

# The Grothendieck-Teichmüller Group and the Operad of Parenthesized Braids

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# *Abstract*

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We introduce the operad of parenthesized braids **PaB**, show that algebras over **PaB** correspond to braided monoidal categories, and describe the action of the Grothendieck-Teichmüller group **GT** on **PaB**.

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# Chapter 1

## Introduction

$\mathcal{M}_{g,n}$  is the moduli stack (i.e. 2-scheme) of genus- $g$  curves over  $\mathbb{Q}$  with  $n$  marked points. In his Sketch of a Program [Gro97] from 1983, Grothendieck proposes that we study  $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  through its action on the Teichmüller Tower — the collection of

$$\pi_1^{\text{geom}}(\mathcal{M}_{g,n}) \stackrel{\text{def}}{=} \pi_1^{\text{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

for all  $g, n \in \mathbb{N}$ . Here,  $\pi_1^{\text{et}}$  takes the étale fundamental group, which is the inverse limit of the automorphism groups (analogous to the groups of deck transformations) of finite étale covers.

In genus-0, it is known that  $\pi_1^{\text{geom}}(\mathcal{M}_{0,4}) \cong \widehat{F}_2$ , the profinite completion of the free group on two generators. It follows from a theorem of Belyi's [Bel80] that  $G_{\mathbb{Q}}$  acts faithfully on  $\pi_1^{\text{geom}}(\mathcal{M}_{0,4})$ , hence also  $\{\mathcal{M}_{0,n}\}_n$ . In [Iha91], Ihara showed that the image of the action of  $G_{\mathbb{Q}}$  includes into the image of  $\widehat{\mathbf{GT}}$ , the profinite completion of the inertia-preserving automorphisms of  $\{\mathcal{M}_{0,n}\}_n$ , i.e. the automorphisms  $F$  such that on  $\widehat{F}_2$ ,  $F$  maps the procyclic subgroups  $\langle x \rangle, \langle y \rangle, \langle (xy)^{-1} \rangle$  to conjugate subgroups, where  $x$  and  $y$  are the generators of  $F_2$  [Loc].

In [Dri91], Drinfeld constructs a pro-unipotent (describe in a few sentences) version of  $\mathbf{GT}$ , and uses its action on braided monoidal categories (i.e.  $\widehat{\mathbf{PaB}}$ -algebras) to prove a result about quasitriangular quasi-Hopf algebras. In [Pet14], Petersen uses the corresponding action on  $\widehat{\mathbf{PaB}}$  to show that  $E_2$  (the operad of little 2-disks) is formal.

# Appendix A

## An Appendix

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## A.1 ChatGPT

*The following will be commented out in the end, but it's here for now because it wouldn't render in the chat interface*

Thanks! I'll look into what it means for  $\pi_1(E_2)$  to be a model for the Teichmüller tower, including how this connection is made precise in algebraic topology and geometry. I'll get back to you with a detailed explanation and any key references that clarify this relationship.

### A.1.1 Little 2-Disks Operad and the Grothendieck-Teichmüller Tower

The Teichmüller tower is the profinite system formed by the fundamental groups of the moduli stacks  $\mathcal{M}_{g,n}$  of algebraic curves (genus  $g$ ,  $n$  marked points) and the natural maps between them (induced by forgetting or gluing marked points). By a theorem of Grothendieck, the geometric (étale) fundamental group of  $\mathcal{M}_{g,n}$  is canonically the profinite completion of the mapping-class group  $\Gamma_{g,n}$  of a genus- $g$  surface with  $n$  punctures [dBHR19]. Thus the Teichmüller tower can be described purely in topological terms as the tower of all profinite completions  $\widehat{\Gamma}_{g,n}$ , together with the natural homomorphisms induced by inclusions of subsurfaces [BR25]. In genus 0 this amounts to the tower of profinite mapping-class groups of spheres with marked points. One then defines the (profinite) Grothendieck-Teichmüller group  $\widehat{\mathrm{GT}}$  as the group of automorphisms of this tower commuting with the operadic/gluing maps [dBHR19]. For example, Ihara showed that the Galois action on  $\pi_1(\mathcal{M}_{0,4}) \cong$

$\widehat{F}_2$  lands in  $\widehat{\mathbf{GT}}$  [dBHR19], reflecting the faithful action of  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the whole Teichmüller tower.

The little 2-disks operad  $E_2$  is the topological operad whose space of  $n$ -ary operations  $E_2(n)$  is the space of configurations of  $n$  disjoint labeled embeddings of the unit disk into itself. Concretely,  $E_2(n)$  is homotopy equivalent to the configuration space of  $n$  labeled points in  $\mathbb{R}^2$  (or  $\mathbb{C}$ ). A classical result (Arnol'd, Fadell-Neuwirth, etc.) is that each such configuration space is a  $K(\pi, 1)$  with fundamental group the braid group on  $n$  strands. Equivalently, each  $E_2(n)$  has no higher homotopy, so the topological operad  $E_2$  is fully captured by its fundamental groupoid operad. In fact one can show:

1. Claim: The space  $E_2(n)$  is a  $K(\pi, 1)$  with  $\pi_1(E_2(n)) \cong B_n$  (the braid group on  $n$  strands). In particular, there are no nontrivial higher homotopy groups. Thus the operad  $E_2$  is (homotopy equivalent to) the operad in groupoids obtained by taking the fundamental groupoid of each  $E_2(n)$  [C14].

This means that, up to homotopy,  $E_2$  is an operad whose  $n$ -ary part is essentially the braid group  $B_n$ , with operadic composition given by “inserting” braids into one another (the usual operadic substitution corresponds to concatenating configuration braids).

### A.1.2 Moduli of genus 0 curves as an operad

The moduli spaces of genus-0 curves themselves carry an operadic structure. More precisely, let  $\overline{\mathcal{M}}_{0,n+1}$  denote the Deligne-Mumford moduli of stable genus-0 curves with  $n + 1$  marked points. Then the collection

$$\{\overline{\mathcal{M}}_{0,n+1}\}_{n \geq 1}$$

forms an operad in topological spaces (with no 0-ary part): the  $n + 1$  points are viewed as  $n$  inputs and one output, and operadic composition is given by gluing the “output” of one curve to an input of another (creating a node) [dBHR19]. In more detail:

1. For each  $n \geq 1$ ,  $\overline{\mathcal{M}}_{0,n+1}$  is a (compact) complex manifold of dimension  $n - 2$ .
2. By a well-known result, for  $n \geq 3$  it is isomorphic (analytically) to the configuration space of  $n + 1$  distinct points on  $\mathbb{CP}^1$  modulo  $\mathrm{PGL}_2(\mathbb{C})$ .

2. The operad structure is given as follows: if we have one stable curve with  $n + 1$  markings and another with  $m + 1$  markings, we glue the marked output of one to the  $k$ -th input of the other. The resulting nodal curve lies in  $\overline{\mathcal{M}}_{0,n+m}$  [dBHR19]. Composition is associative and  $\Sigma_n$ -equivariant in the obvious way.

Thus  $\overline{\mathcal{M}}_{0,\bullet+1}$  is a modular operad (or in fact a cyclic operad) encoding the combinatorics of stable genus-0 curves. At the level of fundamental groups, this operad structure corresponds to the usual inclusions and gluing of mapping-class groups of spheres with punctures.

A key connection to the little disks operad was established by Drummond-Cole: there is a homotopy pushout diagram of operads relating the framed little 2-disks operad  $FD$  to the moduli operad  $\overline{\mathcal{M}}_{0,\bullet+1}$  [dBHR19]. Concretely,  $FD(n)$  differs from the non-framed  $E_2(n)$  by allowing each little disk to rotate (so  $FD(n) \simeq E_2(n) \times (S^1)^n$ ). Drummond-Cole showed that “killing” the  $S^1$ -rotation in arity 1 in  $FD$  yields an operadic quotient equivalent to the moduli operad [dBHR19]. In other words, the genus-0 surface operad  $M$  (classifying spaces of spheres with boundary circles) maps onto  $\overline{\mathcal{M}}_{0,n+1}$  by collapsing the boundary circles; by a theorem of Drummond-Cole, this map is an operadic homotopy quotient obtained by contracting the circle in arity 1 [dBHR19]. Equivalently, one can say the genus-0 surface operad  $M$  (or  $FD$ ) and the moduli operad  $\overline{\mathcal{M}}_{0,\bullet+1}$  become equivalent once one trivializes the  $S^1$ -factor.

Because of this equivalence, the fundamental group operad of  $E_2$  (or of framed  $E_2$ ) “models” the tower of mapping class groups for genus 0. Indeed, as noted above,  $E_2(n)$  has  $\pi_1$  the  $n$ -strand braid group  $B_n$ . Topologically, a sphere with  $n + 1$  boundary components has a mapping class group which fits in a split extension

$$1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma_{0,n}(\text{with boundary}) \rightarrow \Gamma_{0,n}(\text{punctured sphere}) \rightarrow 1,$$

where the  $\mathbb{Z}^n$  comes from rotating each boundary circle [BR25]. Collapsing the boundaries recovers the pure mapping-class group of the sphere with  $n$  punctures. But as operads these spheres-with-boundaries  $\Gamma_{0,n}$  assemble exactly into an operad homotopy equivalent to the little disks operad [BR25][dBHR19]. Put succinctly:

1. The mapping-class groups of genus-0 surfaces with boundaries form a modular operad under gluing of boundary components [BR25]. Its genus-0 part (spheres with  $n + 1$  boundary components) is often called the genus-0 surface operad.
2. This surface operad  $M$  is homotopy equivalent to the (framed) little disks operad  $FD$  (and hence to  $E_2$  up to the extra  $S^1$ -factors) [dBHR19].



Since each  $E_2(n) \simeq FD(n)$  is  $K(\pi, 1)$  with  $\pi = B_n$ , it follows that taking fundamental groupoids recovers the pure braid operad (or more precisely the operad of pure braids, which is also the operad of classifying spaces of these mapping-class groups). In this sense  $\pi_1(E_2)$  is exactly an operadic model for the genus-0 Teichmüller tower: its  $n$ th piece is the (profinite) mapping class group of a sphere with  $n + 1$  points (or boundaries). Hatcher-Lochak-Schneps originally described an “operadic” version of the Teichmüller tower by using boundary-based mapping-class groups, and one recovers the same structure from  $E_2$  [BR25] [dBHR19].

### A.1.3 Profinite completion and the Grothendieck-Teichmüller group

Because each  $E_2(n)$  (and each moduli space  $\overline{\mathcal{M}}_{0,n+1}$ ) is defined over  $\mathbb{Q}$ , one can consider its profinite completion. The result is that  $\widehat{E_2}$  (the collection of profinite homotopy types of  $E_2(n)$ ) is an operad in profinite spaces whose fundamental progroups are the profinite braid groups  $\widehat{B}_n$  (equivalently, the profinite mapping class groups  $\widehat{\Gamma}_{0,n+1}$ ). A remarkable result of Boavida de Brito-Horel-Robertson is that the profinite automorphism group of this entire profinite operad is isomorphic to the profinite Grothendieck-Teichmüller group [dBHR19]. Equivalently, any automorphism of the little-disks operad up to homotopy (in the profinite or completed sense) comes from an element of  $\widehat{\mathbf{GT}}$ . In their words: “the group  $\widehat{\mathbf{GT}}$  is isomorphic to the group of homotopy automorphisms of the profinite completion of the genus zero surface operad” [dBHR19]. This shows that  $\pi_1(E_2)$  (profinite fundamental groupoid of  $E_2$ ) indeed encodes the full genus-0 Teichmüller tower and its  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action. In particular,  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts faithfully on the profinite  $\pi_1$  of  $E_2$  (equivalently on the profinite braids operad) via its embedding into  $\widehat{\mathbf{GT}}$  [dBHR19].

**Summarizing**, the little 2-disks operad  $E_2$  (or its framed version) provides a topological model for the genus-0 part of the Teichmüller tower. Its operadic structure mirrors the gluing of genus-0 curves, its homotopy groups are the braid/mapping-class groups, and its profinite completion carries the  $\widehat{\mathbf{GT}}$ -action. For further details see, e.g., Boavida de Brito-Horel-Robertson, [dBHR19], and Hatcher-Lochak-Schneps (2000) or survey articles on Grothendieck-Teichmüller theory.

**Key references:** The operadic realization of the Teichmüller tower is explained in Hatcher-Lochak-Schneps, “On the Teichmüller tower of mapping class groups,” J. Reine Angew. Math. 521 (2000), and in lecture notes by Borghi-Robertson [BR25]. The work of Boavida de Brito-Horel-Robertson (2020) establishes rigorously that  $\widehat{\mathbf{GT}}$  equals the automorphism group of the profinite little-disks operad [dBHR19]. Finally, the fact that  $E_2(n)$  is  $K(\pi, 1)$  with  $\pi$  a braid group is classical [C14]. These

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and other sources give a thorough account of how  $\pi_1(E_2)$  encodes the genus-zero Grothendieck-Teichmüller structures.

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