

# Sheaves in Geometry and Logic — Solutions

zin3724

# Chapter 1

## 1.1

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## 1.2

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## 1.7 zin3724

Let  $G$  be the Lie group  $S^1 := \{\mathbb{R} \bmod 2\pi, +\}$ . Then each  $\theta$  induces a map  $S^1 \xrightarrow{+\theta} S^1$  as  $G$ -spaces, given by

$$+\theta(\omega) = \omega + \theta \bmod 2\pi$$

for all  $\omega \in S^1$ . Here,  $G$  acts by left multiplication in both cases. The equalizer of  $\{+\theta \mid \theta \in [0, 2\pi)\}$  in the category of  $G$ -spaces is  $S^1 \xrightarrow{\text{id}_{S^1}}$ , but each nonzero  $+\theta$  has no fixed points, so in **Sets**, the equalizer is  $\emptyset$ , which isn't the underlying set of the  $G$ -set  $S^1$ , so we have a counterexample to the claim that the forgetful functor  $U: \mathbf{BG} \rightarrow \mathbf{Sets}$  preserves limits.

## 1.8 zin3724

This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that  $D_B$  is the forgetful functor from  $\mathbf{C}/B$  to  $\mathbf{C}$  that takes each object in  $\mathbf{C}/B$  to its domain.

Suppose  $\phi$  is a natural transformation in  $\widehat{\mathbf{C}}(R \times P, Q)$ , then we define for each  $B \in \text{obj}(\mathbf{C})$  the component  $\phi'_B$  such that for each  $u \in R(B)$ , the image  $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$  has its component at each  $(C \xrightarrow{c} B) \in \text{obj}(\mathbf{C}/B)$  defined as

$$(\phi'_B(u))_c : P(C) \rightarrow Q(C) : y \mapsto \phi_C(u \cdot c, y) = \phi_C(R(c)(u), y)$$

To see that the components  $(\phi'_B(u))_c$  indeed form a natural transformation in  $\widehat{\mathbf{C}/B}(P_B, Q_B)$ , suppose  $k : G \rightarrow H$  is a morphism from  $G \xrightarrow{g} B$  to  $H \xrightarrow{h} B$  in  $\mathbf{C}/B$ , then for any  $u \in R(B)$  and  $y \in P(H)$

$$(\phi'_G(u))_g \circ P(k)(y) = \phi_G(u \cdot g, P(k)(y)) = \phi_G(R(g)(u), P(k)(y))$$

on the other hand, by the naturality of  $\phi$  and the functoriality of  $R$ ,

$$\begin{aligned} Q(k) \circ (\phi'_H(u))_h(y) &= Q(k) \circ \phi_H(u \cdot h, y) = Q(k) \circ \phi_H(R(h)(u), y) \\ &= \phi_G(R(k) \circ R(h)(u), P(k)(y)) = \phi_G(R(h \circ k)(u), P(k)(y)) \\ &= \phi_G(R(g)(u), P(k)(y)) = (\phi'_G(u))_g \circ P(k)(y) \end{aligned}$$

Now, we define the evaluation map  $ev$  from  $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$  to  $Q$ , by defining for each  $B \in \text{obj}(\mathbf{C})$ , its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \rightarrow Q(B) : (\alpha, w) \mapsto \alpha_{1_B}(w)$$



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so as an example, for  $(u, w) \in R(B) \times P(B)$ , we have

$$\begin{aligned} (ev)_B(\phi'_B(u), w) &= (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w) \\ &= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w) \end{aligned}$$

hence each  $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$  factors through  $ev$ . Now, it follows from the way we constructed each  $\phi'_B$  and the constraint that  $\phi'$  has to be a natural transformation from  $R$  to  $\widehat{\mathbf{C}}/(-)(P_{(-)}, Q_{(-)})$ , that  $\phi$  factors uniquely through  $ev$ , so  $\widehat{\mathbf{C}}/(-)(P_{(-)}, Q_{(-)})$  can be regarded as the exponential  $Q^P$  in  $\widehat{\mathbf{C}}$ .

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**1.11**