Block (3/3)

The Finite Element Method for Applications in Electrical Engineering

EE4375 - FEM For EE Applications

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Last updated July 18, 2025

Modeling of Permanent Magnet Machines

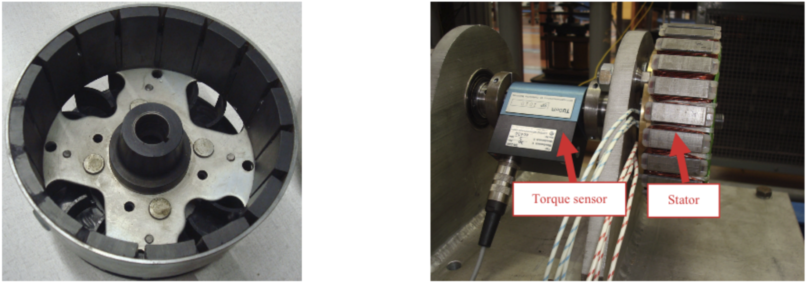


Fig. 2: Rotor of permanent magnet machine under study and test setup.

Modeling of Permanent Magnet Machines

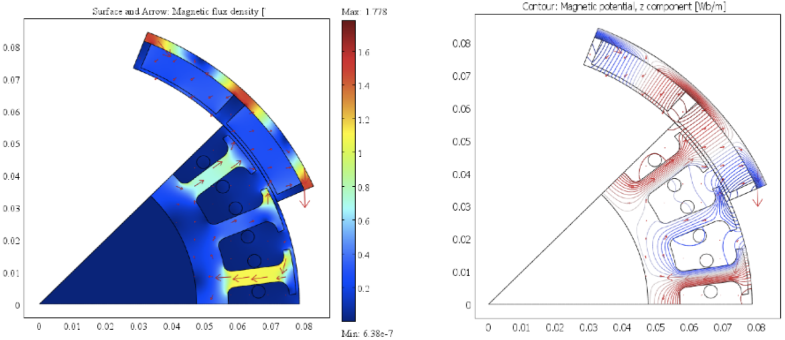


Fig. 6: Flux density and flux contour of the PM machine during load.

Modeling of Permanent Magnet Machines

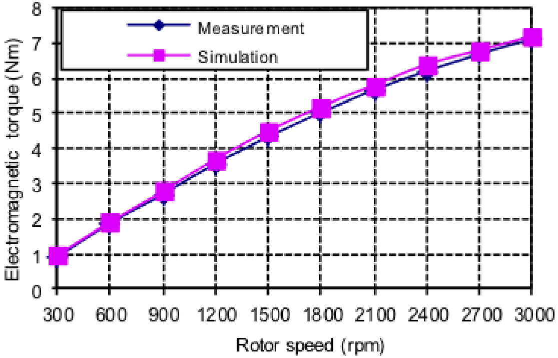
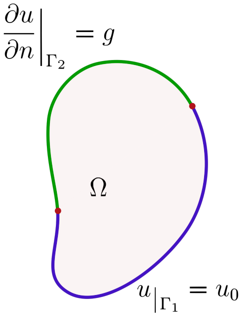


Fig.16. Mean electromagnetic torque vs. rotor speed.

Two-Dimensional Problem to Solve (1/4)

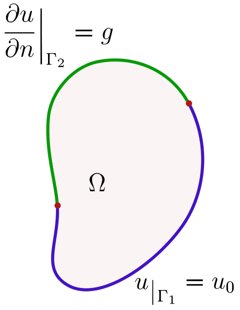
Geometry - Domain of Computation

- $(x, y) \in \Omega$ bounded domain in flat space



Two-Dimensional Problem to Solve (2/4)

Two Types of Boundary Conditions: $\Gamma = \Gamma_D \cup \Gamma_N$



- Dirichlet condition on Γ_D

  fix u

  (equivalent of $x = 0$ in 1D)

- Neumann condition on Γ_N

  fix $\frac{\partial u}{\partial n}$

  (equivalent of $x = 1$ in 1D)

Two-Dimensional Problem to Solve (3/4)

Boundary Value Problem for Second Order Differential Equations

- given $(x, y) \in \Omega$ with $\Gamma = \Gamma_D \cup \Gamma_N$ the boundary of Ω
- given: $f(x, y)$ given function and α given number
- find: $u(x, y)$ such that

$$
\begin{aligned}
-\triangle u(x,y) &= f(x,y) \text{ for } (x,y) \in \Omega \text{ (differential equation on } \Omega) \\
u &= 0 \text{ (Dirichlet boundary condition on } \Gamma_D) \\
\frac{\partial u}{\partial n} &= \alpha \text{ (Neumann boundary condition on } \Gamma_N)
\end{aligned}
$$

- differential equation invalid on Γ - boundary conditions valid on Γ

Two-Dimensional Problem to Solve (4/4)

Residual function r(x,y)

- definition: $r(x,y) = \triangle u(x,y) + f(x,y)$
- quality of the solution:
  small (large) residual is indication of good (poor) approximation
- solve $- \triangle u(x,y) = f(x,y)$ + b.c. equivalent to
- find $u(x,y)$ such that $r(x,y) = 0$ and $u(x,y)$ satisfies b.c.

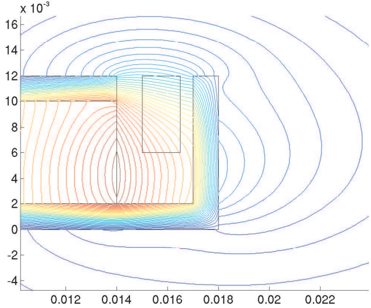
Mesh Generation (1/19)

Mesh Ω^h on Ω

- nodes: $\mathbf{x}_i = (x_i, y_i)$
- edges: edge_i
- triangular elements: e_i

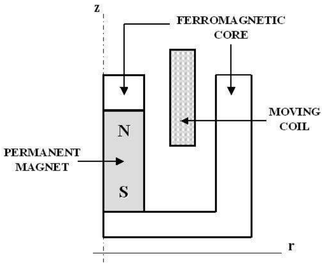
Mesh Generation (2/19)

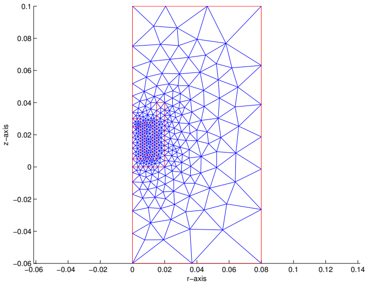
Loudspeaker Example



Mesh Generation (3/19)

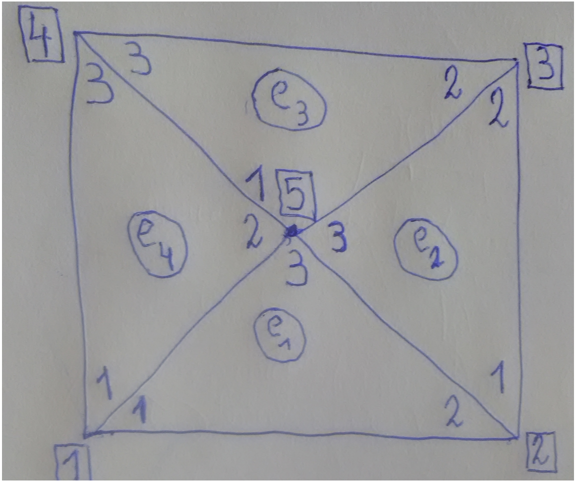
Loudspeaker Example





Mesh Generation (4/19)

Four-Element Five-Node Mesh Example



Mesh Generation (5/19)

Four-Element Five-Node Mesh Example

Gmsh Output for First Order Elements

- 5 nodes: 4 boundary nodes plus one interior node
- 8 edges (4 boundary edges plus 4 interior edges) not reportedly separately by Gmsh
- 12 elements: 4 triangles + 8 edges

Mesh Generation (6/19)

Local and Global Numbering of Nodes

- local numbering: numbering from 1 to 3 on each triangle e_i
- global numbering: numbering from 1 to nnodes on the mesh Ω^h
- local-to-global mapping:

  on each element e_i: given local number, find global number
- see Second Homework Assignment: matrix e
- valuable bookkeeping tool for assembly of matrix and vector

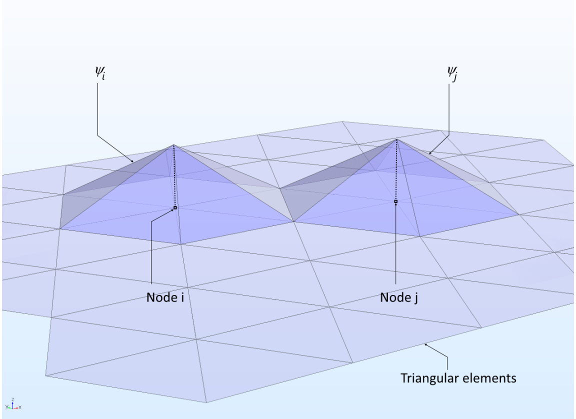
Mesh Generation (7/19)

Example of Local and Global Numbering of Nodes
Four-Element Five-Node Mesh Example

- on element e_1
  local nrs 1, 2 and 3 corresponds to global nrs 1, 2 and 5
- on element e_2
  local nrs 1, 2 and 3 corresponds to global nrs 2, 3 and 5
- on element e_3
  local nrs 1, 2 and 3 corresponds to global nrs 5, 3 and 4
- on element e_4
  local nrs 1, 2 and 3 corresponds to global nrs 1, 5 and 4

Mesh Generation (8/19)

Shape Functions



Mesh Generation (9/19)

Definition of shape function $\phi_i(\mathbf{x}) = \phi_i(x, y)$

- linear Lagrange interpolation function $\phi_i(x, y)$
- linear means that $\phi_i(x, y) = C_1\, x + C_2\, y + C_3$
- each node $\mathbf{x}_i = (x_i, y_i)$ (including boundary nodes) has its $\phi_i(x, y)$
- $\phi_i(\mathbf{x}_j) = \delta_{ij}$ (see figure)

Mesh Generation (10/19)

What is the Buzz on Element e_i?

- element e_i has three nodes $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ (local numbering)
- elements e_i ”sees” three linear basis function (local numbering)

$$
\begin{aligned}
\phi_1(x,y) &= a_1\, x + b_1\, y + c_1 \\
\phi_2(x,y) &= a_2\, x + b_2\, y + c_2 \\
\phi_3(x,y) &= a_3\, x + b_3\, y + c_3
\end{aligned}
$$

- what are a_1, b_1 and c_1? (same for $\phi_2(x,y)$ and $\phi_3(x,y)$)
- impose $\phi_1(\mathbf{x}_1) = 1$, $\phi_1(\mathbf{x}_2) = 0$ and $\phi_1(\mathbf{x}_3) = 0$
  (same for $\phi_2(x,y)$ and $\phi_3(x,y)$)

Mesh Generation (11/19)

What is the Buzz on Element e_i?

- three conditions $\phi_1(\mathbf{x}_1) = 1$, $\phi_1(\mathbf{x}_2) = 0$ and $\phi_1(\mathbf{x}_3) = 0$ read

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- same for $\phi_2(x, y)$ and $\phi_3(x, y)$

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Important messsage: coordinate $x_1, \ldots, y_3$ uniquely fix $a_1, \ldots, c_3$!

Mesh Generation (12/19)

Exercise

- assume $\phi_i(x, y) = a_i\, x + b_i\, y + c_i$ for $1 \leq i \leq 3$
- compute $\nabla\phi_i(x, y)$ for $1 \leq i \leq 3$
- compute $\nabla\phi_i(x, y) \cdot \nabla\phi_j(x, y)$ for $1 \leq i, j \leq 3$
- compute $A_{e_k} = \int_{e_k} \nabla\phi_i(x, y) \cdot \nabla\phi_j(x, y)\, d\Omega$

Mesh Generation (13/19)

Exercise Solution

- assume $\phi_i(x, y) = a_i\, x + b_i\, y + c_i$ for $1 \leq i \leq 3$
- compute $\nabla \phi_i(x, y) = \left(\frac{\partial \phi_i(x,y)}{\partial x}, \frac{\partial \phi_i(x,y)}{\partial y} \right) = (a_i, b_i)$ for $1 \leq i \leq 3$
- compute $\nabla \phi_i(x, y) \cdot \nabla \phi_j(x, y) = \underbrace{a_i\, a_j + b_i\, b_j}_{constant-in-x-and-y}$ for $1 \leq i, j \leq 3$
- compute for $1 \leq i, j \leq 3$

$$A_{e_k} = \int_{e_k} \nabla \phi_i(x, y) \cdot \nabla \phi_j(x, y)\, d\Omega = \text{area}(e_k)\, [a_i\, a_j + b_i\, b_j]$$

- A_{e_k}: contribution to global matrix A on element e_k
- observe that A_{e_k} is independent of c_1, c_2 and c_3

Mesh Generation (14/19)

How to Compute the Elementary Matrix Contribution (1/2)

- define on each element the 3-by-3 matrix

$$Emat = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

- compute *Emat* by solving the linear system

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}}_{=Emat} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- set third row of *Emat* equal to zero or $Emat[3,:].=0$

$$Emat = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}$$

Mesh Generation (15/19)

How to Compute the Elementary Matrix Contribution (2/2)

- then

$$Emat^T\ Emat = [a_i\, a_j + b_i\, b_j] \text{ for } 1 \leq i, j \leq 3$$

- and therefore contribution to global matrix A on element e_k

$$\text{area}(e_k)\ Emat^T\ Emat = \text{area}(e_k)\,[a_i\, a_j + b_i\, b_j] \text{ for } 1 \leq i, j \leq 3$$

Mesh Generation (16/19)

Computation of the Coefficients $a_1, \ldots, b_3$ Revisited (1/4)

- motivation: solving the linear system for *Emat* on each element likely computationally expensive
- alternative: solve symbolically using Cramer's Rule instead
- linear system for a_1 and b_1 (coefficient c_1 not required)

$$\underbrace{\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}}_{=X} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- $$a_1 = \frac{\det(X_1)}{\det(X)} \text{ and } b_1 = \frac{\det(X_2)}{\det(X)} \text{ and } c_1 \text{ not required}$$

Mesh Generation (17/19)

Computation of the Coefficients $a_1, \ldots, b_3$ Revisited (2/4)

- $$\begin{aligned} \det(X) &= \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \det \begin{pmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{pmatrix} \\ &= (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) \\ &= 2\,\text{signed-area}(e_k) \end{aligned}$$

- $\det(X_1) = \det \begin{pmatrix} 1 & y_1 & 1 \\ 0 & y_2 & 1 \\ 0 & y_3 & 1 \end{pmatrix} = y_2 - y_3$

  $\det(X_2) = \det \begin{pmatrix} x_1 & 1 & 1 \\ x_2 & 0 & 1 \\ x_3 & 0 & 1 \end{pmatrix} = -(x_2 - x_3) = x_3 - x_2$

Mesh Generation (18/19)

How to Compute the Coefficients $a_1, \ldots, b_3$ (3/4)

- thus

$$a_1 = \frac{\det(X_1)}{\det(X)} = \frac{y_2 - y_3}{2\,\text{signed-area}(e_k)}$$

- and

$$b_1 = \frac{\det(X_2)}{\det(X)} = \frac{x_3 - x_2}{2\,\text{signed-area}(e_k)}$$

- similarly for a_2, a_3, b_2 and b_3

Mesh Generation (19/19)

How to Compute the Coefficients $a_1, \ldots, b_3$ (4/4)

- local matrix A_{e_k} can be computed from nodal coordinates without solving 3-by-3 linear systems

Linear Combination of Shape Functions (1/3)

Linear Combination of Shape Functions

- set of functions $\{\phi_i(x) | 1 \leq i \leq n\}$ where $n = nnodes$
- linear combinations of these functions $\phi_i(x)$ can be made
- $V_0^h(\Omega)$: function space defined by all linear combinations

$$V_0^h(\Omega) = \text{span}\{\phi_1(x,y), \ldots, \phi_n(x,y)\}$$

- $u^h(x,y) \in V_0^h(\Omega)$: there exists coordinates $c_1, \ldots, c_n$ such that

$$u^h(x,y) = c_1\, \phi_1(x,y) + \ldots + c_n\, \phi_n(x,y)$$

Linear Combination of Shape Functions (2/3)

Application to Finite Elements

- $u(x,y)$: exact solution of the boundary value problem
- $u^h(x,y)$: finite element approximation to $u(x)$ computed on Ω^h
- $u^h(x,y) \in V_0^h(\Omega) = \text{span}\{\phi_1(x,y), \ldots, \phi_n(x,y)\}$
- $u^h(x,y) = 0$ on Γ_D by definition of $V_0^h(\Omega)$
- expansion of $u^h(x)$ as linear combination of shape function

$$u^h(x,y) = c_1\,\phi_1(x,y) + \ldots + c_n\,\phi_n(x,y)$$

Linear Combination of Shape Functions (3/3)

Application to Finite Elements

- expansion of $u^h(x)$ as linear combination of shape function

$$u^h(x,y) = c_1\,\phi_1(x,y) + \ldots + c_n\,\phi_n(x,y)$$

- $c_1, \ldots, c_n$ coordinates - $\phi_1(x,y), \ldots, \phi_n(x,y)$ basis functions

- basis functions: unique determined by the mesh

- coordinates: to by determined by solving a linear system

  one coordinate for each node $\mathbf{x}_i$ in the mesh

Strong vs. Weak Equal to Zero (1/2)

Weak or Variational Formulation

- n basis functions $\phi_i(x, y)$ defined by the mesh Ω^h
- inner product: $< g(x, y), \phi_i(x, y) >= \int_\Omega g(x, y)\, \phi_i(x, y)\, d\Omega$
- $< g(x, y), \phi_i(x, y) >$ coordinate of $g(x, y)$ along the basis function $\phi_i(x, y)$
- numerically: essential in remainder of the course

$$\boxed{g(x, y) = 0 \text{ in discrete weak form}}$$

$$\boxed{\Leftrightarrow \forall \mathbf{x}_i \in \Omega^h :< g(x, y), \phi_i(x, y) >= 0}$$

n equations indexed by i where $n = nnodes$

Strong vs. Weak Equal to Zero (2/2)

Applied to Finite Elements

- choose $g(x,y) = r(x,y) = \triangle u(x,y) + f(x,y)$
- enforce $g(x,y) = 0$ plus boundary conditions in discrete weak form

$$
\begin{aligned}
< g(x,y), \phi_i(x,y) > &= 0 \quad \text{for } 1 \leq i \leq n \\
\Leftrightarrow \int_\Omega g(x,y)\, \phi_i(x,y)\, d\Omega &= 0 \quad \text{for } 1 \leq i \leq n \\
\Leftrightarrow \int_\Omega -\triangle\, u(x,y)\, \phi_i(x,y)\, d\Omega &= \int_\Omega f(x,y)\, \phi_i(x,y)\, d\Omega \quad \text{for } 1 \leq i \leq n
\end{aligned}
$$

where $n = nnodes$

Calculus of functions in 2 vars (1/12)

Two Small Exercises

- recap: Block (2/3) of this course: $F(x) = v\frac{du}{dx}$
- suppose now that $\mathbf{F} = v\,\nabla u = v\,\text{grad}(u)$
- Exercise 1: compute $\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \nabla \cdot (v\,\nabla u)$
- Exercise 2: compute $\mathbf{F} \cdot \mathbf{n} = (v\,\nabla u) \cdot \mathbf{n}$

Calculus of functions in 2 vars (2/12)

Two Small Exercises: Exercise One

- $\nabla u = \text{grad}(u)$ is a vector function

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}$$

- $\mathbf{F} = v\,\nabla u$ is scalar times vector and thus again a vector function (multiply each component of vector ∇u with scalar v)

$$\mathbf{F} = v\,\nabla u = v\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \left(v\,\frac{\partial u}{\partial x}, v\,\frac{\partial u}{\partial y}\right)$$

- note that $\mathbf{F}$ is a vector function with two components

$$F_x = v\,\frac{\partial u}{\partial x} \quad \text{and} \quad F_y = v\,\frac{\partial u}{\partial y}$$

Calculus of functions in 2 vars (3/12)

Two Small Exercises: Exercise One

- $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \nabla \cdot (v \nabla u)$ is a scalar function

$$
\begin{aligned}
\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} &= \nabla \cdot \left(v \frac{\partial u}{\partial x}, v \frac{\partial u}{\partial y} \right) \\
&= \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial y} \right) \\
&= v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \\
&= v \triangle u + \nabla u \cdot \nabla v \\
&= v \operatorname{Laplacian}(u) + \underbrace{\operatorname{grad}(u) \cdot \operatorname{grad}(v)}_{inner-product-of-vectors}
\end{aligned}
$$

Calculus of functions in 2 vars (4/12)

Two Small Exercises: Exercise 2

- $\mathbf{F} \cdot \mathbf{n}$ is the inner product of $\mathbf{F}$ and $\mathbf{n}$

$$
\begin{aligned}
\mathbf{F} \cdot \mathbf{n} &= (v \, \nabla u) \cdot \mathbf{n} \quad \text{(definition of } \mathbf{F}\text{)} \\
&= v \underbrace{(\nabla u \cdot \mathbf{n})}_{inner-product-of-vectors} \quad \text{(change order of operations)} \\
&= v \, (\frac{\partial u}{\partial x} \, n_x + \frac{\partial u}{\partial y} \, n_y) \quad \text{(inner product explicitly)} \\
&= v \, \frac{\partial u}{\partial n} \quad \text{(definition of the normal product)}
\end{aligned}
$$

Calculus of functions in 2 vars (5/12)

Two Small Exercises: Summary

- suppose that $\mathbf{F} = v \nabla u$
- Ex 1: solution $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \nabla u \cdot \nabla v + v \bigtriangleup u$
- Ex 2: solution $\mathbf{F} \cdot \mathbf{n} = (v \nabla u) \cdot \mathbf{n} = v (\nabla u \cdot \mathbf{n}) = v \frac{\partial u}{\partial n}$
- these results will be used in the next two slides

Calculus of functions in 2 vars (6/12)

Recap: Integration of Function in One Variable

- assume $0 < x < 1$ or $x \in \Omega = (0,1)$ and $F'(x) = \dfrac{d\,F(x)}{dx}$

$$\int_{\Omega} F'(x)\,dx = \underbrace{\int_0^1 F'(x)\,dx}_{1D-line-integral} = [F(x)]_0^1 = \underbrace{F(1) - F(0)}_{0D-point-evaluation}$$

- choose $F(x) = v(x)\,u'(x)$ (u has prime - v has no prime)
- arrive at integration by parts formula

$$-\int_0^1 u''(x)\,v(x)\,dx = \int_0^1 u'(x)\,v'(x)\,dx - [u'(x)\,v(x)]_0^1$$

- wish - goal - dream - ambition: repeat in 2D

Calculus of functions in 2 vars (7/12)

Integration by Parts in two variables

- Gauss Integration Theorem or Divergence Theorem

  $(x, y) \in \Omega$ with boundary Γ

$$\underbrace{\int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega}_{2D-surface-integral} = \underbrace{\int_{\Gamma} \mathbf{F} \cdot \mathbf{n} \, ds}_{1D-line-integral}$$

- see calculus textbook or wiki
- choose: $\mathbf{F} = v \, \nabla u = (v \, \frac{\partial u}{\partial x}, v \, \frac{\partial u}{\partial y})$
- requires $\nabla \cdot \mathbf{F}$ and $\mathbf{F} \cdot \mathbf{n}$ from exercise

Calculus of functions in 2 vars (8/12)

Integration by Parts in two variables

- then

$$\int_\Omega \nabla \cdot \mathbf{F}\, d\Omega = \int_\Omega \nabla \cdot (v\, \nabla u)\; d\Omega = \int_\Omega \left[\nabla u \cdot \nabla v + v\, \triangle u\right]\, d\Omega$$
$$= \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, ds = \int_\Gamma \frac{\partial u}{\partial n}\, v\, ds$$

- after rearranging terms:

$$\int_\Omega (-\triangle u)\, v\, d\Omega = \int_\Omega \nabla u \cdot \nabla v\, d\Omega - \int_\Gamma \frac{\partial u}{\partial n}\, v\, ds$$

Calculus of functions in 2 vars (9/12)

Derivative: Integration by Parts in two variables

- integration by parts formula becomes

$$\boxed{\int_{\Omega} (- \triangle u) \, v \, d\Omega = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma} \frac{\partial u}{\partial n} \, v \, ds}$$

- observe that as before:

  LHS: double derivatives in u - no derivatives on v

  LHS: minus sign

  RHS: first order derivatives on both u and v - additional term on the boundary

Calculus of functions in 2 vars (10/12)

Quadrature by Trapezoidal Rule

- trapezoidal rule: e_k triangle with vertices $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ (local numbering)

$$\int_{e_k} g(x,y)\, d\Omega \approx \frac{\text{area}(e_k)}{3}\,[g(\mathbf{x}_1) + g(\mathbf{x}_2) + g(\mathbf{x}_3)]$$

- $\text{area}(e_k)$ can be computed using $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ as input
- more accurate rules exist (e.g. Gauss quadrature, see references)

Calculus of functions in 2 vars (11/12)

Exercise

- assume that $g(x, y) = f(x, y)\, \phi_i(x, y)$ for $1 \leq i \leq 3$
- compute

$$\begin{pmatrix} \int_{e_k} f(x, y)\, \phi_1(x, y)\, d\Omega \\ \int_{e_k} f(x, y)\, \phi_2(x, y)\, d\Omega \\ \int_{e_k} f(x, y)\, \phi_3(x, y)\, d\Omega \end{pmatrix}$$

Calculus of functions in 2 vars (12/12)

Exercise Solution

- assume that $g(x,y) = f(x,y)\,\phi_i(x,y)$ for $1 \leq i \leq 3$
- compute

$$\begin{pmatrix} \int_{e_k} f(x,y)\,\phi_1(x,y)\,d\Omega \\ \int_{e_k} f(x,y)\,\phi_2(x,y)\,d\Omega \\ \int_{e_k} f(x,y)\,\phi_3(x,y)\,d\Omega \end{pmatrix} \approx \frac{\text{area}(e_k)}{3} \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ f(\mathbf{x}_3) \end{pmatrix}$$

- contribution to global vector **f** on element e_k

Discrete Weak Form (1/6)

Apply Integration by Parts on Weak Formulation

- earlier we set the residual $r(x,y) = \triangle u''(x,y) + f(x,y)$ to zero in weak form and arrived at

$$\int_\Omega (-\triangle u)\,\phi_i(x,y)\,d\Omega = \int_\Omega f(x,y)\,\phi_i(x,y)\,d\Omega \quad \text{for all } 1 \leq i \leq n$$

- apply integration by part to the LHS

$$\int_\Omega \nabla u \cdot \nabla \phi_i(x,y)\,d\Omega = \int_\Omega f(x,y)\,\phi_i(x,y)\,d\Omega + \int_\Gamma \frac{\partial u}{\partial n}(x,y)\,\phi_i(x,y)\,ds$$

for all $1 \leq i \leq n$ where n=nnodes

observe: minus sign in LHS disappeared

- first order derivative on both $u(x)$ and $\phi_i(x)$

Discrete Weak Form (2/6)

- Dirichlet boundary conditions: $u = 0$ on Γ_D
- Neumann boundary conditions: $\dfrac{\partial u}{\partial n} = \alpha$ on Γ_N
- Dirichlet and Neumann boundary conditions treated differently
- boundary term in the RHS of the weak form

$$\int_{\Gamma} \frac{\partial u}{\partial n}(x, y)\, \phi_i(x, y)\, ds = \int_{\Gamma_D} \frac{\partial u}{\partial n}(x, y)\, \phi_i(x, y)\, ds + \int_{\Gamma_N} \underbrace{\frac{\partial u}{\partial n}(x, y)}_{=\alpha}\, \phi_i(x, y)\, ds$$

- we thus obtain that

$$\int_{\Gamma} \frac{\partial u}{\partial n}(x, y)\, \phi_i(x, y)\, ds = \int_{\Gamma_D} \frac{\partial u}{\partial n}(x, y)\, \phi_i(x, y)\, ds + \int_{\Gamma_N} \alpha\, \phi_i(x, y)\, ds$$

Discrete Weak Form (3/6)

- Dirichlet boundary conditions: impose that $\phi_i(x, y) = 0$ on Γ_D
- we thus obtain that

$$\int_{\Gamma} \frac{\partial u}{\partial n}(x, y)\, \phi_i(x, y)\, ds = \int_{\Gamma_N} \alpha\, \phi_i(x, y)\, ds$$

- discrete weak form becomes

$$\int_{\Omega} \nabla u \cdot \nabla \phi_i(x, y)\, d\Omega = \int_{\Omega} f(x, y)\, \phi_i(x, y)\, d\Omega + \int_{\Gamma_N} \alpha\, \phi_i(x, y)\, ds$$

for all $1 \leq i \leq n$

Discrete Weak Form (4/6)

Discrete Weak Form Becomes for $1 \leq i \leq n$

$$\int_\Omega \nabla u(x,y) \cdot \nabla \phi_i(x,y)\, d\Omega = \int_\Omega f(x,y)\, \phi_i(x,y)\, d\Omega + \int_{\Gamma_N} \alpha\, \phi_i(x,y)\, ds$$

- assume $u(x,y)$ approximate by $u^h(x,y)$ where $u^h(x,y) = \sum_{j=1}^n c_j\, \phi_j(x,y)$
- thus $u'(x)$ approximate by $\nabla u^h(x,y) = \sum_{j=1}^n c_j\, \nabla\phi_j(x,y)$
- then for $1 \leq i \leq n$

$$\sum_{j=1}^n \int_\Omega \nabla\phi_j(x,y) \cdot \nabla\phi_i(x,y)\, dx\, c_j = \int_\Omega f\, \phi_i\, d\Omega + \int_{\Gamma_N} \alpha\, \phi_i(x,y)\, ds$$

Discrete Weak Form (6/6)

Discrete Weak Form Becomes

- for $1 \leq i \leq n$ where $n = nnodes$

$$\sum_{j=1}^{n} \int_{\Omega} \nabla \phi_j(x,y) \cdot \nabla \phi_i(x,y) \, d\Omega \, c_j = \int_{\Omega} f(x,y) \, \phi_i(x,y) \, d\Omega$$

$$+ \int_{\Gamma_N} \alpha \, \phi_i(x,y) \, ds$$

- can be written in the form: for $1 \leq i \leq n$
  index i counts equations - index j counts unknowns

$$\sum_{j=1}^{n} A_{ij} \, c_j = f_i$$

- and thus as a n by n linear system

$$A \mathbf{c} = \mathbf{f}$$

Linear System Formulation (1/3)

Expression for Matrix and Vector Elements

- Matrix elements:

$$A_{ij} = \int_{\Omega} \nabla\phi_j(x,y) \cdot \nabla\phi_i(x,y)\, d\Omega \text{ for } 1 \le i,j \le n$$

- Vector elements:

$$f_i = \int_{\Omega} f(x,y)\,\phi_i(x,y)\, d\Omega + \int_{\Gamma_N} \alpha\,\phi_i(x,y)\, ds \text{ for } 1 \le i \le n$$

Linear System Formulation (2/3)

Properties of Matrix A

- A is large

  $n > 1e6$ in 3D applications in no exception

- A is sparse

  A contains many zero elements (cfr. 2D finite difference method)

- A many other cool properties $\Rightarrow$ fast solvers for $A\mathbf{c} = \mathbf{f}$ exist

Linear System Formulation (3/3)

Summary: Matrix A and Right-Hand Vector $\mathbf{f}$

Treatment of the Boundary Conditions

- Dirichlet boundary conditions: $u(x, y) = 0$:

  modify equations corresponding to the boundary nodes in linear system

  see finite difference method

- Neumann boundary conditions: $\frac{\partial u}{\partial n} = \alpha$ on Γ_N

  add term $\int_{\Gamma_N} \alpha\, \phi_i(x, y)\, ds$ to vector $\mathbf{f}$

  see lab sessions for details

Element-by-Element Construction of the Vector (1/2)

How does element e_k contribute to the vector $\mathbf{f}$?

- $f_{e_k} \in \mathbb{R}^3$ contribution of element e_i to global vector $\mathbf{f}$

  using local numbering of the three nodes on the element e_k

$$f_{e_k} = \begin{pmatrix} \int_{e_k} f(x,y)\,\phi_1(x,y)\,dx \\ \int_{e_k} f(x,y)\,\phi_2(x,y)\,dx \\ \int_{e_k} f(x,y)\,\phi_3)x,y)\,dx \end{pmatrix}$$

- use trapezoidal rule of integration (see earlier)

$$f_{e_k} = \begin{pmatrix} \int_{e_k} f(x,y)\,\phi_1(x,y)\,d\Omega \\ \int_{e_k} f(x,y)\,\phi_2(x,y)\,d\Omega \\ \int_{e_k} f(x,y)\,\phi_3(x,y)\,d\Omega \end{pmatrix} \approx \frac{\text{area}(e_k)}{3} \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ f(\mathbf{x}_3) \end{pmatrix}$$

- given mesh Ω^h and source $f(x)$, f_{e_k} for each e_k can be computed

Element-by-Element Construction of the Vector (2/2)

Finite Element Assembly of the Vector $\mathbf{f}$

- loop over all of the N elements e_k in the mesh Ω^h
- on e_k compute the local element vector $f_{e_k} \in \mathbb{R}^3$
  the local element vector has three components
- add local element vector to the global vector $\mathbf{f} \in \mathbb{R}^n$
  the global vector $\mathbf{f}$ has n components where $n = nnodes$
- $\mathbf{f} = \mathbf{f} + f_{e_k}$
  assembly requires taking the mesh connectivity into account
  connectivity here refers to mapping from local to global numbering on the element e_k

Element-by-Element Construction of the Matrix (1/2)

How does element e_k contribute to the vector A?

- $A_{e_k} \in \mathbb{R}^{3\times 3}$ contribution of element e_j to global vector A

$$A_{e_k} = \left(\int_{e_k} \nabla\phi_i(x,y) \cdot \nabla\phi_j(x,y)\, d\Omega\right)_{1\leq i,j\leq 3}$$

- using derivative of the shape functions (see earlier)

$$A_{e_k} = \text{area}(e_k)\left(a_i\, a_j + b_i\, b_j\right)_{1\leq i,j\leq 3}$$

Element-by-Element Construction of the Matrix (2/2)

Finite Element Assembly of the Matrix A

- loop over all of the N elements e_k in the mesh Ω^h
- on e_i compute the local element matrix $A_{e_k} \in \mathbb{R}^{3 \times 3}$
  the local element matrix has three by three components
- add local element matrix to the global matrix $A \in \mathbb{R}^{n \times n}$
  the global matrix A has n by n components where $n = nnnodes$
- $A = A + A_{e_k}$
  assembly requires taking the mesh connectivity into account
  connectivity here refers to mapping from local to global numbering on the element e_k

Computation on Reference Element (1/8)

Area of Triangle in Mesh

- triangle **t** in mesh with nodes $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$ and $\mathbf{x}_1 = (x_3, y_3)$
- direction vectors $\mathbf{x}_{12} = \mathbf{x}_2 - \mathbf{x}_1$ and $\mathbf{x}_{13} = \mathbf{x}_3 - \mathbf{x}_1$
- area-triangle equal to $.5\|\mathbf{x}_{12} \times \mathbf{x}_{13}\|$
- $\mathbf{x}_{12} \times \mathbf{x}_{13} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} =$ $0\,\mathbf{i} + 0\,\mathbf{j} + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\,\mathbf{k}$
- area-triangle $= .5\,\|\mathbf{x}_{12} \times \mathbf{x}_{13}\| = .5\,|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$

Computation via Ref Element (2/8)

Coordinate Transformation from Configuration Space (ξ, η)
to Physical Phase (x, y) (need figure)

- triangle **t** in mesh with nodes
  $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$ and $\mathbf{x}_3 = (x_3, y_3)$
- triangle $\widehat{\mathbf{t}}$ in configuration space or (ξ, η)-space with nodes
  $(\xi, \eta) = (0, 0)$, $(\xi, \eta) = (1, 0)$ and $(\xi, \eta) = (0, 1)$
- mapping from reference space to physical space
  $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(\xi, \eta) \\ y(\xi, \eta) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$
- maps $(\xi, \eta) = (0, 0)$ to $(x, y) = (x_1, y_1)$,
  $(\xi, \eta) = (1, 0)$ to $(x, y) = (x_2, y_2)$ and
  $(\xi, \eta) = (0, 1)$ to $(x, y) = (x_3, y_3)$
- observe similarity with the one-dimensional case

Computation via Ref Element (3/8)

Inverse Coordinate Transformation and Jacobians

- mapping from reference space to physical space

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix}^{-1} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}$$

- Jacobian $J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}$

- determinant of the Jacobian
$\det(J) = \left| \frac{\partial(x,y)}{\partial(\xi,\eta)} \right| = |(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$ or
$\det(J) = 2\,$area-triangle

- Jacobian of inverse transformation or inverse Jacobian

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} y_3 - y_1 & -(y_2 - y_1) \\ -(x_3 - x_1) & x_2 - x_1 \end{pmatrix}$$

Computation via Ref Element (4/8)

Basis Functions on Reference and Physical Element

- on the reference element: $\widehat{\phi}_1(\xi, \eta) = \xi$
- on the physical element: $\phi_1(x, y) = \phi_1(x(\xi, \eta), y(\xi, \eta)) = \widehat{\phi}_1(\xi, \eta)$
- similar for $\phi_2(x, y)$ and $\phi_3(x, y)$

Computation via Ref Element (5/8)

Chain Rule and Jacobian Transformations

- chain rule $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}$
- thus $\nabla_{(x,y)} = J\,\nabla_{(\xi,\eta)}$

Computation via Ref Element (6/8)

Integration over mesh triangle **t** via Coordinate Transformation
and integration over $\widehat{\mathbf{t}}$

- $\int_{\mathbf{t}} g(x,y)\,dx\,dy = \int_{\widehat{\mathbf{t}}} g(x(\xi,\eta),y(\xi,\eta)) \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| d\xi\,d\eta$
- $\int_{\mathbf{t}} g(x,y)\,dx\,dy = 2\,\text{area-triangle} \int_{\widehat{\mathbf{t}}} g(x(\xi,\eta),y(\xi,\eta))\,d\xi\,d\eta$
- is it sufficient to have the Jacobian and the basis functions on the reference element ?

Computation via Ref Element (7/8)

Small Example - Proof of the Pudding

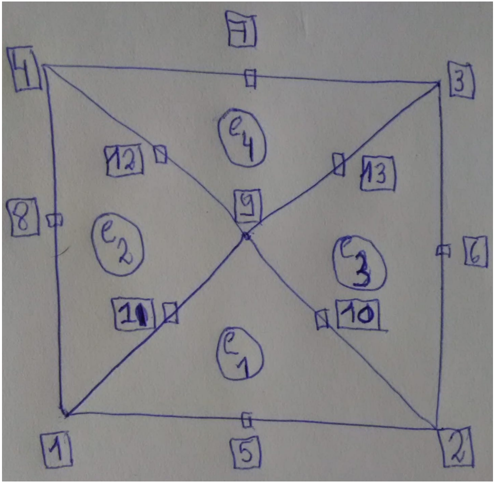
Computation via Ref Element (8/8)

Linear Basis Function on the Reference Element

- basis functions
  $\psi_1(\xi,\eta) = 1 - \xi - \eta$, $\psi_2(\xi,\eta) = \xi$ and $\psi_3(\xi,\eta) = \eta$
- basis function gradients
  $\nabla\psi_1(\xi,\eta) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\nabla\psi_2(\xi,\eta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\nabla\psi_3(\xi,\eta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- basis function gradients pairwise inner products
  $\nabla\psi_i \cdot \nabla\psi_j = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ for $1 \leq i,j \leq 3$
  constant on the element
- integral over triangle of pairwise inner products - area reference triangle is 0.5 - $\int_{\text{triangle}} \nabla\psi_i \cdot \nabla\psi_j \, dx\, dy = 0.5 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$
  $1 \leq i,j \leq 3$

Second Order Elements

Second Order Four-Element Five-Node Mesh Example



Exercises

- to be defined