CS/ECE/ME532 Activity 6

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Estimated Time: 15 min for P1, 30 min for P2, 25 min for P3

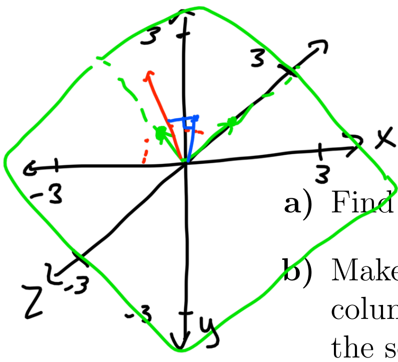
$$\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

1. Consider the following matrix and vector:

$$\hat{w} = (A^T A)^{-1} A^T d$$

$$\boldsymbol{A} = \overset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}, \quad \boldsymbol{d} = \overset{3\times 1}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}.$$

$$\hat{w} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$



a) Find the solution $\widehat{\boldsymbol{w}}$ to $\min_{\boldsymbol{w}} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{w}\|_2$.

b) Make a sketch of the geometry of this particular problem in $\mathbb{R}^3$, showing the columns of $\boldsymbol{A}$, the plane they span, the target vector $\boldsymbol{d}$, the residual vector and the solution $\widehat{\boldsymbol{d}} = \boldsymbol{A}\widehat{\boldsymbol{w}}$.

2. Recall the cereal calorie prediction problem discussed previously. Assume the data matrix for this problem is

$$\boldsymbol{A} = \begin{bmatrix} 25 & 0 & 1 \\ 20 & 1 & 2 \\ 40 & 1 & 6 \end{bmatrix}.$$

Each column contains the grams/serving of carbohydrates, fat, and protein, and each row corresponds to a different cereal (*Frosted Flakes, Froot Loops, Grape-Nuts*). The total calories per serving for each cereal are

$$\boldsymbol{b} = \begin{bmatrix} 110 \\ 110 \\ 210 \end{bmatrix}.$$

You will find it helpful to solve the problems numerically. Relevant Python commands include `numpy.linalg.inv` and `numpy.linalg.matrix_rank`.

$$x = \begin{bmatrix} 4.25 \\ 17.5 \\ 3.75 \end{bmatrix}$$

a) Write a small program that solves the system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Recall the solution $\boldsymbol{x}$ gives the calories/gram of carbohydrate, fat, or protein. What is the solution? The solution may not agree with the known calories/gram, which are 4 for carbs, 9 for fat and 4 for protein.

b) Now suppose that you use a more refined breakdown of carbohydrates, into total carbohydrates, complex carbohydrates and sugars (simple carbs). This gives 5 features to predict calories (the three carb features + fat and protein). The grams of these features in 5 different cereals is measured to obtain this data matrix

$$\boldsymbol{A} = \begin{bmatrix} 25 & 15 & 10 & 0 & 1 \\ 20 & 12 & 8 & 1 & 2 \\ 40 & 30 & 10 & 1 & 6 \\ 30 & 15 & 15 & 0 & 3 \\ 35 & 20 & 15 & 2 & 4 \end{bmatrix}.$$

Here the first column represents grams of total carbohydrates per serving, the second column complex carbohydrates, the third column sugars, the fourth column fat, and the fifth column protein. The total calories in a serving of each cereal are

$$\boldsymbol{b} = \begin{bmatrix} 104 \\ 97 \\ 193 \\ 132 \\ 174 \end{bmatrix}.$$

Consider the problem $\boldsymbol{Ax} = \boldsymbol{b}$ with 5 features.

i. Does an exact solution exist? Why or why not? *Hint:* The condition for an exact solution was studied in the previous period activity. Yes. $\text{Rank}\{A\} = \text{Rank}\{[A \vdots b]\}$ $4 = 4$

ii. Does a unique solution exist? Why or why not? *Hint:* The condition for a unique solution was studied in the previous period activity. No $\text{Rank}\{A\} < \dim(x)$ $4 < 5$

iii. Suppose you ignore (remove) the total carbohydrates per serving (first column of $\boldsymbol{A}$). Find a unique solution to the modified least-squares problem and the resulting squared error.

$(A^T A)^{-1} A^T b = x = \begin{bmatrix} 4 \\ 4 \\ 9 \\ 4 \end{bmatrix}$ error $= (Ax - b)^2 = 0$

3. Suppose the four-by-three matrix $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{W}^T$ where $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix}$ and $\boldsymbol{W}^T = \begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \end{bmatrix}$. Further, let $\boldsymbol{t}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$, $\boldsymbol{t}_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$, and $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

$T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ $W = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$

a) What is the rank of $\boldsymbol{A}$? $\text{rank}\{A\} = 2$

b) What is the dimension of the subspace spanned by the columns of $\boldsymbol{T}$?

dimension = 2

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -0.5 & 1.5 & 1.5 & -0.5 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$$

c) Is $\boldsymbol{Q} = \boldsymbol{A}^T \boldsymbol{A} \succ 0$?

(positive definite)

$$Q = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$ is not positive definite

d) Suppose $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$. Does the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ have a unique solution?

Rank$\{A\} = 2$ dim $x = 3$

$2 < 3$ so no unique solution

e) Suppose we force $\boldsymbol{x}$ to lie in the subspace spanned by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, that is, we constrain $\boldsymbol{x} = \boldsymbol{W}\tilde{\boldsymbol{x}}$ where $\tilde{\boldsymbol{x}}$ is a two-by-one vector. Does the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ have a unique solution for $\tilde{\boldsymbol{x}}$? Find at least one solution. Note that the numbers are chosen in this problem so you can easily do the calculations on paper.

$\tilde{x} = ((AW)^T (AW))\ (AW)^T b$

There is a unique solution as Rank$\{A\} = \dim \tilde{x}$

$$AW = \begin{bmatrix} 1.5 & -3 \\ 1.5 & -3 \\ 1.5 & -3 \\ 1.5 & -3 \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} 1 \\ 0.1667 \end{bmatrix}$$