Sheaves in Geometry and Logic — Solutions

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Chapter 1

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Let G be the Lie group $S^1 := \{\mathbb{R} \mod 2\pi, +\}$. Then each θ induces a map $S^1 \xrightarrow{+\theta} S^1$ as G-spaces, given by

$$+\theta(\omega) = \omega + \theta \mod 2\pi$$

for all $\omega \in S^1$. Here, G acts by left multiplication in both cases. The equalizer of $\{+\theta \mid \theta \in [0, 2\pi)\}$ in the category of G-spaces is $S^1 \xrightarrow{\mathrm{id}_{S^1}}$, but each nonzero $+\theta$ has no fixed points, so in **Sets**, the equalizer is $\emptyset$, which isn't the underlying set of the G-set S^1, so we have a counterexample to the claim that the forgetful functor $U\colon \mathbf{B}G \to \mathbf{Sets}$ preserves limits.

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This is almost exactly the same as the proof of Proposition 1.6.1 on pp. 46-47. Note that D_B is a forgetful-like functor from $\mathbf{C}/B$ to $\mathbf{C}$ that takes each object in $\mathbf{C}/B$ to its domain.

Suppose ϕ is a natural trasformation in $\widehat{\mathbf{C}}(R \times P, Q)$, then we define for each $B \in \mathrm{obj}(\mathbf{C})$ the component ϕ'_B such that for each $u \in R(B)$, the image $\phi'_B(u) \in \widehat{\mathbf{C}/B}(P_B, Q_B)$ has its component at each $\left(C \xrightarrow{c} B\right) \in \mathrm{obj}(\mathbf{C}/B)$ defined as

$$(\phi'_B(u))_c : P(C) \to Q(C) : y \mapsto \phi_C(u \cdot c, y) = \phi_C(R(c)(u), y)$$

To see that the components $(\phi'_B(u))_c$ indeed form a natural trasformation in $\widehat{\mathbf{C}/B}(P_B, Q_B)$, suppose $k : G \to H$ is a morphism from $G \xrightarrow{g} B$ to $H \xrightarrow{h} B$ in $\mathbf{C}/B$, then for any $u \in R(B)$ and $y \in P(H)$

$$(\phi'_G(u))_g \circ P(k)(y) = \phi_G(u \cdot g, P(k)(y)) = \phi_G(R(g)(u), P(k)(y))$$

on the other hand, by the naturality of ϕ and the functoriality of R,

$$Q(k) \circ (\phi'_H(u))_h(y) = Q(k) \circ \phi_H(u \cdot h, y) = Q(k) \circ \phi_H(R(h)(u), y)$$

$$= \phi_G(R(k) \circ R(h)(u), P(k)(y)) = \phi_G(R(h \circ k)(u), P(k)(y))$$
$$= \phi_G(R(g)(u), P(k)(y)) = (\phi'_G(u))_g \circ P(k)(y)$$

For brevity, we now write Q^P to be the functor $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$. To see that Q^P is indeed a functor in $\mathbf{Sets}^{\mathbf{C}^{op}}$, note that if $(A \xrightarrow{f} B) \in \mathbf{C}$, then

Now, we define the evaluation map ev from $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)}) \times P$ to Q, by defining for each $B \in \text{obj}(\mathbf{C})$, its component as

$$(ev)_B : \widehat{\mathbf{C}/B}(P_B, Q_B) \times P(B) \to Q(B) : (\alpha, w) \mapsto \alpha_{1_B}(w)$$

so as an example, for $(u, w) \in R(B) \times P(B)$, we have

$$(ev)_B(\phi'_B(u), w) = (\phi'_B(u))_{1_B}(w) = \phi_B(u \cdot 1_B, w) = \phi_B(R(1_B)(u), w)$$

$$= \phi_B(1_{R(B)}(u), w) = \phi_B(u, w)$$

hence each $\phi \in \widehat{\mathbf{C}}(R \times P, Q)$ factors through ev. Now, it follows from the way we costructed each ϕ'_B and the constraint that ϕ' has to be a natural trasformation from R to $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$, that ϕ factors uniquely through ev, so $\widehat{\mathbf{C}/(-)}(P_{(-)}, Q_{(-)})$ can be regarded as the exponential Q^P in $\widehat{\mathbf{C}}$.

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