equations of planes

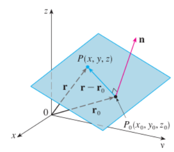

Figure 1: vector equation of the plane

- vector equation of the palne

$$n \cdot (r - r_0) = 0$$
$$n \cdot r = n \cdot r_0$$

$$n = \langle a, b, c \rangle, \quad r = \langle x, y, z \rangle, r_0 = \langle x_0, y_0, z_0 \rangle$$
$$\langle a, b, c, \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

problem)

Plane through origin with normal vector $\vec{N} = \langle 1, 5, 10 \rangle$?

P is in plane

$$\Leftrightarrow \overrightarrow{OP} \cdot \vec{N} = 0$$

$$\Leftrightarrow x + 5y + 10z = 0$$

problem)

plane through $P_0(2, 1, -1)$ and $\perp \vec{N} = \langle 1, 5, 10 \rangle$

P is in plane

$$\Leftrightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0$$

$$\Leftrightarrow \langle x - 2, y - 1, z + 1 \rangle \cdot \langle 1, 5, 10 \rangle = 0$$

$$\Leftrightarrow (x - 2) + 5(y - 1) + 10(z + 1) = 0$$

$$\Leftrightarrow x + 5y + 10z = -3$$

In equation $ax + by + cz = d, \quad \langle a, b, c, \rangle =$ normal vector$\vec{N}$

equations of lines

- line

  intersection of 2 planes or trajectory of a moving point "parametric equation"

ex)

line through

$$Q_0 = (-1, 2, 2)$$
$$Q_1 = (1, 3, -1)$$

$Q(t) =$ moving point, at $t = 0$ it's at Q_0 moves at constant speed on the line

What is the position at time t, $Q(t)$?

$$\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1} = t\langle 2, 1, -3\rangle$$
$$Q(t) = (x(t), y(t), z(t))$$

$$x(t) + 1 = 2t \quad y(t) - 2 = t \quad z(t) - 2 = -3t$$

$$\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$$
$$x(t) = -1 + 2t$$
$$y(t) = 2 + t$$
$$z(t) = 2 - 3t$$
$$Q(t) = Q_0 + t\overrightarrow{Q_0Q_1}$$

- cycloid

  wheel of radious a rolling on floor(x-axis) p = a point on rim of wheel starts at 0. what happen?

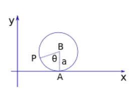

Figure 2: cycloid

Question: position $(x(\theta), y(\theta))$ of the point P? as a function of the angle θ by which the wheel has rotated

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$ $\quad \overrightarrow{OA} = \langle a\theta, 0\rangle$ $\quad \overrightarrow{OA} =$ arclength from A to P

$$\overrightarrow{AB} = \langle 0, a \rangle$$

$$\overrightarrow{BP} \langle -a \sin\theta, -a \cos theta \rangle$$

$$\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$$

Question: what happens near bottom? Answer: take length unit = radius a=1

$$\begin{aligned}
x(\theta) &= \theta - \sin\theta \\
y(\theta) &= 1 - \cos\theta \\
\sin(\theta) &\sim \theta \text{ for } \theta \text{ small} \\
\cos(\theta) &\sim 1
\end{aligned}$$

Taylor approximation for t small

$$f(t) \approx f^{'}(0) + tf^{'}(0) + \frac{t^2}{2} f^{''}(0) + \frac{t^3}{6} f^{'''}(0)$$

$$\begin{aligned}
sin(\theta) &\approx \theta - \frac{\theta}{6} \\
\cos(\theta) &\approx 1 - \frac{\theta^2}{2} \\
x(\theta) &\approx \theta - \left(\theta - \frac{\theta}{6}\right) \approx \frac{\theta^3}{6} \\
y(\theta) &\approx 1 - \left(1 - \frac{\theta^2}{2}\right) \approx \frac{\theta^2}{2}
\end{aligned}$$

x 값이 y값의 크기에 비해 훨씬 작다

$\frac{y}{x} \approx \quad = \frac{3}{\theta} \to \infty$ when $\theta \to 0$ slope at orighin is ∞

parametric equations

position of a moving point

$$F(t) = \langle x(t), y(t), z(t) \rangle$$

ex) cycloid (wheel radius 1, at unit speed)

$$\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t)$$

velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

ex) for cycloid

$$\vec{v} = \langle 1 - \cos(t), \sin(t) \rangle$$

at t=0: $\vec{v} = 0$

speed (scalar) $|\vec{v}| = \sqrt{(1 - \cos(t)^2 + \sin^2(t)} = \sqrt{2 - 2\cos(t)}$

acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

ex) cycloid $\vec{a} = \langle \sin(t), \cos(t) \rangle$ at t=0: $\vec{a} = \langle 0, 1 \rangle$

$$|\frac{d\vec{r}}{dt} \neq \frac{d|\vec{r}|}{dt}$$

arc length s=distance travelled alng trajectory

s versus t?

$$\frac{ds}{dt} = speed = |\vec{v}|$$

ex) length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos(t)dt}$

unit tangent vector

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{T}\frac{ds}{dt}$$

velocity has direction(tangent to traj = $\hat{T}$), lenght(speed = $\frac{ds}{dt}$)

partial derivative

partial derivative of f with respect to x at (a,b)

$$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) - f(x,b)$$

$$f_x(a,b) = \lim_{h\to\infty} \frac{f(a+h,b) - f(a,b)}{h}$$

partial_derivative of f with respect to y at (a,b)

$$f_y(a,b) = \lim_{h\to\infty} \frac{f(a,b+h) - f(a,b)}{h}$$

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to\infty} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to\infty} \frac{f(x,y+h) - f(x,y)}{h}$$

notiation for partial derivatives If $z = f(f,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_x = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Approximation formula

If we change $x \to x + \Delta x \quad y \to y + \Delta y$

$z = f(x,y)$ then $\Delta z \approx f_x \Delta x + f_y \Delta y$

Tangent Planes $f_x\, f_x$ are slopes of 2 tangent lines.

$$\frac{\partial f}{\partial x}(x_0,y_0) = a \Rightarrow \quad L_1 = \begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = b \Rightarrow \quad L_2 = \begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$$

L_1, L_2 are both tangent to the graph $z = f(x, y)$

Together they determine a plane.

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

Approximation formula says: graph of f is close to its tangent plane.

Application of partial derivatives

Optimization problems - find min/max of a function $f(x, y)$ At a local min or max, $f_x = 0$ and $f_y = 0 \Leftrightarrow$ tangent plane to graph $z = f(x, y)$ is horizontal

critical point (x_0, y_0) is a critical point of f if $f_x(x_0, y_0) = 0$ and $f_{y}(x_{0}, y_{0})=0

ex)

$$f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$$
$$f_x = 2x - 2y + 2 = 0$$
$$f_y = -2x + 6y - 2 = 0$$
$$\therefore \text{ critical point } (x, y) = (-1, 0)$$

possible local min, local max, saddle. complete the square

$$f(x, y) = (x - y)^2 + 2y^2 + 2x - 2y$$

$$f(x, y) = ((x - y) + 1)^2 + 2y^2 - 1 \geqslant -1 = f(-1, 0)$$

critical point (-1, 0) is a minimum.

Least squares interpolation Given experimental data $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ find “best fit” line $y = ax + b$

find minimizing total squere deviation a and b.

deviation for each data point $y_i - (ax_i + b)$

Minimize $D(a, b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0$$

$$\begin{cases} \sum_{i=1}^{n}(x_i^2 a + x_i b x_i y_i) = 0 \\ \sum_{i=1}^{n}(x_i a + b - y_i) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\sum_{i=1}^{n} x_i^2)a + (\sum_{i=1}^{n} x_i)b = (\sum_{i=1}^{n} x_i y_i) \\ (\sum_{i=1}^{n} x_i)a + nb = \sum_{i=1}^{n} y_i \end{cases}$$

2x2 linear system. solve for (a, b)

Best exponential fit

$$y = ce^{ax} \Leftrightarrow \ln(y) = \ln(c) + ax$$

Best quadratic fit

$$D(a,b,c) = \sum_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2$$

Implicit differentiation

$$y = f(x)$$
$$dy = f'(x)dx$$

Ex :

$$y = \sin^{-1}(x)$$
$$x = \sin(y)$$
$$dx = \cos(y)dy$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

if $f(x,y,z)$

$$\begin{aligned} df &= f_x dx + f_y dy + f_z dz \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \end{aligned}$$

df is not ∇f

- encode how change in x, y, z affect f
- placeholder for small variations $\nabla x, \nabla y, \nabla z$ to get approx formula $\nabla f \approx f_x \nabla x + f_y \nabla y + f_z \nabla z$
- divide by something like dt to get a rate of change when x = x(t), y=y(t), z=z(t)

$$\frac{df}{dt} = f_x \frac{dx}{dz} + f_y \frac{dy}{dt} + f_z dzdt$$

Gradient

at some point (x, y, z)

$$\nabla w = < W_x, W_y, W_z >$$

Theorem $\nabla w \perp$ level surface; w = constant

ex)

$$w = a_1 x + a_2 y + a_3 z$$

$$\nabla w = < a_1, a_2, a_3 >; \quad a_1 = \frac{\partial w}{\partial x}$$

level surface

$$a_1 x + a_2 + a_3 z = c \qquad \text{pane with normal } < a_1, a_2, a_3 >$$

directional derivatives

The directional derivative of f at $(X_0 + y_0)$ in the direction of a unit vector $u = < a, b >$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

The partial derivatives of f with respect to x and y are just special cases of the directional derivative.

If f is a differentiable function of x and y, then f has a directinal derivative in the direction of any unit vector $u = < a, b >$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

LAGRANGE MULTIPLIERS

if $\nabla g(x_0, y_0, z_0) \neq 0$, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ is called a Lagrange multiplier.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subjet to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]

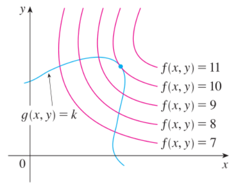

FIGURE 1

Figure 3: Lagrange Multipliers

- Find all values of x, y, z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) g(x,y,z) = k$$

- Evaluate f at all the points(x, y, z) that result from previous step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

vector calculus

The fundamental Theorem for line integrals

Let C be a smooth curve given by the vector function r(t), $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whos gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Suppose F is a vector field that is continuous on an open connected region D. If $\int_C F \cdot dr$ is independent of path in D, then F is a conservative vector field on D; that is, there exists a function f such that ∇f = F$.

How is it possible to determine whether or not a vector field F is conservative? Suppose it is known that $F = P\hat{i} + Q\hat{j}$ is conservative, where P and Q have continuous first-order partial derivatives. then there is a function f such that $F = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Curl

If $F = p\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q and R all exist, then the curl of F is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

define the vector differential operator ∇("del") as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{curl } F = \nabla \times F$$

If f is a function of three variables that has continuous second-order partial derivatives, then

$$curl(\nabla f) = 0$$

Since a conservative vector field is one for which $F = \nabla f$, above can be rephrased as follows : If F is conservative, then cur F = 0.

Divergence

If $F = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the divergence of F is the function of three variables defined by

$$div \quad \text{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$div \quad \text{F} = \nabla \cdot F$$

If $F = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$div \quad curl \quad \text{F} = 0$$