Sheaves in Geometry and Logic — Solutions

zin3724

Chapter 1

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Let G be the Lie group $S^1 := \{\mathbb{R} \mod 2\pi, +\}$. Then each θ induces a map $S^1 \xrightarrow{+\theta} S^1$ as G-spaces, given by

$$+\theta(\omega) = \omega + \theta \mod 2\pi$$

for all $\omega \in S^1$. Here, G acts by left multiplication in both cases. The equalizer of $\{+\theta \mid \theta \in [0, 2\pi)\}$ in the category of G-spaces is $S^1 \xrightarrow{\mathrm{id}_{S^1}}$, but each nonzero $+\theta$ has no fixed points, so in **Sets**, the equalizer is $\emptyset$, which isn't the underlying set of the G-set S^1, so we have a counterexample to the claim that the forgetful functor $U\colon \mathbf{B}G \to \mathbf{Sets}$ preserves limits.

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dedicated to sharp2229, previously known as Dir, Darkrifts

This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that D_B is a forgetful-like functor from $\mathbf{C}/B$ to $\mathbf{C}$ that takes each object in $\mathbf{C}/B$ to its domain.

Suppose ϕ is a natural trasformation in $\widehat{\mathbf{C}}(R \times P, Q)$, then we define for each $B \in \mathrm{obj}(\mathbf{C})$ the component ϕ'_B such that for each $u \in R(B)$, the image $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$ has its component at each $\left(C \xrightarrow{c} B\right) \in \mathrm{obj}(\mathbf{C}/B)$ defined as

$$(\phi'_B(u))_c : P(C) \to Q(C) : y \mapsto \phi_C(u \cdot c, y) = \phi_C(R(c)(u), y)$$

To see that the components $(\phi'_B(u))_c$ indeed form a natural trasformation in $\widehat{\mathbf{C}/B}(P_B, Q_B)$, suppose $k : G \to H$ is a morphism from $G \xrightarrow{g} B$ to $H \xrightarrow{h} B$ in $\mathbf{C}/B$, then for any $u \in R(B)$ and $y \in P(H)$

$$(\phi'_G(u))_g \circ P(k)(y) = \phi_G(u \cdot g, P(k)(y)) = \phi_G(R(g)(u), P(k)(y))$$

on the other hand, by the naturality of ϕ and the functoriality of R,

$$\begin{aligned} Q(k) \circ (\phi'_H(u))_h(y) &= Q(k) \circ \phi_H(u \cdot h, y) = Q(k) \circ \phi_H(R(h)(u), y) \\ &= \phi_G(R(k) \circ R(h)(u), P(k)(y)) = \phi_G(R(h \circ k)(u), P(k)(y)) \\ &= \phi_G(R(g)(u), P(k)(y)) = (\phi'_G(u))_g \circ P(k)(y) \end{aligned}$$

For brevity, we now write Q^P to be $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) : \mathbf{C}^{op} \to \mathbf{Sets}$. Q^P maps each morphism $(B \xrightarrow{f^{op}} A)$ in $\mathbf{C}^{op}$ by mapping $\beta \in Q^P(B)$ to $\alpha \in Q^P(A)$ componentwise via

$$\alpha_{x^{op}} := \beta_{(f \circ x)^{op}}$$

for all $(A \xrightarrow{x^{op}} X) \in (\mathbf{C}/A)^{op}$.

To see where that came from, since each $(X \xrightarrow{x} A) \in \mathrm{obj}(\mathbf{C}/A)$ can be mapped to $X \xrightarrow{f \circ x} B \in \mathrm{obj}(\mathbf{C}/B)$ such that they have the same underlying domain in $\mathbf{C}$, and since each morphism $Y \xrightarrow{g^{op}} X$ such that

$$\begin{tikzcd} X & Y \arrow[l, "g^{op}"] \\ A \arrow[u, "x^{op}"] \arrow[ur, "y^{op}"'] & \end{tikzcd}$$

commutes is also a morphism such that

$$\begin{tikzcd} X & Y \arrow[l, "g^{op}"] \\ A \arrow[u, "x^{op}"] \arrow[ur, "y^{op}"'] & \\ B \arrow[u, "f^{op}"] & \end{tikzcd}$$

commutes, it follows that each morphism in $(\mathbf{C}/A)^{op}$ is identically a morphism in $(\mathbf{C}/B)^{op}$. Now, we have a functor $(\mathbf{C}/f)^{op} : (\mathbf{C}/A)^{op} \to (\mathbf{C}/B)^{op}$. Clearly, $Q^P(f) : Q^P(B) \to Q^P(A)$ is the function $\beta \mapsto \beta((\mathbf{C}/f)^{op})$, which takes β and precomposes it horizontally with the functor $(\mathbf{C}/f)^{op}$ ("prewhiskering" in Emily Riehl's terminology?). We can now conclude from the assumption $\beta \in Q^P(B)$, that α is indeed a natural transformation in $Q^P(A)$.

To see that Q^P is a functor in $\mathbf{Sets}^{\mathbf{C}^{op}}$, note that for all $B \in \mathrm{obj}(\mathbf{C})$, $(\mathbf{C}/\,\mathrm{id}_B)^{op}$ is the identity functor on $(\mathbf{C}/B)^{op}$, so $Q^P(\mathrm{id}_B) = \mathrm{id}_{Q^P(B)}$. Now, suppose that $(A \xrightarrow{f} B), (B \xrightarrow{g} C) \in \mathbf{C}$, then $(\mathbf{C}/(g \circ f))^{op} = (\mathbf{C}/f)^{op}(\mathbf{C}/g)^{op}$, so $Q^P((g \circ f)^{op}) = Q^P(f^{op}) \circ Q^P(g^{op})$, all because of how precomposing functors work.

To see that $\phi' : R \to Q^P$ is natural, suppose $u \in R(B)$, and $(A \xrightarrow{f} B) \in \mathbf{C}$. Then $\phi'_A \circ R(f)(u)$ is the natural transformation such that its component at $X \xrightarrow{x} A$ is

$$\phi_X(R(x)(R(f)(u)), -) = \phi_X(R(f \circ x)(u), -)$$

on the other hand, $Q^P(f)(\phi'_B(u)) = \phi'_B(u)((\mathbf{C}/f)^{op})$ is the natural transformation such that its component at x is

$$(\phi'_B(u))_{f \circ x} = \phi_X(R(f \circ x)(u), -)$$

so it follows that $Q^P(f) \circ \phi'_B = \phi'_A \circ Q^P(f)$ as required.

Now, we define the evaluation map ev from $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$ to Q, by defining for each $B \in \mathrm{obj}(\mathbf{C})$, its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \to Q(B) : (\alpha, w) \mapsto \alpha_{1_B}(w)$$

so as an example, for $(u, w) \in R(B) \times P(B)$, we have

$$(ev)_B(\phi'_B(u), w) = (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w)$$

$$= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w)$$

hence each $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$ factors through ev. Now, it follows from the way we costructed each ϕ'_B and the constraint that ϕ' has to be a natural trasformation from R to $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$, that ϕ factors uniquely through ev, so $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$ can be regarded as the exponential Q^P in $\widehat{\mathbf{C}}$.

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