Neyman-Pearson Lemma

Let $X_1, \dots, X_n$ be i.i.d. with likelihood function $L(x_1, \dots, x_n | \theta)$

Consider simple hypotheses

$$H_0: \theta = \theta_0 \text{ and } H_1: \theta = \theta_1$$

Let T be the test for H_0 against H_1 with rejection region

$$R = \{(x_1, \dots, x_n) \in \mathbb{R}^n: \Lambda(x_1, \dots, x_n) \leq t\}$$

where t is a constant

Then there is no test for H_0 against H_1 of the same size which has larger power than T

Preparations for Proof

Recall that if f is the joint PDF of $X_1, \ldots, X_n$ then for $R \subseteq \mathbb{R}^n$

$$P((X_1, \ldots, X_n) \in R) = \int_R f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

If $X_1, \ldots, X_n$ are i.i.d. then their likelihood function $L(x_1, \ldots, x_n|\theta)$ is equal to their joint PDF

$$P((X_1, \ldots, X_n) \in R) = \int_R L(x_1, \ldots, x_n|\theta)\, dx_1 \ldots dx_n$$

Preparations for Proof

Some notations,

Let

$$x = (x_1, \dots, x_n)$$

Write $\int_R L(x, \theta)$ for $\int_R L(x_1, \dots, x_n | \theta)\, dx_1 \dots dx_n$

For $R \subseteq \mathbb{R}^n$, let $\bar{R} = \mathbb{R}^n \backslash R$, the compliment set of R

Proof of Neyman-Pearson Lemma

Recall that for the test T:

The rejection region is,

$$R = \{x \in \mathbb{R}^n : \Lambda(x) \leq t\} = \left\{x \in \mathbb{R}^n : \frac{L(x, \theta_0)}{L(x, \theta_1)} \leq t\right\}$$

And its size is,

$$\alpha = P((X_1, \ldots, X_n) \in R | H_0) = \int_R L(x, \theta_0)$$

With power,

$$P((X_1, \ldots, X_n) \in R | H_1) = \int_R L(x, \theta_1)$$

Proof, continued

Let T_1 be another test for H_0 against H_1 with rejection region R_1 and the same size,

$$\alpha = P((X_1, \dots, X_n) \in R_1 | H_0) = \int_{R_1} L(x, \theta_0)$$

We need to that T_1 does NOT have higher power than T:

$$\int_{R_1} L(x, \theta_1) \leq \int_{R} L(x, \theta_1) \quad ------ -(*)$$

The idea is to break up the LHS $\int_{R_1} L(x, \theta_1)$ into several parts and us the fact that:

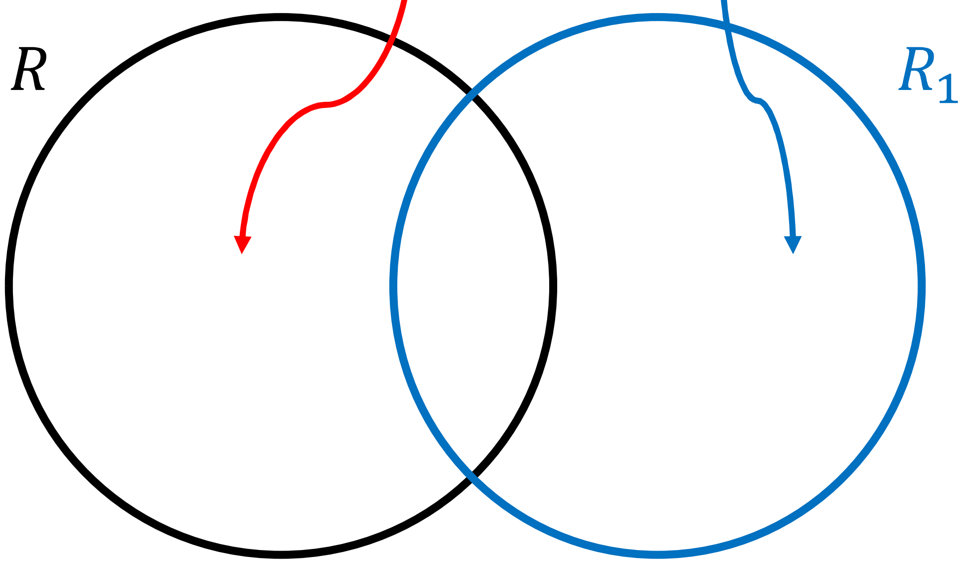
$$\frac{L(x,\theta_0)}{L(x,\theta_1)} \leq t \text{ for } x \in R, \text{ and } \frac{L(x,\theta_0)}{L(x,\theta_1)} > t \text{ for } x \in \bar{R} \;--- (**)$$

Proof, continued

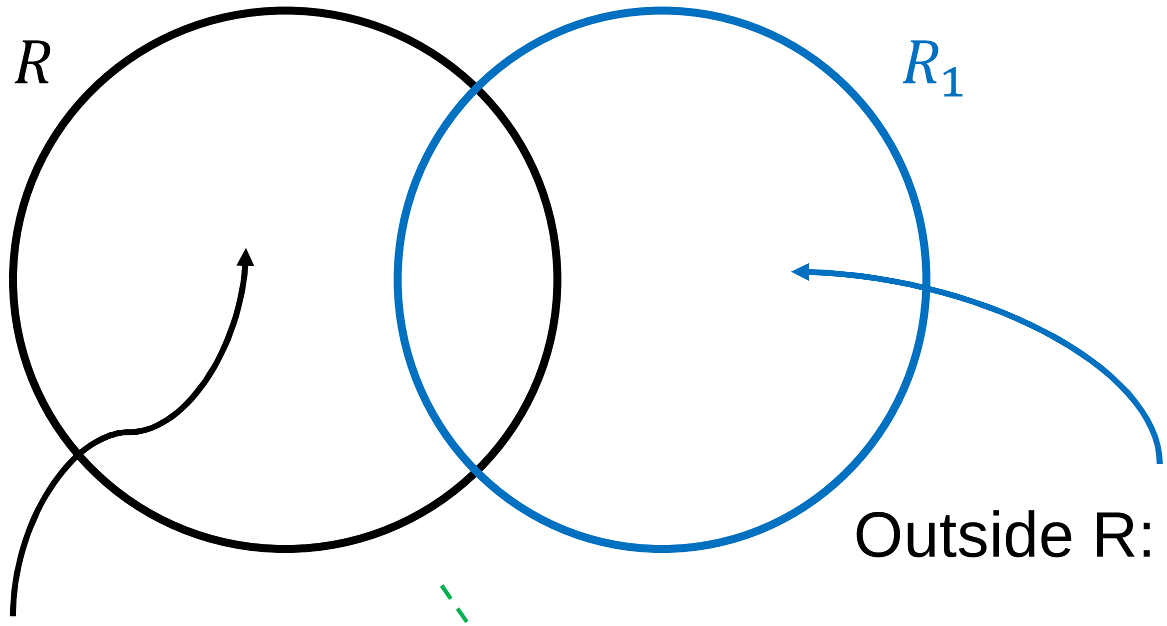
$$LHS = \int_{R_1} L(x,\theta_1) = \int_{R_1 \cap R} L(x,\theta_1) + \int_{R_1 \cap \bar{R}} L(x,\theta_1)$$

$$= \int_{R} L(x,\theta_1) - \int_{\bar{R}_1 \cap R} L(x,\theta_1) + \int_{R_1 \cap \bar{R}} L(x,\theta_1)$$

$$R_1 = R \backslash (R \cap \bar{R}_1) \cup (R_1 \cap \bar{R})$$



$$R_1 = R \backslash (R \cap \bar{R}_1) \cup (R_1 \cap \bar{R})$$



Outside R: $\frac{L(x,\theta_0)}{L(x,\theta_1)} > t$

$$\Leftrightarrow L(x,\theta_1) < \frac{1}{t} L(x,\theta_0)$$

$$\Rightarrow \int_{R_1 \cap \bar{R}} L(x,\theta_1) < \frac{1}{t} \int_{R_1 \cap \bar{R}} L(x,\theta_0)$$

Inside R: $\frac{L(x,\theta_0)}{L(x,\theta_1)} \leq t$

$$\Leftrightarrow L(x,\theta_1) \geq \frac{1}{t} L(x,\theta_0)$$

$$\Rightarrow -\int_{\bar{R}_1 \cap R} L(x,\theta_1) \leq -\frac{1}{t} \int_{\bar{R}_1 \cap R} L(x,\theta_0)$$

Proof, continued

$$LHS = \int_{R_1} L(x,\theta_1) = \int_{R_1 \cap R} L(x,\theta_1) + \int_{R_1 \cap \bar{R}} L(x,\theta_1)$$

$$= \int_{R} L(x,\theta_1) - \int_{\bar{R}_1 \cap R} L(x,\theta_1) + \int_{R_1 \cap \bar{R}} L(x,\theta_1)$$

$$\leq \int_{R} L(x,\theta_1) - \frac{1}{t}\int_{\bar{R}_1 \cap R} L(x,\theta_0) + \frac{1}{t}\int_{R_1 \cap \bar{R}} L(x,\theta_0)$$

To see that the last two integrals sum up to zero, we add and minus the integration over the intersection, $\int_{R_1 \cap R} L(x,\theta_0)$

Proof, continued

$$LHS = \int_{R_1} L(x,\theta_1) \le \int_{R} L(x,\theta_1) - \frac{1}{t}\int_{\bar{R}_1\cap R} L(x,\theta_0) + \frac{1}{t}\int_{R_1\cap \bar{R}} L(x,\theta_0)$$

$$= \int_{R} L(x,\theta_1) - \frac{1}{t}\int_{\bar{R}_1\cap R} L(x,\theta_0) + \frac{1}{t}\int_{R_1\cap \bar{R}} L(x,\theta_0)$$

$$-\frac{1}{t}\int_{R_1\cap R} L(x,\theta_0) + \frac{1}{t}\int_{R_1\cap R} L(x,\theta_0)$$

$$= \int_{R} L(x,\theta_1) - \frac{1}{t}\int_{R} L(x,\theta_0) + \frac{1}{t}\int_{R_1} L(x,\theta_0)$$

Proof, continued

$$LHS = \int_{R_1} L(x,\theta_1) \le \int_R L(x,\theta_1) - \frac{1}{t}\int_R L(x,\theta_0) + \frac{1}{t}\int_{R_1} L(x,\theta_0)$$

$$= \int_R L(x,\theta_1) - \frac{\alpha}{t} + \frac{\alpha}{t} = \int_R L(x,\theta_1) = RHS$$

This completes the proof of the Neyman-Pearson Lemma.