

微积分蓝皮书近几年秋学期期末与先修考试选填解析

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2012 期末

一、1、解:

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = \lim_{x \rightarrow 0} \frac{f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + o(x^2)}{x^2} = -1 \Rightarrow f(0) = f'(0) = 0, f''(0) = -2$$

$$\text{故 } k = \frac{|f''(0)|}{\{1+[f'(0)]^2\}^{\frac{3}{2}}} = 2;$$

$$\text{一、2、解: } x^2 + (y-2)^2 = 1 \Rightarrow y = 2 \pm \sqrt{1-x^2},$$

$$V = \pi \int_{-1}^1 (2 + \sqrt{1-x^2})^2 - (2 - \sqrt{1-x^2})^2 dx = 8\pi \int_{-1}^1 \sqrt{1-x^2} dx = 4\pi^2;$$

$$\text{一、3、解: 两边求导得 } f'(x) \cdot g[f(x)] = \frac{1}{2}\sqrt{x}, \text{ 因为 } f(x), g(x) \text{ 互为反函数, 故}$$

$$f'(x)g[f(x)] = xf'(x) = \frac{1}{2}\sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f(x) = \sqrt{x} + C \Rightarrow g(x) = (x-C)^2,$$

$$\text{于是 } \int_1^{f(x)} g(t)dt = \int_1^{\sqrt{x}+C} (t-C)^2 dt = \dots = \frac{x^2}{3} + \frac{C^3}{3} - \frac{1}{3} + C - C^2 = \frac{1}{3}(x^2 - 8), \text{ 化简得}$$

$$(C-1)^3 = -8 \Rightarrow C = -1 \Rightarrow f(x) = \sqrt{x} - 1;$$

$$\text{一、4、解: } \int_2^{+\infty} \frac{3}{4+x^2} = \frac{3}{2} \arctan \frac{x}{2} \Big|_2^{+\infty} = \frac{3\pi}{4} - \frac{3\pi}{8} = \frac{3\pi}{8};$$

$$\text{一、5、解: 注意到 } \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx,$$

$$\text{故 } \int_0^{\frac{\pi}{2}} \frac{e^{\cos x} - e^{\sin x} + \cos^2 x}{2} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{2} dx = \frac{\pi}{8};$$

$$\text{二、1、解: } F(x) = \int_0^x xf(x-t)dt = x \int_0^x f(x-t)dt = x \int_x^0 f(u)(-du) = x \int_0^x f(t)dt,$$

$$f(0) = 0, f'(x) > 0 \Rightarrow f(x) > 0 (x > 0), \text{ 则当 } x \in (0, +\infty) \text{ 时:}$$

$$F'(x) = \int_0^x f(t)dt + xf(x) = xf(c) + xf(x) > 0 \Rightarrow F(x) \uparrow$$

$$F''(x) = 2f(x) + xf'(x) > 0 \Rightarrow F(x) \text{ 下凸, 选 D;}$$

二、2、解: 因 $f(x)$ 有 $2n+1$ 个极值点, 所以 $f'(x)=0$ 至少有 $2n+1$ 个实根, 连续运用罗尔定理, $f^{(n)}(x)=0$ 至少有 $n+2$ 个实根, 选 D;

二、3、解: 当 $0 < x < \frac{\pi}{4}$ 时, $\cot x = \frac{\cos x}{\sin x} > \cos x > \sin x$, 且 $f(x) = \ln x$ 单调, 故 $I < K < J$, 选 C;

二、4、解: $0 \leq x \leq 2$ 时, $f(x) = \lim_{n \rightarrow \infty} \frac{1-x^{2n}}{1+x^{2n}} = \begin{cases} 1, 0 \leq x < 1 \\ 0, x = 1 \\ -1, 1 < x \leq 2 \end{cases} \Rightarrow \int_0^2 f(x) dx = 0$, 选 A;

二、5、解: $x \rightarrow x + \Delta x \Rightarrow \Delta F = \frac{km(u\Delta x)}{(x+a)^2} \Rightarrow F = \sum \frac{km(u\Delta x)}{(x+a)^2} = \int_0^a \frac{kmu}{(x+a)^2} dx$, 选 D;

2013 期末

一、1、解: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\ln \frac{2-x}{2+x} + \cos^2 x \right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx = \frac{\pi}{2}$;

一、2、解: $y = x2^x \Rightarrow y' = 2^x(1+x\ln 2)$, $y'' = 2^x(2\ln 2 + x\ln^2 2) > 0$, 令 $y'(x) = 0$ 得 $x_0 = -\frac{1}{\ln 2}$ 是极小值;

一、3、解: $\frac{\sqrt{1}}{n\sqrt{n+1}} + \frac{\sqrt{2}}{n\sqrt{n+1}} + \cdots + \frac{\sqrt{n}}{n\sqrt{n+1}} < L < \frac{\sqrt{1}}{n\sqrt{n}} + \frac{\sqrt{2}}{n\sqrt{n}} + \cdots + \frac{\sqrt{n}}{n\sqrt{n}}$

左边 $\Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt{1}}{n\sqrt{n+1}} + \cdots + \frac{\sqrt{n}}{n\sqrt{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} \left(\frac{\sqrt{1}}{\sqrt{n}} + \cdots + \frac{\sqrt{n}}{\sqrt{n}} \right) = \lim_{n \rightarrow \infty} \left(\frac{\sqrt{1}}{\sqrt{n}} + \cdots + \frac{\sqrt{n}}{\sqrt{n}} \right)$

右边 $\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{\sqrt{1}}{\sqrt{n}} + \cdots + \frac{\sqrt{n}}{\sqrt{n}} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt{1}}{\sqrt{n}} + \frac{\sqrt{2}}{\sqrt{n}} + \cdots + \frac{\sqrt{n}}{\sqrt{n}} = \int_0^1 \sqrt{x} dx = \frac{2}{3}$, 故原式 $= \frac{2}{3}$

一、4、解: 两边求导: $xy = 2x + y' \Rightarrow \frac{dy}{dx} - xy = -2x$,

常数变易法: $y = e^{\int x dx} \left[C + \int (-2x)e^{-\int x dx} dx \right] = 2 + Ce^{\frac{x^2}{2}}$, 故 $y(0) = 0 \Rightarrow C = -2 \Rightarrow 2 + Ce^{\frac{x^2}{2}}$

一、5、解: $\int_0^a x\varphi''(x) dx = \int_0^a x d[\varphi'(x)] = x\varphi'(x) \Big|_0^a - \int_0^a \varphi'(x) dx = x\varphi'(x) - \varphi(x) \Big|_0^a$, 又 $\varphi(x)$

在 $x=a$ 处取得极大值 $\varphi(a)=0$, 故 $\varphi'(a)=0$, 于是 $\int_0^a x\varphi''(x) dx = \varphi(0) = b$;

二、1、解: BCD 显然错, $F(x) = \int f(x) dx = \int -f(-x) dx = \int f(-x) d(-x) = F(-x)$,

故 $F(x)$ 是偶函数, 选 A;

二、2、解: 令 $P(x) = (x-2)^3(x-3)^2(x-4)$, $Q(x) = (x-1)^4(x-3)^2(x-4)$,

$$y' = (x-1)^4 P'(x) + 4(x-1)^3 P(x), y'' = (x-1)^4 P''(x) + 8(x-1)^3 P'(x) + 12(x-1)^2 P(x)$$

$y''(1) = 0, y'''(1) = 0$, 故 $x=1$ 不是拐点;

$$y' = (x-2)^3 Q'(x) + 3(x-2)^2 Q(x), y'' = (x-2)^3 Q''(x) + 6(x-2)^2 Q'(x) + 6(x-2) Q(x)$$

$y''(2) = 0, y'''(2) = -12 \neq 0$, 故 $x=2$ 是拐点;

显然 $x=3, 4$ 不可能是拐点, 故 ACD 错误, 选 B;

二、3、解: $\int_0^1 |f(x)| dx = \int_0^1 |f(x) - f(0)| dx = \int_0^1 |f'(c)x| dx \leq \int_0^1 Mx dx = \frac{M}{2}$, 选 C;

二、4、解: 令 $f(x) = 1$ 可排除 ACD, 又 $\int_0^T - \int_{-T}^0 = \int_0^T - \int_0^T = 0$, 于是

$$\int_0^{x+T} - \int_{-x-T}^0 = \int_0^x + \int_x^{x+T} - \int_{-x}^0 - \int_{-x-T}^{-x} = \int_0^x + \int_0^T - \int_{-x}^0 - \int_{-T}^0 = \int_0^x - \int_{-x}^0, \text{ 选 B;}$$

二、5、解: $f(x) = \int_0^{x^2} \ln(2+t) dt \Rightarrow f'(x) = 2x \ln(2+x^2), f''(x) > 0$, 又 $f'(0) = 0$, 故 $x=0$ 是唯一零点, 选 C;

2014 期末

一、1、解: $f(1) = 3, f'(1) = 2, f'(1)g'(3) = 1 \Rightarrow g'(3) = \frac{1}{2}$;

一、2、解: $x=0 \Rightarrow \lim_{x \rightarrow 0} f(x) < \infty \Rightarrow x=0$ 不是铅直渐近线,

$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 2, \lim_{x \rightarrow \infty} f(x) - 2x = 0 \Rightarrow y = 2x$ 是唯一渐近线;

一、3、解: $5 = \int_0^\pi [f(x) + f''(x)] \sin x dx = \int_0^\pi f(x) \sin x dx + \int_0^\pi f''(x) \sin x dx$

$$= I + \int_0^\pi \sin x d[f'(x)] = I + \sin x f'(x) \Big|_0^\pi - \int_0^\pi \cos x f'(x) dx = I - \cos x f(x) \Big|_0^\pi - I = f(0) + 3$$

故 $f(0) = 2$;

一、4、解: $\int f(x) dx = \begin{cases} \frac{x^3}{3} + C_1, & x \leq 1 \\ \frac{x^4}{4} + C_2, & x > 1 \end{cases}$, 因 $\int f(x) dx$ 在 $x=1$ 处连续, 得 $C_2 = \frac{1}{12} + C_1$,

$$\text{故 } \int f(x) dx = \begin{cases} \frac{x^3}{3} + C, & x \leq 1 \\ \frac{x^4}{4} + C + \frac{1}{12}, & x > 1 \end{cases};$$

一、5、解: $\frac{dy}{dx} = \frac{1+y^2}{xy+y} \Rightarrow \frac{dx}{dy} = \frac{xy+y}{1+y^2} \Rightarrow \frac{dx}{dy} - \frac{y}{1+y^2}x = \frac{y}{1+y^2},$

常数变易法: $x = e^{\int \frac{y}{1+y^2} dy} (C + \int \frac{y}{1+y^2} e^{-\int \frac{y}{1+y^2} dy} dy) = C e^{\frac{1}{2} \ln(1+y^2)} - 1 \Rightarrow 1+y^2 = C(x+1)^2;$

2015 期末

一、1、解: 两边取极限得: $\lim_{x \rightarrow 0} f(x) = 1 + 2 \lim_{x \rightarrow 0} f(x) \Rightarrow \lim_{x \rightarrow 0} f(x) = -1,$

于是 $f(x) = 2e^x - \frac{\sin x}{x} - 2 \cos x \Rightarrow \lim_{x \rightarrow 1} f(x) = 2e - \sin 1 - 2 \cos 1;$

一、2、解: $\int x \arctan x dx = \frac{1}{2} \int \arctan x dx^2 = \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$
 $= \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C;$

一、3、解: $f(x)$ 任意阶可导, 于是 $f(x) = \frac{\sin x}{x} \Rightarrow f'(x) = \begin{cases} \frac{x \cos x - \sin x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$

故 $f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{-x \sin x}{3x^2} = -\frac{1}{3};$

一、4、解: $\int_{-1}^1 (|x| + \sin x^3 \sqrt{\cos(1+x^2)}) dx = \int_{-1}^1 |x| dx = 2 \int_0^1 x dx = 1;$

一、5、解: 易知 $f(x) = f(-x) \Rightarrow f'(x) = -f'(-x) \Rightarrow f'(-1) = -f'(1) = -2016;$

一、6、解: $S = \int_0^1 \ln^2 x dx = x(\ln^2 x - 2 \ln x + 2) \Big|_0^1 = 2;$

2016 期末

一、1、解: $\lim_{x \rightarrow 0} \frac{x^3 - \int_0^x \cos t dt}{x^9} = \lim_{x \rightarrow 0} \frac{3x^2 - 3x^2 \cos x^3}{9x^8} = \lim_{x \rightarrow 0} \frac{1 - \cos x^3}{3x^6} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} x^6}{3x^6} = \frac{1}{6};$

一、2、解: $\int x \tan^2 x dx = \int x(\tan^2 x + 1) dx - \int x dx = \int x d \tan x - \frac{x^2}{2}$

$= x \tan x - \int \tan x - \frac{x^2}{2} = x \tan x + \ln |\cos x| - \frac{x^2}{2} + C;$

一、3、解: $f'(0^+) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = 3a - 2 + \lim_{x \rightarrow 0^+} x^{a-1} \sin \frac{1}{x},$

$f'(0^-) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{a^2 x}{x} = a^2, \quad f(x) \text{ 在 } x=0 \text{ 可导} \Rightarrow f'(0^+) = f'(0^-),$

$$\text{故 } a^2 = 3a - 2 + \lim_{x \rightarrow 0^+} x^{a-1} \sin \frac{1}{x} \Rightarrow a = 2;$$

$$\text{一、4、解: } \int_{-1}^1 \left(\frac{x^2}{1+e^x} + \sin^3 x \right) dx = \int_{-1}^1 \frac{x^2}{1+e^x} dx = \int_0^1 \left(\frac{x^2}{1+e^x} + \frac{x^2 e^x}{1+e^x} \right) dx = \int_0^1 x^2 dx = \frac{1}{3};$$

$$\text{一、5、解: } f(x) \text{ 在 } x=0 \text{ 处的泰勒公式为: } f(x) = (x+1) \left[(-x) + \frac{(-x)^2}{2!} + \dots + \frac{(-1)^n x^n}{n!} \right],$$

$$\text{则 } f^{(2017)}(0) = \left[\frac{(-1)^{2016} x^{2017}}{2016!} + \frac{(-1)^{2017} x^{2017}}{2017!} \right]^{(2017)} \Big|_{x=0} = \frac{2017!}{2016!} - \frac{2017!}{2017!} = 2016;$$

$$\text{一、6、解: } y'(1) = 3, y''(1) = -1, R = \frac{1}{k} = \frac{(1+y'(1))^{\frac{3}{2}}}{|y''(1)|} = 10\sqrt{10};$$

$$\text{一、7、解: 原式} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \ln(x+\sqrt{1+x^2}) \Big|_0^1 = \ln(1+\sqrt{2});$$

$$\text{一、8、解: 两边求导得: } f'(x) = e^{2x}(4+4x) - 2 \lim_{x \rightarrow 0} f'(x), \text{ 两边取极限得:}$$

$$\lim_{x \rightarrow 0} f'(x) = 4 - 2 \lim_{x \rightarrow 0} f'(x) \Rightarrow \lim_{x \rightarrow 0} f'(x) = \frac{4}{3} \Rightarrow f(x) = e^{2x}(1+2x) - \frac{8}{3} \Rightarrow f(1) = 3e^2 - \frac{8}{3};$$

2017 期末

$$\text{一、1、解: } \lim_{x \rightarrow 0} \frac{(e^x - 1 - x) \ln(1-x)}{x(\cos x - 1)} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} x^2 (-x)}{x(-\frac{1}{2} x^2)} = 1;$$

$$\text{一、2、解: } \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{1-x^2}} \Rightarrow y = f(x) = \arcsin x + C_1,$$

$$\text{故 } \int f(x) dx = x \arcsin x - \sqrt{1-x^2} + C_1 x + C_2;$$

$$\text{一、3、解: } y = \ln(x + \sqrt{1+x^2}) \Rightarrow dy \Big|_{x=0} = \frac{dx}{\sqrt{1+x^2}} \Big|_{x=0} = dx;$$

$$\text{一、4、解: } x = \tan t \Rightarrow dx = (\tan^2 t + 1) dt \Rightarrow \int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^4)} = \int_0^{\frac{\pi}{2}} \frac{dt}{1+\tan^4 t},$$

$$\text{注意到 } \int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx,$$

$$\text{故 } \int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^4)} = \int_0^{\frac{\pi}{2}} \frac{dt}{1+\tan^4 t} = \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\tan^4 t} + \frac{1}{1+\cot^4 t} \right) dt = \int_0^{\frac{\pi}{4}} dt = \frac{\pi}{4};$$

一、5、解: 两边求导得: $2x - y - xy' + 2yy' = 0 \Rightarrow y' = \frac{y-2x}{2y-x}$, 令 $y' = 0 \Rightarrow y = 2x$,

代回得 $x^2 - 2x^2 + 4x^2 = a^2 \Rightarrow 3x^2 = a^2 \Rightarrow x = \frac{\sqrt{3}}{3}a \Rightarrow y = \frac{2\sqrt{3}}{3}a$ (部分步骤省略);

一、6、解: 原式 $= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} e^{\frac{2k}{n}} = \int_0^1 x e^{2x} dx = \frac{1}{4}(e^2 + 1)$;

一、7、解: $x \rightarrow 0 \Rightarrow g(x) = \tan^n x \sim x^n$,

$x \rightarrow 0 \Rightarrow f(x) = \dots = 3x - 4(x - \frac{x^3}{6} + \frac{x^5}{120}) + (x - \frac{x^3}{6} + \frac{x^5}{120})(1 - \frac{x^2}{2} + \frac{x^4}{24}) + o(x^5) \sim x^5$,

故 $n = 5$;

一、8、解: 设 $a = \lim_{x \rightarrow \infty} \frac{y}{x}, b = \lim_{x \rightarrow \infty} (y - ax)$,

$y^3 + x^3 - 3xy = 0 \Rightarrow \lim_{x \rightarrow \infty} \left[\left(\frac{y}{x} \right)^3 + 1 - \frac{3y}{x^2} \right] = 0 \Rightarrow a^3 + 1 = 0 \Rightarrow a = -1$,

将 b 代回, 同除 x^2 并取极限得 $\lim_{x \rightarrow \infty} \frac{3(1+b)x^2 - 3bx(1+b) + b^3}{x^2} = 0 \Rightarrow b = -1$,

故渐近线方程为 $y = -x - 1$;

一、1、解: 两边取极限并化简:

$\lim_{x \rightarrow 0} f(x) = 1 + 2 \lim_{x \rightarrow 0} f(x) \Rightarrow \lim_{x \rightarrow 0} f(x) = -1 \Rightarrow f(x) = 2e^x - \frac{\sin(\sin x)}{x} - x - 2$;

一、2、解: $\int x \ln(1+x^2) dx = \frac{1}{2} \int \ln(1+x^2) d(1+x^2) = \frac{(1+x^2)}{2} \ln(1+x^2) - \frac{x^2}{2} + C$;

一、3、解: 因 $f(x)$ 有任意阶导数, $f'(x) = \begin{cases} \frac{x - (1+x^2) \arctan x}{(1+x^2)x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, 故

$f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0} \frac{x - (1+x^2) \arctan x}{x^3 + x^5} = \lim_{x \rightarrow 0} \frac{1 - 2x \arctan x - 1}{3x^2} = -\frac{2}{3}$;

一、4、解: $\lim_{n \rightarrow \infty} \left(\frac{1^3}{n^4} + \frac{2^3}{n^4} + \dots + \frac{n^3}{n^4} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n} \right)^3 = \int_0^1 x^3 dx = \frac{1}{4}$;

一、5、解: 由 $e^x + e^{-x} \geq 2 + ax^2$, 当 $x = 0$ 时显然成立, 当 $x \neq 0$ 时, 有

$$a \leq \frac{e^x + e^{-x} - 2}{x^2} \Rightarrow \frac{e^x + e^{-x} - 2}{x^2} \geq \frac{1 + \frac{x^2}{2} + 1 + \frac{(-x)^2}{2} - 2}{x^2} = 1 \Rightarrow a \leq 1;$$

一、6、解: $x \rightarrow 0 \Rightarrow \lim_{x \rightarrow 0} (2x + x^2 \sin \frac{1}{x}) < \infty \Rightarrow x = 0$ 不是铅直渐近线,

$$\lim_{x \rightarrow \infty} \frac{2x + x^2 \sin \frac{1}{x}}{x} = 3, \lim_{x \rightarrow \infty} (y - 3x) = 0 \Rightarrow y = 3x \text{ 是唯一的渐近线};$$

2017 先修

一、1、解: $\lim_{x \rightarrow 0} \left(\frac{\sin x^2}{x^2} + \frac{\sin(x+1)}{x+1} \right) = 1 + \sin 1;$

一、2、解: $(f(f(x)))' = f'(x) \cdot f'(f(x)) \xrightarrow{x=e} f'(e) f'(\frac{1}{2}) = \frac{1}{2e} \times 2 = \frac{1}{e};$

一、3、解: $x \rightarrow 0 \Rightarrow \lim_{x \rightarrow 0} x(2018 + \arcsin \frac{8}{x}) < \infty \Rightarrow x = 0$ 不是铅直渐近线,

$$\lim_{x \rightarrow \infty} \frac{x(2018 + \arcsin \frac{8}{x})}{x} = 2018, \lim_{x \rightarrow \infty} (y - 2018x) = 8 \Rightarrow y = 2018x + 8 \text{ 是唯一的渐近线};$$

一、4、解: $\int 2x^3 e^{x^2} dx = \int x^2 e^{x^2} dx^2 = (x^2 - 1)e^{x^2} + C;$

一、5、解: 原式 = $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x \cos x}{e^x + 1} dx = \int_0^{\frac{\pi}{4}} \left(\frac{e^x \cos x}{e^x + 1} + \frac{\cos x}{e^x + 1} \right) dx = \int_0^{\frac{\pi}{4}} \cos x dx = \frac{\sqrt{2}}{2};$

一、6、解: 原式 = $\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} \cdot \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \sin \frac{k}{n} = \int_0^1 x \sin x dx = \sin 1 - \cos 1;$

一、7、解: 因 $\int_1^{+\infty} f(x) dx$ 收敛, 知 $\int_1^e f(x) dx, \int_e^{+\infty} f(x) dx$ 均收敛, 从而有

$$\begin{cases} a-1 < 1 \\ a+1 > 1 \end{cases} \Rightarrow 0 < a < 2;$$

一、8、解: $f(x)$ 在 $x=0$ 处的泰勒公式为 $f(x) = ex^2 e^x = ex^2 \left(x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right),$

故 $f^{(5)}(0) = \left(\frac{e}{3!} x^5 \right)^{(5)} = \frac{5!e}{3!} = 20e;$