$$\text{proj}_{u_1} x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

CS/ECE/ME532 Activity 7

Estimated Time: 15 min for each problem

$$\tilde{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \tilde{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_{u_1} x_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix}$$

1. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a) Use the Gram-Schmidt orthogonalization procedure and hand calculation to find an orthonormal basis for the space spanned by the columns of $\boldsymbol{X}$. What geometric object is described by the span of these bases?

This is a plane.

$$\text{basis} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix}$$

b) Now interchange the columns of $\boldsymbol{X}$, that is, define $\tilde{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

i. Do the columns of $\boldsymbol{X}$ span the same space as the columns of $\tilde{\boldsymbol{X}}$? Yes

ii. Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\boldsymbol{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a? The projection is still a plane.

$$\text{proj}_{u_1} x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

iii. Are the bases vectors you found for $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ the same? Does the space spanned by the columns of a matrix depend on the order of the columns?

$$\tilde{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{\frac{3}{2}}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \qquad \text{basis} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{\frac{3}{2}}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

The bases are different, but the space spanned should be the same.

2. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the previous problem.

a) Place the orthonormal bases you found as columns of a matrix $\boldsymbol{U}$. $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix}$

b) Find $\boldsymbol{U}^T\boldsymbol{U}$. $= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) Since $\boldsymbol{U}$ contains a basis for space spanned by the columns of $\boldsymbol{X}$ you decide to write each column of $\boldsymbol{X}$ as a linear combination of the columns of $\boldsymbol{U}$: $\boldsymbol{X} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$. What is the dimension of $\boldsymbol{a}_1$? Briefly describe the meaning of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

X is 3×2
U is 3×2
$[a_1; a_2]$ is 2×2
a_1 is 2×1

a_1 will only multiply with u_1, a_2 with u_2

a_1 and a_2 will scale the columns of U to the columns of X

d) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$ so that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{A}$. Multiply both sides of this equation by $\boldsymbol{U}^T$ and solve for $\boldsymbol{A}$.

$$U^T X = U^T U A$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{\frac{3}{2}}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \qquad A = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

b) $X(X^TX)^{-1}X^T = UT((UT)^T UT)^{-1}(UT)^T$

$= UT(UT)^{-1}((UT)^T)^{-1}(UT)^T$

$= U\cancel{T}\,\cancel{T^{-1}}U^{-1}(U^T)^{-1}\cancel{(T^T)^{-1}}\,\cancel{T^T}U^T$

$= UU^{-1}(U^T)^{-1}U^T$

$\underline{U(U^TU)^{-1}U^T}$

3. Let the columns of an n-by-p ($n > p$) matrix $\boldsymbol{X}$ be linearly independent and $\boldsymbol{U}$ be an orthonormal basis for the p-dimensional space spanned by the columns of $\boldsymbol{X}$.

$x_i = Ut_i$

a) It can be shown that $\boldsymbol{X} = \boldsymbol{UT}$ where $\boldsymbol{T}$ is a p-by-p invertible matrix. Briefly explain why $\boldsymbol{T}$ should be invertible without resorting to a mathematical proof. That is, explain why this result is intuitively reasonable. X is linearly independent, so it has rank P, multiplying matrices have to have the same rank, so U and T have rank p. Then T is PxP with rank P, so it is invertible.

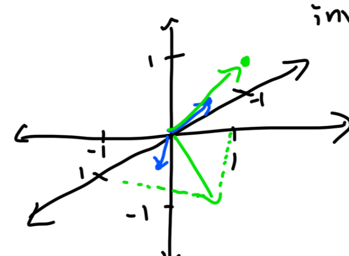
b) Use the result in the previous item to show that the projection onto the space spanned by $\boldsymbol{X}$ is identical to that onto the space spanned by $\boldsymbol{U}$. That is, show $\boldsymbol{P}_x = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}_U = \boldsymbol{U}(\boldsymbol{U}^T\boldsymbol{U})^{-1}\boldsymbol{U}^T$. *Hint:* Recall that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

c) Express $\boldsymbol{P}_U$ without a matrix inverse. $U^TU = I$

$P_u = U(I)^{-1}U^T = UU^T$

4. Consider the matrix and vector

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3/2}} \end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$



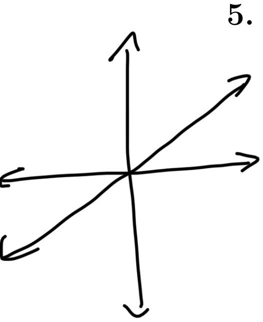
Note that $\boldsymbol{X}$ is defined identically in the preceding problems.

a) Make a sketch of the orthonormal bases $\boldsymbol{U}$ and the columns of $\boldsymbol{X}$ in three dimensions.

b) Use $\boldsymbol{U}$ and the result of the previous problem to compute the LS estimate $\hat{\boldsymbol{b}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{b}$.

$X(X^TX)^{-1}X^T = UU^T$ $\quad \hat{b} = UU^Tb = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3/2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3/2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

5. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and define $\boldsymbol{Q} = \boldsymbol{z}\boldsymbol{z}^T$. $= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



a) Sketch the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

$y = [x_1, x_2]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1+x_2 \quad x_1+x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1(x_1+x_2) + x_2(x_1+x_2)$

$\Rightarrow x_1^2 + x_2^2 + 2x_1x_2$

b) Let $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the subspace spanned by $\boldsymbol{z}$ and the subspace spanned by $\boldsymbol{w}$ on your sketch of the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$.

c) Does the problem $\min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ have a unique solution?

d) Is $\boldsymbol{Q} \succ 0$? Is $\boldsymbol{Q} \succeq 0$?