) - B(v, wB(v,u)) = 0$$

$$B(v, uB(v,w) - wB(v,u)) = 0$$

$$B(uB(v,w) - wB(v,u), v) = 0$$

Lemma 1.

Any reflexive bilinear form $B\colon V \times V \longrightarrow F$ **satisfies the relation**

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

Proof:

For all $u, v, w \in V$ **we have**

$$B(v,u)B(v,w) - B(v,u)B(v,w) = 0$$

$$B(v, uB(v,w)) - B(v, wB(v,u)) = 0$$

$$B(v, uB(v,w) - wB(v,u)) = 0$$

$$B(uB(v,w) - wB(v,u), v) = 0$$

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

$\blacksquare$

Lemma 1.

Any reflexive bilinear form $B : V \times V \longrightarrow F$ satisfies the relation

$$B(u,v)B(v,w) - B(v,u)B(w,v) = 0$$

$w = v \Downarrow$

Corollary 1.

Let $B : V \times V \longrightarrow F$ be a reflexive bilinear form. Then for all $u, v \in V$ at lest one of the following conditions holds

a) $B(v,v) = B(u,u) = 0$ **b)** $B(v,u) = B(u,v)$

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u, v) \neq B(v, u)$.

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u,v) \neq B(v,u)$.

Claim: $B(w,w) = 0$ for all $w \in V$. Assume the opposite, i.e. there is $w \in V$ such that $B(w,w) \neq 0$.

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u,v) \neq B(v,u)$.

Claim: $B(w,w) = 0$ for all $w \in V$. Assume the opposite, i.e. there is $w \in V$ such that $B(w,w) \neq 0$. Applying Corollary 1 to pairs (v,w) and (u,w) we get

$$B(v,w) = B(w,v) \quad \text{and} \quad B(u,w) = B(w,u).$$

Applying Lemma 1 to the triple (w,v,u) we obtain the identity

$$B(w,v)(B(v,u) - B(u,v)) = 0,$$

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u,v) \neq B(v,u)$.

Claim: $B(w,w) = 0$ for all $w \in V$. Assume the opposite, i.e. there is $w \in V$ such that $B(w,w) \neq 0$. Applying Corollary 1 to pairs (v,w) and (u,w) we get

$$B(v,w) = B(w,v) \quad \text{and} \quad B(u,w) = B(w,u).$$

Applying Lemma 1 to the triple (w,v,u) we obtain the identity

$$B(w,v)(B(v,u) - B(u,v)) = 0,$$

which in its turn implies that $B(w,v) = B(v,w) = 0$. Since v and u play symmetric roles we also have $B(w,u) = B(u,w) = 0$.

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u, v) \neq B(v, u)$.

Claim: $B(w, w) = 0$ for all $w \in V$. Assume the opposite, i.e. there is $w \in V$ such that $B(w, w) \neq 0$. Applying Corollary 1 to pairs (v, w) and (u, w) we get

$$B(v, w) = B(w, v) \quad \text{and} \quad B(u, w) = B(w, u).$$

Applying Lemma 1 to the triple (w, v, u) we obtain the identity

$$B(w, v)(B(v, u) - B(u, v)) = 0,$$

which in its turn implies that $B(w, v) = B(v, w) = 0$. Since v and u play symmetric roles we also have $B(w, u) = B(u, w) = 0$. Observe that

$$B(v + w, u) = B(v, u) \neq B(u, v) = B(u, v + w).$$

Proof of Theorem 1:

Assume that B is not symmetric. Then we can find (and fix) some $u, v \in V$ such that $B(u, v) \neq B(v, u)$.

Claim: $B(w, w) = 0$ for all $w \in V$. Assume the opposite, i.e. there is $w \in V$ such that $B(w, w) \neq 0$. Applying Corollary 1 to pairs (v, w) and (u, w) we get

$$B(v, w) = B(w, v) \quad \text{and} \quad B(u, w) = B(w, u).$$

Applying Lemma 1 to the triple (w, v, u) we obtain the identity

$$B(w, v)(B(v, u) - B(u, v)) = 0,$$

which in its turn implies that $B(w, v) = B(v, w) = 0$. Since v and u play symmetric roles we also have $B(w, u) = B(u, w) = 0$. Observe that

$$B(v + w, u) = B(v, u) \neq B(u, v) = B(u, v + w).$$

Therefore, by Corollary 1 we have

$$B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) = 0.$$

This leads to the desired contradiction $B(w, w) = 0$. ∎

Theorem 1.

Let $B\colon V \times V \longrightarrow F$ be a reflexive bilinear form on a vector space V over F with $\mathrm{char}(F) \neq 2$. Then B is either symmetric or skew-symmetric.

Theorem 1.

Let $B\colon V \times V \longrightarrow F$ be a reflexive bilinear form on a vector space V over F with $\mathrm{char}(F) \neq 2$. Then B is either symmetric or skew-symmetric.

Theorem 2.

$B: V \times \bar{V} \to \mathbb{C}$

Let $B\colon V \times V \longrightarrow \mathbb{C}$ be a reflexive sesquilinear form on a vector space V over $\mathbb{C}$. Then B is either Hermitian or anti-Hermitian.

$$B(u,v) = B(v,u) \qquad B(u,v) = -B(v,u)$$

$$B(u,v) = \overline{B(v,u)} \qquad \cancel{B(u,v) = -\overline{B(v,u)}}$$

$$\overline{B(u,v)} = B(v,u) \Leftrightarrow \overline{iB(u,v)} = -iB(v,u)$$

Part II.

Naturality.

$x \longmapsto ev_x$

$ev_x(f) = f(x)$

V is **naturally** isomorphic to V^{**}

V is isomorphic to V^{*} but **not naturally**

$\{v_i\}_1^n$ is a basis for V, $v_i \mapsto v^i$, $v^i(v_j) = \delta_{ij}$

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

Definition 1.

A category $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of objects.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$

whose elements are called "morhpisms".

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called "**morhpisms**".

(3) for every triple A, B, C a **composition law**

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called "**morhpisms**".

(3) for every triple A, B, C a **composition law**

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

(4) for every object A a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called "**morhpisms**".

(3) for every triple A, B, C a **composition law** $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$

(4) for every object A a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$

subject to:

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called “**morhpisms**”.

(3) for every triple A, B, C a **composition law**

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

(4) for every object A a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$

subject to:

(1) for any $f\colon A \longrightarrow B,\ g\colon B \longrightarrow C,\ h\colon C \longrightarrow D$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(2) for any $f\colon A \longrightarrow B,\ g\colon B \longrightarrow C$

$$1_B \circ f = f, \quad g \circ 1_B = g$$

STEPAN MAXIMOV

Definition 1.

A **category** $\mathcal{C}$ consists of the following data:

(1) a collection $|\mathcal{C}|$ of **objects**.

(2) for every pair A, B a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called "**morhpisms**".

(3) for every triple A, B, C a **composition law**

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

(4) for every object A a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$

(1) for any $f\colon A \longrightarrow B,\ g\colon B \longrightarrow C,\ h\colon C \longrightarrow D$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(2) for any $f\colon A \longrightarrow B,\ g\colon B \longrightarrow C$

$$1_B \circ f = f, \quad g \circ 1_B = g$$

F - Fin Vect

Obj: f.d. v. spaces over F

Morphisms: linear transformations + standard composition.

$(F\text{-}\mathrm{Fin\,Vect})^{op}$

Obj: f.d. v. spaces over F

Morphism: $\mathrm{Hom}_{-''-^{op}}(A, B) = \mathrm{Hom}_{-''-}(B, A)$

$$A \xrightarrow{f^*} B \quad \leftrightarrow \quad B \xrightarrow{f} A$$

Definition 2.

A (covariant) functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following data:

Definition 2.

A (covariant) **functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following data:

(1) a mapping $|\mathcal{A}| \longrightarrow |\mathcal{B}|, \ A \longmapsto FA$

Definition 2.

A (covariant) functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following data:

(1) a mapping $|\mathcal{A}| \longrightarrow |\mathcal{B}|, \; A \longmapsto FA$

(2) for every pair $A, A' \in |\mathcal{A}|$ a mapping

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$
$$(f \colon A \longrightarrow A') \longmapsto (Ff \colon FA \longrightarrow FA')$$

Definition 2.

A (covariant) functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following data:

(1) a mapping $|\mathcal{A}| \longrightarrow |\mathcal{B}|,\ A \longmapsto FA$

(2) for every pair $A, A' \in |\mathcal{A}|$ a mapping

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$
$$(f\colon A \longrightarrow A') \longmapsto (Ff\colon FA \longrightarrow FA')$$

subject to:

(1) for any $f : A \longrightarrow A',\ g : A' \longrightarrow A''$

$$F(g \circ f) = Fg \circ Ff$$

(2) for any object $A \in |\mathcal{A}|$

$$F(1_A) = 1_{FA}$$

Definition 2.

A (covariant) functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following data:

(1) a mapping $|\mathcal{A}| \longrightarrow |\mathcal{B}|,\ A \longmapsto FA$

(2) for every pair $A, A' \in |\mathcal{A}|$ a mapping

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$
$$(f\colon A \longrightarrow A') \longmapsto (Ff\colon FA \longrightarrow FA')$$
$$(f\colon A \to A') \mapsto (Ff\colon FA' \to FA)$$

(1) for any $f\colon A \longrightarrow A',\ g\colon A' \longrightarrow A''$

$F(g\circ f) = Ff \circ Fg$ $\quad F(g \circ f) = Fg \circ Ff$

(2) for any object $A \in |\mathcal{A}|$

$$F(1_A) = 1_{FA}$$

$$Id\colon \underline{F\text{-}FinVect} \to \underline{F\text{-}FinVect}$$
$$V \longmapsto V$$
$$f \longmapsto f$$

$$Hom(-, F)\colon \underline{F\text{-}FinVect} \longrightarrow \underline{F\text{-}FinVect}$$
$$V \longmapsto V^* = Hom(V, F)$$
$$(A \xrightarrow{f} B) \longmapsto B^* \xrightarrow{f^*} A^*$$
$$f^*(g) = g \circ f$$

Definition 3.

Consider two functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$

A **natural transformation** $\alpha : F \Longrightarrow G$

is a collection of morphisms $(\alpha_A : FA \longrightarrow GA)_{A \in |\mathcal{A}|}$

such that the following diagram commutes

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\alpha_{A'}"'] & GA' \end{tikzcd}$$

We say that a map between individual objects is natural if it can be defined in a coherent way on the entire category (it defines a natural transformation of functors)

$$\varphi_V : V \xrightarrow{\sim} V^{**}, \quad \varphi_V(x) = ev_x$$

$$\begin{tikzcd} V \arrow[r, "\varphi_V"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & V^{**} \arrow[d, "f^{**}"] \\ U \arrow[r, "\varphi_U"'] & U^{**} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\circlearrowleft !"] & ev_x \arrow[d, "f^{**}"] \\ f(x) \arrow[r, mapsto] & ev_{f(x)} \end{tikzcd}$$

$$\underbrace{f^{**}(ev_x)}_{\in U^{**}}(h) = (ev_x \circ f^*)(h) = ev_x \circ h \circ f = h(f(x)) = ev_{f(x)}(h)$$

$$V \xrightarrow{\sim} V^{*} \qquad V, \ V^{*} \qquad V \to V^{*}$$

$$0 = f \qquad \begin{tikzcd} V \arrow[r, "\alpha_V"] \arrow[d, "f"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & V^{*} \\ U \arrow[r, "\alpha_U"'] & U^{*} \arrow[u, "f^{*}"'] \end{tikzcd}$$

$$\begin{tikzcd} 0 \neq x \arrow[r, mapsto] \arrow[d, "0"'] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & \alpha_V(x) = 0 \\ 0 \arrow[r, mapsto] & 0 \arrow[u, "0"'] \end{tikzcd}$$

(Co)End