Sheaves in Geometry and Logic — Solutions

zin3724

Chapter 1

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Let G be the Lie group $S^1 := \{\mathbb{R} \mod 2\pi, +\}$. Then each θ induces a map $S^1 \xrightarrow{+\theta} S^1$ as G-spaces, given by

$$+\theta(\omega) = \omega + \theta \mod 2\pi$$

for all $\omega \in S^1$. Here, G acts by left multiplication in both cases. The equalizer of $\{+\theta \mid \theta \in [0, 2\pi)\}$ in the category of G-spaces is $S^1 \xrightarrow{\mathrm{id}_{S^1}}$, but each nonzero $+\theta$ has no fixed points, so in **Sets**, the equalizer is $\emptyset$, which isn't the underlying set of the G-set S^1, so we have a counterexample to the claim that the forgetful functor $U\colon \mathbf{B}G \to \mathbf{Sets}$ preserves limits.

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dedicated to sharp2229, previously known as Dir (see the cursed music theory lecture "Abstracted Systems of Music", on YouTube), Darkrifts

This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that D_B is a forgetful-like functor from $\mathbf{C}/B$ to $\mathbf{C}$ that takes each object in $\mathbf{C}/B$ to its domain.

Suppose ϕ is a natural trasformation in $\widehat{\mathbf{C}}(R \times P, Q)$, then we define for each $B \in \mathrm{obj}(\mathbf{C})$ the component ϕ'_B such that for each $u \in R(B)$, the image $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$ has its component at each $\left(C \xrightarrow{c} B\right) \in \mathrm{obj}(\mathbf{C}/B)$ defined as

$$(\phi'_B(u))_c : P(C) \to Q(C) : y \mapsto \phi_C(u \cdot c, y) = \phi_C(R(c)(u), y)$$

To see that the components $(\phi'_B(u))_c$ indeed form a natural trasformation in $\widehat{\mathbf{C}/B}(P_B, Q_B)$, suppose $k : G \to H$ is a morphism from $G \xrightarrow{g} B$ to $H \xrightarrow{h} B$ in $\mathbf{C}/B$, then for any $u \in R(B)$ and $y \in P(H)$

$$(\phi'_G(u))_g \circ P(k)(y) = \phi_G(u \cdot g, P(k)(y)) = \phi_G(R(g)(u), P(k)(y))$$

on the other hand, by the naturality of ϕ and the functoriality of R,

$$Q(k) \circ (\phi'_H(u))_h(y) = Q(k) \circ \phi_H(u \cdot h, y) = Q(k) \circ \phi_H(R(h)(u), y)$$
$$= \phi_G(R(k) \circ R(h)(u), P(k)(y)) = \phi_G(R(h \circ k)(u), P(k)(y))$$
$$= \phi_G(R(g)(u), P(k)(y)) = (\phi'_G(u))_g \circ P(k)(y)$$

For brevity, we now write Q^P to be $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) : \mathbf{C}^{op} \to \mathbf{Sets}$, regardless of whether it is such an exponential object, or not. Q^P maps each morphism $(B \xrightarrow{f^{op}} A)$ in $\mathbf{C}^{op}$ by mapping $\beta \in Q^P(B)$ to $\alpha \in Q^P(A)$ componentwise via

$$\alpha_{x^{op}} := \beta_{(f\circ x)^{op}}$$

for all $(A \xrightarrow{x^{op}} X) \in (\mathbf{C}/A)^{op}$.

To see where that came from, since each $(X \xrightarrow{x} A) \in \mathrm{obj}(\mathbf{C}/A)$ can be mapped to $X \xrightarrow{f\circ x} B \in \mathrm{obj}(\mathbf{C}/B)$ such that they have the same underlying domain in $\mathbf{C}$, and since each morphism $Y \xrightarrow{g^{op}} X$ such that

$$\begin{tikzcd}
X & Y \arrow[l, "g^{op}"'] \\
A \arrow[u, "x^{op}"] \arrow[ur, "y^{op}"'] &
\end{tikzcd}$$

commutes is also a morphism such that

$$\begin{tikzcd}
X & Y \arrow[l, "g^{op}"'] \\
A \arrow[u, "x^{op}"] \arrow[ur, "y^{op}"'] & \\
B \arrow[u, "f^{op}"] &
\end{tikzcd}$$

commutes, it follows that each morphism in $(\mathbf{C}/A)^{op}$ is identically a morphism in $(\mathbf{C}/B)^{op}$. Now, we have a functor $(\mathbf{C}/f)^{op} : (\mathbf{C}/A)^{op} \to (\mathbf{C}/B)^{op}$. Clearly, $Q^P(f) : Q^P(B) \to Q^P(A)$ is the function $\beta \mapsto \beta((\mathbf{C}/f)^{op})$, which takes β and precomposes it horizontally with the functor $(\mathbf{C}/f)^{op}$ ("pre-whiskering" in Emily Riehl's terminology?). We can now conclude from the assumption $\beta \in Q^P(B)$, that α is indeed a natural transformation in $Q^P(A)$.

To see that Q^P is a functor in $\mathbf{Sets}^{\mathbf{C}^{op}}$, note that for all $B \in \mathrm{obj}(\mathbf{C})$, $(\mathbf{C}/\,\mathrm{id}_B)^{op}$ is the identity functor on $(\mathbf{C}/B)^{op}$, so $Q^P(\mathrm{id}_B) = \mathrm{id}_{Q^P(B)}$. Now, suppose that $(A \xrightarrow{f} B), (B \xrightarrow{g} C) \in \mathbf{C}$, then $(\mathbf{C}/(g\circ f))^{op} = (\mathbf{C}/f)^{op}(\mathbf{C}/g)^{op}$, so $Q^P((g\circ f)^{op}) = Q^P(f^{op}) \circ Q^P(g^{op})$, all because of how precomposing functors work.

To see that $\phi' : R \to Q^P$ is natural, suppose $u \in R(B)$, and $(A \xrightarrow{f} B) \in \mathbf{C}$. Then $\phi'_A \circ R(f)(u)$ is the natural transformation such that its component at $X \xrightarrow{x} A$ is

$$\phi_X(R(x)(R(f)(u)), -) = \phi_X(R(f\circ x)(u), -)$$

on the other hand, $Q^P(f)(\phi'_B(u)) = \phi'_B(u)((\mathbf{C}/f)^{op})$ is the natural transformation such that its component at x is

$$(\phi'_B(u))_{f\circ x} = \phi_X(R(f\circ x)(u), -)$$

so it follows that $Q^P(f) \circ \phi'_B = \phi'_A \circ Q^P(f)$ as required.

Now, we define the evaluation map ev from $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$ to Q, by defining for each $B \in \mathrm{obj}(\mathbf{C})$, its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \to Q(B) : (\beta, w) \mapsto \beta_{1_B}(w)$$

so as an example, for $(u, w) \in R(B) \times P(B)$, we have

$$(ev)_B(\phi'_B(u), w) = (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w)$$

$$= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w)$$

hence each component of $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$ factors through the corresponding component of ev.

To see that ev is natural, for any $(A \xrightarrow{f} B) \in \mathbf{C}$ and $(\beta, w) \in Q^P(B) \times P(B)$, note that

$$ev_A \circ (Q^P(f) \times P(f))(\beta, w) = ev_A(\beta((\mathbf{C}/f)^{op}), P(f)(w))$$

$$= (\beta((\mathbf{C}/f)^{op}))_{1_A}(P(f)(w)) = \beta_{f^{op}}(P(f)(w)) \in Q(A)$$

and on the other hand,

$$Q(f) \circ ev_B(\beta, w) = Q(f)\beta_{1_B}(w)$$

moreover, it follows from the naturality of β (as a natural transformation in $\widehat{\mathbf{C}/B}(P_B, Q_B)$) that $Q(f)\beta_{1_B} = \beta_{f^{op}}P(f)$, so we indeed have $ev_A \circ (Q^P(f) \times P(f)) = Q(f) \circ ev_B$.

Now, it follows from the way we costructed each ϕ'_B and the constraint that ϕ' has to be a natural trasformation from R to $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$, that ϕ factors uniquely through ev, so $Q^P(-) = \widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$ is indeed such an exponential in $\widehat{\mathbf{C}}$.

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