Assignment 2

1. Answer the following questions. Justify your answers.

a) Are the columns of the following matrix linearly independent?

$$A = \begin{array}{cc} a_1 & a_2 \\ \end{array}$$
$$A = \begin{bmatrix} +0.92 & +0.92 \\ -0.92 & +0.92 \\ +0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$$

The columns are linearly independent. Only $\emptyset$ weights would satisfy $a_1w_1 + a_2w_2 = \underline{0}$.

$$w_1 \begin{bmatrix} 0.92 \\ -0.92 \\ 0.92 \\ -0.92 \end{bmatrix} + w_2 \begin{bmatrix} 0.92 \\ 0.92 \\ -0.92 \\ -0.92 \end{bmatrix} = \begin{bmatrix} 0.92(w_1+w_2) \\ 0.92(w_2-w_1) \\ 0.92(w_1-w_2) \\ -0.92(w_1+w_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} w_1+w_2 \\ w_2-w_1 \\ w_1-w_2 \\ -w_1-w_2 \end{bmatrix} \begin{array}{l} \rightarrow w_1 = -w_2 \\ \rightarrow w_1 = w_2 \\ \rightarrow w_1 = w_2 \\ \rightarrow w_1 = -w_2 \end{array}$$

$-w_1$ would have to simultaneously equal w_2 and $-w_2$, which can only happen when $w_1 = w_2 = \emptyset$

b) Are the columns of the following matrix linearly independent?

$$A = \begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$
$$A = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$$

Yes, the columns are linearly independent. The only way to get $\underline{0} = a_1w_1 + a_2w_2 + a_3w_3$ is if the weights are 0.

$$a_1w_1 + a_2w_2 + a_3w_3 = \begin{bmatrix} w_1 + w_2 + w_3 \\ -w_1 + w_2 - w_3 \\ w_1 - w_2 - w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{rl} & w_1 + w_2 + w_3 = 0 \\ + & w_1 - w_2 - w_3 = 0 \\ \hline & 2w_1 = 0 \\ & w_1 = 0 \end{array}$$

$0 + w_2 - w_3 = 0$

$w_2 = w_3$

$0 + w_3 + w_3 = 0$

$2w_3 = 0$

$w_2 = w_3 = 0$

All weights are 0.

c) Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

No, the columns are not linearly independent. There are a set of weights that result in the zero vector that is not 0.

$w = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$

$$Aw = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 1+1-2 \\ 3+2-5 \\ 5+3-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

d) What is the rank of the following matrix?

$$A = \begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix} \end{array}$$

rank = num of linearly independent columns

rank$\{A\} = 2$

There are 2 linearly independent columns.

$$a_1 w_1 + a_2 w_2 = \begin{bmatrix} 5w_1 + 2w_2 \\ -5w_1 + 2w_2 \\ 5w_1 - 2w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 5w_1 + 2w_2 &= 0 \\ + \; -5w_1 + 2w_2 &= 0 \\ \hline 4w_2 &= 0 \\ w_2 &= 0 \end{aligned}$$

$$\begin{aligned} 5w_1 - 2(0) &= 0 \\ 5w_1 &= 0 \\ w_1 &= 0 \end{aligned}$$

weights are 0, so columns are independent.

e) Suppose the matrix in part d is used in to solve the system of linear equations $A^T A w = d$. Does a unique solution exist? Explain why.

rank$\{A\}$ = rank$\{A^T\}$ = rank$\{A^TA\}$ = 2.

The vector d must be linearly dependent on the columns of A^TA, since it is a linear combination of them. Thus rank$\{[A^TA \,\vdots\, d]\} = 2$.

A^TA is 2x2 so $\underline{w}$ must be 2x1 $\longrightarrow$ dim$\{w\} = 2$

Since rank$\{A^TA\}$ = rank$\{[A^TA \,\vdots\, d]\}$ = dim$\{w\}$,

There exists a unique solution.

2. *Norm additivity.* Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$.

a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_a + \|\boldsymbol{x}\|_b$ is also a norm on $\mathbb{R}^n$.

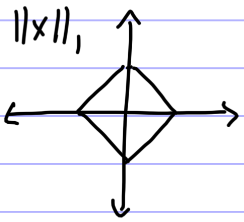
Conditions for a norm:
1. $\|\underline{x}\| \geq 0$ for all $\underline{x}$
2. $\|\underline{x}\| = 0$ if and only if $x = 0$
3. $\|b\underline{x}\| = |b|\,\|\underline{x}\|$ for all $b \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^N$
4. Triangle inequality $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

1. Since $\|\cdot\|_a$ and $\|\cdot\|_b$ are already norms, they are both ≥ 0. Adding two values that are ≥ 0 will give a result that is ≥ 0 so $f(x) \geq 0$.
2. $\|x\|_a$ and $\|x\|_b$ are norms on the same vector so when $x=0$, $\|x\|_a + \|x\|_b = 0 + 0 = f(x)$. So the condition holds
3. Let $c \in \mathbb{R}$ and $\|cx\|_a = |c|\,\|x\|_a$ and $\|cx\|_b = |c|\,\|x\|_b$
   Then $\|cx\|_a + \|cx\|_b = |c|\,\|x\|_a + |c|\,\|x\|_b$
   $= |c|\,(\|x\|_a + \|x\|_b)$
   Therefore $c f(x) = |c|(\|x\|_a + \|x\|_b)$ and the third condition holds.
4. $f(x) + f(y) = \|x\|_a + \|x\|_b + \|y\|_a + \|y\|_b$
   $f(x+y) = \|x+y\|_a + \|x+y\|_b$
   Since $\|x+y\|_a \leq \|x\|_a + \|x\|_b$ and
   $\|x+y\|_b \leq \|y\|_a + \|y\|_b$
   adding them will not change the inequality. Thus $f(x+y) \leq f(x) + f(y)$ and the last condition holds.

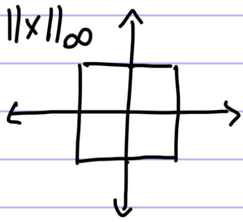
All conditions are satisfied with $f(x) = \|x\|_a + \|x\|_b$ so $f(x)$ is a norm on $\mathbb{R}^n$.

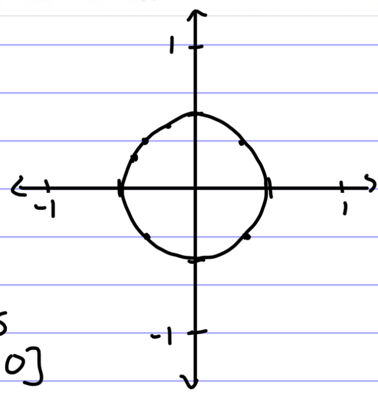
b) The "norm ball" is defined as the set of $\boldsymbol{x}$ for which an (arbitrary) norm $f(\boldsymbol{x}) = 1$. Sketch the norm ball in $\mathbb{R}^2$ for the norm $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

$\|x\|_1$



$\|x\|_\infty$





1 = sum of elements + max of elements

vectors on the norm ball = $[0.5, 0]$

$[\frac{1}{3}, \frac{1}{3}]$

$[0.4, 0.2]$