

Sheaves in Geometry and Logic — Solutions

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Chapter 1

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Let G be the Lie group $S^1 := \{\mathbb{R} \bmod 2\pi, +\}$. Then each θ induces a map $S^1 \xrightarrow{+\theta} S^1$ as G -spaces, given by

$$+\theta(\omega) = \omega + \theta \bmod 2\pi$$

for all $\omega \in S^1$. Here, G acts by left multiplication in both cases. The equalizer of $\{+\theta \mid \theta \in [0, 2\pi)\}$ in the category of G -spaces is $S^1 \xrightarrow{\text{id}_{S^1}}$, but each nonzero $+\theta$ has no fixed points, so in **Sets**, the equalizer is \emptyset , which isn't the underlying set of the G -set S^1 , so we have a counterexample to the claim that the forgetful functor $U: \mathbf{BG} \rightarrow \mathbf{Sets}$ preserves limits.

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dedicated to sharp2229, previously known as Dir, Darkrifts

This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that D_B is a forgetful-like functor from \mathbf{C}/B to \mathbf{C} that takes each object in \mathbf{C}/B to its domain.

Suppose ϕ is a natural transformation in $\widehat{\mathbf{C}}(R \times P, Q)$, then we define for each $B \in \text{obj}(\mathbf{C})$ the component ϕ'_B such that for each $u \in R(B)$, the image $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$ has its component at each $(C \xrightarrow{c} B) \in \text{obj}(\mathbf{C}/B)$ defined as

$$(\phi'_B(u))_c : P(C) \rightarrow Q(C) : y \mapsto \phi_C(u \cdot c, y) = \phi_C(R(c)(u), y)$$

To see that the components $(\phi'_B(u))_c$ indeed form a natural transformation in $\widehat{\mathbf{C}/B}(P_B, Q_B)$, suppose $k : G \rightarrow H$ is a morphism from $G \xrightarrow{g} B$ to $H \xrightarrow{h} B$ in \mathbf{C}/B , then for any $u \in R(B)$ and $y \in P(H)$

$$(\phi'_G(u))_g \circ P(k)(y) = \phi_G(u \cdot g, P(k)(y)) = \phi_G(R(g)(u), P(k)(y))$$

on the other hand, by the naturality of ϕ and the functoriality of R ,

$$\begin{aligned} Q(k) \circ (\phi'_H(u))_h(y) &= Q(k) \circ \phi_H(u \cdot h, y) = Q(k) \circ \phi_H(R(h)(u), y) \\ &= \phi_G(R(k) \circ R(h)(u), P(k)(y)) = \phi_G(R(h \circ k)(u), P(k)(y)) \\ &= \phi_G(R(g)(u), P(k)(y)) = (\phi'_G(u))_g \circ P(k)(y) \end{aligned}$$

For brevity, we now write Q^P to be $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) : \mathbf{C}^{op} \rightarrow \mathbf{Sets}$. Q^P maps each morphism $(B \xrightarrow{f^{op}} A)$ in \mathbf{C}^{op} by mapping $\beta \in Q^P(B)$ to $\alpha \in Q^P(A)$ componentwise via

$$\alpha_{x^{op}} := \beta_{(f \circ x)^{op}}$$

for all $(A \xrightarrow{x^{op}} X) \in (\mathbf{C}/A)^{op}$.

To see where that came from, since each $(X \xrightarrow{x} A) \in \text{obj}(\mathbf{C}/A)$ can be mapped to $X \xrightarrow{f \circ x} B \in \text{obj}(\mathbf{C}/B)$ such that they have the same underlying domain in \mathbf{C} , and since each morphism $Y \xrightarrow{g^{op}} X$ such that

$$\begin{array}{ccc} X & \xleftarrow{g^{op}} & Y \\ x^{op} \uparrow & \nearrow y^{op} & \\ A & & \end{array}$$

commutes is also a morphism such that

$$\begin{array}{ccc} X & \xleftarrow{g^{op}} & Y \\ x^{op} \uparrow & \nearrow y^{op} & \\ A & & \\ f^{op} \uparrow & & \\ B & & \end{array}$$

commutes, it follows that each morphism in $(\mathbf{C}/A)^{op}$ is identically a morphism in $(\mathbf{C}/B)^{op}$. Now, we have a functor $(\mathbf{C}/f)^{op} : (\mathbf{C}/A)^{op} \rightarrow (\mathbf{C}/B)^{op}$. Clearly, $Q^P(f) : Q^P(B) \rightarrow Q^P(A)$ is the function $\beta \mapsto \beta((\mathbf{C}/f)^{op})$, which takes β and precomposes it horizontally with the functor $(\mathbf{C}/f)^{op}$ ("prewhiskering" in Emily Riehl's terminology?). We can now conclude from the assumption $\beta \in Q^P(B)$, that α is indeed a natural transformation in $Q^P(A)$.

To see that Q^P is a functor in $\mathbf{Sets}^{\mathbf{C}^{op}}$, note that for all $B \in \text{obj}(\mathbf{C})$, $(\mathbf{C}/\text{id}_B)^{op}$ is the identity functor on $(\mathbf{C}/B)^{op}$, so $Q^P(\text{id}_B) = \text{id}_{Q^P(B)}$. Now, suppose that $(A \xrightarrow{f} B) \in \mathbf{C}$, then

Now, we define the evaluation map ev from $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$ to Q , by defining for each $B \in \text{obj}(\mathbf{C})$, its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \rightarrow Q(B) : (\alpha, w) \mapsto \alpha_{1_B}(w)$$

so as an example, for $(u, w) \in R(B) \times P(B)$, we have

$$\begin{aligned} (ev)_B(\phi'_B(u), w) &= (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w) \\ &= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w) \end{aligned}$$

hence each $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$ factors through ev . Now, it follows from the way we constructed each ϕ'_B and the constraint that ϕ' has to be a natural transformation from R to $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$, that ϕ factors uniquely through ev , so $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$ can be regarded as the exponential Q^P in $\widehat{\mathbf{C}}$.

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