The Grothendieck-Teichmüller Group and the Operad of Parenthesized Braids

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Abstract

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We introduce the operad of parenthesized braids PaB, show that algebras over PaB correspond to braided monoidal categories, and describe the action of the Grothendieck-Teichmüller group GT on PaB.

Contents

Chapter 1

Introduction

$\mathcal{M}_{g,n}$ is the moduli stack (i.e. 2-scheme) of genus-g curves over $\mathbb{Q}$ with n marked points. In his Sketch of a Program [Gro97] from 1983, Grothendieck proposes that we study $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its action on the Teichmüller Tower — the collection of

$$\pi_1^{\text{geom}}(\mathcal{M}_{g,n}) \stackrel{\text{def}}{=} \pi_1^{\text{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

for all $g, n \in \mathbb{N}$. Here, π_1^{et} takes the étale fundamental group, which is the inverse limit of the automorphism groups (analogous to the groups of deck transformations) of finite étale covers.

In genus-0, it is known that $\pi_1^{\text{geom}}(\mathcal{M}_{0,4}) \cong \widehat{\mathsf{F}}_2$, the profinite completion of the free group on two generators. It follows from a theorem of Belyi's [Bel80] that $G_{\mathbb{Q}}$ acts faithfully on $\pi_1^{\text{geom}}(\mathcal{M}_{0,4})$, hence also $\{\mathcal{M}_{0,n}\}_n$. In [Iha91], Ihara showed that the image of the action of $G_{\mathbb{Q}}$ includes into the image of $\widehat{\mathsf{GT}}$, the profinite completion of the inertia-preserving automorphisms of $\{\mathcal{M}_{0,n}\}_n$, i.e. the automorphisms ϕ such that on $\widehat{\mathsf{F}}_2$, ϕ maps the procyclic subgroups $\langle x\rangle$, $\langle y\rangle$, $\langle (xy)^{-1}\rangle$ to conjugate subgroups, where x and y are the generators of F_2 [Loc].

In [Dri91], Drinfeld constructs a pro-unipotent version of GT, and uses its action on braided monoidal categories (i.e. $\widehat{PaB}$-algebras) to prove a result about quasitriangular quasi-Hopf algebras. In [Pet14], Petersen uses the corresponding action on $\widehat{PaB}$ to show that E_2 (the operad of little 2-disks) is formal.

Chapter 2

Algebras of PaB

2.1 Braided Monoidal Categories

Definition 2.1 (monoidal category)**.** A category $\mathcal{C}$ with a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the monoidal, or tensor product), unit object $\mathbf{1}$, and natural isomorphisms

$$\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -) \tag{2.1}$$

$$\lambda : \mathbf{1} \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}} \qquad \rho : - \otimes \mathbf{1} \Rightarrow \mathrm{id}_{sC} \tag{2.2}$$

respectively called the associator, left unitor, right unitor, is *monoidal* if when we write $\otimes$ to act componentwise on natural transformations,

1. the triangle identity is satisfied, i.e. $\rho \otimes \mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes \lambda) \circ \alpha$
2. the pentagon identity holds, i.e. for all $w, x, y, z \in \mathrm{Obj}(\mathcal{C})$, the following diagram commutes

$$\begin{tikzcd}
 & (w \otimes x) \otimes (y \otimes z) \arrow[dr] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur] \arrow[d] & & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[r] & w \otimes ((x \otimes y) \otimes z) \arrow[ur] &
\end{tikzcd}$$

   or as natural transformations, $\alpha \circ \alpha = (\mathrm{id}_{\mathcal{C}} \otimes \alpha) \circ \alpha \circ (\alpha \otimes \mathrm{id}_{\mathcal{C}})$

A monoidal category is *strict* if the components of the associator and unitors are identity morphisms.

Definition 2.2 (braided monoidal category)**.** If $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is a monoidal category with a natural isomorphism $\beta_{x,y} : x \otimes y \to y \otimes x$, then it is *braided* if it satisfies the hexagon identities, i.e. the following diagrams commute

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "\alpha"] \arrow[d, "\beta\otimes\mathrm{id}_{\mathcal{C}}"] & x \otimes (y \otimes z) \arrow[r, "\beta"] & (y \otimes z) \otimes x \arrow[d, "\alpha"] \\
(y \otimes x) \otimes z \arrow[r, "\alpha"] & y \otimes (x \otimes z) \arrow[r, "\mathrm{id}_{\mathcal{C}}\otimes\beta"] & y \otimes (z \otimes x)
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}\otimes\beta"] & (x \otimes y) \otimes z \arrow[r, "\beta"] & z \otimes (x \otimes y) \arrow[d, "\alpha^{-1}"] \\
x \otimes (z \otimes y) \arrow[r, "\alpha^{-1}"] & (x \otimes z) \otimes y \arrow[r, "\beta\otimes\mathrm{id}_{\mathcal{C}}"] & (z \otimes x) \otimes y
\end{tikzcd}$$

We will now build up to Mac Lane's Coherence Theorem

Definition 2.3 (monoidal functor, or strong monoidal functor at nLab)**.** Suppose $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$, and $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$, are monoidal categories, then a monoidal functor consists of a functor $F : \mathcal{C} \to \mathcal{C}'$ such that $\mathbf{1}' \cong F(\mathbf{1})$, and a natural isomorphism $m : F(-) \otimes F(-) \to F(- \otimes -)$ such that the following diagram commutes

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\alpha'"] \arrow[d, "m\otimes\mathrm{id}_{\mathcal{C}}"] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id}_{\mathcal{C}}\otimes m"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "m"] & F(X) \otimes F(Y \otimes Z) \arrow[d, "m"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha)"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

A monoidal functor that happens to be an equivalence of categories is called an *equivalence of monoidal categories.*

Theorem 2.4 (Mac Lane's Strictness Theorem)**.** *For any monoidal category* $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$*, there exists a* strict *monoidal category* $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$ *and an equivalence of monoidal categories* $L : \mathcal{C} \to \mathcal{C}'$*.*

Proof. see p.20 of [Eti] from the 2009 lecture notes to "Topics in Lie Theory". □

The next theorem is a corollary of Mac Lane's Strictness Theorem.

Theorem 2.5 (Mac Lane's Coherence Theorem)**.** *Suppose* P *and* P' *are parenthesizations of* $X_1 \otimes \cdots \otimes X_n$ *with arbitrary insertions of* $\mathbf{1}$*, and* $f, g : P \to P'$ *are isomorphisms built by composing associator and unitor components, then* $f = g$*.*

Proof. see p.1 of [Eti09] □

2.2 The Operad PaB of Parenthesized Braids

Operads are an unbiased way of keeping track of n-ary operations and how they compose. The "elements" of $P(n)$, we imagine to be corollas, i.e., rooted trees of depth 1 with $n+1$ half-edges. We can treat $\otimes$ in this "unbiased" way (i.e. ignoring how products are parenthesized, and products with units) because of Mac Lane's Coherence Theorem 2.5.

Definition 2.6 (planar operad, symmetric operad)**.** A *planar operad* P in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a collection of objects $\{P(n)\}_n$, with operadic composition

$$c_{m;n_1,\cdots,n_m} : P(m) \otimes P(n_1) \otimes \cdots \otimes P(n_m) \to P(n_1 + \cdots + n_m)$$

such that 1. and 2. of following are satisfied. If each object $P(n)$ has a right S_n-action (S_n being the nth symmetric group), and 3. is staisfied, then P is a *symmetric operad*, which is what "operad" refers to from now on.

1. Associativity: if
   (a) $\sum_{j=1}^{m_i} n_{i_j} = n_i$ for each $1 \le i \le k$,
   (b) $\sum_{i=1}^{k} n_i = l$,
   (c) $\sum_{i=1}^{k} m_i = m$,

   then the following commutes

$$\begin{tikzcd}
P(k) \otimes \bigotimes_{i=1}^k \left(P(m_i) \otimes \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[rr, "\cong"'] \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \bigotimes_{i=1}^k c_{m_i;n_{i_1},\cdots,n_{i_{m_i}}}"] & & \left(P(k) \otimes \bigotimes_{i=1}^k P(m_i)\right) \otimes \left(\bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "\circ_{k;m_1,\cdots,m_k} \otimes \mathrm{id}_{\mathcal{C}}^{\otimes m}"'] \\
P(k) \otimes \bigotimes_{i=1}^k P(n_i) \arrow[d, "\circ_{k;n_1,\cdots,n_k}"] & & P(m) \otimes \left(\bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "\circ_{m;n_{1_1},\cdots,n_{k_{(m_k)}}}"'] \\
P(l) \arrow[rr, "="] & & P(l)
\end{tikzcd}$$

2. Unit: there exists a map $e : \mathbf{1} \to P(1)$ such that the following hold for all n

$$\begin{tikzcd}
P(n) \arrow[r, "\lambda^{-1}"] \arrow[rrr, bend right, "="'] & \mathbf{1} \otimes P(n) \arrow[r, "e\otimes\mathrm{id}_{\mathcal{C}}"] & P(1) \otimes P(n) \arrow[r, "\circ_{1;n}"] & P(n)
\end{tikzcd}$$

$$\begin{tikzcd}
P(n) \arrow[r, "(\rho^{-1})^{\circ n}"] \arrow[rrr, bend right, "="'] & P(n) \otimes \mathbf{1}^{\otimes n} \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes e^{\otimes n}"] & P(n) \otimes P(1)^{\otimes n} \arrow[r, "\circ_{n;1,\cdots,1}"] & P(n)
\end{tikzcd}$$

3. Equivariance: the right action $- \cdot \sigma : P(n) \to P(n)$ of $\sigma \in \mathsf{S}_n$ on $P(n)$ is given by σ permuting the n inputs of each "element" of $P(n)$, so that on operadic composition maps, we have

$$\circ_{n;m_1,\cdots,m_n} \circ (- \cdot \sigma) = \circ_{n;m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)}},$$

then equivariance is the condition that the following commutes

$$\begin{tikzcd} P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes (-)\sigma"'] \arrow[d, "- \cdot \sigma \otimes \mathrm{id}_{\mathcal{C}}^{\otimes n}"] & P(n) \otimes \bigotimes_{i=1}^n P(m_{\sigma^{-1}(i)}) \arrow[d, "\circ_{n;m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)}}"'] \\ P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "\circ_{n;m_1,\cdots,m_n}"] & P(m_1 + \cdots + m_n) \end{tikzcd}$$

(where $(-)\sigma$ is the isomorphism given by σ permuting the n factors of $\bigotimes_{i=1}^n P(m_i)$)

Example 2.1 (the endofunctor operad)**.** *If $\mathcal{C}$ is a category, then the endofunctor operad, defined by* $\mathrm{End}(\mathcal{C})(n) \stackrel{\text{def}}{=} \mathrm{Fun}\,(\mathcal{C}^{\times n}, \mathcal{C})$ *for $n \in \mathbb{N}_{\geq 0}$, is a symmetric operad in* $(\mathbf{Cat}, \times, \{*\})$*, with*

1. $\mathrm{id}_{\mathcal{C}}$ *as the unit, and $\times$ as the monoidal product.*
2. *the standard composition as the operadic composition,*
3. *$\sigma \in \mathsf{S}_n$ acting on each element $\theta \in \mathrm{Fun}(\mathcal{C}^{\times n}, \mathcal{C})$ by permuting the inputs, i.e. $\theta \cdot \sigma(x_1, \cdots, x_n) = \theta(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$.*

The endofunctor operad is a special case of the endomorphism operad ; it is an endomorphism operad in $(\mathbf{Cat}, \times, \{*\})$, a symmetric monoidal category with internal hom.

Example 2.2 (the operad of parenthesized braids)**.** *This construction is similar to the one in pp.18-19 of [Fre18]. PaB is an operad in* **Grpd** *(the category of groupoids).*

The objects of $PaB(n)$ are parenthesized permutations of n elements, i.e.

$$\mathrm{Obj}(PaB(n)) = \mathsf{S}_n \times M_n,$$

where M_n is the set of words of length n in the free magma generated by one element. If $(\sigma, w) \in PaB(n)$, then for $1 \leq i \leq n$, we label the ith factor of w (which must be the generating element) with $\sigma(n)$ to get the parenthesized permutation.

The morphisms of $PaB(n)$ are braids from (σ, w) to (τ, v) such that if the l factor of w and the mth factor of v are the ends of a strand, then $\sigma(l) = \tau(m)$.

PaB has the following operadic properties:

1. *The unit, which happens to be the only morphism of $PaB(1)$, is the braid from $(1,x)$ to $(1,x)$.*

2. *The operadic composition is given by cabling:*

3. *The action of S_n on $PaB(n)$ is given by $(\sigma, w) \overset{\tau}{\mapsto} (\tau\sigma, w)$ on objects.*

Definition 2.7 (operad morphism)**.** A morphism $f : P \to Q$ of operads is a collection of maps $\{f_n : P(n) \to Q(n)\}_n$ such that

1. The unit is preserved, i.e. $f_1 : P(1) \to Q(1)$ satisfies $f_1 \circ e^P = e^Q$.

2. Composition is preserved, i.e. for all $k, n_1, \cdots, n_k \in \mathbb{N}_{\geq 1}$ with $l = \sum_{i=1}^k n_i$, the following commutes

$$\begin{tikzcd} P(k) \otimes \bigotimes_{i=1}^k P(n_i) \arrow[r, "f_k \otimes \bigotimes_{i=1}^k f_{n_i}"'] \arrow[d, "\circ^P_{k;n_1,\cdots,n_k}"] & Q(k) \otimes \bigotimes_{i=1}^k Q(n_i) \arrow[d, "\circ^Q_{k;n_1,\cdots,n_k}"] \\ P(l) \arrow[r, "f_l"'] & Q(l) \end{tikzcd}$$

3. The S_n-actions are preserved, i.e. for all $n \in \mathbb{N}_{\geq 1}$, $\sigma \in \mathsf{S}_n$, $f \circ (- \cdot^P \sigma) = (- \cdot^Q \sigma) \circ f$.

Definition 2.8 (operad algebra)**.** For an operad P in $(\mathbf{Cat}, \times, \{*\})$, a morphism of operads $P \to \mathrm{End}(\mathcal{C})$ gives $\mathcal{C}$ the structure of a P-*algebra*.

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