CASE STUDY: Portfolio Credit-Risk Optimization Modeled by Scenarios and Mixtures of Normal Distributions (var_risk, cvar_risk, avg_var_risk_ni, avg_cvar_risk_ni, avg_pr_pen_ni, avg_pm_pen_ni)

Background

This case study evaluates several alternative formulations of optimization problems for minimizing credit risk of a portfolio of financial contracts with different counterparties. The formulations and numerical runs (both models and data) are motivated by paper Iscoe et al. (2009). This paper considers various approximations to the conditional portfolio loss distribution and formulate VaR and CVaR minimization problems for each case. Formulations exploit the conditional independence of counterparties under a structural credit risk model. The case studies consider the "conventional" scenarios and the mixture of normal distribution approaches for modeling the conditional loss distribution.

Similar to Iscoe et al. (2009) we find four optimal portfolios for minimization problems with different but closely risk measures. Problems 1,2 consider quantile-based risk measures, var_risk, cvar_risk with "conventional" scenarios and Problems 3,4 consider avg_var_risk_ni, avg_cvar_risk_ni calculating mixtures of normal independent distributions.

Additionally, Problems 3,4 involving avg_var_risk_ni and avg_cvar_risk_ni were equivalently reformulated with functions avg_pr_pen_ni, avg_pm_pen_ni. These formulations help to understand relation between "avg_..." functions.

Case study demonstrates that PSG optimization provides results comparable to Iscoe, I., et al. (2009).

References

- Iscoe, I. et al. (2009): Portfolio Credit-Risk Optimization. McMaster University, Hamilton, Ontario, Canada, AdvOl-Report No. 2009/06

Notations

J = number of scenarios;
j = index of scenarios $\{1,\ldots, J\}$;
I = number of stocks;
i = index of stocks in the portfolio $\{1,\ldots, I\}$;
x_i = fraction of capital invested in stock i;
$L(x)$ = random loss function given by scenarios $(L_1(x), L_2(x), \ldots, L_J(x))$:

$$L_j(x) = -\sum_{i=1}^{I} x_i l_{ji},\ \ j=1,\ldots, J;$$

l_{ji} = mean value of random coefficient of function $L_j(x)$;
$L_j(x)$ = mean value of loss function on scenario j;
$p_j = 1/J$ = scenarios probabilities for $j = 1,\ldots,J$ (we assume that all scenarios have equal probabilities);
v_{ji} = variance of random coefficient of function $L_j(x)$;
$V_j(x)$ = variance of loss function $L_j(x)$: $V_j(x) = \sum_{i=1}^{I} x_i v_{ji},\ j=1,\ldots, J$;
$V(x)$ = vector of variances for scenarios $j=1,\ldots, J$;
α = confidence level for calculating quantile-based risk measures;
x_{var} = additional variable which is equal to VaR in optimal solutions of Problems 5 and 6;
$L'(x')$ = random loss function with additional variable x_{var} given by scenarios $\left(L'_1(x'), L'_2(x'), \ldots, L'_J(x')\right)$:

$$L'_j(x') = x_{var} - \sum_{i=1}^{I} x_i l_{ji},\ \ j=1,\ldots, J;$$

x' = vector of decision variables for Problems 5 and 6: $x' = (x_{var},\ x_1, x_2, \ldots, x_I)$;
x^0 = initial point;
$var\_risk_\alpha(L(x))$ = Value-at-Risk;
$cvar\_risk_\alpha(L(x))$ = Conditional Value-at-Risk;
$avg\_var\_risk\_ni_\alpha(L(x), V(x))$ = Average Value-at-Risk for Loss Normal Independent with confidence level α;

$avg\_cvar\_risk\_ni_{\alpha}(L(x), V(x))$ = Average Conditional Value-at-Risk for Loss Normal Independent with confidence level α;
$avg\_pr\_pen\_ni_{0}(L'(x'), V(x))$ = Average Probability Exceeding Penalty for Loss Normal Independent with level 0;
$avg\_pm\_pen\_ni_{0}(L'(x'), V(x))$ = Average Partial Moment Penalty for Loss Normal Independent with level 0.

Optimization Problem 1

minimizing Value-at-Risk

$$\min_x var\_risk_{\alpha}(L(x)) \quad \text{(CS.1)}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.2)}$$

portfolio return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.3)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \; i = 1, \dots, I \quad \text{(CS.4)}$$

Optimization Problem 2

minimizing Conditional Value-at-Risk

$$\min_x cvar\_risk_{\alpha}(L(x)) \quad \text{(CS.4)}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.5)}$$

portfolio return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.6)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \; i = 1, \dots, I \quad \text{(CS.7)}$$

Optimization Problem 3

minimizing Average Value-at-Risk for Normal Independent Distribution

$$\min_x avg\_var\_risk\_ni_{\alpha}(L(x), V(x)) \quad \text{(CS.8)}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.9)}$$

return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.10)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \ i = 1, \dots, I \quad \text{(CS.11)}$$

Optimization Problem 4

minimizing Average Conditional Value-at-Risk for Normal Independent Distribution

$$\min_x avg\_cvar\_risk\_ni_{\alpha}(L(x), V(x)) \quad \text{(CS.12)}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.13)}$$

portfolio return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.14)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \ i = 1, \dots, I \quad \text{(CS.15)}$$

Optimization Problem 5

minimizing Average Value-at-Risk for Normal Independent Distribution using alternative formulation of Problem 3

$$\min_{x\prime} \ x_{var} \quad \text{(CS.16)}$$

subject to

probability constraint

$$avg\_pr\_pen\_ni_0(L'(x), V(x)) \leq 1 - \alpha \quad \text{(CS.17)}$$

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.18)}$$

portfolio return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.19)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \; i = 1, \dots, I \quad \text{(CS.20)}$$

Optimization Problem 6

minimizing Average Conditional Value-at-Risk for Normal Independent Distribution using alternative formulation of Problem 4

$$\min_{x'} \left\{ x_{var} + \frac{1}{1-\alpha} avg\_pm\_pen\_ni_0(L'(x), V(x)) \right\} \quad \text{(CS.21)}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.22)}$$

portfolio return constraint

$$\sum_{i=1}^{I} r_i x_i \geq const \quad \text{(CS.23)}$$

bounds on decision variables

$$0 \leq x_i \leq 2x_i^0, \; i = 1, \dots, I \quad \text{(CS.24)}$$