

# The Grothendieck-Teichmüller Group and the Operad of Parenthesized Braids

by  
Yaxin Li

A thesis submitted in partial fulfillment for the  
degree of Master of Science

to the  
Faculty of Science  
School of Mathematics and Statistics  
**THE UNIVERSITY OF MELBOURNE**

July 2025

THE UNIVERSITY OF MELBOURNE

# *Abstract*

Faculty of Science  
School of Mathematics and Statistics

Master of Science

by [Yaxin Li](#)

We introduce the operad of parenthesized braids  $PaB$ , show that algebras over  $PaB$  correspond to braided monoidal categories, and describe the action of the Grothendieck-Teichmüller group  $GT$  on  $PaB$ .

# Contents

<b>Abstract</b>	<b>i</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Algebras of <math>PaB</math></b>	<b>2</b>
2.1 Braided Monoidal Categories . . . . .	2
2.2 The Operad $PaB$ of Parenthesized Braids . . . . .	4
<b>Bibliography</b>	<b>7</b>

# Chapter 1

## Introduction

$\mathcal{M}_{g,n}$  is the moduli stack (i.e. 2-scheme) of genus- $g$  curves over  $\mathbb{Q}$  with  $n$  marked points. In his Sketch of a Program [Gro97] from 1983, Grothendieck proposes that we study  $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  through its action on the Teichmüller Tower — the collection of

$$\pi_1^{\text{geom}}(\mathcal{M}_{g,n}) \stackrel{\text{def}}{=} \pi_1^{\text{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

for all  $g, n \in \mathbb{N}$ . Here,  $\pi_1^{\text{et}}$  takes the étale fundamental group, which is the inverse limit of the automorphism groups (analogous to the groups of deck transformations) of finite étale covers.

In genus-0, it is known that  $\pi_1^{\text{geom}}(\mathcal{M}_{0,4}) \cong \widehat{F}_2$ , the profinite completion of the free group on two generators. It follows from a theorem of Belyi's [Bel80] that  $G_{\mathbb{Q}}$  acts faithfully on  $\pi_1^{\text{geom}}(\mathcal{M}_{0,4})$ , hence also  $\{\mathcal{M}_{0,n}\}_n$ . In [Iha91], Ihara showed that the image of the action of  $G_{\mathbb{Q}}$  includes into the image of  $\widehat{\text{GT}}$ , the profinite completion of the inertia-preserving automorphisms of  $\{\mathcal{M}_{0,n}\}_n$ , i.e. the automorphisms  $\phi$  such that on  $\widehat{F}_2$ ,  $\phi$  maps the procyclic subgroups  $\langle x \rangle, \langle y \rangle, \langle (xy)^{-1} \rangle$  to conjugate subgroups, where  $x$  and  $y$  are the generators of  $F_2$  [Loc].

In [Dri91], Drinfeld constructs a pro-unipotent version of  $\text{GT}$ , and uses its action on braided monoidal categories (i.e.  $\widehat{PaB}$ -algebras) to prove a result about quasitriangular quasi-Hopf algebras. In [Pet14], Petersen uses the corresponding action on  $\widehat{PaB}$  to show that  $E_2$  (the operad of little 2-disks) is formal.

# Chapter 2

## Algebras of $PaB$

### 2.1 Braided Monoidal Categories

**Definition 2.1** (monoidal category). A category  $\mathcal{C}$  with a bifunctor  $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (the monoidal, or tensor product), unit object  $\mathbf{1}$ , and natural isomorphisms

$$\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -) \quad (2.1)$$

$$\lambda : \mathbf{1} \otimes - \Rightarrow \text{id}_{\mathcal{C}} \quad \rho : - \otimes \mathbf{1} \Rightarrow \text{id}_{\mathcal{C}} \quad (2.2)$$

respectively called the associator, left unitor, right unitor, is *monoidal* if when we write  $\otimes$  to act componentwise on natural transformations,

1. the triangle identity is satisfied, i.e.  $\rho \otimes \text{id}_{\mathcal{C}} = (\text{id}_{\mathcal{C}} \otimes \lambda) \circ \alpha$
2. the pentagon identity holds, i.e. for all  $w, x, y, z \in \text{Obj}(\mathcal{C})$ , the following diagram commutes

$$\begin{array}{ccccc}
 & & (w \otimes x) \otimes (y \otimes z) & & \\
 & \nearrow & & \searrow & \\
 ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\
 & \searrow & & \nearrow & \\
 (w \otimes (x \otimes y)) \otimes z & \longrightarrow & & & w \otimes ((x \otimes y) \otimes z)
 \end{array}$$

or as natural transformations,  $\alpha \circ \alpha = (\text{id}_{\mathcal{C}} \otimes \alpha) \circ \alpha \circ (\alpha \otimes \text{id}_{\mathcal{C}})$

A monoidal category is *strict* if the components of the associator and unitors are identity morphisms.

**Definition 2.2** (braided monoidal category). If  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$  is a monoidal category with a natural isomorphism  $\beta_{x,y} : x \otimes y \rightarrow y \otimes x$ , then it is *braided* if it satisfies the hexagon identities, i.e. the following diagrams commute

$$\begin{array}{ccc}
 (x \otimes y) \otimes z & \xrightarrow{\alpha} & x \otimes (y \otimes z) \xrightarrow{\beta} (y \otimes z) \otimes x \\
 \downarrow \beta \otimes \text{id}_e & & \downarrow \alpha \\
 (y \otimes x) \otimes z & \xrightarrow{\alpha} & y \otimes (x \otimes z) \xrightarrow{\text{id}_e \otimes \beta} y \otimes (z \otimes x)
 \end{array}$$
  

$$\begin{array}{ccc}
 x \otimes (y \otimes z) & \xrightarrow{\alpha^{-1}} & (x \otimes y) \otimes z \xrightarrow{\beta} z \otimes (x \otimes y) \\
 \downarrow \text{id}_e \otimes \beta & & \downarrow \alpha^{-1} \\
 x \otimes (z \otimes y) & \xrightarrow{\alpha^{-1}} & (x \otimes z) \otimes y \xrightarrow{\beta \otimes \text{id}_e} (z \otimes x) \otimes y
 \end{array}$$

We will now build up to Mac Lane's Coherence Theorem

**Definition 2.3** (monoidal functor, or strong monoidal functor at nLab). Suppose  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ , and  $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$ , are monoidal categories, then a monoidal functor consists of a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $\mathbf{1}' \cong F(\mathbf{1})$ , and a natural isomorphism  $m : F(-) \otimes F(-) \rightarrow F(- \otimes -)$  such that the following diagram commutes

$$\begin{array}{ccc}
 (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha'} & F(X) \otimes (F(Y) \otimes F(Z)) \\
 \downarrow m \otimes \text{id}_e & & \downarrow \text{id}_e \otimes m \\
 F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
 \downarrow m & & \downarrow m \\
 F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha)} & F(X \otimes (Y \otimes Z))
 \end{array}$$

A monoidal functor that happens to be an equivalence of categories is called an *equivalence of monoidal categories*.

**Theorem 2.4** (Mac Lane's Strictness Theorem). *For any monoidal category  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ , there exists a strict monoidal category  $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$  and an equivalence of monoidal categories  $L : \mathcal{C} \rightarrow \mathcal{C}'$ .*

*Proof.* see p.20 of [Eti] from the 2009 lecture notes to "Topics in Lie Theory".  $\square$

The next theorem is a corollary of Mac Lane's Strictness Theorem.

**Theorem 2.5** (Mac Lane's Coherence Theorem). *Suppose  $P$  and  $P'$  are parenthesizations of  $X_1 \otimes \cdots \otimes X_n$  with arbitrary insertions of  $\mathbf{1}$ , and  $f, g : P \rightarrow P'$  are isomorphisms built by composing associator and unitor components, then  $f = g$ .*

*Proof.* see p.1 of [Eti09] □

## 2.2 The Operad $PaB$ of Parenthesized Braids

Operads are an unbiased way of keeping track of  $n$ -ary operations and how they compose. The "elements" of  $P(n)$ , we imagine to be corollas, i.e., rooted trees of depth 1 with  $n + 1$  half-edges. We can treat  $\otimes$  in this "unbiased" way (i.e. ignoring how products are parenthesized, and products with units) because of Mac Lane's Coherence Theorem 2.5.

**Definition 2.6** (planar operad, symmetric operad). A *planar operad*  $P$  in a monoidal category  $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$  is a collection of objects  $\{P(n)\}_n$ , with operadic composition

$$c_{m;n_1, \dots, n_m} : P(m) \otimes P(n_1) \otimes \dots \otimes P(n_m) \rightarrow P(n_1 + \dots + n_m)$$

such that 1. and 2. of following are satisfied. If each object  $P(n)$  has a right  $S_n$ -action ( $S_n$  being the  $n$ th symmetric group), and 3. is satisfied, then  $P$  is a *symmetric operad*, which is what "operad" refers to from now on.

1. Associativity: if

- (a)  $\sum_{j=1}^{m_i} n_{i_j} = n_i$  for each  $1 \leq i \leq k$ ,
- (b)  $\sum_{i=1}^k n_i = l$ ,
- (c)  $\sum_{i=1}^k m_i = m$ ,

then the following commutes

$$\begin{array}{ccc}
 P(k) \otimes \bigotimes_{i=1}^k \left( P(m_i) \otimes \bigotimes_{j=1}^{m_i} P(n_{i_j}) \right) & \xrightarrow{\cong} & \left( P(k) \otimes \bigotimes_{i=1}^k P(m_i) \right) \otimes \left( \bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j}) \right) \\
 \downarrow \text{id}_{\mathcal{C}} \otimes \bigotimes_{i=1}^k c_{m_i; n_{i_1}, \dots, n_{i_{m_i}}} & & \downarrow \circ_{k; m_1, \dots, m_k} \otimes \text{id}_{\mathcal{C}}^{\otimes m} \\
 P(k) \otimes \bigotimes_{i=1}^k P(n_i) & & P(m) \otimes \left( \bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j}) \right) \\
 \downarrow \circ_{k; n_1, \dots, n_k} & & \downarrow \circ_{m; n_{1_1}, \dots, n_{k(m_k)}} \\
 P(l) & \xrightarrow{=} & P(l)
 \end{array}$$

2. Unit: there exists a map  $e : \mathbf{1} \rightarrow P(1)$  such that the following hold for all  $n$

$$\begin{array}{c}
 P(n) \xrightarrow{\lambda^{-1}} \mathbf{1} \otimes P(n) \xrightarrow{e \otimes \text{id}_{\mathcal{C}}} P(1) \otimes P(n) \xrightarrow{\circ_{1; n}} P(n) \\
 \hline
 P(n) \xrightarrow{(\rho^{-1})^{\circ n}} P(n) \otimes \mathbf{1}^{\otimes n} \xrightarrow{\text{id}_{\mathcal{C}} \otimes e^{\otimes n}} P(n) \otimes P(1)^{\otimes n} \xrightarrow{\circ_{n; 1, \dots, 1}} P(n) \\
 \hline
 \end{array}$$

3. Equivariance: the right action  $-\cdot\sigma : P(n) \rightarrow P(n)$  of  $\sigma \in S_n$  on  $P(n)$  is given by  $\sigma$  permuting the  $n$  inputs of each "element" of  $P(n)$ , so that on operadic composition maps, we have

$$\circ_{n;m_1,\dots,m_n} \circ (-\cdot\sigma) = \circ_{n;m_{\sigma^{-1}(1)},\dots,m_{\sigma^{-1}(n)}},$$

then equivariance is the condition that the following commutes

$$\begin{array}{ccc} P(n) \otimes \bigotimes_{i=1}^n P(m_i) & \xrightarrow{\text{id}_e \otimes (-)\sigma} & P(n) \otimes \bigotimes_{i=1}^n P(m_{\sigma^{-1}(i)}) \\ \downarrow -\cdot\sigma \otimes \text{id}_e^{\otimes n} & & \downarrow \circ_{n;m_{\sigma^{-1}(1)},\dots,m_{\sigma^{-1}(n)}} \\ P(n) \otimes \bigotimes_{i=1}^n P(m_i) & \xrightarrow{\circ_{n;m_1,\dots,m_n}} & P(m_1 + \dots + m_n) \end{array}$$

(where  $(-)\sigma$  is the isomorphism given by  $\sigma$  permuting the  $n$  factors of  $\bigotimes_{i=1}^n P(m_i)$ )

**Example 2.1** (the endofunctor operad). *If  $\mathcal{C}$  is a category, then the endofunctor operad, defined by  $\text{End}(\mathcal{C})(n) \stackrel{\text{def}}{=} \text{Fun}(\mathcal{C}^{\times n}, \mathcal{C})$  for  $n \in \mathbb{N}_{\geq 0}$ , is a symmetric operad in  $(\mathbf{Cat}, \times, \{*\})$ , with*

1.  $\text{id}_{\mathcal{C}}$  as the unit, and  $\times$  as the monoidal product.
2. the standard composition as the operadic composition,
3.  $\sigma \in S_n$  acting on each element  $\theta \in \text{Fun}(\mathcal{C}^{\times n}, \mathcal{C})$  by permuting the inputs, i.e.  $\theta \cdot \sigma(x_1, \dots, x_n) = \theta(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$ .

The endofunctor operad is a special case of the endomorphism operad ; it is an endomorphism operad in  $(\mathbf{Cat}, \times, \{*\})$ , a symmetric monoidal category with internal hom.

**Example 2.2** (the operad of parenthesized braids). *This construction is similar to the one in pp.18-19 of [Fre18]. PaB is an operad in  $\mathbf{Grpd}$  (the category of groupoids).*

*The objects of  $\text{PaB}(n)$  are parenthesized permutations of  $n$  elements, i.e.*

$$\text{Obj}(\text{PaB}(n)) = S_n \times M_n,$$

*where  $M_n$  is the set of words of length  $n$  in the free magma generated by one element. If  $(\sigma, w) \in \text{PaB}(n)$ , then for  $1 \leq i \leq n$ , we label the  $i$ th factor of  $w$  (which must be the generating element) with  $\sigma(i)$  to get the parenthesized permutation.*

*The morphisms of  $\text{PaB}(n)$  are braids from  $(\sigma, w)$  to  $(\tau, v)$  such that if the  $l$  factor of  $w$  and the  $m$ th factor of  $v$  are the ends of a strand, then  $\sigma(l) = \tau(m)$ .*

*PaB has the following operadic properties:*



1. The unit, which happens to be the only morphism of  $\text{PaB}(1)$ , is the braid from  $(1, x)$  to  $(1, x)$ .
2. The operadic composition is given by cabling:
3. The action of  $\mathbf{S}_n$  on  $\text{PaB}(n)$  is given by  $(\sigma, w) \mapsto (\tau\sigma, w)$  on objects.

**Definition 2.7** (operad morphism). A morphism  $f : P \rightarrow Q$  of operads is a collection of maps  $\{f_n : P(n) \rightarrow Q(n)\}_n$  such that

1. The unit is preserved, i.e.  $f_1 : P(1) \rightarrow Q(1)$  satisfies  $f_1 \circ e^P = e^Q$ .
2. Composition is preserved, i.e. for all  $k, n_1, \dots, n_k \in \mathbb{N}_{\geq 1}$  with  $l = \sum_{i=1}^k n_i$ , the following commutes

$$\begin{array}{ccc}
 P(k) \otimes \bigotimes_{i=1}^k P(n_i) & \xrightarrow{f_k \otimes \bigotimes_{i=1}^k f_{n_i}} & Q(k) \otimes \bigotimes_{i=1}^k Q(n_i) \\
 \downarrow \circ_{k; n_1, \dots, n_k}^P & & \downarrow \circ_{k; n_1, \dots, n_k}^Q \\
 P(l) & \xrightarrow{f_l} & Q(l)
 \end{array}$$

3. The  $\mathbf{S}_n$ -actions are preserved, i.e. for all  $n \in \mathbb{N}_{\geq 1}$ ,  $\sigma \in \mathbf{S}_n$ ,  $f \circ (- \cdot^P \sigma) = (- \cdot^Q \sigma) \circ f$ .

**Definition 2.8** (operad algebra). For an operad  $P$  in  $(\mathbf{Cat}, \times, \{*\})$ , a morphism of operads  $P \rightarrow \text{End}(\mathcal{C})$  gives  $\mathcal{C}$  the structure of a  $P$ -algebra.

# Bibliography

- [Bel80] G V Belyĭ. On galois extensions of a maximal cyclotomic field. *Mathematics of the USSR-Izvestiya*, 14(2):247–256, April 1980.
- [Dri91] Vladimir Gershonovich Drinfeld. On quasitriangular quasi-hopf algebras and on a group that is closely connected with  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . *Leningrad Mathematical Journal*, 2(4):829–860, 1991.
- [Eti] Pavel Etingof. Monoidal functors.
- [Eti09] Pavel Etingof. *MacLane Coherence Theorem, Tensor and Multitensor Categories*. 2009.
- [Fre18] Benoit Fresse. Little discs operads, graph complexes and grothendieck–teichmüller groups. (arXiv:1811.12536), November 2018. arXiv:1811.12536 [math].
- [Gro97] Alexandre Grothendieck. *Esquisse d’un Programme*, page 7–48. Cambridge University Press, 1 edition, July 1997.
- [Iha91] Y. Ihara. *Braids, Galois Groups, and Some Arithmetic Functions*. Kyoto University. Research Institute for Mathematical Sciences [RIMS]. Kyoto University. Research Institute for Mathematical Sciences [RIMS], 1991.
- [Loc] P Lochak. Automorphism groups of profinite complexes of curves and the grothendieck-teichmüller group.
- [Pet14] Dan Petersen. Minimal models, gt-action and formality of the little disk operad. *Selecta Mathematica*, 20(3):817–822, July 2014.