CS/ECE/ME532 Activity 4

Estimated Time: 15 min for P1, 10 min for P2, 15 min for P3

1) *Matrix Rank.* Let $\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) What is the rank of $\boldsymbol{X}$? Rank = 2

b) Find a set of linearly independent columns in $\boldsymbol{X}$. Is there more than one set? How many sets of linearly independent columns can you find?

$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ $\quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ $\quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}$

c) A matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & -1 \end{bmatrix}$. Find the relationship between b and a so that rank$\{\boldsymbol{A}\} = 2$. *Hint:* find a, b so that the third column is a weighted sum of the first two columns. Note that there are many choices for a, b that result in rank 2.

$c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\-1\end{bmatrix}$ $\quad \begin{bmatrix}c_1\\c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\c_2\end{bmatrix} = \begin{bmatrix}a\\b\\-1\end{bmatrix}$ $\quad a = 2$, $b = 1$

2) *Solution Existence.* A system of linear equations is given by $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Suppose $\boldsymbol{b} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

$\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}8\\6\\-2\end{bmatrix}$

$\begin{bmatrix}x_1\\x_1+x_2\\x_2\end{bmatrix} = \begin{bmatrix}8\\6\\-2\end{bmatrix} = \begin{bmatrix}8\\8-2\\-2\end{bmatrix}$ $\quad x = \begin{bmatrix}8\\-2\end{bmatrix}$

b) Suppose $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

$\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4\\6\\1\end{bmatrix}$

$\begin{bmatrix}x_1\\x_1+x_2\\x_2\end{bmatrix} = \begin{bmatrix}4\\6\\1\end{bmatrix}$ $\quad \begin{bmatrix}4\\4+1\\1\end{bmatrix} = \begin{bmatrix}4\\5\\1\end{bmatrix}$ No solution exists.

c) Consider the general system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$. This equation says that $\boldsymbol{b}$ is a weighted sum of the columns of $\boldsymbol{A}$. Assume $\boldsymbol{A}$ is full rank. Use the definition of linear independence to find the condition on rank$\{[\ \boldsymbol{A}\ \ \boldsymbol{b}\ ]\}$ that guarantees a solution exists.

Rank$\{A\}$ = Rank$\{[A \vdots b]\}$

b must be linearly dependent on the matrix A for it to be a weighted sum of the columns of A.

3) *Non Unique Solutions.*

a) Consider $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

i) Does this system of equations have a solution? Justify your answer.

ii) Is the solution unique? Justify your answer.

iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

b) If the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on rank$\{\boldsymbol{A}\}$ that determines whether there is more than one solution.

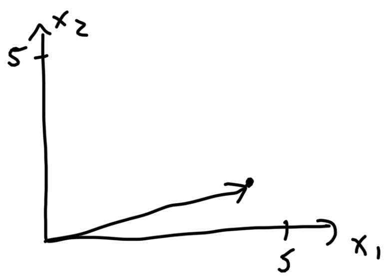
a) i. $\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$

$$\begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 2x_2 \\ -2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 - 2(1) \\ -4 + 2(1) \\ -2(-4) + 4(1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$$

Solution: $x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

ii. The solution is unique. Rank$\{A\} = 2$ which is the dimension of the matrix. (aka. num of columns)

iii.



b) $A(x+w) = b$

$Ax + Aw = b$

$\underbrace{(Ax - b)}_{Ax = b} + Aw = 0$

$Aw = 0$

Non unique if columns of A are linearly dependent so if rank$\{A\}$ < dim(A)