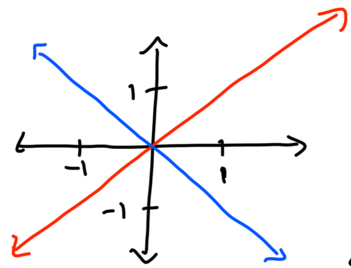
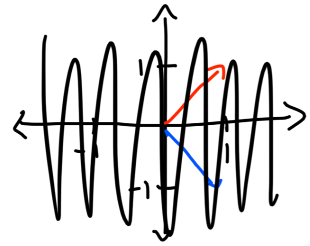
CS/ECE/ME532 Classroom Activity

1. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.



a) Sketch the subspace spanned by $\boldsymbol{z}$ in $\mathbb{R}^2$.

b) Sketch the subspace spanned by $\boldsymbol{w}$ in $\mathbb{R}^2$.



c) Sketch span $\{\boldsymbol{z}, \boldsymbol{w}\}$ in $\mathbb{R}^2$.

d) Are $\boldsymbol{z}$ and $\boldsymbol{w}$ orthogonal? Why or why not? $z^T w = [1 \ \ 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$ they are orthogonal

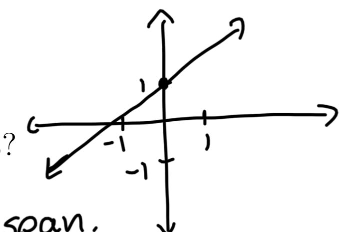
e) Do $\{\boldsymbol{z}, \boldsymbol{w}\}$ form an orthonormal basis? Why or why not? If not, can you modify $\boldsymbol{z}$ and $\boldsymbol{w}$ to form an orthonormal basis?

$||z|| = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\quad z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$||w|| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ $\quad w = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. Consider the line in $\mathbb{R}^2$ defined by the equation $x_2 = x_1 + 1$.

a) Sketch the line in $\mathbb{R}^2$.



b) Does this line define a subspace of $\mathbb{R}^2$? Why or why not?

This is not a subspace because the point (0, 0) is not part of the span.

3. You collect ratings of three space-related science fiction movies and two romance movies from seven friends on a scale of 1-10.

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita
Star Trek	4	7	2	8	7	4	2
Pride and Prejudice	9	3	5	6	10	5	5
The Martian	4	8	3	7	6	4	1
Sense and Sensibility	9	2	6	5	9	5	4
Star Wars: Empire Strikes	4	9	2	8	7	4	1

You put this data into a matrix $\boldsymbol{X}$ (available in the file `movie.mat`) and decide to model (approximate) as the product of a rank-r taste matrix with orthonormal columns and a weight matrix. That is, $\boldsymbol{X} \approx \boldsymbol{T}\boldsymbol{W}$.

a) What is the rank of $\boldsymbol{X}$? Relevant Python commands are `numpy.linalg.matrix_rank()`. 5

b) What are the dimensions of $\boldsymbol{T}$ and $\boldsymbol{W}$ (in terms of r)? $T = 5 \times r$

X needs to be 5×7 $\quad W = r \times 7$

$\frac{30}{5}$ $\frac{29}{5}$ $\frac{18}{5}$ $\frac{34}{5}$ $\frac{39}{5}$ $\frac{22}{5}$ $\frac{13}{5}$

c) You know that each user's ratings have an average value that is greater than zero because the scale is 1-10. And you suspect the baseline (average) rating may differ from user to user. To account for this you decide your first basis vector in the taste matrix should be

$$\boldsymbol{t}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$X_{:,j} = t_1 w_{1j} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} w_{1j}$$

Choose w_{1j} so that each element of the vector $\boldsymbol{t}_1 w_{1j}$ equals the average value j^{th} column of $\boldsymbol{X}$, denoted as $\boldsymbol{X}_{:,j}$. Find an expression for w_{1j} that depends on $\boldsymbol{t}_1$ and $\boldsymbol{X}_{:,j}$.

$$w_{1j} = \sqrt{5}\, X_{:,j} = \frac{1}{t_1} X_{:,j}$$

d) Define $\boldsymbol{w}_1^T = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17} \end{bmatrix}$ and find the rank-1 approximation to $\boldsymbol{X}$ that reflects the baseline ratings of each friend, $\boldsymbol{t}_1 \boldsymbol{w}_1^T$.

$$w_1^T = \begin{bmatrix} \frac{30}{\sqrt{5}} & \frac{29}{\sqrt{5}} & \frac{18}{\sqrt{5}} & \frac{34}{\sqrt{5}} & \frac{39}{\sqrt{5}} & \frac{22}{\sqrt{5}} & \frac{13}{\sqrt{5}} \end{bmatrix}$$

e) Which friend has the highest baseline rating? Which friend has the lowest baseline rating? highest: Ioan lowest: Juanita

f) Find the residual not modeled by $\boldsymbol{t}_1 \boldsymbol{w}_1^T$, that is, $\boldsymbol{X} - \boldsymbol{t}_1 \boldsymbol{w}_1^T$. Do you see any patterns in the residual? Briefly describe them qualitatively.

This problem is continued in a homework assignment.

$$X - t_1 w_1^T = \begin{bmatrix} 4 & 7 & 2 & 8 & 7 & 4 & 2 \\ 9 & 3 & 5 & 6 & 10 & 5 & 5 \\ 4 & 8 & 3 & 7 & 6 & 4 & 1 \\ 9 & 2 & 6 & 5 & 9 & 5 & 4 \\ 4 & 9 & 2 & 8 & 7 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1.2 & -1.6 & 1.2 & -0.8 & -0.4 & -0.6 \\ 3 & -2.8 & 1.4 & -0.8 & 2.2 & 0.6 & 2.4 \\ -2 & 2.2 & -0.6 & 0.2 & -1.8 & -0.4 & -1.6 \\ 3 & -3.8 & 2.4 & -1.8 & 1.2 & 0.6 & 1.4 \\ -2 & 3.2 & -1.6 & 1.2 & -0.8 & -0.4 & -1.6 \end{bmatrix}$$

The signs alternate in each column. So if started with negative then the next value in the column will be positive. So this model alternates between underestimating and overestimating ratings.