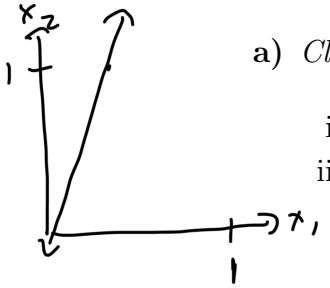
CS/ECE/ME532 Activity 8

Estimated Time: 15 min for P1, 15 min for P2, 20 min for P3, 5 min for P4, 15 min for P5

1. **Binary linear classifiers.** Assume there are two possible labels, $y = 1$ or $y = -1$ associated with two features x_1 and x_2. We consider several different linear classifiers $\hat{y} = \text{sign}\{\boldsymbol{x}^T\boldsymbol{w}\}$ where $\boldsymbol{x}$ is derived from x_1 and x_2 and $\boldsymbol{w}$ are the classifier weights. Define the decision boundary of the classifier as the set $\{x_1, x_2\}$ for which $\boldsymbol{x}^T\boldsymbol{w} = 0$. Let x_2 be the vertical axis in your sketches and depict the interval $0 \le x_1, x_2 \le 1$.

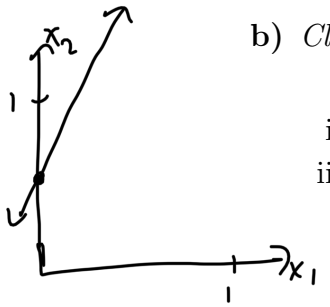


   a) *Classifier 1.* Let $\boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and assume $\boldsymbol{w} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

      i. Sketch the decision boundary in the x_1-x_2 plane. $0 = 5x_1 - 2x_2 \quad x_2 = \frac{5}{2}x_1$

      ii. Does the decision boundary represent a subspace in $\mathbb{R}^2$? Why or why not? If it represents a subspace, then find an orthonormal basis for the subspace.

      yes, it is a line that goes through the origin. basis: $\begin{bmatrix} \frac{5}{\sqrt{29}} \\ \frac{-2}{\sqrt{29}} \end{bmatrix}$

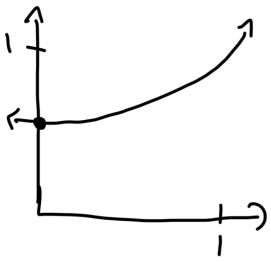


   b) *Classifier 2.* Let $\boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 & 1 \end{bmatrix}$ and assume $\boldsymbol{w} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

      i. Sketch the decision boundary in the x_1-x_2 plane. $0 = 5x_1 - 2x_2 + 1 \quad x_2 = \frac{5}{2}x_1 + \frac{1}{2}$

      ii. Does the decision boundary represent a subspace in $\mathbb{R}^2$? Why or why not? If it represents a subspace, then find an orthonormal basis for the subspace.

      No, the origin is not in the span of the decision boundary.



   c) *Classifier 3.* Let $\boldsymbol{x}^T = \begin{bmatrix} x_1^2 & x_2 & 1 \end{bmatrix}$ and assume $\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

      i. Sketch the decision boundary in the x_1-x_2 plane. $0 = x_1^2 - 2x_2 + 1 \quad x_2 = \frac{x_1^2}{2} + \frac{1}{2}$

      ii. Does the decision boundary represent a subspace in $\mathbb{R}^2$? Why or why not? If it represents a subspace, then find an orthonormal basis for the subspace.

      Not a subspace, the origin is not a point on the line.

2. *Linear Classifier.* Download the script and the data file `classifier_data.mat`. This code trains linear classifiers using least squares. The scripts provided steps through the problems below.

   Make sure to 'publish' your results and store them as a PDF file for submission.

   a) *Classifier 1.* Let $\boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. Briefly comment on the fit of the classifier to the decision boundary apparent in the evaluation data. Also identify the percent error based on the ratio of misclassified evaluation data points to the total number of evaluation data points.

      misclassed points: 1102

      total points: 10000

      ratio: $\frac{1102}{10000} = 0.1102$

      The classifier is linear, however the actual boundary is more like a curve.

The classifier is now more similar to the curve in the eval data, just a little shorter. Misclassifications occur in the gap where the classifier is a little short.

b) *Classifier 2.* Consider squaring the original features, and also using them for classification, so that: $\boldsymbol{x}^T = \begin{bmatrix} x_1^2 & x_2^2 & x_1 & x_2 & 1 \end{bmatrix}$. This will allow for a curved decision boundary. Briefly comment on the fit of the classifier to the decision boundary apparent in the evaluation data. Also identify the percent error based on the ratio of misclassified evaluation data points to the total number of evaluation data points. Misclass: 542 total: 10000 ratio: $\frac{542}{10000} = 0.0542$

c) *Shortcoming of training using least squares as a loss function.* Training a classifier using the squared error as a loss function can fail when correctly labeled data points lie far from the decison boundary. A new dataset consisting of the first dataset, plus 1000 (correctly labeled) datapoints at $x_1 = 0, x_2 = 3$ is created. What happens to the decision boundary when these new data points are included in training? What happens to the error rate if you move the 1000 data points to $x_1 = 0, x_2 = 10$? Why does this happen? The classifier is much steeper than the actual boundary. As the data points get farther from the rest of the data, the boundary continues to get steeper.

The classifier takes into account all points and finds an average for all points. Thus a lot of extreme outliers will pull the line towards the outliers and create a boundary with higher error.

3. *Overfitting.* Download the dataset `overfitting_data.mat`. You may find it helpful to adapt the code from the previous problem. The dataset has 50 data points for training, and 10,000 data points to be used for evaluation of the classifier. Each data point consists a two-dimensional feature vector $\boldsymbol{x}$ and a label $y \in \{-1, 1\}$. The feature vector is a "noisy" version of the true underlying feature, which blurs the boundary between classes.

a) Plot the training data using a scatter plot. Indicate the points $y = -1$ using one color, and the points with $y = 1$ with another.

b) Plot the evaluation data using a scatter plot. Indicate the points labeled -1 using one color, and the points labeled $+1$ with another.

c) Classifier 1. As before, $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, and $y = \text{sign}(\boldsymbol{x}^T \boldsymbol{w})$.

i. Train the classifier using least squares to find the classifier weights $\boldsymbol{w}$. Apply the classifier to the evaluation data, and plot the data points using a scatter plot with different colors for different predicted labels.

ii. Plot the correctly predicted evaluation data points using one color, and incorrectly predicted points using a second color. How many errors are there?

errors: 759

d) Classifier 2. Let $\boldsymbol{x} = \begin{bmatrix} x_1^2 & x_2^2 & x_1 & x_2 & 1 \end{bmatrix}^T$, and $y = \text{sign}(\boldsymbol{x}^T \boldsymbol{w})$.

i. Train the classifier using least squares to find the classifier weights $\boldsymbol{w}$. Apply the classifier to the evaluation data, and plot the data points using a scatter plot with different colors for different predicted labels.

ii. Plot the correctly predicted evaluation data points using one color, and incorrectly predicted points using a second color. How many errors are there?

errors: 1066

e) Classifier 3. Let $\boldsymbol{x} = \begin{bmatrix} x_1^6 & x_2^6 & x_1^5 & x_2^5 & \dots & x_1 & x_2 & 1 \end{bmatrix}^T$, and $y = \text{sign}(\boldsymbol{x}^T\boldsymbol{w})$.

i. Train the classifier using least squares to find the classifier weights $\boldsymbol{w}$. Apply the classifier to the evaluation data, and plot the data points using a scatter plot with different colors for different predicted labels.

ii. Plot the correctly predicted evaluation data points using one color, and incorrectly predicted points using a second color. How many errors are there?

errors: 1677

f) Of the three classifiers, which one performs worse, and why?

In order of performance: 1 → 2 → 3

The complex classifier may represent the 50 training data well, but there is only a small sample and the actual population does not reflect the complexity found by classifier 3.

4. A binary linear classifier based on three features x_1, x_2, and x_3 is $\hat{y} = \text{sign}\{\boldsymbol{x}^T\boldsymbol{w}\}$ where $\boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$. Hence the decision boundary is the set $\{x_1, x_2, x_3\}$ for which $\boldsymbol{x}^T\boldsymbol{w} = 0$.

The decision boundary for a two-dimensional classifier is a line. What type of geometric object is the decision boundary in three dimensions?

A plane.

5. A decision boundary for a classification problem involving features x_1, x_2, and x_3 is defined as $\boldsymbol{x}^T\boldsymbol{w} = 0$ where $\boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix}$. Find $\boldsymbol{w}$ so that the decision boundary is parallel to the x_1-x_2 plane and includes the point $(x_1, x_2, x_3) = (0, 0, 1)$.

$$x^T w = \begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = 0$$

$$w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$0 = w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4$$

$$0 = w_1(0) + w_2(0) + w_3(1) + w_4$$

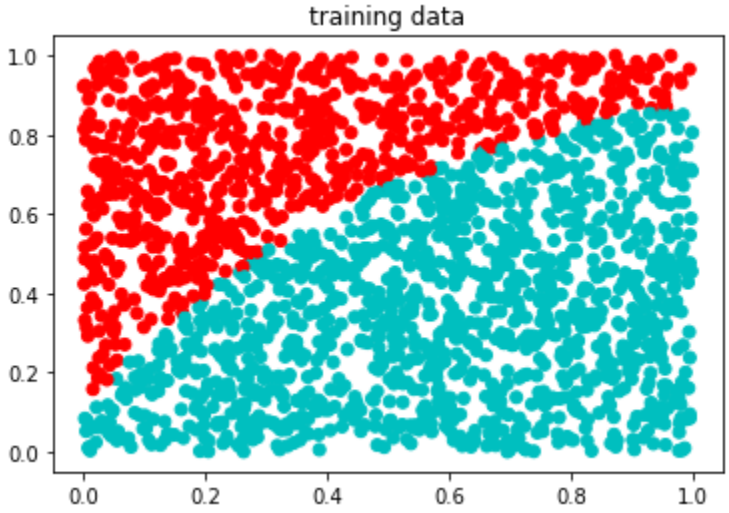
$$0 = w_3 + w_4$$

$$w_4 = -w_3$$

2a)

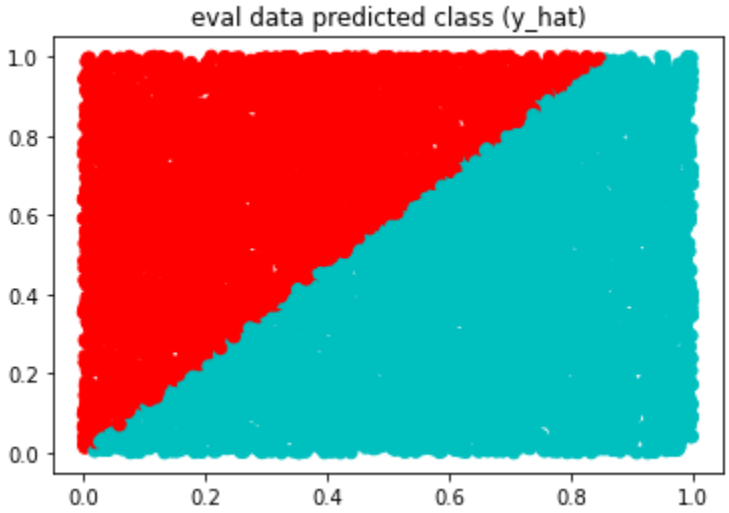
```
from google.colab import drive
#drive.mount('/content/drive')
import numpy as np
from scipy.io import loadmat
import matplotlib.pyplot as plt

in_data = loadmat('/content/drive/My Drive/Colab Notebooks/classifier_data.mat')
#print([key for key in in_data]) # -- use this line to see the keys in the dictionary data stru

x_train = in_data['x_train']
x_eval = in_data['x_eval']
y_train = in_data['y_train']
y_eval = in_data['y_eval']

n_eval = np.size(y_eval)
n_train = np.size(y_train)

plt.scatter(x_train[:,0],x_train[:,1], color=['c' if i==-1 else 'r' for i in y_train[:,0]])
plt.title('training data')
plt.show()
```

```
Mounted at /content/drive
```

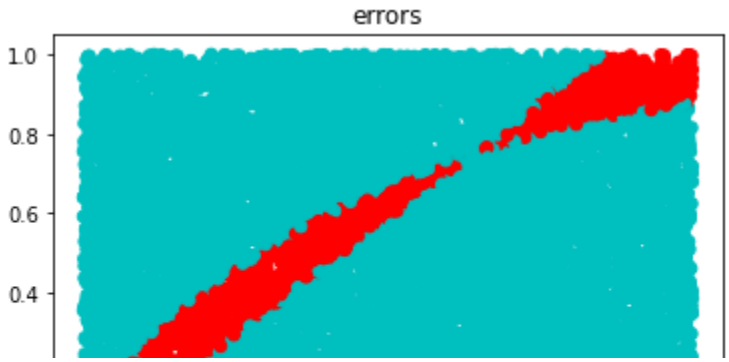



```
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_eval[:,0]])
plt.title('eval data true class')
plt.show()
```


```
## Classifier 1

# w = (X^T X)^(-1)X^T y
w_opt = np.linalg.inv(x_train.transpose()@x_train)@x_train.transpose()@y_train
y_hat = np.sign(x_eval@w_opt)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat[:,0]])
plt.title('eval data predicted class (y_hat)')
plt.show()
```

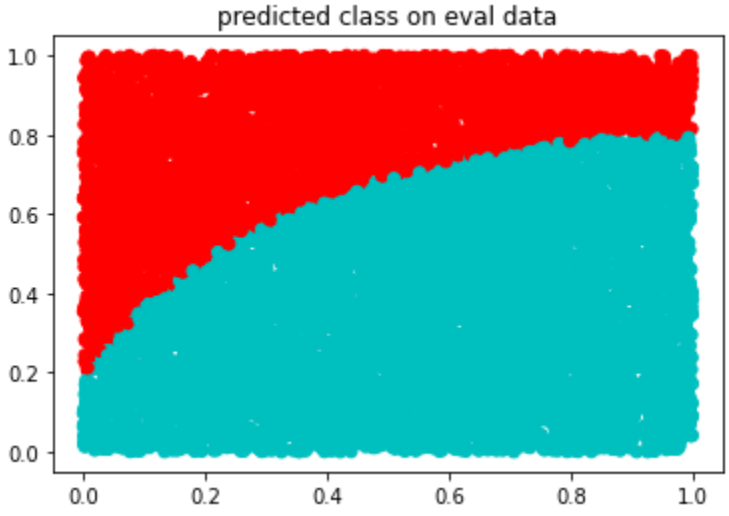



```
error_vec = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec])
plt.title('errors')
plt.show()

print('Errors: '+ str(sum(error_vec)))
print(len(x_eval))
print(sum(error_vec)/len(x_eval))
```

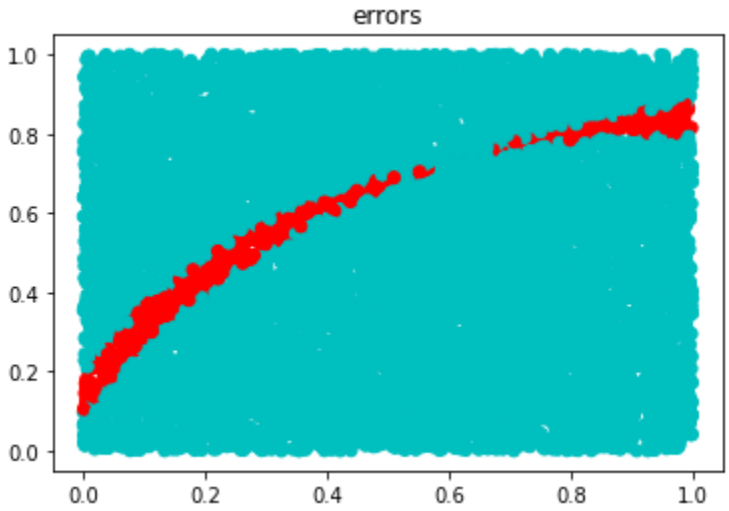


2b)

```
## Classifier 2
x_train_2 = np.hstack((x_train**2, x_train, np.ones((n_train,1)) ))
x_eval_2 = np.hstack((x_eval**2, x_eval, np.ones((n_eval,1)) ))

w_opt_2 = np.linalg.inv(x_train_2.transpose()@x_train_2)@x_train_2.transpose()@y_train
y_hat_2 = np.sign(x_eval_2@w_opt_2)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat_2[:,0]])
plt.title('predicted class on eval data')
plt.show()
```



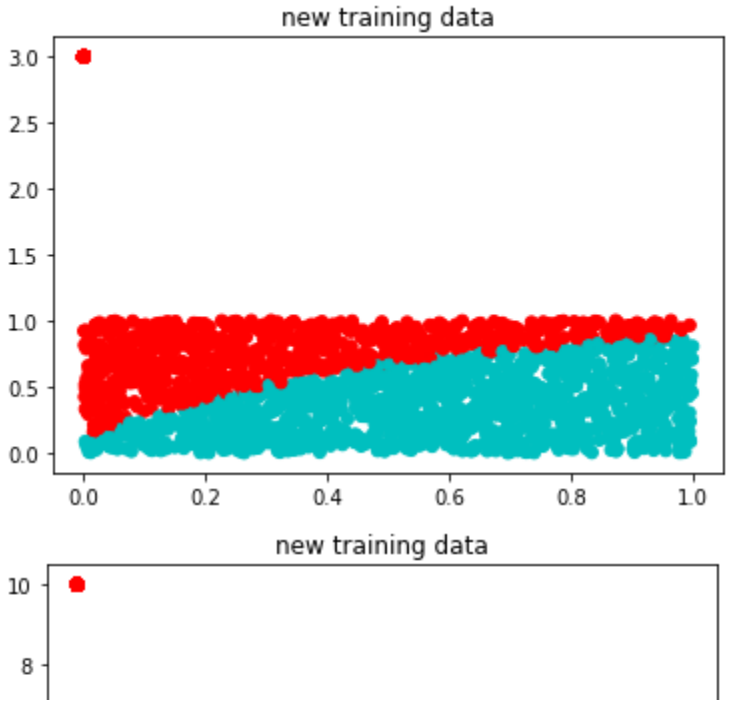

```
error_vec_2 = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat_2, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec_2])
plt.title('errors')
plt.show()

print('Error: '+ str(sum(error_vec_2)))
print(len(x_eval))
print(sum(error_vec_2)/len(x_eval))
```



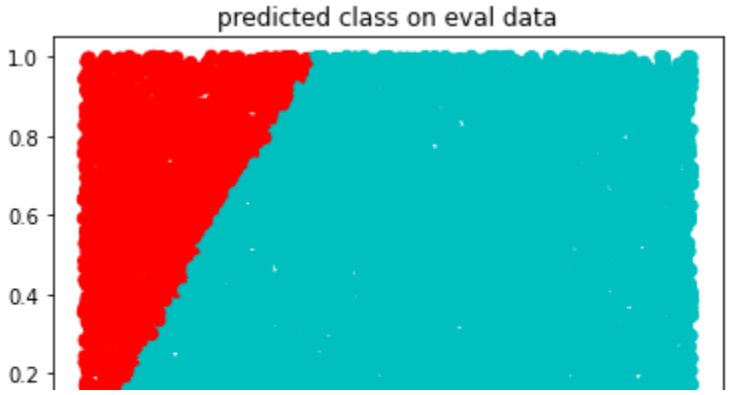
Error: 542

2c)

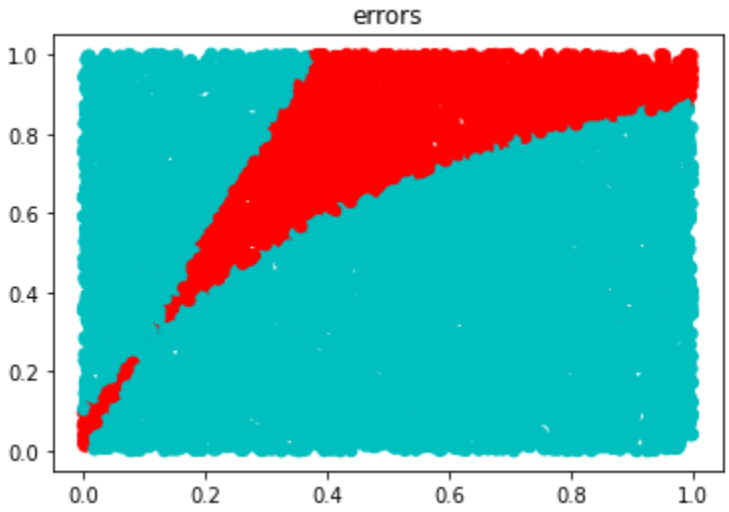
```
## create new, correctly labeled points
n_new = 1000 #number of new datapoints
x_train_new = np.hstack((np.zeros((n_new,1)), 3*np.ones((n_new,1))))
y_train_new = np.ones((n_new,1))

## add these to the training data
x_train_outlier = np.vstack((x_train,x_train_new))
y_train_outlier = np.vstack((y_train,y_train_new))
plt.scatter(x_train_outlier[:,0],x_train_outlier[:,1], color=['c' if i==-1 else 'r' for i in y_
plt.title('new training data')
plt.show()

# x2=10
n_new = 1000 #number of new datapoints
x_train_new_2 = np.hstack((np.zeros((n_new,1)), 10*np.ones((n_new,1))))
y_train_new_2 = np.ones((n_new,1))

## add these to the training data
x_train_outlier_2 = np.vstack((x_train,x_train_new_2))
y_train_outlier_2 = np.vstack((y_train,y_train_new_2))
plt.scatter(x_train_outlier_2[:,0],x_train_outlier_2[:,1], color=['c' if i==-1 else 'r' for i i
plt.title('new training data')
plt.show()
```




```
#train with new data
w_opt_outlier = np.linalg.inv(x_train_outlier.transpose()@x_train_outlier)@x_train_outlier.tra
y_hat_outlier = np.sign(x_eval@w_opt_outlier)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat_outlier[:,0]]
plt.title('predicted class on eval data')
plt.show()

# x2=10
w_opt_outlier_2 = np.linalg.inv(x_train_outlier_2.transpose()@x_train_outlier_2)@x_train_outli
y_hat_outlier_2 = np.sign(x_eval@w_opt_outlier_2)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat_outlier_2[:,0
plt.title('predicted class on eval data')
plt.show()
```



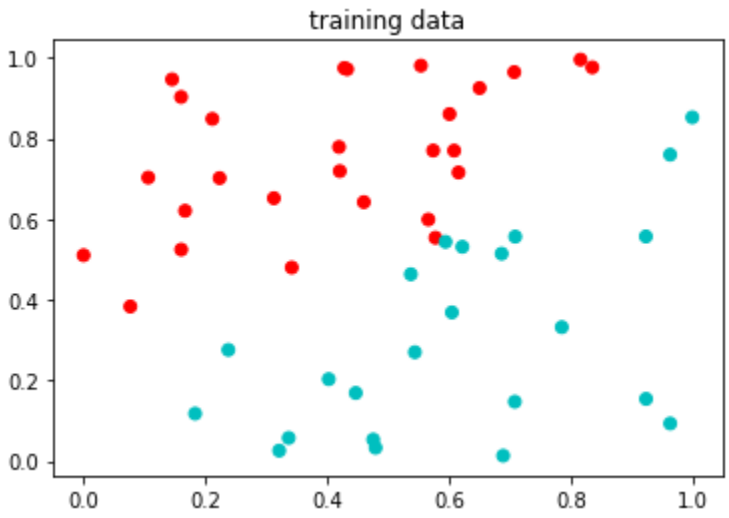

```
error_vec = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat_outlier, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec])
plt.title('errors')
plt.show()

print('Errors: '+ str(sum(error_vec)))
```




```
Errors: 2134
```

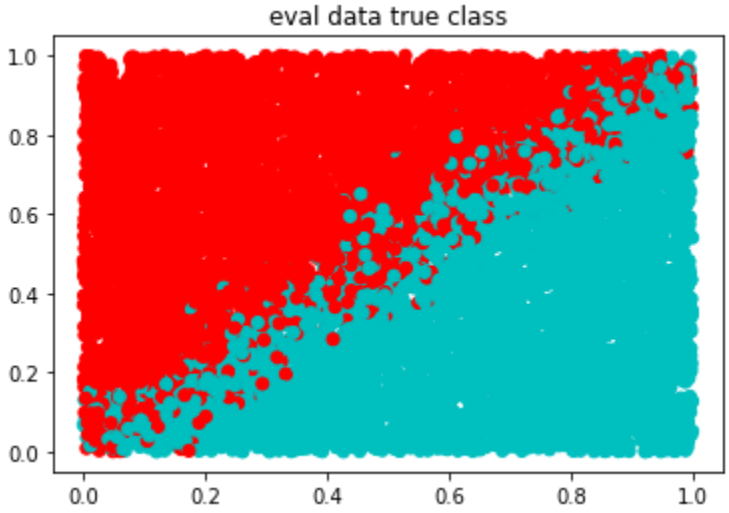
3a)

```
in_data = loadmat('/content/drive/My Drive/Colab Notebooks/overfitting_data.mat')
#print([key for key in in_data]) # -- use this line to see the keys in the dictionary data stru

x_train = in_data['x_train']
x_eval = in_data['x_eval']
y_train = in_data['y_train']
y_eval = in_data['y_eval']

n_eval = np.size(y_eval)
n_train = np.size(y_train)

plt.scatter(x_train[:,0],x_train[:,1], color=['c' if i==-1 else 'r' for i in y_train[:,0]])
plt.title('training data')
plt.show()
```



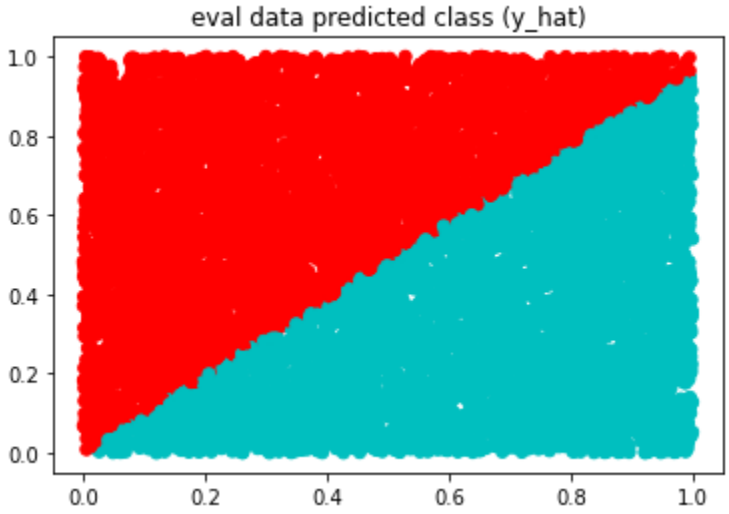
3b)

```
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_eval[:,0]])
plt.title('eval data true class')
plt.show()
```

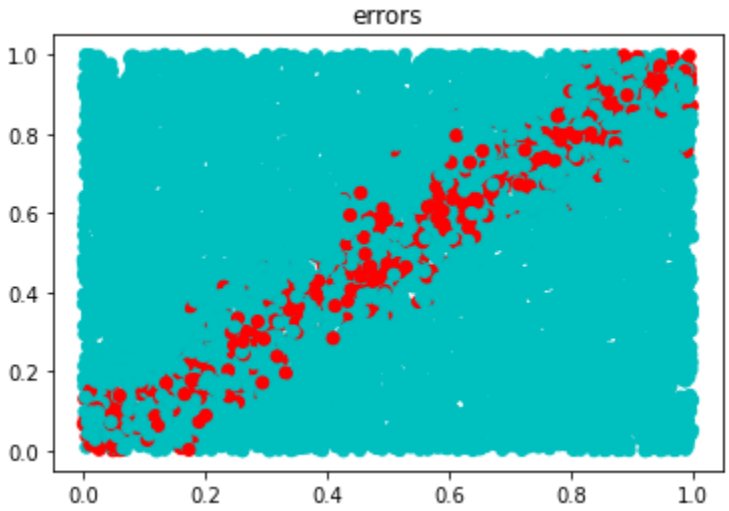


3c)

```
# Classifier 1

# w = (X^T X)^(-1)X^T y
w_opt = np.linalg.inv(x_train.transpose()@x_train)@x_train.transpose()@y_train
y_hat = np.sign(x_eval@w_opt)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat[:,0]])
plt.title('eval data predicted class (y_hat)')
plt.show()
```




```
# errors

error_vec = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec])
plt.title('errors')
plt.show()

print('Errors: '+ str(sum(error_vec)))
```

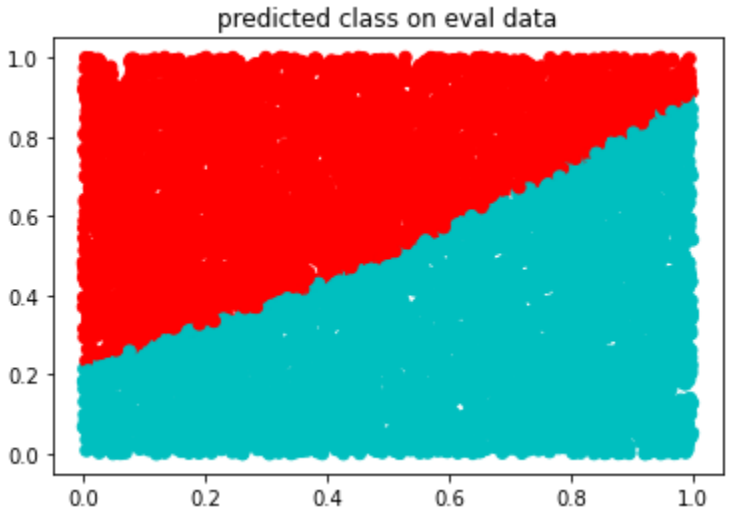



```
Errors: 759
```

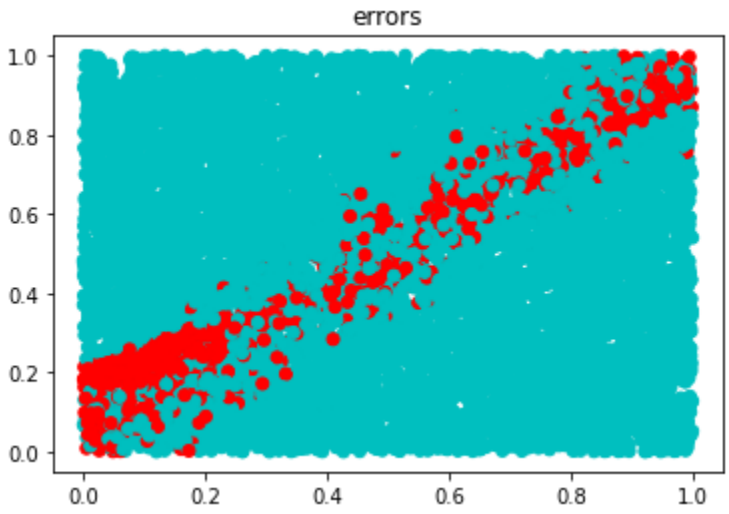
3d)

```
# Classifier 2

x_train_2 = np.hstack((x_train**2, x_train, np.ones((n_train,1)) ))
x_eval_2 = np.hstack((x_eval**2, x_eval, np.ones((n_eval,1)) ))

w_opt_2 = np.linalg.inv(x_train_2.transpose()@x_train_2)@x_train_2.transpose()@y_train
y_hat_2 = np.sign(x_eval_2@w_opt_2)
```

```
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat_2[:,0]])
plt.title('predicted class on eval data')
plt.show()
```




```
# errors

error_vec_2 = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat_2, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec_2])
plt.title('errors')
plt.show()

print('Error: '+ str(sum(error_vec_2)))
```

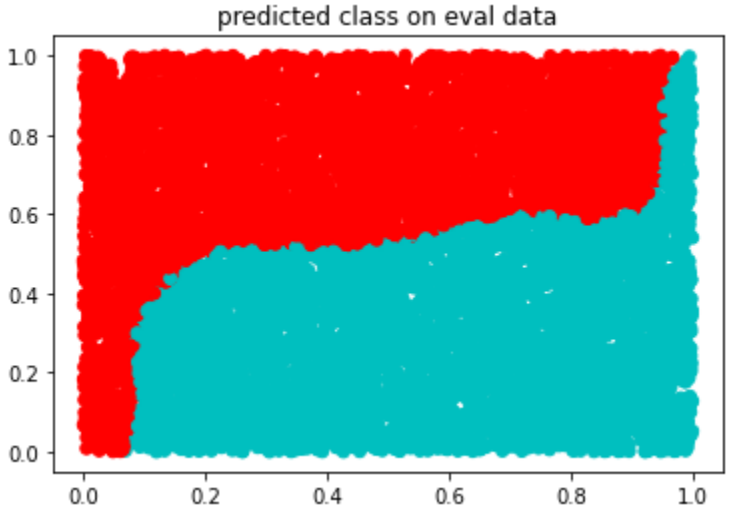



```
Error: 1066
```

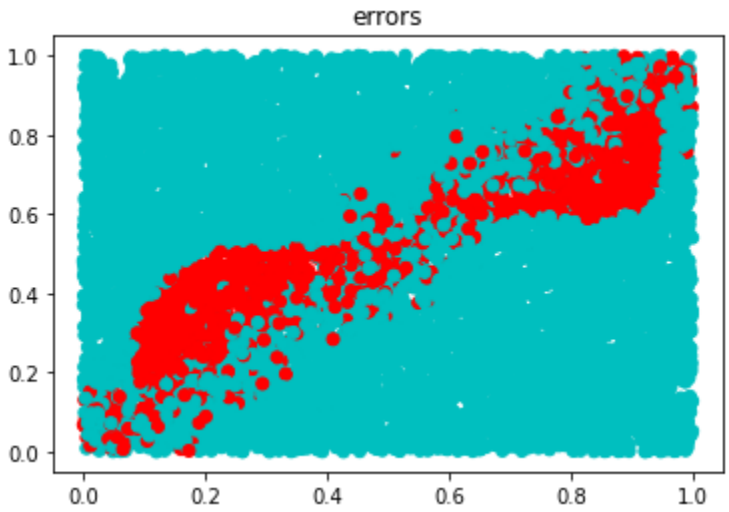
3e)

```
# Classifier 3

x_train_3 = np.hstack((x_train**6, x_train**5, x_train**4, x_train**3, x_train**2, x_train, np
x_eval_3 = np.hstack((x_eval**6, x_eval**5, x_eval**4, x_eval**3, x_eval**2., x_eval, np.ones(
```

```
w_opt_3 = np.linalg.inv(x_train_3.transpose()@x_train_3)@x_train_3.transpose()@y_train
y_hat_3 = np.sign(x_eval_3@w_opt_3)

plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==-1 else 'r' for i in y_hat_3[:,0]])
plt.title('predicted class on eval data')
plt.show()
```




```
# errors

error_vec_3 = [0 if i[0]==i[1] else 1 for i in np.hstack((y_hat_3, y_eval))]
plt.scatter(x_eval[:,0],x_eval[:,1], color=['c' if i==0 else 'r' for i in error_vec_3])
plt.title('errors')
plt.show()

print('Error: '+ str(sum(error_vec_3)))
```




```
Error: 1677
```

Colab paid products - Cancel contracts here

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