The Grothendieck-Teichmüller Group and the Operad of Parenthesized Braids

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CHAPTER 1

Introduction

$\mathcal{M}_{g,n}$ is the moduli stack (i.e. 2-scheme) of genus-g curves over $\mathbb{Q}$ with n marked points. In his Sketch of a Program [Gro97] from 1983, Grothendieck proposes that we study $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its action on the Teichmüller Tower — the collection of

$$\pi_1^{\text{geom}}(\mathcal{M}_{g,n}) \stackrel{\text{def}}{=} \pi_1^{\text{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

for all $g, n \in \mathbb{N}$. Here, π_1^{et} takes the étale fundamental group, which is the inverse limit of the automorphism groups (analogous to the groups of deck transformations) of finite étale covers.

In genus-0, it is known that $\pi_1^{\text{geom}}(\mathcal{M}_{0,4}) \cong \widehat{\mathsf{F}}_2$, the profinite completion of the free group on two generators. It follows from a theorem of Belyi's [Bel80] that $G_{\mathbb{Q}}$ acts faithfully on $\pi_1^{\text{geom}}(\mathcal{M}_{0,4})$, hence also $\{\mathcal{M}_{0,n}\}_n$. In [Iha91], Ihara showed that the image of the action of $G_{\mathbb{Q}}$ includes into the image of $\widehat{\mathsf{GT}}$, the profinite completion of the inertia-preserving automorphisms of $\{\mathcal{M}_{0,n}\}_n$, i.e. the automorphisms ϕ such that on $\widehat{\mathsf{F}}_2$, ϕ maps the procyclic subgroups $\langle x\rangle, \langle y\rangle, \langle (xy)^{-1}\rangle$ to conjugate subgroups, where x and y are the generators of F_2 [Loc].

In [Dri91], Drinfeld constructs a pro-unipotent version of GT, and uses its action on braided monoidal categories (i.e. $\widehat{PaB}$-algebras) to prove a result about quasitriangular quasi-Hopf algebras. In [Pet14], Petersen uses the corresponding action on $\widehat{PaB}$ to show that E_2 (the operad of little 2-disks) is formal.

CHAPTER 2

Algebras of PaB

1. Braided Monoidal Categories

DEFINITION 1 (monoidal category). A category $\mathcal{C}$ with a bifunctor $-\otimes - : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (the monoidal, or tensor product), unit object $\mathbf{1}$, and natural isomorphisms

$$\alpha : (-\otimes -)\otimes - \Rightarrow -\otimes(-\otimes -) \tag{1}$$

$$\lambda : \mathbf{1}\otimes - \Rightarrow \mathrm{id}_{\mathcal{C}} \qquad\qquad \rho : -\otimes \mathbf{1} \Rightarrow \mathrm{id}_{sC} \tag{2}$$

respectively called the associator, left unitor, right unitor, is *monoidal* if when we write $\otimes$ to act componentwise on natural transformations,

(1) the triangle identity is satisfied, i.e. $\rho\otimes\mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}}\otimes\lambda)\circ\alpha$
(2) the pentagon identity holds, i.e. for all $w,x,y,z\in\mathrm{Obj}(\mathcal{C})$, the following diagram commutes

$$\begin{tikzcd}
 & (w\otimes x)\otimes(y\otimes z) \arrow[dr] & \\
((w\otimes x)\otimes y)\otimes z \arrow[ur] \arrow[d] & & w\otimes(x\otimes(y\otimes z)) \\
(w\otimes(x\otimes y))\otimes z \arrow[rr] & & w\otimes((x\otimes y)\otimes z) \arrow[u]
\end{tikzcd}$$

or as natural transformations, $\alpha\circ\alpha = (\mathrm{id}_{\mathcal{C}}\otimes\alpha)\circ\alpha\circ(\alpha\otimes\mathrm{id}_{\mathcal{C}})$

A monoidal category is *strict* if the components of the associator and unitors are identity morphisms.

DEFINITION 2 (braided monoidal category). If $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho)$ is a monoidal category with a natural isomorphism $\beta_{x,y} : x\otimes y\to y\otimes x$, then it is *braided* if it satisfies the hexagon identities, i.e. the following diagrams commute

$$\begin{tikzcd}
(x\otimes y)\otimes z \arrow[r, "\alpha"] \arrow[d, "\beta\otimes\mathrm{id}_{\mathcal{C}}"] & x\otimes(y\otimes z) \arrow[r, "\beta"] & (y\otimes z)\otimes x \arrow[d, "\alpha"] \\
(y\otimes x)\otimes z \arrow[r, "\alpha"] & y\otimes(x\otimes z) \arrow[r, "\mathrm{id}_{\mathcal{C}}\otimes\beta"] & y\otimes(z\otimes x)
\end{tikzcd}$$

$$\begin{array}{ccccc} x\otimes(y\otimes z) & \xrightarrow{\alpha^{-1}} & (x\otimes y)\otimes z & \xrightarrow{\beta} & z\otimes(x\otimes y) \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}}\otimes\beta} & & & & \downarrow{\scriptstyle \alpha^{-1}} \\ x\otimes(z\otimes y) & \xrightarrow{\alpha^{-1}} & (x\otimes z)\otimes y & \xrightarrow{\beta\otimes\mathrm{id}_{\mathcal{C}}} & (z\otimes x)\otimes y \end{array}$$

We will now build up to Mac Lane's Coherence Theorem

Definition 3 (monoidal functor, or strong monoidal functor at nLab). Suppose $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho)$, and $(\mathcal{C}',\otimes',\mathbf{1}',\alpha',\lambda',\rho')$, are monoidal categories, then a monoidal functor consists of a functor $F:\mathcal{C}\to\mathcal{C}'$ such that $\mathbf{1}'\cong F(\mathbf{1})$, and a natural isomorphism $m: F(-)\otimes F(-)\to F(-\otimes -)$ such that the following diagram commutes

$$\begin{array}{ccc} (F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{\alpha'} & F(X)\otimes(F(Y)\otimes F(Z)) \\ \downarrow{\scriptstyle m\otimes\mathrm{id}_{\mathcal{C}}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}}\otimes m} \\ F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ F((X\otimes Y)\otimes Z) & \xrightarrow{F(\alpha)} & F(X\otimes(Y\otimes Z)) \end{array}$$

A monoidal functor that happens to be an equivalence of categories is called an *equivalence of monoidal categories.*

Theorem 4 (Mac Lane's Strictness Theorem). *For any monoidal category*
$(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho)$, *there exists a* strict *monoidal category* $(\mathcal{C}',\otimes',\mathbf{1}',\alpha',\lambda',\rho')$ *and an equivalence of monoidal categories* $L:\mathcal{C}\to\mathcal{C}'$.

Proof. see p.20 of [Eti] from the 2009 lecture notes to "Topics in Lie Theory". □

The next theorem is a corollary of Mac Lane's Strictness Theorem.

Theorem 5 (Mac Lane's Coherence Theorem). *Suppose P and P' are parenthesizations of $X_1\otimes\cdots\otimes X_n$ with arbitrary insertions of $\mathbf{1}$, and $f,g:P\to P'$ are isomorphisms built by composing associator and unitor components, then $f=g$.*

Proof. see p.1 of [Eti09] □

2. The Operad PaB of Parenthesized Braids

Operads are an unbiased way of keeping track of n-ary operations and how they compose. The "elements" of $P(n)$, we imagine to be corollas, i.e., rooted trees of depth 1 with $n+1$ half-edges. We can treat $\otimes$ in this "unbiased" way (i.e. ignoring how products are parenthesized, and products with units) because of Mac Lane's Coherence Theorem 5.

Definition 6 (planar operad, symmetric operad). A *planar operad* P in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a collection of objects $\{P(n)\}_n$, with operadic composition

$$c_{m;n_1,\cdots,n_m} : P(m) \otimes P(n_1) \otimes \cdots \otimes P(n_m) \to P(n_1 + \cdots + n_m)$$

such that 1. and 2. of following are satisfied. If each object $P(n)$ has a right S_n-action (S_n being the nth symmetric group), and 3. is staisfied, then P is a *symmetric operad*, which is what "operad" refers to from now on.

(1) Associativity: if
   (a) $\sum_{j=1}^{m_i} n_{i_j} = n_i$ for each $1 \leq i \leq k$,
   (b) $\sum_{i=1}^{k} n_i = l$,
   (c) $\sum_{i=1}^{k} m_i = m$,
   then the following commutes

$$\begin{tikzcd}
P(k) \otimes \bigotimes_{i=1}^k \left(P(m_i) \otimes \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[r, "\cong"'] \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \bigotimes_{i=1}^k c_{m_i;n_{i_1},\cdots,n_{i_{m_i}}}"] & \left(P(k) \otimes \bigotimes_{i=1}^k P(m_i)\right) \otimes \left(\bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "\circ_{k;m_1,\cdots,m_k} \otimes \mathrm{id}_{\mathcal{C}}^{\otimes m}"'] \\
P(k) \otimes \bigotimes_{i=1}^k P\left(n_i\right) \arrow[d, "\circ_{k;n_1,\cdots,n_k}"] & P(m) \otimes \left(\bigotimes_{i=1}^k \bigotimes_{j=1}^{m_i} P(n_{i_j})\right) \arrow[d, "\circ_{m;n_{1_1},\cdots,n_{k_{(m_k)}}}"'] \\
P(l) \arrow[r, "="] & P(l)
\end{tikzcd}$$

(2) Unit: there exists a map $e : \mathbf{1} \to P(1)$ such that the following hold for all n

$$\begin{tikzcd}
P(n) \arrow[r, "\lambda^{-1}"] \arrow[rrr, bend right, "="'] & \mathbf{1} \otimes P(n) \arrow[r, "e \otimes \mathrm{id}_{\mathcal{C}}"] & P(1) \otimes P(n) \arrow[r, "\circ_{1;n}"] & P(n)
\end{tikzcd}$$

$$\begin{tikzcd}
P(n) \arrow[r, "(\rho^{-1})^{\circ n}"] \arrow[rrr, bend right, "="'] & P(n) \otimes \mathbf{1}^{\otimes n} \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes e^{\otimes n}"] & P(n) \otimes P(1)^{\otimes n} \arrow[r, "\circ_{n;1,\cdots,1}"] & P(n)
\end{tikzcd}$$

(3) Equivariance: the right action $- \cdot \sigma : P(n) \to P(n)$ of $\sigma \in \mathsf{S}_n$ on $P(n)$ is given by σ permuting the n inputs of each "element" of $P(n)$, so that on operadic composition maps, we have

$$\circ_{n;m_1,\cdots,m_n} \circ (- \cdot \sigma) = \circ_{n;m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)}},$$

then equivariance is the condition that the following commutes

$$\begin{tikzcd}
P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "\mathrm{id}_{\mathcal{C}} \otimes (-)\sigma"'] \arrow[d, "- \cdot \sigma \otimes \mathrm{id}_{\mathcal{C}}^{\otimes n}"] & P(n) \otimes \bigotimes_{i=1}^n P(m_{\sigma^{-1}(i)}) \arrow[d, "\circ_{n;m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)}}"'] \\
P(n) \otimes \bigotimes_{i=1}^n P(m_i) \arrow[r, "\circ_{n;m_1,\cdots,m_n}"] & P(m_1 + \cdots + m_n)
\end{tikzcd}$$

(where $(-)\sigma$ is the isomorphism given by σ permuting the n factors of $\bigotimes_{i=1}^n P(m_i)$)

EXAMPLE 7 (the endofunctor operad). If $\mathcal{C}$ is a category, then the endofunctor operad, defined by $\mathrm{End}(\mathcal{C})(n) \overset{\text{def}}{=} \mathrm{Fun}\,(\mathcal{C}^{\times n}, \mathcal{C})$ for $n \in \mathbb{N}_{\geq 0}$, is a symmetric operad in $(\mathbf{Cat}, \times, \{*\})$, with

(1) $\mathrm{id}_{\mathcal{C}}$ as the unit, and $\times$ as the monoidal product.
(2) the standard composition as the operadic composition,
(3) $\sigma \in \mathsf{S}_n$ acting on each element $\theta \in \mathrm{Fun}(\mathcal{C}^{\times n}, \mathcal{C})$ by permuting the inputs, i.e. $\theta \cdot \sigma(x_1, \cdots, x_n) = \theta(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$.

The endofunctor operad is a special case of the endomorphism operad ; it is an endomorphism operad in $(\mathbf{Cat}, \times, \{*\})$, a symmetric monoidal category with internal hom.

EXAMPLE 8 (the operad of parenthesized braids). This construction is similar to the one in pp.18-19 of [Fre18]. PaB is an operad in **Grpd** (the category of groupoids).

The objects of $PaB(n)$ are parenthesized permutations of n elements, i.e.

$$\mathrm{Obj}(PaB(n)) = \mathsf{S}_n \times M_n,$$

where M_n is the set of words of length n in the free magma generated by one element. If $(\sigma, w) \in PaB(n)$, then for $1 \leq i \leq n$, we label the ith factor of w (which must be the generating element) with $\sigma(n)$ to get the parenthesized permutation.

The morphisms of $PaB(n)$ are braids from (σ, w) to (τ, v) such that if the l factor of w and the mth factor of v are the ends of a strand, then $\sigma(l) = \tau(m)$.

PaB has the following operadic properties:

(1) The unit, which happens to be the only morphism of $PaB(1)$, is the braid from $(1, x)$ to $(1, x)$.
(2) The operadic composition is given by cabling:
(3) The action of S_n on $PaB(n)$ is given by $(\sigma, w) \overset{\tau}{\mapsto} (\tau\sigma, w)$ on objects.

DEFINITION 9 (operad morphism). A morphism $f : P \to Q$ of operads is a collection of maps $\{f_n : P(n) \to Q(n)\}_n$ such that

(1) The unit is preserved, i.e. $f_1 : P(1) \to Q(1)$ satisfies $f_1 \circ e^P = e^Q$.
(2) Composition is preserved, i.e. for all $k, n_1, \cdots, n_k \in \mathbb{N}_{\geq 1}$ with $l = \sum_{i=1}^k n_i$, the following commutes

$$\begin{tikzcd} P(k) \otimes \bigotimes_{i=1}^k P(n_i) \arrow[r, "f_k \otimes \bigotimes_{i=1}^k f_{n_i}"'] \arrow[d, "\circ^P_{k;n_1,\cdots,n_k}"] & Q(k) \otimes \bigotimes_{i=1}^k Q(n_i) \arrow[d, "\circ^Q_{k;n_1,\cdots,n_k}"] \\ P(l) \arrow[r, "f_l"'] & Q(l) \end{tikzcd}$$

(3) The S_n-actions are preserved, i.e. for all $n \in \mathbb{N}_{\geq 1}$, $\sigma \in \mathsf{S}_n$, $f \circ (- \cdot^P \sigma) = (- \cdot^Q \sigma) \circ f$.

Definition 10 (operad algebra). For an operad P in $(\mathbf{Cat}, \times, \{*\})$, a morphism of operads $P \to \mathrm{End}(\mathcal{C})$ gives $\mathcal{C}$ the structure of a *P-algebra*.

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