ANALYZING EFFECT OF UNCERTAINTY IN OPTIMIZATION PROBLEMS SUCH AS SCHEDULING AND PRODUCTION PLANNING PROBLEM AND WAYS TO HANDLE THEM EFFECTIVELY

By Mohit Kumar Sharma & Dev Jyoti

1.0 ABSTRACT

This project delves into the critical realm of optimization problems under uncertainty, with a specific focus on the challenges presented in scheduling and production planning. The study addresses the inherent complexity of decision-making when faced with incomplete knowledge of future outcomes, emphasizing the ubiquity of uncertainty in diverse sectors such as electrical power generation, reservoir operation, inventory management, portfolio selection, and more. By exploring various modeling philosophies, including stochastic programming and fuzzy programming, the research aims to contribute to the evolving methodologies for robust decision-making in the presence of uncertainty. The mathematical formulation of a Production Planning Problem (PPP) serves as the foundation for two distinct analytical approaches: a naive method involving multiple runs and a robust optimization approach. The findings provide insights into the impact of uncertainty on profits and the trade-off between robustness and optimality.

2.0 INTRODUCTION

The most prominent example of optimization under uncertainty difficulties are fluctuations in the commodities market and general stock buying and selling issues. These problems involve making decisions without fully understanding the consequences of those decisions. These kinds of issues arise across a wide range of application domains and pose intriguing conceptual and computational challenges in the quest for dependable solutions. Most of the optimization literature makes the assumption that all data have known, constant values. However, in practice, uncertainty is common in many production, scheduling, and other problems because of the lack of precise process models and the variability of environmental and process data. As a result, the meaning of a decisive optimal solution is lost, and instead, the focus shifts to identifying the most likely optimal solution or making a confident prediction about it. In order to solve this issue, produce dependable/probable findings that are still achievable in the face of parameter uncertainty, and more realistically simulate real-world scenarios, a new field of study has already begun to take shape. Different approaches, such as stochastic, probabilistic, and fuzzy programming methods, have been used to solve the scheduling under uncertainty problem. We will discuss these approaches as we go along and outline our plan to address the problem analytically while also making adjustments to the algorithm, which we researched while drawing ideas from previously suggested strategies.

During our modelling of uncertainty and discussion we'll make certain assumptions justifying their applicability in certain use cases validating our uncertainty model. **Note the purpose of this project is to depict how introducing uncertainty causes deviation from classic mathematical techniques and discuss ways to mitigate or incorporate uncertain parameters within our**

model making naïve attempts as of now to produce results which remain feasible even under variability of defining parameters with certain level of confidence.

Before Beginning our discussion on we'll like to highlight some areas where need to optimize under uncertainty arises greatly further highlighting importance and need of this project.

2.1 Uncertainty:

Making decisions frequently requires facing uncertainty. Decisions made now will have repercussions that won't become completely apparent for some time. However, as more information becomes available, there might be opportunities for subsequent corrective action or even multiple opportunities for recourse.

Numerous modeling philosophies, such as expectation minimization, minimization of goal deviations, minimization of maximum costs, and optimization over soft constraints, have been applied to optimization under uncertainty. The primary methods for optimizing under uncertainty are as follows: stochastic dynamic programming, fuzzy programming (possibilistic and flexible programming), and stochastic programming (recourse models, robust stochastic programming, and probabilistic models). We can also discuss applications, the state-of-the-art in computations, and significant algorithmic advancements made by the process systems engineering community.

Since our solution is based on modeling parameter uncertainty using the concept of random variables, we primarily focus on stochastic modeling.

2.2 Modelling of uncertainty:

A problem's uncertainties must be represented so that their implications on current decision-making can be appropriately considered. This is an intriguing and difficult topic, but our understanding of probability can help.

Stochastic modeling: The theory of probability can be used to model the uncertain components of a problem as random variables. Such elements need to have a "known" probability distribution in order to serve this purpose.

Degrees of knowledge: As with weather variables, such a distribution may be derived from statistical data, or it may simply be "subjective" probabilities based on educated guesswork, as in the case of interest rates or election prospects. The mathematical approach is the same in either case.

Stochastic formulations incorporate uncertainty by taking advantage of the fact that probability distributions governing the data are known or can be estimated. The goal then is to find an optimal solution that is feasible for all, or almost all, of the instances of the uncertain parameters while maximizing the expectation of some function of the problem variables and the random variables.

3. MATHEMATICAL FORMULATION / SOLUTION STRATEGY:

We begin with the use of existing Mathematical formulation of PPP as discussed in class and try two approaches and visualize their result.

3.1 MATHEMETICAL MODELING OF PPP:

$$PROFIT = \sum_j SP(j) * X(j) - \sum_j (PC_l(j) * L(j) + PC_m(j) * M(j) + PC_h(j) * H(j))$$

$$X(j) = cl(j) * L(j) + cm(j) * M(j) + ch(j) * H(j)$$

$$H(j) = 1 - Y(j)$$

$$L(j) \leq Y(j)$$

$$L(j) + M(j) + H(j) = Z(j)$$

$$X(j) \leq U * z(j)$$

$$\sum_j rm(k,j) * X(j) \leq R(k)$$

$$\sum_j (IC_l(j) * L(j) + IC_m(j) * M(j) + IC_h(j) * H(j)) \leq B$$

3.1.1 Naïve approach to solve uncertainty with multiple runs:

In this approach we,

- Execute the optimization model with different scenarios of raw material availability.
- Observe variations in profits to understand the impact of uncertainty.
- Analyze the results to identify patterns or trends in the relationship between raw material availability and profit.

We model the random perturbation as:

$$\tilde{a}_{lm} = (1 + \epsilon \xi_{lm}) a_{lm}$$
$$\tilde{b}_{lk} = (1 + \epsilon \xi_{lk}) b_{lk}$$
$$\tilde{p}_l = (1 + \epsilon \xi_l) p_l$$

where ξlm, ξlk and ξl are independent random variables and $\epsilon > 0$ is a given (relative) uncertainty level.

We see the variation of profit w.r.t. to variation in availability of raw materials, trying to identify whether such variation will be linear or not does it actually even impact the problem and if it does than to under what extent. Irrespective of type, it will provide sufficient analytic data to visualize the impact of uncertainty in raw materials giving analytical intuition to choose the result from the most feasible region.

$$R_{lk} = (1 + \epsilon \cdot \xi) \cdot R_{lk}$$

To achieve Normal distribution in Gams we used,

$$R(r1) = ROrignal(r1) * (1 + 0.5 \cdot Normal(0,0.2))$$

To achieve Uniform distribution in Gams we used,

$$R(r1) = ROrignal(r1) * (1 + 0.5 \cdot Uniform(-1,1))$$

3.1.2 Robust optimization approach and formulation:

Robust optimization is a field of mathematical optimization theory that deals with optimization problems in which a certain measure of robustness is sought against uncertainty that can be represented as deterministic variability in the value of the parameters of the problem itself and/or its solution.

The underlying mathematical framework is based on a robust optimization methodology first introduced for linear programming (LP) problems by and extended in Lin et al. (2004) and this work for mixed-integer linear programming problems. The approach produces "robust" solutions for uncertainties in both the coefficients and right-hand-side parameters of the linear inequality constraints and can be applied to address the problem of production scheduling with uncertain parameters.

Consider the following generic mixed-integer linear programming problem

$$
\begin{aligned}
\underset{x,y}{\text{Min/Max}} \quad & c^T x + d^T y \\
\text{s.t.} \quad & Ex + Fy = e \\
& Ax + By \leq p \\
& \underline{x} \leq x \leq \bar{x} \\
& y = 0, 1.
\end{aligned}
$$

Assume that the uncertainty arises from both the coefficients and the right-hand-side parameters of the inequality constraints, namely, a_{lm}, b_{lm} and p_l where l is the index of the uncertain inequality, m is the index of the continuous terms, and k is the index of the binary terms. Thus, we are concerned about the feasibility of the following inequality

$$\sum_m \tilde{a}_{lm} x_m + \sum_k \tilde{b}_{lk} y_k \leq \tilde{p}_l$$

Where a_{lm}, b_{lk} and p_l are the nominal values of the uncertain parameters and $\tilde{a}_{lm}, \tilde{b}_{lk}$, and $\tilde{p}_l$ are the "true" values of the uncertain parameters.

Assume that for inequality constraint l, the true values of the uncertain parameters are obtained from their nominal values by random perturbations

$$\tilde{a}_{lm} = (1 + \epsilon \xi_{lm}) a_{lm}$$
$$\tilde{b}_{lk} = (1 + \epsilon \xi_{lk}) b_{lk}$$
$$\tilde{p}_l = (1 + \epsilon \xi_l) p_l$$

where ξ_{lm}, ξ_{lk} and ξ_l are independent random variables and $\epsilon > 0$ is a given (relative) uncertainty level.

In this situation, we call a solution (x, y) robust if it satisfies the following

(i) (x, y) is feasible for the nominal problem, and

(ii) for every inequality l, the probability of violation of the uncertain inequality in Eq. (2) (i.e., the left-hand-side exceeds the right-hand-side) is at most κ,

The formulation allows for control over the degree of conservatism of the solution in terms of the probabilistic bounds on constraint violation producing results which remain feasible even under uncertainty.

Based on the above we incorporate the formulation by relaxing inequality of availability of raw materials:

$$R_{lk} = (1 + \epsilon \cdot \xi) \cdot R_{lk}$$

$$\sum_j rm(k,j) * X(j) \leq R(\widetilde{k})$$

$$\sum_j rm(k,j) * X(j) \leq R(k) - \varepsilon * (1 - 2 * k) * R(k) + \delta * \max(1, R(k))$$

This ensures that the results produced remain feasible under given uncertainties with a certain level of tolerance (δ) and reliability (k) resulting in effective and reliable decision making. While also ensuring less computational load compared to naïve approach where each uncertainty we would have to reiterate resulting in greater computational loss.
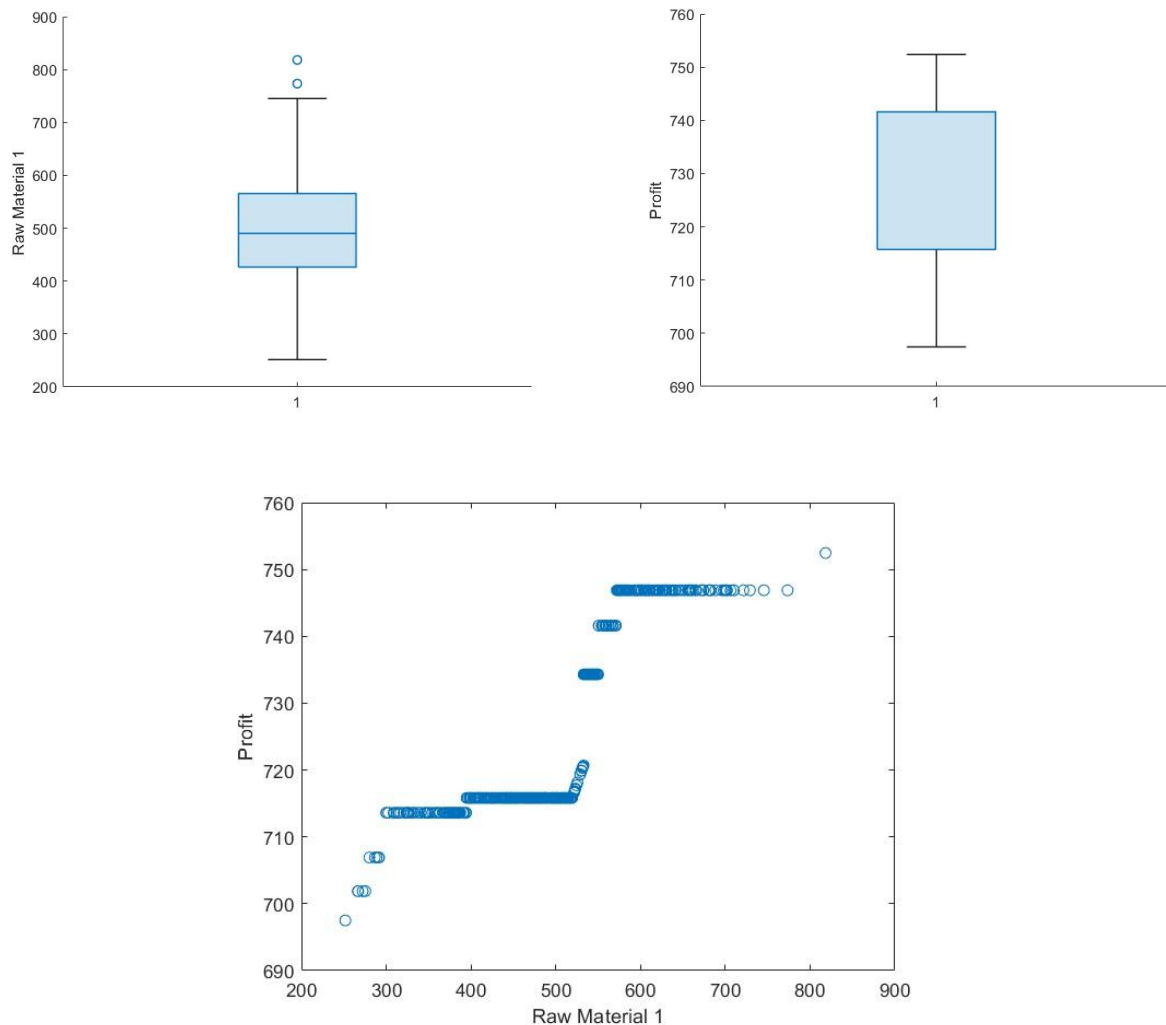
4.0 Results & Discussions:

4.1 Naïve approach

4.1.1. UNDER UNIFORM DISTRIBUTION (-0.5 to 0.5) OF UNCERTAINITY IN RAW MATERIAL
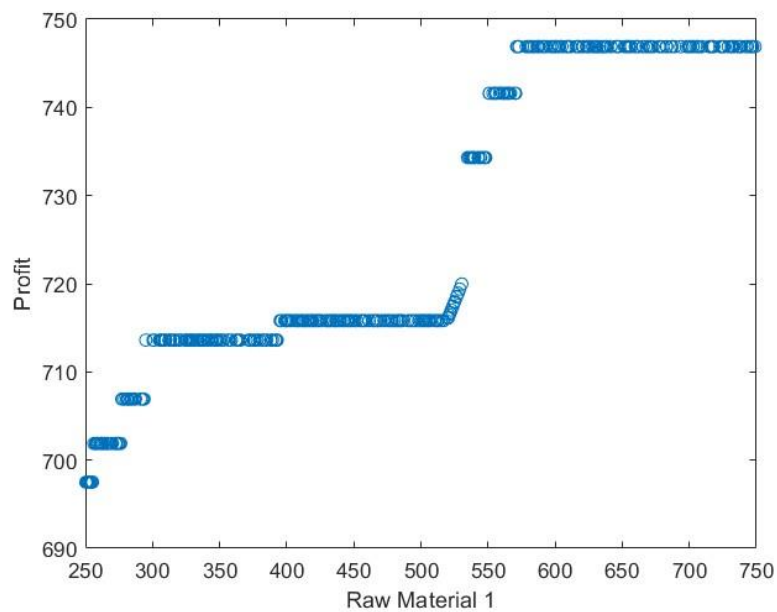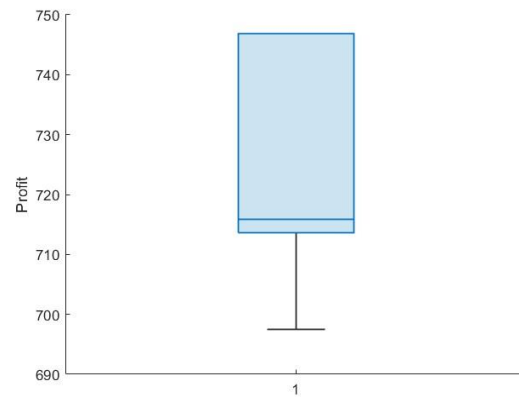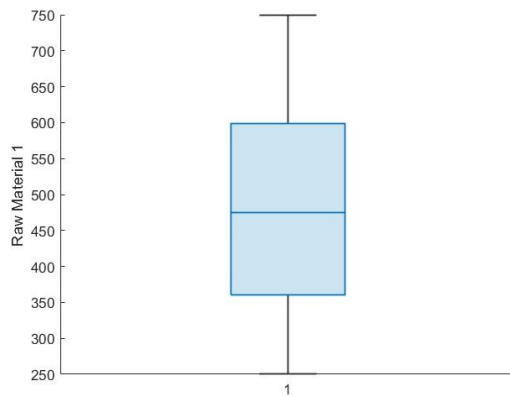
4.1.2. UNDER NORMAL DISTRIBUTION (MEAN 0, STD 0.2) OF UNCERTAINITY

Intuitively, we expect that as our Model has Linear dependency on Raw materials, we expect them to be a linearly increasing graph. But we find it to be constant within certain bands. So, if we find our raw martial supply to vary within one of the bands, we can rest assure that our profit will not change

## 4.2. Robust optimization

4.2.1 Uncertainty Modeling:

o We define the random perturbations in uncertain parameters using independent random variables and assuming the distribution of such uncertainty.

o Specify the uncertainty level (ϵ) to control the degree of conservatism.

4.2.2 Degree of Conservatism:

The term "conservatism" refers to the cautious approach taken to ensure that solutions remain feasible even in the face of uncertain parameters. It represents a deliberate trade-off between pursuing optimal outcomes and safeguarding against potential risks introduced by the variability of parameters.

- o We discuss how adjusting the uncertainty level (ϵ) allows for tuning the conservatism of the solution.
- o Highlight the trade-off between robustness and optimality in the face of uncertainty.

Intuitively, we can expect that as we increase uncertainty in the raw material the expected profit we can obtain should decrease as provide safe side result which remain feasible with greater probability should be a lower value more over as we allow the deviation from given bounds of certain inequality the resulting value should increase at the expense of violation in certain case.
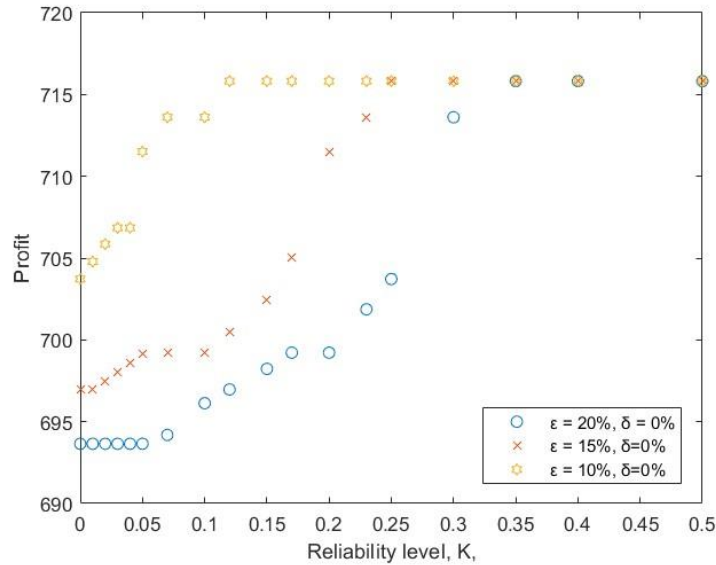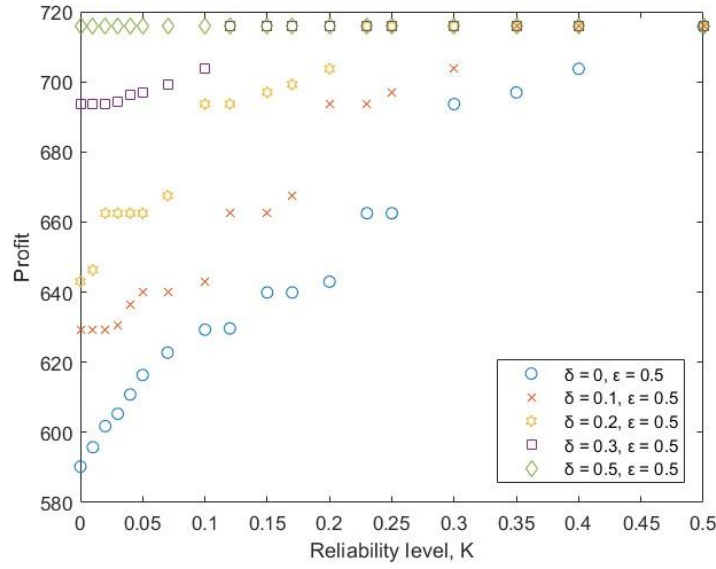
The above graph does agree with our intuitive understanding, moreover if we look at given reliability level K, we find that the band variation in profit still holds true. Though Important point to note here is that while naïvely we need to do multiple runs to arrive at a conclusive result and analytical analysis, this approach does produce satisfactory results by incorporating uncertainty bounds inside deterministic MILP formulation saving us from expensive re-computations and analytical procedure giving us satisfactory results in single run. The result obtained can be considered as a worst-case result under given ε, δ and k.

5.0 CONCLUSION AND FUTURE WORKS:

This project contributes to the field of optimization under uncertainty, specifically addressing challenges in production planning production, taking inspiration from available literature on scheduling problems. The study explores two distinct analytical approaches: a naive method involving multiple runs and a robust optimization approach. The naive approach provides valuable insights into the impact of uncertainty on profits, showcasing variations in different scenarios. On the other hand, the robust optimization approach introduces a systematic and computationally efficient methodology that ensures feasibility under specified uncertainty levels.

Through the mathematical formulation of a Production Planning Problem (PPP), this research illuminates the complexities of decision-making in the presence of incomplete knowledge. The naive approach, relying on multiple runs with varied scenarios, sheds light on the variability of profits under different conditions. However, it lacks the systematic control over robustness that the robust optimization approach offers.

The robust optimization approach, with its foundation in mathematical rigor, introduces a trade-off between optimality and robustness. Decision-makers can fine-tune the degree of conservatism, providing solutions that remain feasible even under uncertain conditions. This

approach is computationally efficient, reducing the need for repetitive computations and offering a reliable framework for decision-making. Though modelling under this approach and probabilistic bounding is quite rigorous and time consuming.

5.1 Naive Approach vs. Robust Optimization Approach: A Comparative Analysis

5.1.1 Basic Principle:

- *Naive Approach:* The naive approach involves executing the optimization model with different scenarios of raw material availability without fundamentally altering the original model. It relies on multiple runs to observe variations in profits and analyze the impact of uncertainty.
- *Robust Optimization Approach:* The robust optimization approach introduces a systematic methodology to address uncertainty. It modifies the optimization model to produce solutions that remain feasible under uncertainty, introducing probabilistic bounds on constraint violation.

5.1.2 Handling Uncertainty:

- *Naive Approach:* Handles uncertainty by running the optimization model multiple times with different scenarios of raw material availability. Provides insights into the variability of profits but may lack a systematic approach to ensure feasibility under uncertainty.
- *Robust Optimization Approach:* Addresses uncertainty through a rigorous mathematical formulation that considers both coefficients and right-hand-side parameters as uncertain. Introduces a level of conservatism to ensure feasibility under a specified level of uncertainty.

5.1.3 Computational Efficiency:

- *Naive Approach:* Involves running the optimization model multiple times, which can be computationally expensive, especially for complex models with numerous parameters.
- *Robust Optimization Approach:* Typically results in a more computationally efficient solution as it incorporates uncertainty bounds within a deterministic framework. Reduces the need for repetitive computations.

5.1.4 Trade-off Between Optimality and Robustness:

- *Naive Approach:* May obtain more optimal results for specific scenarios but lacks a systematic control over robustness. The obtained solutions may not be feasible under certain conditions of uncertainty.
- *Robust Optimization Approach:* Provides a trade-off between optimality and robustness. Decision-makers can control the degree of conservatism by adjusting parameters, allowing them to choose solutions that are both optimal and resilient to uncertainty.

5.1.5 Application to Decision-Making:

- *Naive Approach:* Offers insights into the impact of uncertainty on profits but may not provide actionable solutions that guarantee feasibility under all conditions.
- *Robust Optimization Approach:* Results in solutions that are guaranteed to be feasible under specified uncertainty levels, offering decision-makers more reliable strategies for production planning and scheduling.

5.1.6 Flexibility in Handling Uncertainty Models:

- *Naive Approach:* Relies on simplistic models of uncertainty, such as normal or uniform distributions, and may not capture the full complexity of real-world uncertainty.
- *Robust Optimization Approach:* Provides a more flexible framework to model uncertainty, allowing for the incorporation of various distribution types and offering a more nuanced understanding of uncertainty patterns.

5.2 Future Works:

- **Refinement of Robust Optimization Model:** Further refinement and fine-tuning of the robust optimization model can be explored. Adjustments in the formulation parameters and exploration of additional constraints may enhance the model's performance in specific scenarios.
- **Integration of Real-Time Data:** Incorporating real-time data and adaptive strategies to update the optimization model dynamically based on the latest information can be explored. This would make the decision-making process more responsive to changing conditions.
- **Comparative Analysis:** A comprehensive comparative analysis between different uncertainty modeling approaches, such as stochastic programming and fuzzy programming, can be conducted. Understanding the strengths and limitations of each approach in various contexts would contribute to the development of a more versatile decision-making framework.
- **Application to Specific Industries:** The research can be extended to focus on specific industries, such as electrical power generation, reservoir operation, or inventory management. Tailoring the optimization models to the unique challenges of each industry would provide more targeted and actionable insights.
- **Incorporation of Multi-Objective Optimization:** Exploring the integration of multi-objective optimization techniques to balance conflicting goals in the face of uncertainty. This would allow decision-makers to consider not only profit but also other critical factors like risk mitigation and resource utilization.
- **Machine Learning Integration:** Investigating the integration of machine learning techniques to predict and adapt to uncertainty patterns over time. This adaptive approach could enhance the robustness of the optimization model in dynamic and evolving environments