The Grothendieck-Teichmüller Group and the Operad of Parenthesized Braids

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Abstract

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We introduce the operad of parenthesized braids **PaB**, show that algebras over **PaB** correspond to braided monoidal categories, and describe the action of the Grothendieck-Teichmüller group **GT** on **PaB**.

Contents

Chapter 1

Introduction

$\mathcal{M}_{g,n}$ is the moduli stack (i.e. 2-scheme) of genus-g curves over $\mathbb{Q}$ with n marked points. In his Sketch of a Program [Gro97] from 1983, Grothendieck proposes that we study $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its action on the Teichmüller Tower — the collection of

$$\pi_1^{\text{geom}}(\mathcal{M}_{g,n}) \stackrel{\text{def}}{=} \pi_1^{\text{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

for all $g, n \in \mathbb{N}$. Here, π_1^{et} takes the étale fundamental group, which is the inverse limit of the automorphism groups (analogous to the groups of deck transformations) of finite étale covers.

In genus-0, it is known that $\pi_1^{\text{geom}}(\mathcal{M}_{0,4}) \cong \widehat{F_2}$, the profinite completion of the free group on two generators. It follows from a theorem of Belyi's [Bel80] that $G_{\mathbb{Q}}$ acts faithfully on $\pi_1^{\text{geom}}(\mathcal{M}_{0,4})$, hence also $\{\mathcal{M}_{0,n}\}_n$. In [Iha91], Ihara showed that the image of the action of $G_{\mathbb{Q}}$ includes into the image of $\widehat{\mathbf{GT}}$, the profinite completion of the inertia-preserving automorphisms of $\{\mathcal{M}_{0,n}\}_n$, i.e. the automorphisms F such that on $\widehat{F_2}$, F maps the procyclic subgroups $\langle x\rangle, \langle y\rangle, \langle (xy)^{-1}\rangle$ to conjugate subgroups, where x and y are the generators of F_2 [Loc].

In [Dri91], Drinfeld constructs a pro-unipotent version of $\mathbf{GT}$, and uses its action on braided monoidal categories (i.e. $\widehat{\mathbf{PaB}}$-algebras) to prove a result about quasi-triangular quasi-Hopf algebras. In [Pet14], Petersen uses the corresponding action on $\widehat{\mathbf{PaB}}$ to show that E_2 (the operad of little 2-disks) is formal.

Chapter 2

Algebras of PaB

2.1 Braided Monoidal Categories

Definition 2.1 (monoidal category)**.** A category $\mathcal{C}$ with a bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ (the monoidal, or tensor product), unit object $\mathbf{1}$, and natural isomorphisms

$$\alpha : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-) \tag{2.1}$$

$$\lambda : \mathbf{1}\otimes- \Rightarrow \mathrm{id}_{\mathcal{C}} \qquad \rho : -\otimes\mathbf{1} \Rightarrow \mathrm{id}_{sC} \tag{2.2}$$

respectively called the associator, left unitor, right unitor, is *monoidal* if when we write $\otimes$ to act componentwise on natural transformations,

1. the triangle identity is satisfied, i.e. $\rho\otimes\mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}}\otimes\lambda)\circ\alpha$

2. the pentagon identity holds, i.e. for all $w, x, y, z \in \mathrm{Obj}(\mathcal{C})$, the following diagram commutes

$$\begin{tikzcd}
 & (w\otimes x)\otimes(y\otimes z) \arrow[dr] & \\
((w\otimes x)\otimes y)\otimes z \arrow[ur] \arrow[d] & & w\otimes(x\otimes(y\otimes z)) \\
(w\otimes(x\otimes y))\otimes z \arrow[r] & w\otimes((x\otimes y)\otimes z) \arrow[ur] &
\end{tikzcd}$$

   or as natural transformations, $\alpha\circ\alpha = (\mathrm{id}_{\mathcal{C}}\otimes\alpha)\circ\alpha\circ(\alpha\otimes\mathrm{id}_{\mathcal{C}})$

A monoidal category is *strict* if the components of the associator and unitors are identity morphisms.

Definition 2.2 (braided monoidal category)**.** If $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is a monoidal category with a natural isomorphism $\beta_{x,y} : x \otimes y \to y \otimes x$, then it is *braided* if it satisfies the hexagon identities, i.e. the following diagrams commute

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "\alpha"] \arrow[d, "\beta\otimes\mathrm{id}_{\mathcal{C}}"] & x \otimes (y \otimes z) \arrow[r, "\beta"] & (y \otimes z) \otimes x \arrow[d, "\alpha"] \\
(y \otimes x) \otimes z \arrow[r, "\alpha"] & y \otimes (x \otimes z) \arrow[r, "\mathrm{id}_{\mathcal{C}}\otimes\beta"] & y \otimes (z \otimes x)
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}\otimes\beta"] & (x \otimes y) \otimes z \arrow[r, "\beta"] & z \otimes (x \otimes y) \arrow[d, "\alpha^{-1}"] \\
x \otimes (z \otimes y) \arrow[r, "\alpha^{-1}"] & (x \otimes z) \otimes y \arrow[r, "\beta\otimes\mathrm{id}_{\mathcal{C}}"] & (z \otimes x) \otimes y
\end{tikzcd}$$

We will now build up to Mac Lane's Coherence Theorem

Definition 2.3 (monoidal functor, or strong monoidal functor at nLab)**.** Suppose $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$, and $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$, are monoidal categories, then a monoidal functor consists of a functor $F : \mathcal{C} \to \mathcal{C}'$ such that $\mathbf{1}' \cong F(\mathbf{1})$, and a natural isomorphism $m : F(-) \otimes F(-) \to F(- \otimes -)$ such that the following diagram commutes

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\alpha'"] \arrow[d, "m\otimes\mathrm{id}_{\mathcal{C}}"] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id}_{\mathcal{C}}\otimes m"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "m"] & F(X) \otimes F(Y \otimes Z) \arrow[d, "m"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha)"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

A monoidal functor that happens to be an equivalence of categories is called an *equivalence of monoidal categories.*

Theorem 2.4 (Mac Lane's Strictness Theorem)**.** *For any monoidal category* $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$*, there exists a* strict *monoidal category* $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$ *and an equivalence of monoidal categories* $L : \mathcal{C} \to \mathcal{C}'$*.*

Proof. see [Eti] from the 2009 lecture notes to "Topics in Lie Theory". □

The next theorem is a corollary of Mac Lane's Strictness Theorem.

Theorem 2.5 (Mac Lane's Coherence Theorem)**.** *Suppose P and P' are parenthesizations of $X_1 \otimes \cdots \otimes X_n$ with arbitrary insertions of* $\mathbf{1}$*, and $f, g : P \to P'$ are isomorphisms built by composing associator and unitor components, then $f = g$.*

Proof. see [Eti09] ☐

2.2 The Operad PaB of Parenthesized Braids

Operads are an unbiased way of keeping track of n-ary operations and how they compose. The "elements" of $P(n)$, we imagine to be corollas, i.e., rooted trees of depth 1 with $n+1$ half-edges. We can treat $\otimes$ in this "unbiased" way (i.e. ignoring how products are parenthesized, and products with units) because of Mac Lane's Coherence Theorem 2.5.

Definition 2.6 (operad)**.** An operad P in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, b)$ (i.e. a braided monoidal category such that $b \circ b = \mathrm{id}_{\mathcal{C}}$) is a collection of objects $\{P(n)\}_n$, with each object $P(n)$ having a right S_n-action (S_n being the nth symmetric group), and operadic composition

$$c_{m;n_1,\cdots,n_m} : P(m) \otimes P(n_1) \otimes \cdots \otimes P(n_m) \to P(n_1 + \cdots + n_m)$$

such that the following are satisfied

1. Associativity: if
   (a) $\sum_{j=1}^{m_i} n_{i_j} = n_i$ for each $1 \leq i \leq k$,
   (b) $\sum_{i=1}^{k} n_i = l$,
   (c) $\sum_{i=1}^{k} m_i = m$,

   then the following commutes

$$\begin{tikzcd}
P(k) \otimes \bigotimes_{i=1}^{k} \left(P(m_i) \otimes \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[r, "\cong"'] \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \bigotimes_{i=1}^{k} c_{m_i;n_{i_1},\cdots,n_{i_{m_i}}}"] & \left(P(k) \otimes \bigotimes_{i=1}^{k} P(m_i)\right) \otimes \left(\bigotimes_{i=1}^{k} \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "c_{k;m_1,\cdots,m_k} \otimes \mathrm{id}_{\mathcal{C}}^{\otimes m}"'] \\
P(k) \otimes \bigotimes_{i=1}^{k} P(n_i) \arrow[d, "c_{k;n_1,\cdots,n_k}"] & P(m) \otimes \left(\bigotimes_{i=1}^{k} \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "c_{m;n_{1_1},\cdots,n_{k_{(m_k)}}}"'] \\
P(l) \arrow[r, "="] & P(l)
\end{tikzcd}$$

2. Unit: there exists a map $e : \mathbf{1} \to P(1)$ such that the following hold for all n

$$\begin{tikzcd}
P(n) \arrow[r, "\lambda^{-1}"] \arrow[rrr, bend right=10, "="'] & \mathbf{1} \otimes P(n) \arrow[r, "e \otimes \mathrm{id}_{\mathcal{C}}"] & P(1) \otimes P(n) \arrow[r, "c_{1;n}"] & P(n)
\end{tikzcd}$$

$$\begin{tikzcd}
P(n) \arrow[r, "(\rho^{-1})^{\circ n}"] \arrow[rrr, bend right=10, "="'] & P(n) \otimes \mathbf{1}^{\otimes n} \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes e^{\otimes n}"] & P(n) \otimes P(1)^{\otimes n} \arrow[r, "c_{n;1,\cdots,1}"] & P(n)
\end{tikzcd}$$

3. Equivariance: the right action $(-)s : P(n) \to P(n)$ of $s \in S_n$ on $P(n)$ is given by s permuting the n inputs of each "element" of $P(n)$, so that on operadic composition maps, we have

$$c_{n;m_1,\cdots,m_n} \circ (-)s = c_{n;m_{s^{-1}(1)},\cdots,m_{s^{-1}(n)}},$$

then equivariance is the condition that the following commutes

$$\begin{tikzcd}
P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes s(-)"'] \arrow[d, "(-)s \otimes \mathrm{id}_{\mathcal{C}}^{\otimes n}"] & P(n) \otimes \bigotimes_{i=1}^n P(m_{s^{-1}(i)}) \arrow[d, "c_{n;m_{s^{-1}(1)},\cdots,m_{s^{-1}(n)}}"'] \\
P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "c_{n;m_1,\cdots,m_n}"] & P(m_1 + \cdots + m_n)
\end{tikzcd}$$

(where $s(-)$ is the isomorphism given by s permuting the n factors of $\bigotimes_{i=1}^n P(m_i)$)

Definition 2.7 (operad morphism)**.** A morphism $f : P \to Q$ of operads is a collection of maps $\{f_n : P(n) \to Q(n)\}_n$ such that

Appendix A

An Appendix

A.1 ChatGPT

The following will be commented out in the end, but it's here for now because it wouldn't render in the chat interface

Thanks! I'll look into what it means for $\pi_1(E_2)$ to be a model for the Teichmüller tower, including how this connection is made precise in algebraic topology and geometry. I'll get back to you with a detailed explanation and any key references that clarify this relationship.

A.1.1 Little 2-Disks Operad and the Grothendieck-Teichmüller Tower

The Teichmüller tower is the profinite system formed by the fundamental groups of the moduli stacks $\mathcal{M}_{g,n}$ of algebraic curves (genus g, n marked points) and the natural maps between them (induced by forgetting or gluing marked points). By a theorem of Grothendieck, the geometric (étale) fundamental group of $\mathcal{M}_{g,n}$ is canonically the profinite completion of the mapping-class group $\Gamma_{g,n}$ of a genus-g surface with n punctures [dBHR19]. Thus the Teichmüller tower can be described purely in topological terms as the tower of all profinite completions $\widehat{\Gamma}_{g,n}$, together with the natural homomorphisms induced by inclusions of subsurfaces [BR25]. In genus 0 this amounts to the tower of profinite mapping-class groups of spheres with marked points. One then defines the (profinite) Grothendieck-Teichmüller group $\widehat{\mathsf{GT}}$ as the group of automorphisms of this tower commuting with the operadic/gluing maps [dBHR19]. For example, Ihara showed that the Galois action on $\pi_1(\mathcal{M}_{0,4}) \cong$

$\widehat{F}_2$ lands in $\widehat{\mathsf{GT}}$ [dBHR19], reflecting the faithful action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the whole Teichmüller tower.

The little 2-disks operad E_2 is the topological operad whose space of n-ary operations $E_2(n)$ is the space of configurations of n disjoint labeled embeddings of the unit disk into itself. Concretely, $E_2(n)$ is homotopy equivalent to the configuration space of n labeled points in $\mathbb{R}^2$ (or $\mathbb{C}$). A classical result (Arnol'd, Fadell-Neuwirth, etc.) is that each such configuration space is a $K(\pi, 1)$ with fundamental group the braid group on n strands. Equivalently, each $E_2(n)$ has no higher homotopy, so the topological operad E_2 is fully captured by its fundamental groupoid operad. In fact one can show:

1. Claim: The space $E_2(n)$ is a $K(\pi, 1)$ with $\pi_1(E_2(n)) \cong B_n$ (the braid group on n strands). In particular, there are no nontrivial higher homotopy groups. Thus the operad E_2 is (homotopy equivalent to) the operad in groupoids obtained by taking the fundamental groupoid of each $E_2(n)$ [C14].

This means that, up to homotopy, E_2 is an operad whose n-ary part is essentially the braid group B_n, with operadic composition given by "inserting" braids into one another (the usual operadic substitution corresponds to concatenating configuration braids).

A.1.2 Moduli of genus 0 curves as an operad

The moduli spaces of genus-0 curves themselves carry an operadic structure. More precisely, let $\overline{\mathcal{M}}_{0,n+1}$ denote the Deligne-Mumford moduli of stable genus-0 curves with $n+1$ marked points. Then the collection

$$\{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 1}$$

forms an operad in topological spaces (with no 0-ary part): the $n+1$ points are viewed as n inputs and one output, and operadic composition is given by gluing the "output" of one curve to an input of another (creating a node) [dBHR19]. In more detail:

1. For each $n \geq 1$, $\overline{\mathcal{M}}_{0,n+1}$ is a (compact) complex manifold of dimension $n-2$. By a well-known result, for $n \geq 3$ it is isomorphic (analytically) to the configuration space of $n+1$ distinct points on $\mathbb{C}P^1$ modulo $\mathrm{PGL}_2(\mathbb{C})$.

2. The operad structure is given as follows: if we have one stable curve with $n+1$ markings and another with $m+1$ markings, we glue the marked output of one to the k-th input of the other. The resulting nodal curve lies in $\overline{\mathcal{M}}_{0,n+m}$ [dBHR19]. Composition is associative and Σ_n-equivariant in the obvious way.

Thus $\overline{\mathcal{M}}_{0,\bullet+1}$ is a modular operad (or in fact a cyclic operad) encoding the combinatorics of stable genus-0 curves. At the level of fundamental groups, this operad structure corresponds to the usual inclusions and gluing of mapping-class groups of spheres with punctures.

A key connection to the little disks operad was established by Drummond-Cole: there is a homotopy pushout diagram of operads relating the framed little 2-disks operad FD to the moduli operad $\overline{\mathcal{M}}_{0,\bullet+1}$ [dBHR19]. Concretely, $FD(n)$ differs from the non-framed $E_2(n)$ by allowing each little disk to rotate (so $FD(n) \simeq E_2(n) \times (S^1)^n$). Drummond-Cole showed that "killing" the S^1-rotation in arity 1 in FD yields an operadic quotient equivalent to the moduli operad [dBHR19]. In other words, the genus-0 surface operad M (classifying spaces of spheres with boundary circles) maps onto $\overline{\mathcal{M}}_{0,n+1}$ by collapsing the boundary circles; by a theorem of Drummond-Cole, this map is an operadic homotopy quotient obtained by contracting the circle in arity 1 [dBHR19]. Equivalently, one can say the genus-0 surface operad M (or FD) and the moduli operad $\overline{\mathcal{M}}_{0,\bullet+1}$ become equivalent once one trivializes the S^1-factor.

Because of this equivalence, the fundamental group operad of E_2 (or of framed E_2) "models" the tower of mapping class groups for genus 0. Indeed, as noted above, $E_2(n)$ has π_1 the n-strand braid group B_n. Topologically, a sphere with $n+1$ boundary components has a mapping class group which fits in a split extension

$$1 \to \mathbb{Z}^n \to \Gamma_{0,n}\big(\text{with boundary}\big) \to \Gamma_{0,n}\big(\text{punctured sphere}\big) \to 1,$$

where the $\mathbb{Z}^n$ comes from rotating each boundary circle [BR25]. Collapsing the boundaries recovers the pure mapping-class group of the sphere with n punctures. But as operads these spheres-with-boundaries $\Gamma_{0,n}$ assemble exactly into an operad homotopy equivalent to the little disks operad [BR25][dBHR19]. Put succinctly:

1. The mapping-class groups of genus-0 surfaces with boundaries form a modular operad under gluing of boundary components [BR25]. Its genus-0 part (spheres with $n+1$ boundary components) is often called the genus-0 surface operad.

2. This surface operad M is homotopy equivalent to the (framed) little disks operad FD (and hence to E_2 up to the extra S^1-factors) [dBHR19].

Since each $E_2(n) \simeq FD(n)$ is $K(\pi, 1)$ with $\pi = B_n$, it follows that taking fundamental groupoids recovers the pure braid operad (or more precisely the operad of pure braids, which is also the operad of classifying spaces of these mapping-class groups). In this sense $\pi_1(E_2)$ is exactly an operadic model for the genus-0 Teichmüller tower: its nth piece is the (profinite) mapping class group of a sphere with $n + 1$ points (or boundaries). Hatcher-Lochak-Schneps originally described an "operadic" version of the Teichmüller tower by using boundary-based mapping-class groups, and one recovers the same structure from E_2 [BR25] [dBHR19].

A.1.3 Profinite completion and the Grothendieck-Teichmüller group

Because each $E_2(n)$ (and each moduli space $\overline{\mathcal{M}}_{0,n+1}$) is defined over $\mathbb{Q}$, one can consider its profinite completion. The result is that $\widehat{E_2}$ (the collection of profinite homotopy types of $E_2(n)$) is an operad in profinite spaces whose fundamental progroups are the profinite braid groups $\widehat{B}_n$ (equivalently, the profinite mapping class groups $\widehat{\Gamma}_{0,n+1}$). A remarkable result of Boavida de Brito-Horel-Robertson is that the profinite automorphism group of this entire profinite operad is isomorphic to the profinite Grothendieck-Teichmüller group [dBHR19]. Equivalently, any automorphism of the little-disks operad up to homotopy (in the profinite or completed sense) comes from an element of $\widehat{\mathsf{GT}}$. In their words: "the group $\widehat{\mathsf{GT}}$ is isomorphic to the group of homotopy automorphisms of the profinite completion of the genus zero surface operad" [dBHR19]. This shows that $\pi_1(E_2)$ (profinite fundamental groupoid of E_2) indeed encodes the full genus-0 Teichmüller tower and its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. In particular, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the profinite π_1 of E_2 (equivalently on the profinite braids operad) via its embedding into $\widehat{\mathsf{GT}}$ [dBHR19].

Summarizing, the little 2-disks operad E_2 (or its framed version) provides a topological model for the genus-0 part of the Teichmüller tower. Its operadic structure mirrors the gluing of genus-0 curves, its homotopy groups are the braid/mapping-class groups, and its profinite completion carries the $\widehat{\mathsf{GT}}$-action. For further details see, e.g., Boavida de Brito-Horel-Robertson, [dBHR19], and Hatcher-Lochak-Schneps (2000) or survey articles on Grothendieck-Teichmüller theory.

Key references: The operadic realization of the Teichmüller tower is explained in Hatcher-Lochak-Schneps, "On the Teichmüller tower of mapping class groups," J. Reine Angew. Math. 521 (2000), and in lecture notes by Borghi-Robertson [BR25]. The work of Boavida de Brito-Horel-Robertson (2020) establishes rigorously that $\widehat{\mathsf{GT}}$ equals the automorphism group of the profinite little-disks operad [dBHR19]. Finally, the fact that $E_2(n)$ is $K(\pi, 1)$ with π a braid group is classical [C14]. These and other sources give a thorough account of how $\pi_1(E_2)$ encodes the genus-zero Grothendieck-Teichmüller structures.

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