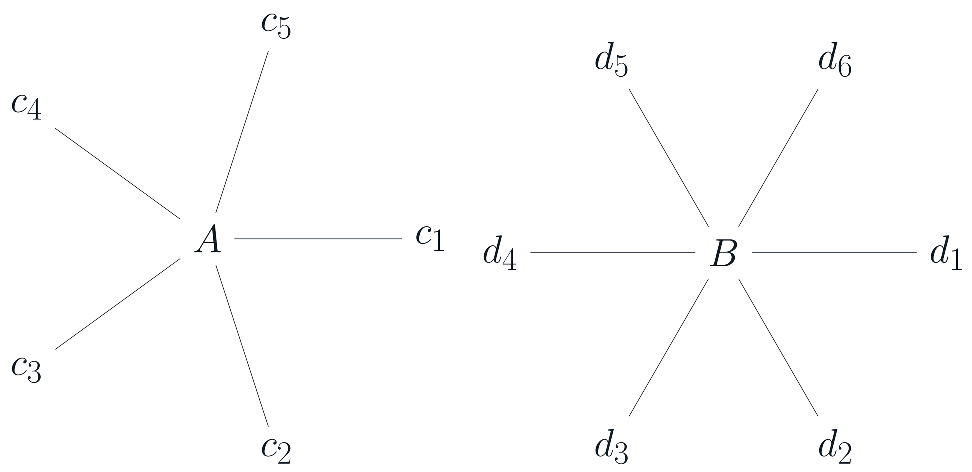
A Simple Example of a Coloured Cyclic Operad on a Non-Strict Compact Closed Category

Yaxin Li, supervised by Marcy Robertson

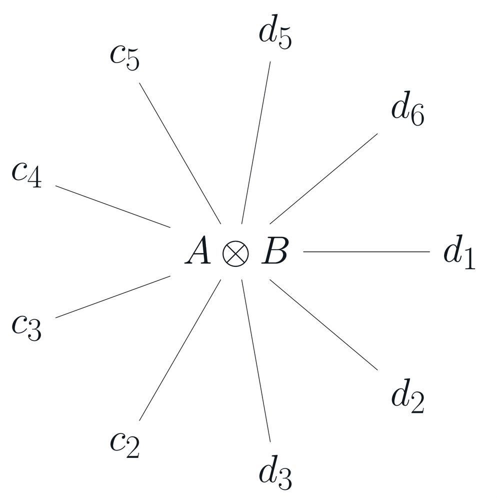
University of Melbourne

This Instance of Coloured Cyclic Operads

This is a vast simplification of the technical, tedious definition from [3]. On a *symmetric monoidal category*, a coloured cyclic operad is pretty much a way of labeling each object A with at least one (possibly empty) ordered list $c_1, \cdots, c_n \in \mathfrak{C}$ of "colours" (i.e. elements of $\mathfrak{C}$, some set of colours with involution given by $(-)^\dagger$). The ordering is always assumed to be cyclic because the corollas do not have distinguished input or output branches. For example, if for A and B as labelled below, $c_1 = d_4^\dagger$,



then we can take the composition $A \circ_4^1 B$ to be the object $A \otimes B$ with the label



. The compositions must satisfy some further conditions, such as being commutative with the action of the appropriately-sized symmetric group permuting the branches, up to isomorphism. Clearly, the unit of the composition, and for this example, $\otimes$ as well, must have the empty list as one of its possible labels.

Compact Closed Category

For this example, the compact closed category is an example of a star-autonomous category such that $\otimes$ coincides with $A \parr B = (A^* \otimes B^*)^*$. It is sufficient for a star-autonomous category (equivalently, a linearly-distributive category with duals [2]) to have $(A \otimes B)^* \cong A^* \otimes B^*$ for all objects A, B, in order for it to be compact closed.

The "Simple Example" $\mathcal{W}$ of a Compact Closed Category

This example is based on the "simple example" of a star-autonomous category with non-strict double-dual, constructed on p.23 of [1]. Let $\mathcal{W}$ be a symmetric monoidal category such that all unitors, associators, swaps, and several other structural maps are identity maps. The set of objects of $\mathcal{W}$ is $\left\{\begin{matrix} & I & \\ L & & R \cong R' \end{matrix}\right\}$, with

1. I the unit and counit of $\otimes$
2. For each object A, $A \otimes A = A$
3. $L \otimes R = L \otimes R' = I$
4. $R \otimes R' = R'$
5. $L = R^* = R'^*, \quad R = L^*$
6. $R \cong R^*$, with $\cong$ not an identity map

The above are enough to ensure that each object of $\mathcal{C}$ satisfy the "triangular equations" of [4] on p.1, which we take to be the definition of a compact closed category.

Finding a Suitable Set of Colours for $\mathcal{W}$

Since the isomorphism between R and R' is not an identity map, we likely cannot put a coloured cyclic operad structure by labelling the objects of $\mathcal{W}$ in a "tautological" way, as in example 2.9 of [3]:

$$P(a_0, \cdots, a_n) \coloneqq \mathcal{V}(a_0 \otimes \cdots \otimes a_n, \perp)$$

where $\mathcal{V}$ is a star-autonomous category with strict double-dual, and the "tautological" is in the sense of the object $\mathcal{V}(a_0 \otimes \cdots \otimes a_n, \perp)$ in **Sets** (and in a way, $a_0 \otimes \cdots \otimes a_n$ as well) being labelled by $a_0, \cdots, a_n$.

The Category of Complemented Objects

This construction comes from [1]. The category of complemented objects $\mathbf{C}(\mathcal{C})$ of the compact closed category $\mathcal{C}$, has

1. as objects, the triples (A, A', τ_A) with $A' \xrightarrow{\tau_A} A^*$ an isomorphism
2. (I, I, id_I) as the unit
3. $(A, A', \tau_A) \otimes (B, B', \tau_B) \coloneqq \left(A \otimes B, A' \parr B', A' \parr B' \xrightarrow{\tau_A \parr \tau_B} A^* \parr B^* \xrightarrow{\cong} (A \otimes B)^* \right)$

For $\mathcal{W}$, we have

$$\mathrm{Obj}(\mathbf{C}(\mathcal{W})) = \left\{\begin{matrix} & (I, I, \mathrm{id}) & & \\ (L, R, \mathrm{id}) & & (R, L, \mathrm{id}) & \\ (L, R', \cong) & & & (R', L, \mathrm{id}) \end{matrix}\right\}$$

which we will use as the involutive set of colours, since $\mathbf{C}(\mathcal{W})$ has a strict double-dual [1]. After brute force calculating the $otimes$-table of $\mathbf{C}(\mathcal{W})$, we can see that if we use as "generators"

$$\begin{aligned} I &= P((I, I, \mathrm{id})) \\ L &= P((L, R, \mathrm{id})) \\ R &= P((R, L, \mathrm{id})) \\ R' &= P((R', L, \mathrm{id})) \end{aligned} \qquad\qquad L = P(L, R', \cong)$$

and define $P(\tilde{A}, \tilde{B}) = P(\tilde{A} \otimes \tilde{B})$ for objects $\tilde{A}, \tilde{B}$, in $\mathbf{C}(\mathcal{W})$, we will see that $\mathcal{W}$ is a $\mathbf{C}(\mathcal{W})$-coloured cyclic operad.

Remarks

Note that in example 2.9 of [3], the coloured cyclic operad structure is on the *image* of a *contravariant* functor, so I might have missed some important things in this choice of $\mathcal{W}$. Perhaps this naive way of using $\mathbf{C}(\mathcal{W})$ as the involutive set of colours would not work in general if $\mathcal{W}$ had more non-identity structural morphisms. These sacrifices were made to fit everything onto one a2 poster.

References

[1] J. R. B. Cockett, M. Hasegawa, and R. A. G. Seely. "COHERENCE OF THE DOUBLE INVOLUTION ON *-AUTONOMOUS CATEGORIES". en. In: *Theory and Applications of Categories* 17.2 (Dec. 2006), pp. 17–29. ISSN: 1201 - 561X.

[2] J. R. B. Cockett and R. A. G. Seely. "Weakly distributive categories". In: *Applications of Categories in Computer Science* (1991), pp. 45–65.

[3] Gabriel C. Drummond-Cole and Philip Hackney. "Dwyer-Kan homotopy theory for cyclic operads". en. In: *Proceedings of the Edinburgh Mathematical Society* 64.1 (Feb. 2021), pp. 29–58. ISSN: 0013-0915, 1464-3839. DOI: 10.1017/S0013091520000267.

[4] G. M. Kelly and M. L. Laplaza. "Coherence for compact closed categories". In: *Journal of Pure and Applied Algebra* 19 (Dec. 1980), pp. 193–213. ISSN: 0022-4049. DOI: 10.1016/0022-4049(80)90101-2.