Assignment 10 – More Network Flow

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Name: Zinnia Nie   Wisc id: 908 319 4044

More Network Flow

1. *Kleinberg, Jon. Algorithm Design (p.416 q.6).* Suppose you're a consultant for the Ergonomic Architecture Commission, and they come to you with the following problem.

   They're really concerned about designing houses that are "user-friendly", and they've been having a lot of trouble with the setup of light fixtures and switches in newly designed houses. Consider, for example, a one-floor house with n light fixtures and n locations for light switches mounted in the wall. You'd like to be able to wire up one switch to control each light fixture, in such a way that a person at the switch can see the light fixture being controlled.

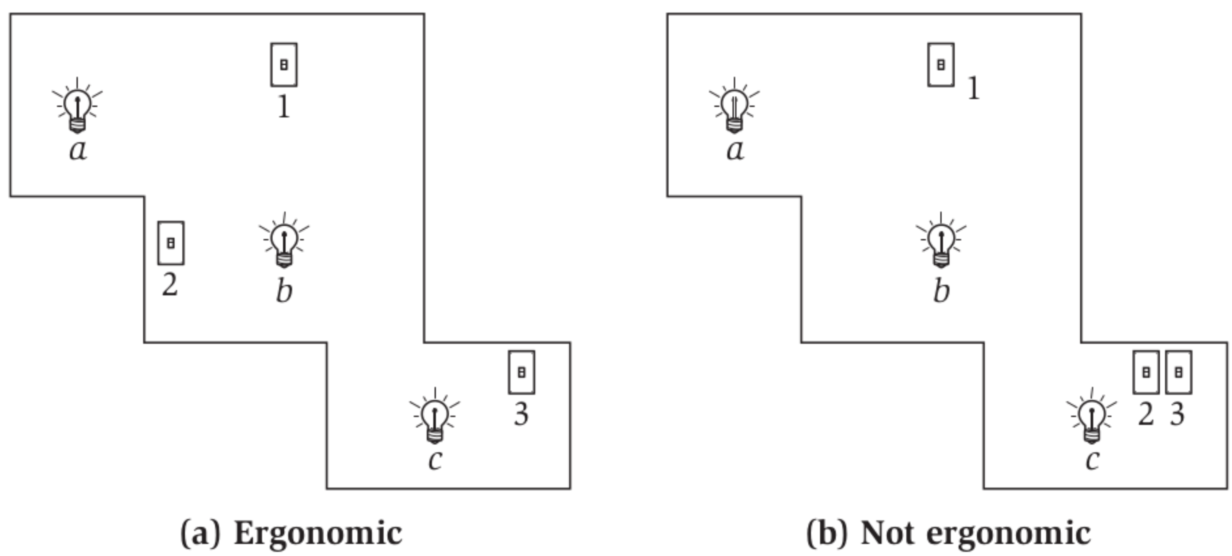


Figure 1: The floor plan in (a) is ergonomic, because we can wire switches to fixtures in such a way that each fixture is visible from the switch that controls it. (This can be done by wiring switch 1 to a, switch 2 to b, and switch 3 to c.) The floor plan in (b) is not ergonomic, because no such wiring is possible.

   Sometimes this is possible and sometimes it isn't. Consider the two simple floor plans for houses in Figure 1. There are three light fixtures (labelled a, b, c) and three switches (labelled 1, 2, 3). It is possible to wire switches to fixtures in Figure 1(a) so that every switch has a line of sight to the fixture, but this is not possible in Figure 1(b).

   Let's call a floor plan, together with n light fixture locations and n switch locations, ergonomic if it's possible to wire one switch to each fixture so that every fixture is visible from the switch that controls it. A floor plan will be represented by a set of m horizontal or vertical line segments in the plane (the walls), where the i-th wall has endpoints $(x_i, y_i), (x'_i, y'_i)$. Each of the n switches and each of the n fixtures is given by its coordinates in the plane. A fixture is visible from a switch if the line segment joining them does not cross any of the walls.

   Give an algorithm to decide if a given floor plan is ergonomic. The running time should be polynomial in m and n. You may assume that you have a subroutine with $O(1)$ running time that takes two line segments as input and decides whether or not they cross in the plane.

Solution:

First determine with the subroutine what walls a line segment between switch i and light j intersect.

Each switch and light is a node in the graph.

The nodes i and j are only connected if the line segment between them do not intersect any walls.

The source connects to all switches, the sink to all lights and all edges have capacity of one, since lights and switches must be one to one.

Then the floor plan is ergonomic if max flow $= n$.

We build the graph in $O(mn^2)$, and run max flow in time $O(n^3)$.

2. *Kleinberg, Jon. Algorithm Design (p.426 q.20).*

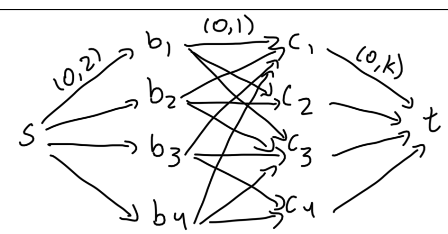
   Your friends are involved in a large-scale atmospheric science experiment. They need to get good measurements on a set S of n different conditions in the atmosphere (such as the ozone level at various places), and they have a set of m balloons that they plan to send up to make these measurements. Each balloon can make at most two measurements. Unfortunately, not all balloons are capable of measuring all conditions, so for each balloon $i = 1, \ldots, m$, they have a set S_i of conditions that balloon i can measure. Finally, to make the results more reliable, they plan to take each measurement from at least k different balloons. (Note that a single balloon should not measure the same condition twice.) They are having trouble figuring out which conditions to measure on which balloon.

   Example. Suppose that $k = 2$, there are $n = 4$ conditions labelled c_1, c_2, c_3, c_4, and there are $m = 4$ balloons that can measure conditions, subject to the limitation that $S_1 = S_2 = c_1, c_2, c_3$, and $S_3 = S_4 = c_1, c_3, c_4$. Then one possible way to make sure that each condition is measured at least $k = 2$ times is to have

   - balloon 1 measure conditions c_1, c_2,
   - balloon 2 measure conditions c_2, c_3,
   - balloon 3 measure conditions c_3, c_4, and
   - balloon 4 measure conditions c_1, c_4.

   (a) Give a polynomial-time algorithm that takes the input to an instance of this problem (the n conditions, the sets S_i for each of the m balloons, and the parameter k) and decides whether there is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions.

Solution:



- Form a bipartite graph with the balloons on one side and the conditions on the other.
- Have edges between the balloons and the conditions they measure with capacity of 1.
  - each balloon should measure each condition once
- Add a source node with edges to each balloon with capacity of 2
  - each balloon can only be used twice
- Add a sink node with edges from each condition with a capacity of k
  - each condition needs to be measured k times
- If max flow = k•n, then there is a way to measure.

(b) You show your friends a solution computed by your algorithm from (a), and to your surprise they reply, "This won't do at all—one of the conditions is only being measured by balloons from a single subcontractor." You hadn't heard anything about subcontractors before; it turns out there's an extra wrinkle they forgot to mention...

Each of the balloons is produced by one of three different subcontractors involved in the experiment. A requirement of the experiment is that there be no condition for which all k measurements come from balloons produced by a single subcontractor.

Explain how to modify your polynomial-time algorithm for part (a) into a new algorithm that decides whether there exists a solution satisfying all the conditions from (a), plus the new requirement about subcontractors.

Solution:

Building on the graph from above, each subcontractor can only take the same measurement k-1 times.

Therefore we create a clone of the condition nodes, and there is an edge between the condition and its clone with capacity k, to guarantee k times to take the measurement.

Then the cloned conditions are connected to the subcontractor nodes with edges with capacity k-1.

Finally subcontractors are connected to the sink with no capacity limit.

If max flow still equals $k \cdot n$, then there is a way to measure while fulfilling conditions.

- This runs in polynomial time if we use Ford-Fulkerson to get runtime of $O(E \cdot C)$
  - $E = m + n + c + \sum_{S_i \in S} S_i + \sum_{M_i \in M} M_i$

3. *Kleinberg, Jon. Algorithm Design (p.442, q.41).*

Suppose you're managing a collection of k processors and must schedule a sequence of m jobs over n time steps.

The jobs have the following characteristics. Each job j has an arrival time a_j when it is first available for processing, a length ℓ_j which indicates how much processing time it needs, and a deadline d_j by which it must be finished. (We'll assume $0 < \ell_j \leq d_j - a_j$.) Each job can be run on any of the processors, but only on one at a time; it can also be preempted and resumed from where it left off (possibly after a delay) on another processor.

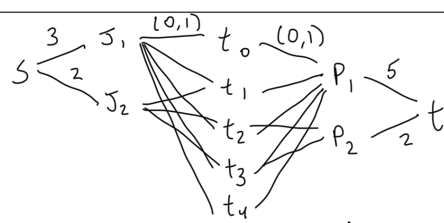
Moreover, the collection of processors is not entirely static either: You have an overall pool of k possible processors; but for each processor i, there is an interval of time $[t_i, t_i']$ during which it is available; it is unavailable at all other times.

Given all this data about job requirements and processor availability, you'd like to decide whether the jobs can all be completed or not. Give a polynomial-time (in k, m, and n) algorithm that either produces a schedule completing all jobs by their deadlines or reports (correctly) that no such schedule exists. You may assume that all the parameters associated with the problem are integers.

Example. Suppose we have two jobs J_1 and J_2. J_1 arrives at time 0, is due at time 4, and has length 3. J_2 arrives at time 1, is due at time 3, and has length 2. We also have two processors P_1 and P_2. P_1 is available between times 0 and 4; P_2 is available between times 2 and 3. In this case, there is a schedule that gets both jobs done.

- At time 0, we start job J_1 on processor P_1.
- At time 1, we preempt J_1 to start J_2 on P_1.
- At time 2, we resume J_1 on P_2. (J_2 continues processing on P_1.)
- At time 3, J_2 completes by its deadline. P_2 ceases to be available, so we move J_1 back to P_1 to finish its remaining one unit of processing there.
- At time 4, J_1 completes its processing on P_1. Notice that there is no solution that does not involve preemption and moving of jobs.

Solution:



- Jobs, time steps, processors are nodes
  - edge between jobs and each time slot between arrival and deadline
  - edge between time and when processors are available
- add source with edge to each job with capacity of the length of the job
- add sink with edge from each job with capacity of length of time processor is available
- If max flow = Σ length of jobs, there is a feasible schedule
- The generated flowgraph from using Ford Fulkerson will create the schedule in $O(E \cdot C)$ where

  $E = m + k + \Sigma$ time each P is available $+ \Sigma$ time Job is available

4. *Kleinberg, Jon. Algorithm Design (p.444, q.45).*

   Consider the following definition. We are given a set of n countries that are engaged in trade with one another. For each country i, we have the value s_i of its budget surplus; this number may be positive or negative, with a negative number indicating a deficit. For each pair of countries i, j, we have the total value e_{ij} of all exports from i to j; this number is always nonnegative. We say that a subset S of the countries is *free-standing* if the sum of the budget surpluses of the countries in S, minus the total value of all exports from countries in S to countries not in S, is nonnegative. Give a polynomial-time algorithm that takes this data for a set of n countries and decides whether it contains a nonempty free-standing subset that is not equal to the full set.

Solution:

- Nodes: countries with edges from country i to j of capacity $e_{i,j}$ representing exports

- Add a source with edges to countries with a surplus. The capacity is the value s_i.
- Add a sink with edges from countries with a deficit. The capacity is the absolute value of s_i.

Find the min cut. The flow between the two sets will be bounded by the sum of all the surpluses.

Therefore, the set with the source node will be a free standing subset, as the sum of surplus − exports will be ≥ 0.

If this set is nonempty, then we have a free standing subset.

5. Implement an algorithm to determine the maximum matching in a bipartite graph and if that matching is perfect (all nodes are matched) in either C, C++, C#, Java, Python, or Rust. Be efficient and use your max-flow implementation from the previous week.

   The input will start with an positive integer, giving the number of instances that follow. For each instance, there will be 3 positive integers m, n, and q. Numbers m and n are the number of nodes in node set A and node set B. Number q is the number of edges in the bipartite graph. For each edge, there will be 2 more positive integers i, and j representing an edge between node $1 \leq i \leq m$ in A and node $1 \leq i \leq n$ in B.

   A sample input is the following:

```
3
2 2 4
1 1
1 2
2 1
2 2
2 3 4
2 3
2 1
1 2
2 2
5 5 10
1 1
1 3
2 1
2 2
2 3
2 4
3 4
4 4
5 4
5 5
```

   The sample input has 3 instances.

   For each instance, your program should output the size of the maximum matching, followed by a space, followed by an N if the matching is not perfect and a Y if the matching is perfect. Each output line should be terminated by a newline. The correct output to the sample input would be:

```
2 Y
2 N
4 N
```