

Sheaves in Geometry and Logic — Solutions

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Chapter 1

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Let G be the Lie group $S^1 := \{\mathbb{R} \bmod 2\pi, +\}$. Then each θ induces a map $S^1 \xrightarrow{+\theta} S^1$ as G -spaces, given by

$$+\theta(\omega) = \omega + \theta \bmod 2\pi$$

for all $\omega \in S^1$. Here, G acts by left multiplication in both cases. The equalizer of $\{+\theta \mid \theta \in [0, 2\pi)\}$ in the category of G -spaces is $S^1 \xrightarrow{\text{id}_{S^1}}$, but each nonzero $+\theta$ has no fixed points, so in **Sets**, the equalizer is \emptyset , which isn't the underlying set of the G -set S^1 , so we have a counterexample to the claim that the forgetful functor $U: \mathbf{BG} \rightarrow \mathbf{Sets}$ preserves limits.

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This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that D_B is the forgetful functor from \mathbf{C}/B to \mathbf{C} that takes each object in \mathbf{C}/B to its domain.

Suppose ϕ is a natural transformation in $\widehat{\mathbf{C}}(R \times P, Q)$, then we define for each $B \in \text{obj}(\mathbf{C})$ the component ϕ'_B such that for each $u \in R(B)$, the image $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$ has its component at each $(C \xrightarrow{f} B) \in \text{obj}(\mathbf{C}/B)$ defined as

$$(\phi'_B(u))_f : P(C) \rightarrow Q(C) : y \mapsto \phi_C(u \cdot f, y) = \phi_C(R(f)(u), y)$$

Now, we define the evaluation map ev from $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$ to Q , by defining for each $B \in \text{obj}(\mathbf{C})$, its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \rightarrow Q(B) : (\alpha, w) \mapsto \alpha_{1_B}(w)$$

so as an example, for $(u, w) \in R(B) \times P(B)$, we have

$$\begin{aligned} (ev)_B(\phi'_B(u), w) &= (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w) \\ &= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w) \end{aligned}$$

hence each $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$ factors through ev . Now, it follows from the way we constructed each ϕ'_B and the constraint that ϕ' has to be a natural transformation from R to $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$, that ϕ factors uniquely through ev , so $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$ can be regarded as the exponential Q^P in $\widehat{\mathbf{C}}$.

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