Elliptic curves, lattices, and some differential geometry – part I

A “modular forms”-y gumbo

Tobias Magnusson

Chalmers University of Technology

November 3, 2020

Introduction

- Modular forms should be viewed geometrically.

Introduction

- Modular forms should be viewed geometrically.
- Why?

Introduction

- Modular forms should be viewed geometrically.
- Why?
  - When doing computations – want to reduce Fourier coefficients modulo p.

Introduction

- Modular forms should be viewed geometrically.
- Why?
  - When doing computations – want to reduce Fourier coefficients modulo p. Need to make sense of modular form with different field of coefficients.

Introduction

- Modular forms should be viewed geometrically.
- Why?
  - When doing computations – want to reduce Fourier coefficients modulo p. Need to make sense of modular form with different field of coefficients.
  - The geometric point of view has much richer theory.

Introduction

- Modular forms should be viewed geometrically.
- Why?
  - When doing computations – want to reduce Fourier coefficients modulo p. Need to make sense of modular form with different field of coefficients.
  - The geometric point of view has much richer theory.
- Goal of these talks:

Introduction

- Modular forms should be viewed geometrically.
- Why?
  - When doing computations – want to reduce Fourier coefficients modulo p. Need to make sense of modular form with different field of coefficients.
  - The geometric point of view has much richer theory.
- Goal of these talks: go from analytic/computational understanding to geometric understanding.

What even is a modular form?

- Function on the upper-half plane $\mathbb{H}$ that is **very** symmetric.

What even is a modular form?

- Function on the upper-half plane $\mathbb{H}$ that is **very** symmetric.
- Defined in terms of two actions by

What even is a modular form?

- Function on the upper-half plane $\mathbb{H}$ that is **very** symmetric.
- Defined in terms of two actions by

$$\mathrm{SL}_2(\mathbb{Z}) = \{(a, b; c, d) : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}.$$

First action

Definition (Möbius action)

Let $\tau \in \mathbb{H}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. Define

First action

Definition (Möbius action)

Let $\tau \in \mathbb{H}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. Define

$$\gamma.\tau = \frac{a\tau + b}{c\tau + d}.$$

First action

Definition (Möbius action)

Let $\tau \in \mathbb{H}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. Define

$$\gamma.\tau = \frac{a\tau + b}{c\tau + d}.$$

Is left action.

Second action

Definition (Slash action)

Let k be an integer.

Second action

Definition (Slash action)

Let k be an integer. For function $f\colon \mathbb{H} \to \mathbb{C}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$, define

Second action

Definition (Slash action)

Let k be an integer. For function $f : \mathbb{H} \to \mathbb{C}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$, define

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma.\tau).$$

Second action

Definition (Slash action)

Let k be an integer. For function $f\colon \mathbb{H} \to \mathbb{C}$ and $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$, define

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma.\tau).$$

Is right action.

Modular forms of level 1

Definition

Let k be an integer.

Modular forms of level 1

Definition

Let k be an integer. Then a modular form of level 1 of weight k is a function $f\colon \mathbb{H} \to \mathbb{C}$ such that

Modular forms of level 1

Definition

Let k be an integer. Then a modular form of level 1 of weight k is a function $f\colon \mathbb{H} \to \mathbb{C}$ such that

(i) f is holomorphic on $\mathbb{H}$

Modular forms of level 1

Definition

Let k be an integer. Then a modular form of level 1 of weight k is a function $f\colon \mathbb{H} \to \mathbb{C}$ such that

(i) f is holomorphic on $\mathbb{H}$

(ii) $f|_k\gamma = f$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

Modular forms of level 1

Definition

Let k be an integer. Then a modular form of level 1 of weight k is a function $f : \mathbb{H} \to \mathbb{C}$ such that

(i) f is holomorphic on $\mathbb{H}$

(ii) $f|_k\gamma = f$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

(iii) $f(\tau)$ is bounded as $\tau \to i\infty$.

Modular forms of level 1

Definition

Let k be an integer. Then a modular form of level 1 of weight k is a function $f: \mathbb{H} \to \mathbb{C}$ such that

(i) f is holomorphic on $\mathbb{H}$

(ii) $f|_k\gamma = f$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

(iii) $f(\tau)$ is bounded as $\tau \to i\infty$.

Set of modular forms of weight k and level 1 is a vector space, denoted by $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Examples

- In level 1: only need to worry about Eisenstein series.

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1, 1; 0, 1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1,1;0,1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma,$$

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1, 1; 0, 1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1, 1; 0, 1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.
- This reorders factors, so is the same.

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1,1;0,1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma]\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k\gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.
- This reorders factors, so is the same.
- Conclusion: $E_k(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1,1;0,1)$ and let $\Gamma_\infty = \langle T, -I\rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma]\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} 1|_k\gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.
- This reorders factors, so is the same.
- Conclusion: $E_k(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.
- In fact

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1, 1; 0, 1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.
- This reorders factors, so is the same.
- Conclusion: $E_k(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.
- In fact

$$\{E_4^a E_6^b : a, b \in \mathbb{Z}_{\geq 0} \text{ and } 4a + 6b = k\},$$

Examples

- In level 1: only need to worry about Eisenstein series.
- Let $T = (1,1;0,1)$ and let $\Gamma_\infty = \langle T, -I \rangle$. Then we define

$$E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k \gamma,$$

- Converges absolutely and locally uniformly on $\mathbb{H}$, so is holomorphic and can move in slash action.
- This reorders factors, so is the same.
- Conclusion: $E_k(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.
- In fact

$$\{E_4^a E_6^b : a, b \in \mathbb{Z}_{\geq 0} \text{ and } 4a + 6b = k\},$$

basis for $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line.

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line. Lattice L generated by (ω_1, ω_2) is

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line. Lattice L generated by (ω_1, ω_2) is
$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line. Lattice L generated by (ω_1, ω_2) is
$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$
- Consider quotient $\mathbb{C}/L = \{\tau + L : \tau \in \mathbb{C}\}$.

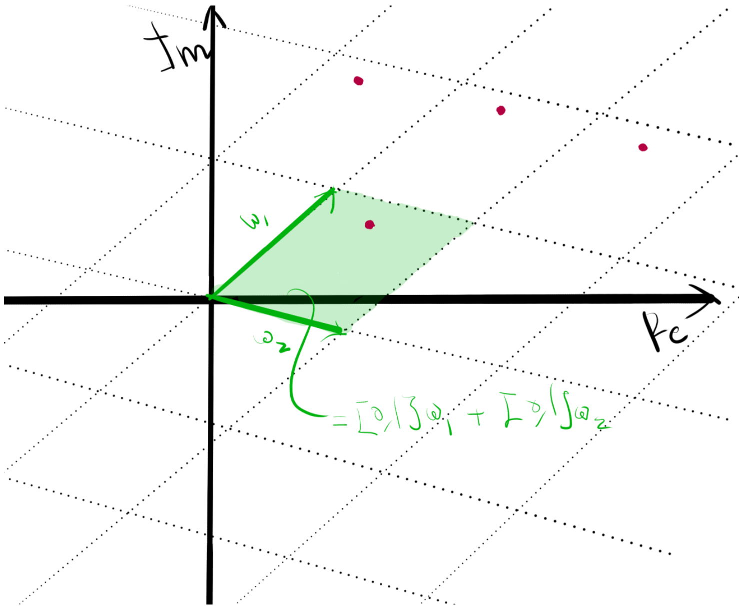
Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line. Lattice L generated by (ω_1, ω_2) is
$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$
- Consider quotient $\mathbb{C}/L = \{\tau + L : \tau \in \mathbb{C}\}$.
- $\tau + L$ has unique representative in "fundamental parallelogram".

Lattices

- Using lattices, can get "simpler" alternative definition of modular forms.
- Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not on the same line. Lattice L generated by (ω_1, ω_2) is
$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$
- Consider quotient $\mathbb{C}/L = \{\tau + L : \tau \in \mathbb{C}\}$.
- $\tau + L$ has unique representative in "fundamental parallelogram".

We denote the set of lattices in $\mathbb{C}$ by $\mathcal{L}(\mathbb{C})$.


Im
Re
ω_1
ω_2
$= [0,1]\omega_1 + [0,1]\omega_2$

Topology

- Homeomorphic to square with sides identified.

Topology

- Homeomorphic to square with sides identified.
- That is, homeomorphic to torus $S^1 \times S^1$.

Modular forms, lattice based definition

Definition

Let k be an integer.

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

$$L(z) = \mathbb{Z}a + \mathbb{Z}z \in \mathcal{L}(\mathbb{C}).$$

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

$$L(z) = \mathbb{Z}a + \mathbb{Z}z \in \mathcal{L}(\mathbb{C}).$$

Then $F \circ L(z)$ holomorphic in z.

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

$$L(z) = \mathbb{Z}a + \mathbb{Z}z \in \mathcal{L}(\mathbb{C}).$$

Then $F \circ L(z)$ holomorphic in z.

(ii) If $\lambda \in \mathbb{C} \setminus \{0\}$ and $L \in \mathcal{L}(\mathbb{C})$, then

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

$$L(z) = \mathbb{Z}a + \mathbb{Z}z \in \mathcal{L}(\mathbb{C}).$$

Then $F \circ L(z)$ holomorphic in z.

(ii) If $\lambda \in \mathbb{C} \setminus \{0\}$ and $L \in \mathcal{L}(\mathbb{C})$, then

$$F(\lambda L) = \lambda^{-k} F(L).$$

Modular forms, lattice based definition

Definition

Let k be an integer. Then a modular form of weight k is function $F : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ satisfying

(i) For $a \in \mathbb{C}$ fixed, define the mapping

$$L(z) = \mathbb{Z}a + \mathbb{Z}z \in \mathcal{L}(\mathbb{C}).$$

Then $F \circ L(z)$ holomorphic in z.

(ii) If $\lambda \in \mathbb{C} \setminus \{0\}$ and $L \in \mathcal{L}(\mathbb{C})$, then

$$F(\lambda L) = \lambda^{-k} F(L).$$

(iii) $F(\mathbb{Z} + \mathbb{Z}\tau)$ is bounded as $\tau \to i\infty$.

Equivalence

Can show is equivalent to previous definition by

Equivalence

Can show is equivalent to previous definition by

$$f(\tau) = F(\mathbb{Z} + \mathbb{Z}\tau),$$

Equivalence

Can show is equivalent to previous definition by

$$f(\tau) = F(\mathbb{Z} + \mathbb{Z}\tau),$$

and

Equivalence

Can show is equivalent to previous definition by

$$f(\tau) = F(\mathbb{Z} + \mathbb{Z}\tau),$$

and

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2),$$

for $\mathrm{Im}(\omega_1/\omega_2) > 0$.

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \Big(\frac{1}{(z-l)^2} - \frac{1}{l^2} \Big).$$

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z; L)$ is L-periodic.

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z; L)$ is L-periodic.
- Can show:

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z; L)$ is L-periodic.
- Can show: $\wp(z; L)$ meromorphic with poles only at $z \in L$.

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z;L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z;L)$ is L-periodic.
- Can show: $\wp(z;L)$ meromorphic with poles only at $z \in L$.
- $\wp'(z;L) = \sum_{l \in L} \frac{-2}{(z-l)^3}$, also L-periodic, poles only at $z \in L$.

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z; L)$ is L-periodic.
- Can show: $\wp(z; L)$ meromorphic with poles only at $z \in L$.
- $\wp'(z; L) = \sum_{l \in L} \frac{-2}{(z-l)^3}$, also L-periodic, poles only at $z \in L$.
- Laurent expansion at $z = 0$:

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right).$$

- $\wp(z; L)$ is L-periodic.
- Can show: $\wp(z; L)$ meromorphic with poles only at $z \in L$.
- $\wp'(z; L) = \sum_{l \in L} \frac{-2}{(z-l)^3}$, also L-periodic, poles only at $z \in L$.
- Laurent expansion at $z = 0$:

$$\wp(z) = z^{-2} + \frac{g_4}{20} z^2 + \frac{g_6}{28} z^4 + O(z^6),$$

Weierstraß $\wp$-function

- Allows us to associate elliptic curve to $\mathbb{C}/L$.
- Let $z \in \mathbb{C}$ and $L \in \mathcal{L}(\mathbb{C})$. Then

$$\wp(z; L) = \frac{1}{z^2} + \sum_{l \in L \setminus \{0\}} \Big(\frac{1}{(z-l)^2} - \frac{1}{l^2} \Big).$$

- $\wp(z; L)$ is L-periodic.
- Can show: $\wp(z; L)$ meromorphic with poles only at $z \in L$.
- $\wp'(z; L) = \sum_{l \in L} \frac{-2}{(z-l)^3}$, also L-periodic, poles only at $z \in L$.
- Laurent expansion at $z = 0$:

$$\wp(z) = z^{-2} + \frac{g_4}{20} z^2 + \frac{g_6}{28} z^4 + O(z^6),$$

where $g_4 = \frac{4}{3}\pi^4 E_4$ and $g_6 = \frac{8}{27}\pi^6 E_6$.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem:

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z;L) = 4\wp(z;L)^3 - g_4\wp(z;L) - g_6 - \wp'(z;L)^2$.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z \to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic:

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z \to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all,

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all, i. e. entire.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact,

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z;L) = 4\wp(z;L)^3 - g_4\wp(z;L) - g_6 - \wp'(z;L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z;L) = 0$.
- Since $E(z;L)$ is L-periodic: no poles at lattice points.
- Hence $E(z;L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact, so $E(z;L)$ attains finite max there.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact, so $E(z; L)$ attains finite max there.
- Since L-periodic,

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z \to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact, so $E(z; L)$ attains finite max there.
- Since L-periodic, $E(z; L)$ bounded everywhere.

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z; L) = 4\wp(z; L)^3 - g_4\wp(z; L) - g_6 - \wp'(z; L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z; L) = 0$.
- Since $E(z; L)$ is L-periodic: no poles at lattice points.
- Hence $E(z; L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact, so $E(z; L)$ attains finite max there.
- Since L-periodic, $E(z; L)$ bounded everywhere.
- Liouville: $E(z; L) = E(L)$ constant,

$\wp$-function, differential equation

- Clever usage of Liouville's theorem: bounded entire function is constant.
- Let $E(z;L) = 4\wp(z;L)^3 - g_4\wp(z;L) - g_6 - \wp'(z;L)^2$.
- Using Laurent: $\lim_{z\to 0} E(z;L) = 0$.
- Since $E(z;L)$ is L-periodic: no poles at lattice points.
- Hence $E(z;L)$ no poles at all, i. e. entire.
- Fundamental parallelogram is compact, so $E(z;L)$ attains finite max there.
- Since L-periodic, $E(z;L)$ bounded everywhere.
- Liouville: $E(z;L) = E(L)$ constant, so

$$E(L) = \lim_{z\to 0} E(z;L) = 0.$$

Mapping tori to elliptic curves

- Define $\psi : \mathbb{C}/L \to \mathbb{C}^2$ by

Mapping tori to elliptic curves

- Define $\psi : \mathbb{C}/L \to \mathbb{C}^2$ by

$$\psi(z + L) = (x = \wp(z; L), y = \wp'(z; L)).$$

Mapping tori to elliptic curves

- Define $\psi : \mathbb{C}/L \to \mathbb{C}^2$ by

$$\psi(z+L) = (x = \wp(z;L), y = \wp'(z;L)).$$

- Then $y^2 = 4x^3 - g_4 x - g_6$ is elliptic curve.

Can go back – lattice of periods

- Next talk!

Can go back – lattice of periods

- Next talk!
- Key: need to associate a differential form to elliptic curve.

Can go back – lattice of periods

- Next talk!
- Key: need to associate a differential form to elliptic curve.
- Can map $(E, \omega) \mapsto L \in \mathcal{L}(\mathbb{C})$.

Katz' idea

- We know:

Katz' idea

- We know: can define modular forms on lattices.

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight?

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.
- Modular form:

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.
- Modular form: rule $\mathbb{F}$ that assigns to (E, ω) a complex number, and that satisfies

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.
- Modular form: rule $\mathbb{F}$ that assigns to (E, ω) a complex number, and that satisfies

$$\mathbb{F}(E, \lambda\omega) = \lambda^{-k}\mathbb{F}(E, \omega),$$

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.
- Modular form: rule $\mathbb{F}$ that assigns to (E, ω) a complex number, and that satisfies

$$\mathbb{F}(E, \lambda\omega) = \lambda^{-k}\mathbb{F}(E, \omega),$$

for $\lambda \in \mathbb{C} \setminus \{0\}$.

Katz' idea

- We know: can define modular forms on lattices.
- Hence: try to define modular forms on elliptic curves.
- How to "see" weight? Consider pairs (E, ω) with E elliptic curve and ω "differential form" on E.
- Modular form: rule $\mathbb{F}$ that assigns to (E, ω) a complex number, and that satisfies

  $$\mathbb{F}(E, \lambda\omega) = \lambda^{-k}\mathbb{F}(E, \omega),$$

  for $\lambda \in \mathbb{C} \setminus \{0\}$.
- Connection: $\mathbb{F}(E, \omega) = F(L(E, \omega))$.

Excerpt from "p-adic Properties [...]"

Equivalently, a modular form of weight k and level 1 is a rule f which assigns to every pair $(E/R,\omega)$ consisting of an elliptic curve over (the spectrum of) a ring R together with a basis ω of $\underline{\omega}_{E/R}$ (i.e., a nowhere vanishing section of $\Omega^1_{E/R}$ on E), an element $f(E/R,\omega) \in R$, such that the following three conditions are satisfied.

1. $f(E/R,\omega)$ depends only on the R-isomorphism class of the pair $(E/R,\omega)$.

2. f is homogeneous of degree -k in the "second variable"; for any $\lambda \in R^{\times}$ (the multiplicative group of R), $f(E,\lambda\omega) = \lambda^{-k} f(E,\omega)$.

3. The formation of $f(E/R,\omega)$ commutes with arbitrary extension of scalars $g: R \longrightarrow R'$ (meaning $f(E_{R'}/R',\omega_{R'}) = g(f(E/R,\omega))$).

Notice in particular that we now can work over other ground fields. Indeed, we can even work over rings!

This talk

- Goal: try to understand $\mathbb{C}/L$ as a complex manifold,

This talk

- Goal: try to understand $\mathbb{C}/L$ as a complex manifold,
- and make sense of $\omega = dz$ as a differential form.

"easy"

Complex geometry

$\mathbb{C}/L$ -- complex torus

a complex manifold

$\omega = dz$ -- differential 1-form

$\mathcal{P}$

lattice of periods

"hard"

Algebraic geometry

$E \to \mathrm{Spec}(R)$, elliptic curve

$\omega = d\frac{x}{y}$ "differential form" on E

The left part

- Complex manifolds.

The left part

- Complex manifolds.
- Complex tori.

The left part

- Complex manifolds.
- Complex tori.
- Differential forms.

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

- A pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to B \subseteq \mathbb{C}^n$ is a homeomorphism is called a complex coordinate system in X.

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

- A pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to B \subseteq \mathbb{C}^n$ is a homeomorphism is called a complex coordinate system in X.
- If $p \in X$ is a point, then every complex coordinate system $(U \ni p, \phi)$ is called a coordinate system at p.

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

- A pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to B \subseteq \mathbb{C}^n$ is a homeomorphism is called a complex coordinate system in X.
- If $p \in X$ is a point, then every complex coordinate system $(U \ni p, \phi)$ is called a coordinate system at p.
- The components of ϕ are called local coordinates with respect to (U, ϕ),

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

- A pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to B \subseteq \mathbb{C}^n$ is a homeomorphism is called a complex coordinate system in X.
- If $p \in X$ is a point, then every complex coordinate system $(U \ni p, \phi)$ is called a coordinate system at p.
- The components of ϕ are called local coordinates with respect to (U, ϕ),
- and in particular the entries of $\phi(p)$ are called the coordinates of p with respect to (U, ϕ).

Complex coordinate system

Let X be Hausdorff and $n \geq 1$ be an integer.

- A pair (U, ϕ) where $U \subseteq X$ is open and $\phi : U \to B \subseteq \mathbb{C}^n$ is a homeomorphism is called a complex coordinate system in X.
- If $p \in X$ is a point, then every complex coordinate system $(U \ni p, \phi)$ is called a coordinate system at p.
- The components of ϕ are called local coordinates with respect to (U, ϕ),
- and in particular the entries of $\phi(p)$ are called the coordinates of p with respect to (U, ϕ).
- If f function in U, we consider it as a function of $z_1, \ldots, z_n$ by

$$(z_1, \ldots, z_n) \mapsto f \circ \phi^{-1}(z_1, \ldots, z_n).$$

Compatible coordinate systems

Definition

Let X be Hausdorff and $\mathcal{U} = (U, \phi)$ and $\mathcal{V} = (V, \psi)$ be two n-dimensional complex coordinate systems in X.

Compatible coordinate systems

Definition

Let X be Hausdorff and $\mathcal{U} = (U, \phi)$ and $\mathcal{V} = (V, \psi)$ be two n-dimensional complex coordinate systems in X. Then $\mathcal{U}$ and $\mathcal{V}$ are called compatible iff $U \cap V = \emptyset$, or if

Compatible coordinate systems

Definition

Let X be Hausdorff and $\mathcal{U} = (U, \phi)$ and $\mathcal{V} = (V, \psi)$ be two n-dimensional complex coordinate systems in X. Then $\mathcal{U}$ and $\mathcal{V}$ are called compatible iff $U \cap V = \emptyset$, or if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

Compatible coordinate systems

Definition

Let X be Hausdorff and $\mathcal{U} = (U, \phi)$ and $\mathcal{V} = (V, \psi)$ be two n-dimensional complex coordinate systems in X. Then $\mathcal{U}$ and $\mathcal{V}$ are called compatible iff $U \cap V = \emptyset$, or if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

is biholomorphic (i. e. bijective holomorphic with holomorphic inverse).

Atlas

Definition

Let X be Hausdorff and $n \geq 1$ be an integer.

Atlas

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. Then a covering (with respect to the first component) of X by pairwise compatible n-dimensional complex coordinate systems is called an n-dimensional atlas on X.

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. Then a covering (with respect to the first component) of X by pairwise compatible n-dimensional complex coordinate systems is called an n-dimensional atlas on X.
Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent

Atlas

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. Then a covering (with respect to the first component) of X by pairwise compatible n-dimensional complex coordinate systems is called an n-dimensional atlas on X.
Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent iff there exists coordinate systems $\mathcal{U} \in \mathcal{A}_1$ and $\mathcal{V} \in \mathcal{A}_2$

Atlas

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. Then a covering (with respect to the first component) of X by pairwise compatible n-dimensional complex coordinate systems is called an n-dimensional atlas on X.
Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent iff there exists coordinate systems $\mathcal{U} \in \mathcal{A}_1$ and $\mathcal{V} \in \mathcal{A}_2$ such that $\mathcal{U}$ and $\mathcal{V}$ are compatible.

Atlas

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. Then a covering (with respect to the first component) of X by pairwise compatible n-dimensional complex coordinate systems is called an n-dimensional atlas on X.
Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent iff there exists coordinate systems $\mathcal{U} \in \mathcal{A}_1$ and $\mathcal{V} \in \mathcal{A}_2$ such that $\mathcal{U}$ and $\mathcal{V}$ are compatible. We then write $\mathcal{A}_1 \sim \mathcal{A}_2$.

Complex structure

Definition

Let X be Hausdorff and $n \geq 1$ be an integer.

Complex structure

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. If $\mathcal{A}$ is an n-dimensional atlas on X,

Complex structure

Definition

Let X be Hausdorff and $n \geq 1$ be an integer. If $\mathcal{A}$ is an n-dimensional atlas on X, then an equivalence class $[\mathcal{A}]_\sim$ is called an n-dimensional complex structure on X.

Complex manifold

Definition

Let $n \geq 1$ be an integer.

Complex manifold

Definition

Let $n \geq 1$ be an integer. Then an n-dimensional complex manifold

Complex manifold

Definition

Let $n \geq 1$ be an integer. Then an n-dimensional complex manifold is a second countable Hausdorff space equipped with an n-dimensional complex structure.

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.
- Let

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.
- Let
$$\mathcal{A} = \{(\mathbb{C}^n, \mathrm{id})\}.$$

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.
- Let
$$\mathcal{A} = \{(\mathbb{C}^n, \mathrm{id})\}.$$
Then $\mathcal{A}$ is an atlas.

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.
- Let
$$\mathcal{A} = \{(\mathbb{C}^n, \mathrm{id})\}.$$
Then $\mathcal{A}$ is an atlas.
- So $[\mathcal{A}]_\sim$ is a complex structure.

Trivial example

- $\mathbb{C}^n$ is second countable Hausdorff with respect to the Euclidean topology.
- Let
$$\mathcal{A} = \{(\mathbb{C}^n, \mathrm{id})\}.$$
Then $\mathcal{A}$ is an atlas.
- So $[\mathcal{A}]_\sim$ is a complex structure.
- Hence $(\mathbb{C}^n, [\mathcal{A}]_\sim)$ is a complex n-dimensional manifold.

Submanifold

- Let $(X, [\mathcal{A}]_\sim)$ be an n-dimensional complex manifold.

Submanifold

- Let $(X, [\mathcal{A}]_{\sim})$ be an n-dimensional complex manifold.
- Let $B \subseteq X$ open.

Submanifold

- Let $(X, [\mathcal{A}]_\sim)$ be an n-dimensional complex manifold.
- Let $B \subseteq X$ open.
- Define $\mathcal{B} = \{(U \cap B, \phi|_{U\cap B}) : (U, \phi) \in \mathcal{A}\}$.

Submanifold

- Let $(X, [\mathcal{A}]_\sim)$ be an n-dimensional complex manifold.
- Let $B \subseteq X$ open.
- Define $\mathcal{B} = \{(U \cap B, \phi|_{U \cap B}) : (U, \phi) \in \mathcal{A}\}$. Atlas on B.

Submanifold

- Let $(X, [\mathcal{A}]_\sim)$ be an n-dimensional complex manifold.
- Let $B \subseteq X$ open.
- Define $\mathcal{B} = \{(U \cap B, \phi|_{U \cap B}) : (U, \phi) \in \mathcal{A}\}$. Atlas on B.
- $(B, [\mathcal{B}]_\sim)$ is n-dimensional complex manifold.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$,

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

so $\pi(U)$ open.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

  $$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying

$$\pi|_U : U \to \pi(U),$$

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying

$$\pi|_U : U \to \pi(U),$$

  bijective (Careful!).

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

  $$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying

  $$\pi|_U : U \to \pi(U),$$

  bijective (Careful!). Then because continuous open bijective, is homeomorphism.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$

  so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying

$$\pi|_U : U \to \pi(U),$$

  bijective (Careful!). Then because continuous open bijective, is homeomorphism.
- Define coordinate system at $\tilde{x}$ by $(\pi(U), \phi)$ where $\phi = \pi|_U^{-1} : \pi(U) \to U$.

Complex tori

- Let $L = \sum_{i=1}^{n} \mathbb{Z}\omega_i$ and let L act on $\mathbb{C}^n$ by translation.
- Consider $\mathbb{C}^n/L$, let $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ by $\pi(z) = z + L$.
- For $U \subset \mathbb{C}^n$ open have
$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in L} U + \lambda,$$
so $\pi(U)$ open.
- Let $\tilde{x} \in \mathbb{C}^n/L$ and take $x \in \mathbb{C}^n$ such that $\pi(x) = \tilde{x}$. Let $U \ni x$ open, satisfying
$$\pi|_U : U \to \pi(U),$$
bijective (Careful!). Then because continuous open bijective, is homeomorphism.
- Define coordinate system at $\tilde{x}$ by $(\pi(U), \phi)$ where $\phi = \pi|_U^{-1} : \pi(U) \to U$.
- Does it make an atlas?

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

- Then

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

- Then

$$\pi|_V^{-1}(\pi(U \cap V)) = U \cap V \text{ and } \pi|_U^{-1}(\pi(U \cap V)) = U \cap V.$$

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

- Then

$$\pi|_V^{-1}(\pi(U \cap V)) = U \cap V \text{ and } \pi|_U^{-1}(\pi(U \cap V)) = U \cap V.$$

- If $a \in U \cap V$, then

$$\pi|_U^{-1} \circ \pi|_V(a) = \pi|_U^{-1}(\pi(a)) = a,$$

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

- Then

$$\pi|_V^{-1}(\pi(U \cap V)) = U \cap V \text{ and } \pi|_U^{-1}(\pi(U \cap V)) = U \cap V.$$

- If $a \in U \cap V$, then

$$\pi|_U^{-1} \circ \pi|_V(a) = \pi|_U^{-1}(\pi(a)) = a,$$

so $\pi|_U^{-1} \circ \pi|_V = \mathrm{id}.$

Is it an atlas?

- Since can form for any $\tilde{x} \in \mathbb{C}^n/L$, it must be is cover.
- Suppose $(\pi(U), \pi|_U^{-1})$ and $(\pi(V), \pi|_V^{-1})$ satisfies

$$\pi(U) \cap \pi(V) = \pi(U \cap V) \neq \emptyset.$$

- Then

$$\pi|_V^{-1}(\pi(U \cap V)) = U \cap V \text{ and } \pi|_U^{-1}(\pi(U \cap V)) = U \cap V.$$

- If $a \in U \cap V$, then

$$\pi|_U^{-1} \circ \pi|_V(a) = \pi|_U^{-1}(\pi(a)) = a,$$

so $\pi|_U^{-1} \circ \pi|_V = \mathrm{id}$.
- Clearly biholomorphic.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective:

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

so $a_1 = a_2$.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U+\lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

so $a_1 = a_2$. Surjective immediate.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff:

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

  $$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that

  $$(U + \lambda) \cap V = \emptyset,$$

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U+\lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that

$$(U+\lambda) \cap V = \emptyset,$$

  for every $\lambda \in L$.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that

$$(U + \lambda) \cap V = \emptyset,$$

  for every $\lambda \in L$.
- Say $[z_1] \neq [z_2]$, pick U, V as above.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U+\lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then
$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$
so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that
$$(U+\lambda) \cap V = \emptyset,$$
for every $\lambda \in L$.
- Say $[z_1] \neq [z_2]$, pick U, V as above. Then if $z \in \pi(U) \cap \pi(V)$ it holds that $z = u + L = v + L$ for some $u \in U$ and $v \in V$, so

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

  $$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that

  $$(U + \lambda) \cap V = \emptyset,$$

  for every $\lambda \in L$.
- Say $[z_1] \neq [z_2]$, pick U, V as above. Then if $z \in \pi(U) \cap \pi(V)$ it holds that $z = u + L = v + L$ for some $u \in U$ and $v \in V$, so

  $$(U + v - u) \ni u + (v - u) = v \in V,$$

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U+\lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then

  $$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$

  so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that

  $$(U+\lambda) \cap V = \emptyset,$$

  for every $\lambda \in L$.
- Say $[z_1] \neq [z_2]$, pick U, V as above. Then if $z \in \pi(U) \cap \pi(V)$ it holds that $z = u + L = v + L$ for some $u \in U$ and $v \in V$, so

  $$(U + v - u) \ni u + (v - u) = v \in V,$$

  so z cannot exist.

π locally bijective and $\mathbb{C}^n/L$ Hausdorff

Follows from:

- Bijective: if $z \in \mathbb{C}^n$, there exists $U \ni z$ open such that $(U + \lambda) \cap U = \emptyset$ unless $\lambda = 0$.
- Take U as above, say $\pi(a_1) = \pi(a_2)$ with $a_1, a_2 \in U$, then
$$(U + a_1 - a_2) \ni a_2 + (a_1 - a_2) = a_1 \in U,$$
so $a_1 = a_2$. Surjective immediate.
- Hausdorff: if $\pi(z_1) \neq \pi(z_2)$ then exists opens $U \ni z_1$ and $V \ni z_2$ such that
$$(U + \lambda) \cap V = \emptyset,$$
for every $\lambda \in L$.
- Say $[z_1] \neq [z_2]$, pick U, V as above. Then if $z \in \pi(U) \cap \pi(V)$ it holds that $z = u + L = v + L$ for some $u \in U$ and $v \in V$, so
$$(U + v - u) \ni u + (v - u) = v \in V,$$
so z cannot exist. Hence $\pi(U) \cap \pi(V) = \emptyset$.

Conclusion

- For proofs of these two facts, see e. g. Fritzsche and Grauert.

Conclusion

- For proofs of these two facts, see e. g. Fritzsche and Grauert.
- Now $\mathbb{C}^n/L$ is complex manifold.

Conclusion

- For proofs of these two facts, see e. g. Fritzsche and Grauert.
- Now $\mathbb{C}^n/L$ is complex manifold.
- What about dz?

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point.

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

$$\mathcal{E}(B, \mathbb{C}) = \{f \colon B \to \mathbb{C} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\},$$

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

$$\mathcal{E}(B, \mathbb{C}) = \{f : B \to \mathbb{C} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\},$$

and

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

$$\mathcal{E}(B, \mathbb{C}) = \{f : B \to \mathbb{C} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\},$$

and

$$\mathcal{E}(B) = \{f : B \to \mathbb{R} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\}.$$

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

$$\mathcal{E}(B, \mathbb{C}) = \{f : B \to \mathbb{C} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\},$$

and

$$\mathcal{E}(B) = \{f : B \to \mathbb{R} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\}.$$

Definition

A derivation on X at a is an $\mathbb{R}$-linear map $v : \mathcal{E}(X) \to \mathbb{R}$ satisfying

Derivations

Let X be an n-dimensional manifold and let $a \in X$ be a point. For $B \subseteq X$ open let

$$\mathcal{E}(B, \mathbb{C}) = \{f : B \to \mathbb{C} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\},$$

and

$$\mathcal{E}(B) = \{f : B \to \mathbb{R} : \forall \text{ c. s. } (U, \phi) \text{ s. t. } U \cap B \neq \emptyset,\ f \circ \phi^{-1} \text{ is smooth}\}.$$

Definition

A derivation on X at a is an $\mathbb{R}$-linear map $v : \mathcal{E}(X) \to \mathbb{R}$ satisfying

$$v[f \cdot g] = v[f] \cdot g(a) + f(a) \cdot v[g].$$

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$,

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$, then $v[f] = 0$ for every derivation on X at a.

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$, then $v[f] = 0$ for every derivation on X at a.
- Means we can work with local coordinates.

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$, then $v[f] = 0$ for every derivation on X at a.
- Means we can work with local coordinates.
- Let $(U, \phi = (z_1, \dots, z_n))$ be local coordinates at a,

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$, then $v[f] = 0$ for every derivation on X at a.
- Means we can work with local coordinates.
- Let $(U, \phi = (z_1, \ldots, z_n))$ be local coordinates at a, write $z_k = x_k + iy_k$ and

Can work locally

- Using cut-off functions, can see if $f|_U = 0$ for some open neighborhood $U \ni a$, then $v[f] = 0$ for every derivation on X at a.
- Means we can work with local coordinates.
- Let $(U, \phi = (z_1, \ldots, z_n))$ be local coordinates at a, write $z_k = x_k + iy_k$ and

$$(\partial/\partial x_k)_a[f] = (f \circ \phi^{-1})_{x_i}(\phi(a))$$
$$(\partial/\partial y_k)_a[f] = (f \circ \phi^{-1})_{y_i}(\phi(a))$$

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.
- For $f \in \mathcal{E}(X, \mathbb{C})$ write $f = g + ih$ and define $v[f] = v[g] + iv[h]$.

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.
- For $f \in \mathcal{E}(X, \mathbb{C})$ write $f = g + ih$ and define $v[f] = v[g] + iv[h]$.
- Define $J(v)[f] = iv[f]$ and $(a + ib)v = av + b \cdot J(v)$.

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.
- For $f \in \mathcal{E}(X, \mathbb{C})$ write $f = g + ih$ and define $v[f] = v[g] + iv[h]$.
- Define $J(v)[f] = iv[f]$ and $(a + ib)v = av + b \cdot J(v)$.
- Then $T_a(X)$ is complexification $(T_a^{\mathbb{R}}(X) \oplus T_a^{\mathbb{R}}(X), J)$.

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.
- For $f \in \mathcal{E}(X, \mathbb{C})$ write $f = g + ih$ and define $v[f] = v[g] + iv[h]$.
- Define $J(v)[f] = iv[f]$ and $(a + ib)v = av + b \cdot J(v)$.
- Then $T_a(X)$ is complexification $(T_a^{\mathbb{R}}(X) \oplus T_a^{\mathbb{R}}(X), J)$.

Has real dimension $2n$ with basis:

Tangent space

- Let $\mathbb{T}_a^{\mathbb{R}}(X)$ be real vector space of derivations on X at a.
- For $f \in \mathcal{E}(X, \mathbb{C})$ write $f = g + ih$ and define $v[f] = v[g] + iv[h]$.
- Define $J(v)[f] = iv[f]$ and $(a + ib)v = av + b \cdot J(v)$.
- Then $T_a(X)$ is complexification $(T_a^{\mathbb{R}}(X) \oplus T_a^{\mathbb{R}}(X), J)$.

Has real dimension $2n$ with basis:

$$\{(\partial/\partial x_1)_a, \dots, (\partial/\partial x_n)_a, (\partial/\partial y_1)_a, \dots, (\partial/\partial y_n)_a\}$$

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_\mathbb{R}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

for $v \in T$.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

for $v \in T$. Then $(df)_x \in F$.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

for $v \in T$. Then $(df)_x \in F$. Example of a differential 1-form on X.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

  for $v \in T$. Then $(df)_x \in F$. Example of a differential 1-form on X.
- Let (U, ϕ) coordinate system at x.

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

  for $v \in T$. Then $(df)_x \in F$. Example of a differential 1-form on X.
- Let (U, ϕ) coordinate system at x. Write

1-forms on a complex manifold

Let X be n-dimensional complex manifold and $x \in X$ a point.

- Write $T = T_x(X)$ and $T^\vee = \mathrm{Hom}_\mathbb{R}(T, \mathbb{R})$.
- Cotangent space: $F = T^\vee \oplus iT^\vee$. Note: is of **complex** dimension $2n$.
- Let f be smooth function (real- or complex-valued) on neighborhood of x, let:

$$(df)_x(v) = v[f],$$

  for $v \in T$. Then $(df)_x \in F$. Example of a differential 1-form on X.
- Let (U, ϕ) coordinate system at x. Write

$$dz_i = (d\phi_i)_x.$$

Conclusion for today

Now we know what

$$(\mathbb{C}/L, dz),$$

means!

Part 2

- Explain “lattice of periods”.

Part 2

- Explain “lattice of periods”. Need integration.

Part 2

- Explain “lattice of periods”. Need integration.
- Consider right part of “big picture”.

Part 2

- Explain "lattice of periods". Need integration.
- Consider right part of "big picture". Generalize dz to differential on elliptic curve (or any scheme).

Part 2

- Explain "lattice of periods". Need integration.
- Consider right part of "big picture". Generalize dz to differential on elliptic curve (or any scheme).
- q-series expansion from a geometric point of view

Part 2

- Explain "lattice of periods". Need integration.
- Consider right part of "big picture". Generalize dz to differential on elliptic curve (or any scheme).
- q-series expansion from a geometric point of view – the Tate curve.

gg,bye