1. Energy generation

We know that $\frac{dL}{dM} = \epsilon$ describes the generation of energy, and this L must be carried to the surface via:

- radiation: $\frac{dT}{dr} \sim -\frac{L}{\kappa}$
- convection: $\frac{dT}{dr} \sim \left.\frac{dT}{dr}\right|_{ad}$

Now we explore the sources and sinks that make up ϵ:

- gravitational contraction (K-H)
- thermonuclear energy
- ν emission

We'll follow Clayton (1983) — it is the reference for reactions

2. Gravitational energy — consider motions on a scale smaller than the whole star

We have: $\frac{dL}{dM} = \epsilon$ w/ ϵ in units of erg/g/s

First law: $dq = de + P\,d(1/\rho)$

following a fluid element:

$$\underbrace{\frac{Dq}{Dt}}_{\uparrow \text{ Lagrangian}} = \frac{De}{Dt} + P\frac{D}{Dt}\left(1/\rho\right) = \underbrace{\epsilon - \frac{dL}{dM}}_{\text{difference from balance is an energy source}}$$

We define $\epsilon_{grav} \equiv -\left[\frac{De}{Dt} + P\frac{D}{Dt}\left(1/\rho\right)\right]$

then $\frac{dL}{dM} = \epsilon + \epsilon_{grav}$

This can be shown to be

$$\epsilon_{grav} = -\frac{P}{\rho(\Gamma_3 - 1)}\left[\frac{D}{Dt}\left(\log\left(\frac{P}{\rho^{\Gamma_1}}\right)\right)\right]$$

↑ if we are isentropic, then $\epsilon_{grav} = 0$

$\therefore$ ϵ_{grav} represents the energy from non-adiabatic contraction

This can arise, for instance, when the core contracts and the outer regions expand — this will give a local ϵ_{grav}

for expansion, $\epsilon_{grav} < 0$ — the shell dM takes in energy
for contraction, $\epsilon_{grav} > 0$ — the shell dM liberates energy
in equilibrium, $\epsilon_{grav} = 0$ — all time dependence is removed

Nuclear reactions (following Clayton)

consider $a + X \rightarrow Y + b$

shorthand notation: $X(a, b)Y$

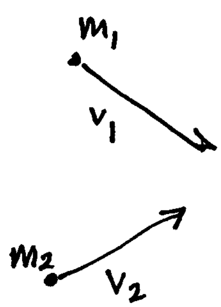
here X, Y are nuclei

a or b can be $p, n, \gamma, \alpha, e, \nu$, or sometimes other nuclei

conserved quantities:

- total E
- linear & angular momentum
- baryon #
- lepton #
- charge

center of mass frame:



$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{V}$$

$$\therefore \quad \vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

relative to CM:

$$m_1(\vec{v}_1 - \vec{V}) = m_1 \frac{\cancel{m_1\vec{v}_1} + m_2\vec{v}_1 - (\cancel{m_1\vec{v}_1} + m_2\vec{v}_2)}{m_1 + m_2}$$

$$= \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2) \equiv \mu\vec{v}$$

reduced mass

similarly: $m_2(\vec{v}_2 - \vec{V}) = -\mu\vec{v}$

Consider ^4He :

from Clayton : $\Delta M_{^4He} = 2.42475$ MeV

$\Delta M_p = 7.289$ MeV

$\Delta M_n = 8.071$ MeV

$\therefore\ B = 2.42475\text{ MeV} - 7.289\text{ MeV}\cdot 2 - 8.071\text{ MeV}\cdot 2 = -28.295\text{ MeV}$

$\sim -7.1\text{ MeV/nucleon}$

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Energy balance gives energy released from each reaction

Energy balance eq. gives energy released from each reaction

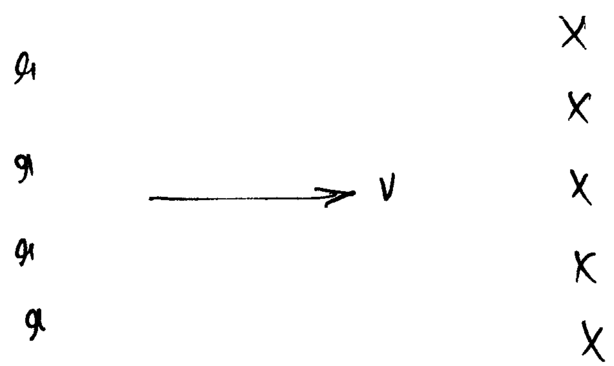
ref: Clayton § 4.2

We now need the energy generation rate

Cross section:

consider $a + X \longrightarrow Y + b$

imaging targets X being bombarded by a flux of a



The cross section is

$$\sigma = \frac{\#\text{ of reactions/nucleus } X/\text{time}}{\#\text{ of incident particles/cm}^2/t} \longleftarrow \text{flux of } a$$

units are cm^2

This is velocity dependent: $\sigma(v)$

relative velocity

you can think of this as the size of the target X presents to the incoming a take n_X = # density of X

then the reaction rate / unit volume is

$$n_X \sigma \cdot (\text{flux of } a)$$

or $r = n_X n_a \sigma v$

since $(n_a v)$ is the flux of incoming a

Note: v is the relative velocity between X and a

In general, there will be a range of v, described by a spectrum $\phi(v)$ with

$$\int \phi(v)\, dv = 1$$

then $\phi(v)\,dv$ is the probability that the relative velocity between X and a is in $[v, v+dv]$

so

$$r = n_a n_x \int_0^\infty v\,\sigma(v)\,\phi(v)\,dv \equiv n_a n_X \langle \sigma v \rangle$$

this is what u needed to find reaction rate

Note: if a and X are identical (e.g. ${}^{12}C + {}^{12}C$), then we need a factor of ½ to avoid double counting

$$r_{aX} = n_a n_X \frac{\langle \sigma v \rangle}{1 + \delta_{aX}}$$

sometimes we write $\lambda \equiv \langle \sigma v \rangle$

What is $\phi(v)$? Maxwell-Boltzmann

- actually a seperate M-B for each particle
- in CM frame, a single M-B for relative v w/ μ

consider:

$$\phi(\vec{v})\, dv_x\, dv_y\, dv_z = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, dv_x dv_y dv_z$$

then our reaction rate has the form

$$r = \iint v \left[n_a \phi(v_a)\right]\left[n_x \phi(v_x)\right] \sigma(v)\, d^3v_a\, d^3v_x$$

it can be shown by a little algebra that

$$r = n_a n_x \int v\,\sigma(v) \left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-\mu v^2/2kT}\, d^3v$$

(there would be a second Maxwellian corresponding to V, but we can integrate that out since only v enters into cross-section)

$\therefore$ our reaction rate is

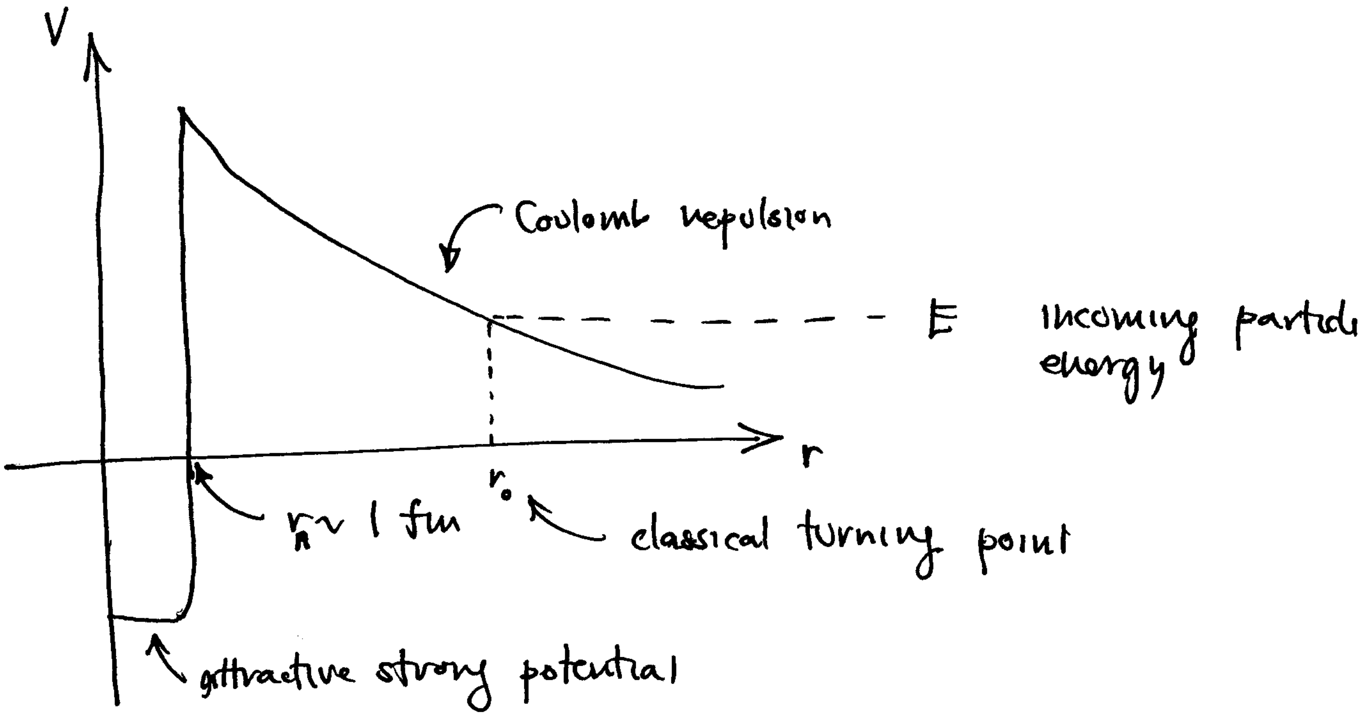
$$\begin{aligned} r_{ax} &= (1+\delta_{ax})^{-1} n_a n_x \langle \sigma v \rangle \\ &= (1+\delta_{ax})^{-1} n_a n_x\, 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v)\, e^{-\mu v^2/2kT}\, dv \end{aligned}$$

switching to spherical

Now we'll focus on the calculation of

$$\lambda = \langle \sigma v \rangle = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v)\, e^{-\mu v^2/2kT}\, dv$$

Consider the reaction process



Classically, the incoming particle needs $E \sim kT >$ Coulomb barrier to fuse — this demands really high T

QM: there is a probability of tunnelling through the coulomb barrier

$$P \sim e^{-r_0/\lambda} \qquad \lambda = \text{de Broglie wavelength}$$

$$r_0 : \quad \frac{1}{2}\mu v^2 = \frac{Z_1 Z_2 e^2}{r_0} \implies r_0 = \frac{2 Z_1 Z_2 e^2}{\mu v^2}$$

$$\lambda = \frac{h}{p} = \frac{h}{\mu v} \quad \therefore$$

$$\therefore \frac{r_0}{\lambda} \sim \frac{2 Z_1 Z_2 e^2}{\mu v^2} \frac{\mu v}{h} = \frac{\text{const}}{v} = \frac{\text{const}}{E^{1/2}}$$

so $P \sim e^{-\text{const}\, E^{-1/2}}$

Gamow showed the true probability has a 2π in it

$$P \propto e^{-2\pi Z_1 Z_2 e^2/\hbar v}$$

(Clayton §4.5 has a detailed derivation)

$$\equiv e^{-bE^{-1/2}}$$

If we take the idea of cross-section representing some target, then the physical size we can imagine is the de Broglie wavelength,

$$\sigma \sim \pi \left(\frac{h}{p}\right)^2 \sim \frac{1}{E} \qquad \left(E = \frac{p^2}{2m}\right)$$

If we put both of these effects together, we have

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}$$

$S(E)$ is everything that is left over — the hope is that we've removed the strongest E terms and $S(E)$ is smooth

$S(E)$ depends on the detailed nuclear properties

If we are away from a <u>resonance</u> in the nuclear structure, then $S(E) \sim$ constant

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for later:

$$b = \frac{2\pi Z_1 Z_2 e^2}{\hbar} \left(\frac{\mu}{2}\right)^{1/2} \quad \text{since} \quad v = \left(\frac{2E}{\mu}\right)^{1/2}$$

14.

M-B distribution

$$\phi(v)\,dv = 4\pi v^2 \left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-\mu v^2/2kT}\,dv$$

taking $E = \frac{1}{2}\mu v^2$ $\qquad v = \left(\frac{2E}{\mu}\right)^{1/2}$

$$dE = \mu v\,dv = \mu\left(\frac{2E}{\mu}\right)^{1/2} dv = (2E\mu)^{1/2}\,dv$$

then

$$\varphi(E)\,dE = 4\pi\left(\frac{2E}{\mu}\right)\left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-E/kT}\,\frac{dE}{(2E\mu)^{1/2}}$$

$$= \frac{2}{\sqrt{\pi}}\,\frac{E^{1/2}}{(kT)^{3/2}}\,e^{-E/kT}\,dE \qquad \left[\phi(v)\,dv\right]$$

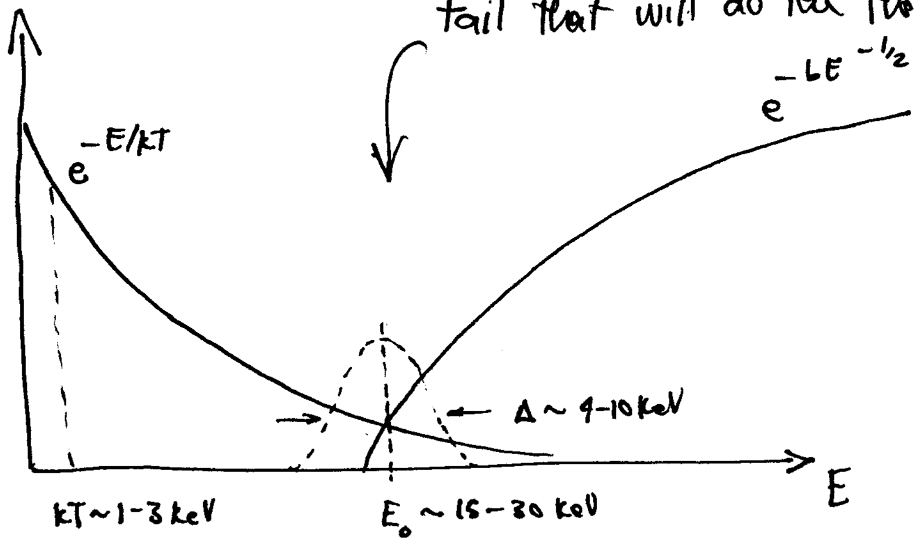
and writing $\lambda = \langle\sigma v\rangle$ in terms of E,

$$\lambda = \langle\sigma v\rangle = \int_0^\infty \sigma(E)\,v(E)\,\varphi(E)\,dE$$

$$= \int_0^\infty \left\{\frac{S(E)}{E}\,e^{-bE^{-1/2}}\right\}\left(\frac{2E}{\mu}\right)^{1/2}\left\{\frac{2}{\sqrt{\pi}}\,\frac{E^{1/2}}{(kT)^{3/2}}\,e^{-E/kT}\right\} dE$$

$$= \left(\frac{8}{\mu\pi}\right)^{1/2}\frac{1}{(kT)^{3/2}}\int_0^\infty S(E)\,e^{\underbrace{-E/kT}_{\text{M-B term}}\;\underbrace{-bE^{-1/2}}_{\text{tunnelling}}}\,dE$$

note that it is the high-velocity tail that will do the fusing



The term $e^{-E/kT - bE^{-1/2}}$ is only "big" in some small range $E_0 \pm \Delta E$

where E_0 is found via

$$\frac{d}{dE}\left(\frac{E}{kT} + bE^{-1/2}\right) = 0 \longrightarrow E_0 = \left(\frac{bkT}{2}\right)^{2/3}$$

If we assume $S(E) = S_0$ = constant then

$$\lambda = \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{S_0}{(kT)^{3/2}} \int_0^{\infty} e^{-E/kT - bE^{-1/2}} \, dE$$

This integral is usually done by approximating the integrand as a gaussian.

Finally, experiments are often done @ $E \gg$ stellar E, and we extrapolate down to stellar energies — ok if S is smooth.

16.

We can show (see Clayton 4.48 and following) that

$$e^{-E/kT - bE^{-1/2}} \sim C e^{-(E-E_0)^2/(\Delta/2)^2}$$

w/ $C = e^{-3E_0/kT} \equiv e^{-\tau}$

$\Delta = \frac{4}{\sqrt{3}} (E_0 kT)^{1/2}$ (this with is choosen to have the same curvature @ maximum as the original function)

and the location of the maximum is E_0 (as we found before)

then

$$\lambda \sim \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{S_0}{(kT)^{3/2}} e^{-\tau} \int_0^\infty e^{-(E-E_0)^2/(\Delta/2)^2} \, dE$$

we can take the lower limit to be $-\infty$ w/o much loss of accuracy.

defining $\xi \equiv \frac{2(E-E_0)}{\Delta}$ $d\xi = \frac{2}{\Delta} dE$

we have

$$\lambda \sim \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{S_0}{(kT)^{3/2}} e^{-\tau} \frac{\Delta}{2} \underbrace{\int_{-\infty}^{\infty} e^{-\xi^2} d\xi}_{=\sqrt{\pi}}$$

then

$$\lambda \sim \left(\frac{8}{\mu}\right)^{1/2} \frac{S_0}{(kT)^{3/2}} e^{-\tau} \frac{\Delta}{2}$$

What is the T dependence here?

$$\tau = \frac{3E_0}{kT} = \frac{3}{kT}\underbrace{\left(\frac{bkT}{2}\right)^{2/3}}_{E_0} = 3\left(\frac{b}{2}\right)^{2/3}(kT)^{-1/3}$$

now $\Delta = \frac{4}{\sqrt{3}}(E_0 kT)^{1/2} = \frac{4}{\sqrt{3}}\left[\left(\frac{b}{2}\right)^{2/3}(kT)^{5/3}\right]^{1/2}$

$$= \frac{4}{\sqrt{3}}\left(\frac{b}{2}\right)^{1/3}(kT)^{5/6}$$

into our λ expression:

$$\lambda \sim \frac{1}{2}\left(\frac{8}{\mu}\right)^{1/2} S_0 e^{-\tau}(kT)^{-3/2}\underbrace{\frac{4}{\sqrt{3}}\left(\frac{b}{2}\right)^{1/3}(kT)^{5/6}}_{=\Delta}$$

$$\sim 2\left(\frac{8}{3\mu}\right)^{1/2} S_0 e^{-\tau}\left(\frac{b}{2}\right)^{1/3}(kT)^{-2/3}$$

This is a rate that goes like

$$\underbrace{S_0\, e^{-aT^{-1/3}}\frac{1}{T^{2/3}}}_{\text{this temperature dependence is representative of non-resonant rates}} \quad \text{w/} \quad a = 3\left(\frac{b}{2}\right)^{2/3} k^{-1/3}$$

Back to our expression for τ, we can solve for T:

$$(kT)^{-1/3} = \frac{1}{3}\left(\frac{2}{b}\right)^{2/3}\tau$$

and then write our rate in terms of τ:

$$\lambda \sim 2\left(\frac{8}{3\mu}\right)^{1/2} S_0 e^{-\tau}\left(\frac{b}{2}\right)^{1/3}\left[\frac{1}{3}\left(\frac{2}{b}\right)^{2/3}\tau\right]^2$$

so

$$\lambda \sim 2\left(\frac{8}{3\mu}\right)^{1/2} S_0 \tau^2 e^{-\tau}\frac{2}{9}\frac{1}{b}$$

putting in b, we have

$$\lambda \sim 2\left(\frac{8}{3\mu}\right)^{1/2} S_0 \tau^2 e^{-\tau}\frac{2}{9}\frac{\hbar}{2\pi Z_1 Z_2 e^2}\left(\frac{2}{\mu}\right)^{1/2}$$

$$\sim \frac{16}{9}\frac{1}{\sqrt{3}}\frac{1}{\mu}\frac{\hbar}{2\pi Z_1 Z_2 e^2} S_0 \tau^2 e^{-\tau}$$

Clayton defines

$$\mu = A m_0$$

$$\text{w/ } A = \frac{A_1 A_2}{A_1 + A_2}$$

putting in #s

$$\lambda \sim 4.5\times 10^{14}\frac{S_0}{A Z_1 Z_2}\tau^2 e^{-\tau}\ \mathrm{cm^3/s}$$

[noting that S_0 has units of erg · cm²]

AKT uses keV – barns

There would be more terms if we consider departure from Gaussian or variations in S

Evaluating some more constants

$$\sigma(E) = \frac{S(E)}{E} e^{-2\pi Z_1 Z_2 e^2/\hbar v}$$

$$E = \frac{1}{2}\mu v^2 \longrightarrow v = \left(\frac{2E}{\mu}\right)^{1/2}$$

and

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}$$

$$\text{w/} \quad b = \frac{2\pi Z_1 Z_2 e^2}{\hbar}\left(\frac{\mu}{2}\right)^{1/2}$$

$$\text{now} \quad \mu = \frac{A_1 A_2}{A_1 + A_2} m_u \equiv A m_u$$

$$\text{so} \quad b = \frac{2\pi Z_1 Z_2 e^2}{\hbar}\left(\frac{A m_u}{2}\right)^{1/2}$$

$$= \frac{2\pi}{\sqrt{2}} \frac{(4.8 \times 10^{-10}\, g^{1/2}\, cm^{3/2}\, s^{-1})^2 (1.66 \times 10^{-24}\, g)^{1/2}}{6.63 \times 10^{-27}\, erg \cdot s/2\pi} Z_1 Z_2 A^{1/2}$$

$$= 1.25 \times 10^{-3}\, erg^{1/2} = 31.2\, keV^{1/2} \quad (1\, keV^{1/2} = 4 \times 10^{-5}\, erg^{1/2})$$

$$\therefore \quad b = 31.2\, keV^{1/2} A^{1/2} Z_1 Z_2$$