Astrofisica Generale II — 2
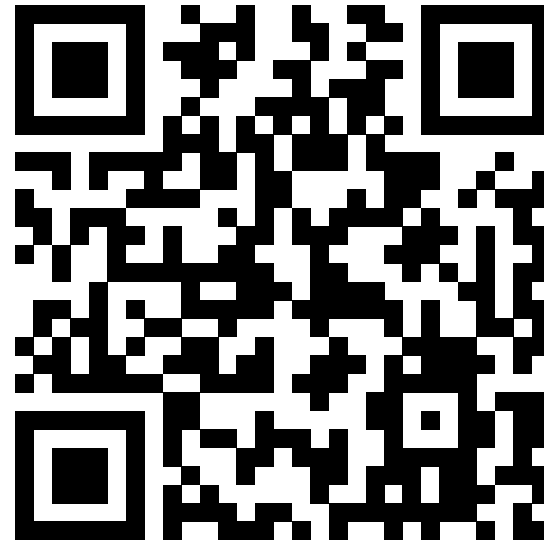
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These slides

- They are available at ziotom78.github.io/lezioni-astronomia/.

- Each time I will make a PDF copy of the slides available on the website.

Relaxation Time

Relaxation Time

- Let's now consider the time required for a cluster to become dynamically relaxed.

- Initially, the stars in a cluster may *not* be relaxed: in this case, the fastest ones ($v > v_f$) escape from the cluster, and this "evaporation" changes the distribution of v.

- Moreover, gravitational interactions cause a redistribution of energy, which leads the cluster towards a relaxed state.

gravatational collapse of spongebob (REUPLOAD)

Gravitational collapse of Spongebob

Relaxation Time

- To quantify the relaxation time, we can assume that it is the time required for each star in the cluster to interact a certain number N_{int} of times with its companions.
- (This is analogous to how one studies an ideal gas reaching thermodynamic equilibrium).

Relaxation Time

- We can define an interaction between two stars as the condition in which the kinetic energy becomes equal to the potential energy between the two (why?):

$$\frac{1}{2}M_* v^2 \sim G\frac{M_*^2}{r}.$$

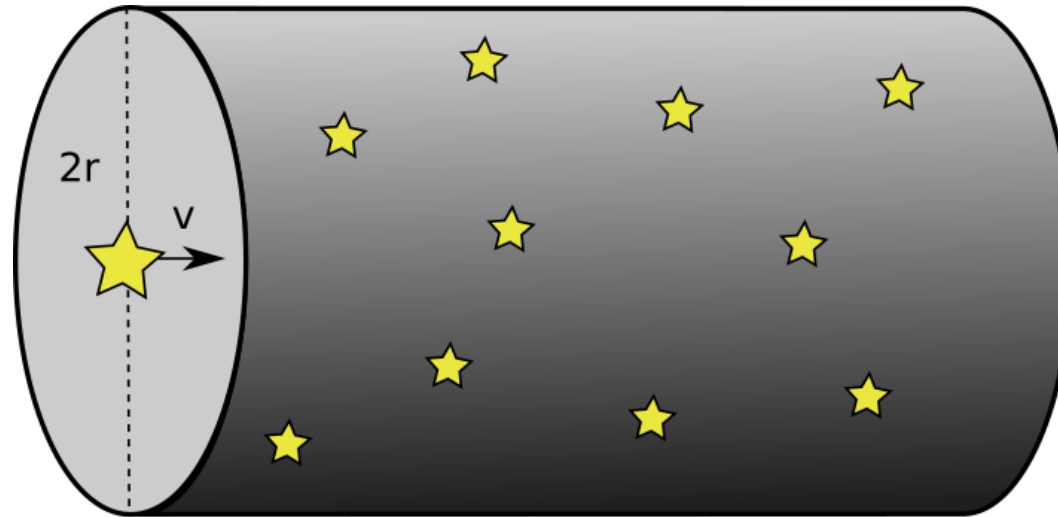- This occurs when the distance between the two stars is

$$r_c \sim 2G\frac{M_*}{v^2}.$$

- The value r_c is called the **collisional radius.**

Relaxation Time

- How likely is a star to interact with others? It depends on how fast the star is moving and the density of its companions:
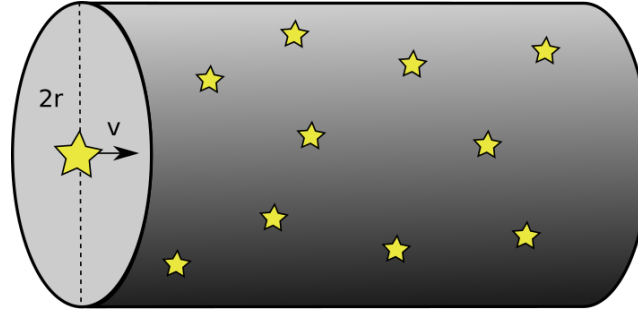

- In the volume $V = \pi r^2 \, \Delta x$ there are $Vn = \left(\pi r^2 \, \Delta x \right) n$ stars (with n being the number density).

Relaxation Time

If the distance traveled by the star is $\Delta x = v\Delta t$, then during the relaxation time Δt_r the star interacts collisionally with the N_{int} stars in the cylinder with base $r = r_c$:

$$\left(\pi r_c^2\, v\, \Delta t_r\right) n = N_{\text{int}} \quad \Rightarrow \quad \Delta t_r = \frac{N_{\text{int}}}{\pi r_c^2\, v\, n}.$$

Relaxation Time

If we now set $N_{\text{int}} \approx 1$ and substitute the expression $r_c \sim 2G\frac{M_*}{v^2}$ into

$$\Delta t_r = \frac{N_{\text{int}}}{\pi r_c^2 \, v \, n}$$

we obtain:

$$\Delta t_r = \frac{v^3}{4\pi G^2 M_*^2 \, n}.$$

Relaxation Time

The expression for Δt_r can be greatly simplified. First, $n = N / \left(\frac{4}{3} \pi R^3 \right)$; furthermore, we can use the virial theorem:

$$K = -\frac{1}{2} U,$$

$$\frac{1}{2} N M_* v^2 = \frac{3}{5} G \frac{(N M_*)^2}{R},$$

$$G M_* N \approx R v^2 \quad \left(\text{assuming } \tfrac{3}{5} \approx \tfrac{1}{2} \right).$$

Relaxation Time

Substituting the expressions for n and GM_*N, we obtain

$$\Delta t_r \approx \frac{NR}{3v}.$$

Therefore, the relaxation time is of the same order of magnitude as the time required to complete N crossings of the cluster (R/v is the time for *one* crossing), with N being the number of stars.

Relaxation Time

- Our estimate leads to a value of Δt_r equal to

$$\Delta t_r \approx \frac{1}{3} \times \frac{10^6 \times 5\,\mathrm{pc}}{16\,\mathrm{km/s}} \approx 100\,\mathrm{Gyr},$$

- But this number is implausible! The universe is less than 14 billion years old, yet most globular clusters appear to be already relaxed.

Relaxation Time

- The problem is that the calculation only considers *short*-range interactions, while long-distance energy exchanges are also relevant.

- We assumed that interaction occurs when the distance between two stars is less than the collision radius r_c given by

$$\frac{1}{2} M_* v^2 \sim G \frac{M_*^2}{r_c}.$$

- But even at distances greater than r_c there are energy exchanges, and we have neglected them.

Average Distance Between Stars

- Let's calculate the average distance λ between two stars in the cluster:

$$\lambda \sim \frac{1}{n^{1/3}} = \left(\frac{4\pi}{3N}\right)^{\frac{1}{3}} R_{\mathrm{core}} \approx 2.4 \times 10^{15}\,\mathrm{m} = 6700\,\mathrm{AU},$$

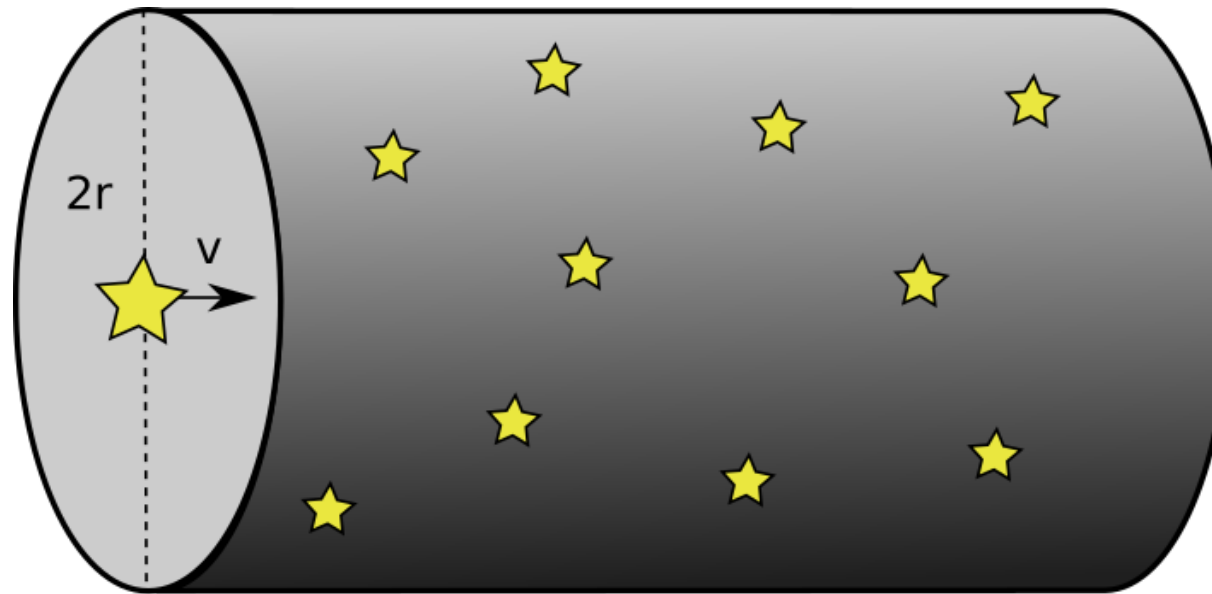if $R_{\mathrm{core}} = 5\,\mathrm{pc}$ and $N = 10^{6}$.

- The collision radius, in the case $v = v_{\mathrm{rms}} = 16\,\mathrm{km/s}$, for a star $M_* = M_\odot/2$, is instead

$$r_c = \frac{2GM_*}{v^2} \approx 5.2 \times 10^{11}\,\mathrm{m} \approx 750\,R_\odot = 0.3\,\mathrm{AU}.$$

Relaxation Time

The problem is that we have only considered short-range interactions, but the previous calculation tells us that they are very rare because the average distance between stars is 10^4 times greater than the collisional radius.
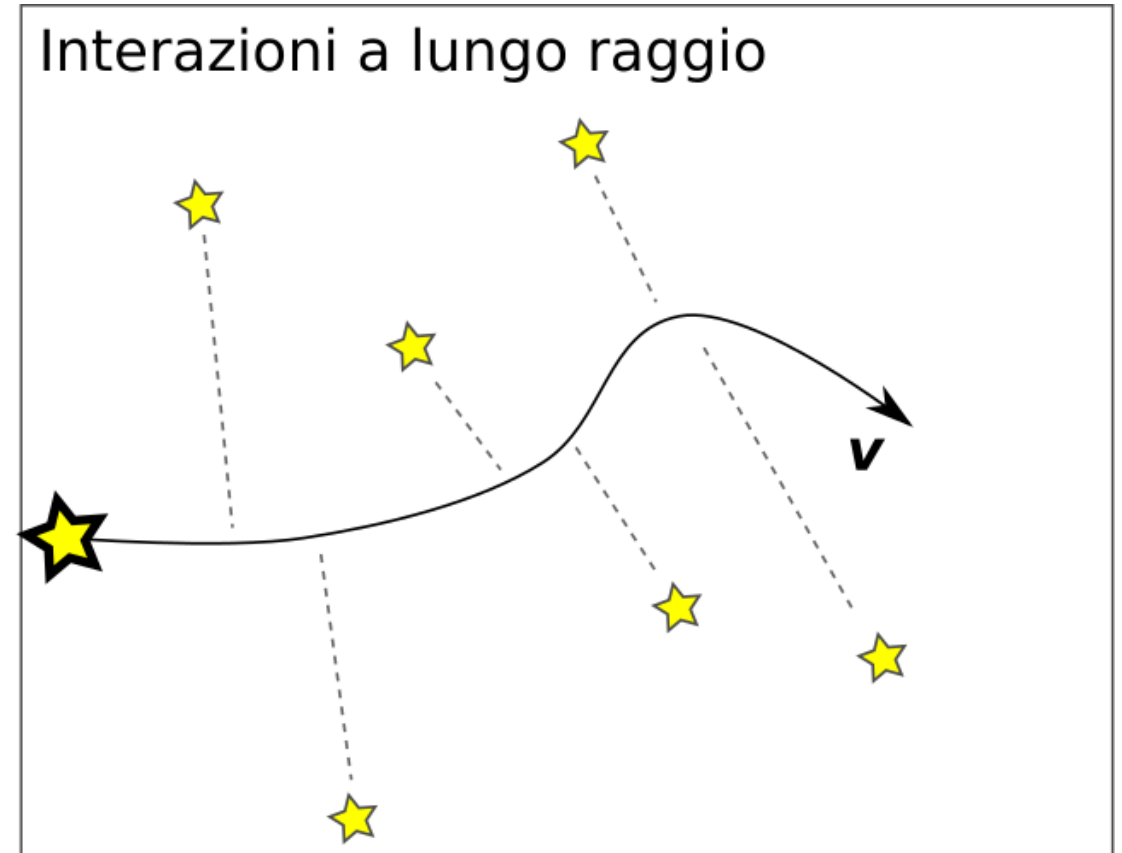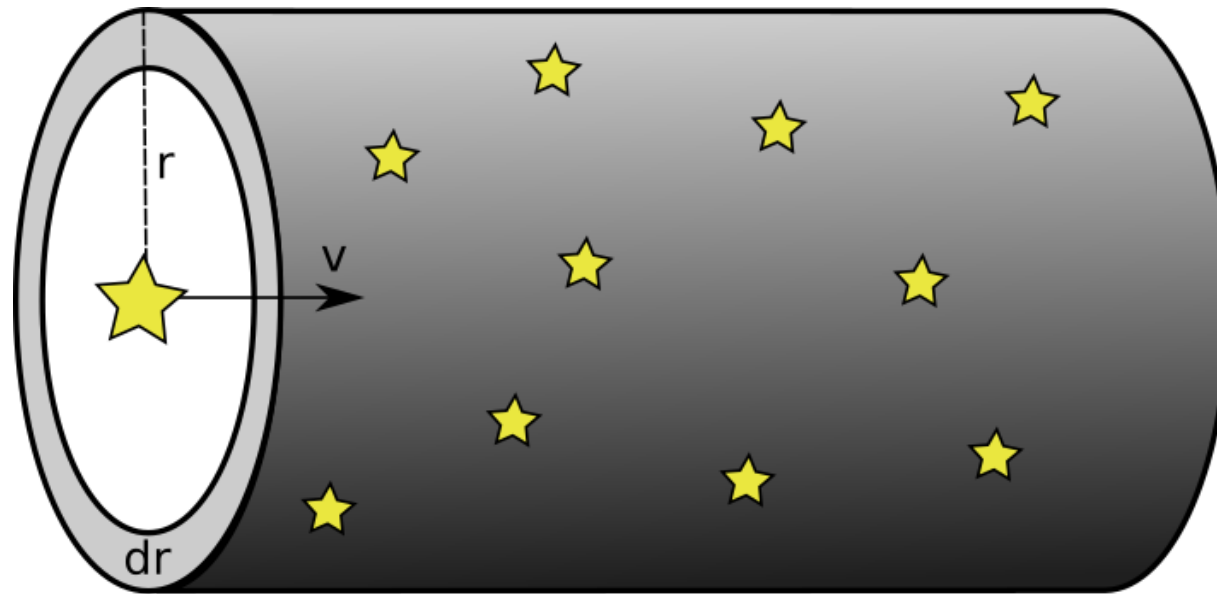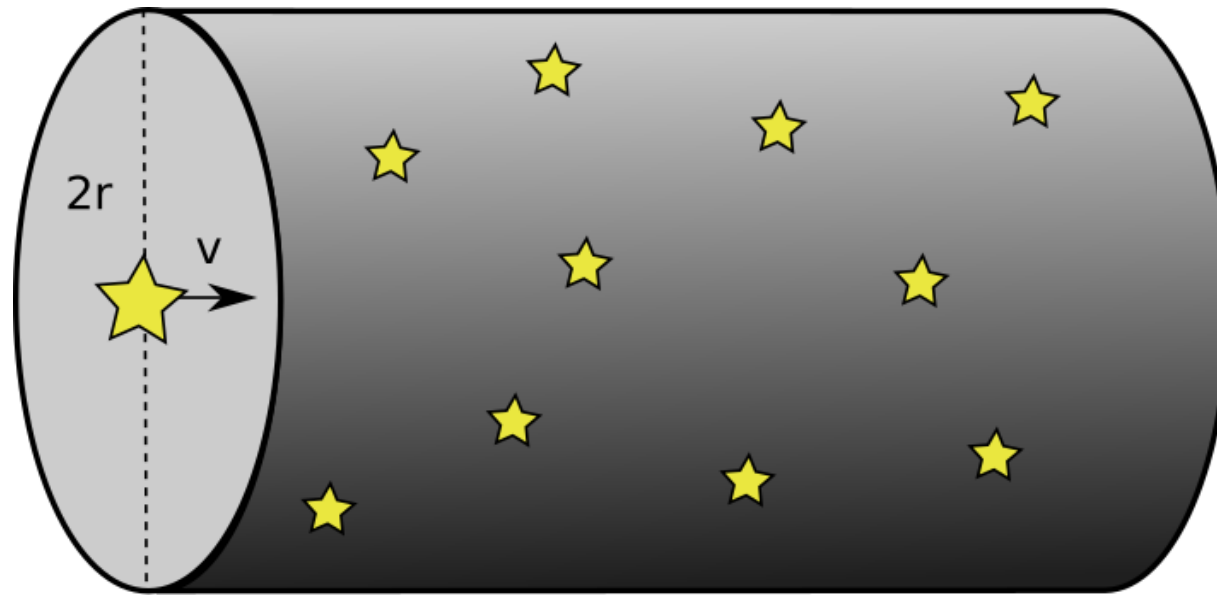
Long-Range Interactions

If a star cluster is virialized, then this is probably mainly determined by long-range interactions:

No interazioni a lungo raggio

Interazioni a lungo raggio

2r

v

r

v

dr

Long-Range Interactions

- We want to estimate the importance of long-range interactions. We will therefore measure the average deflection $\vec{\Delta v}_\perp$ caused by these interactions.
- Obviously, successive interactions will have different values of $\Delta v_\perp$, both in magnitude and in direction. The resulting motion will be Brownian, such that

$$\left\langle \vec{\Delta v}_\perp \right\rangle = 0$$

(Brownian motion has no preferential direction).

Long-Range Interactions

- However, we are not interested in characterizing the direction of $\Delta v_\perp$, only in understanding the order of magnitude of the velocity $\Delta v_\perp$. The most appropriate quantity is the *root mean square* $\langle \Delta v_\perp^2 \rangle$.
- It has the following properties:
  1. It is related to the amount of energy the star acquires due to long-range interactions (because $K \propto v^2$);
  2. It increases with time;
  3. It becomes important when $\langle \Delta v_\perp^2 \rangle \approx v^2$.

Long-Range Interactions

What happens in an interaction where $r > r_c$?
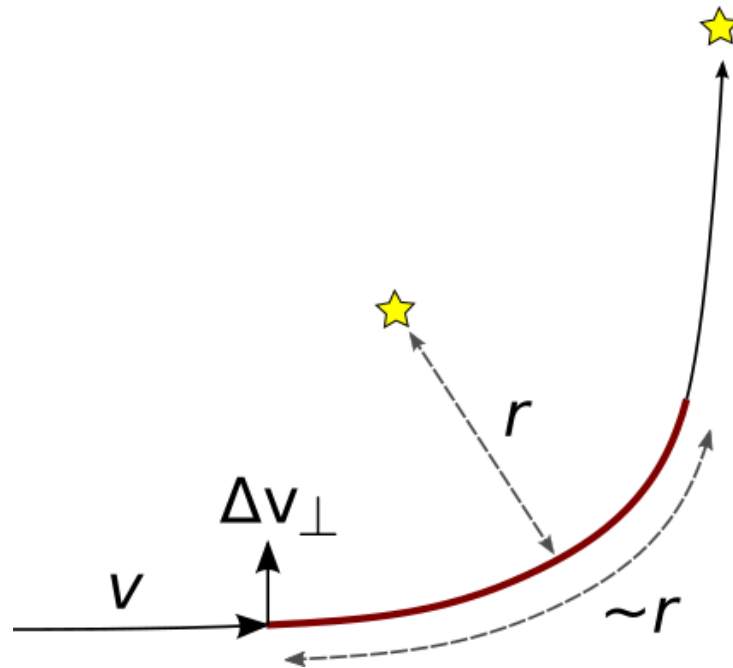
Let's assume that the energy exchange occurs only along the red line, and that it introduces a perpendicular component $\Delta v_\perp$ to the velocity v.

Long-Range Interactions

We can estimate the order of magnitude of $\Delta v_\perp$ using Newton's laws:

$$M_* \frac{\Delta v_\perp}{\Delta t} = G \frac{M_*^2}{r^2},$$

from which

$$\Delta v_\perp = G \frac{M_*}{r^2/\Delta t} = G \frac{M_*}{r\,v}$$

(Δt is the time taken to travel the red segment, which has length $\sim r$: therefore $r/\Delta t \approx v$).
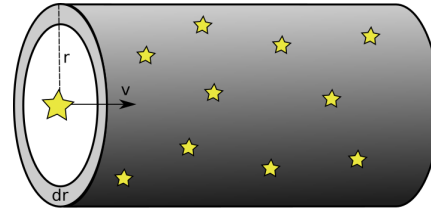
Long-Range Interactions

- What are plausible values to use for r? We have already seen that in globular clusters short-range interactions ($r \leq r_c$) matter little; but obviously $r < R$ (with R being the size of the cluster).

- In general $r_c < r < R$.

- Since $\Delta v_\perp$ is a function of r, we must calculate an «average» value of $\Delta v_\perp$.

Long-Range Interactions

We calculate the mean square value of $\Delta v_\perp$ (the star's trajectory in the cluster can be approximated as Brownian motion):

$$\langle (\Delta v_\perp)^2 \rangle = \int_{r_c}^{R} (2\pi r \, \mathrm{d}r) \, (v\Delta t) \, n \left(G \, \frac{M_*}{r \, v} \right)^2 =$$

$$= \frac{2\pi n G^2 M_*^2}{v} \, \Delta t \, \log \left(\frac{R}{r_c} \right).$$

Long-Range Interactions

For large values of Δt, the component $\Delta v_\perp$ becomes important. After how long does $\Delta v_\perp \approx v$?

$$\left\langle (\Delta v_\perp)^2 \right\rangle \approx v^2$$

$$\frac{2\pi n G^2 M_*^2}{v} \, \Delta t \, \log N \approx v^2$$

$$\Delta t \approx \frac{v^3}{2\pi n G^2 M_*^2 \, \log N}.$$

Long-Range Interactions

Using the usual substitutions

$$n = \frac{N}{\frac{4}{3}\pi R^3}, \quad NM_*G \approx Rv^2,$$

the expression for Δt becomes

$$\Delta t \approx \frac{N}{\log N} \frac{R}{v}.$$

Compared to our initial estimate $\Delta t_r \sim NR/v$, here $\log N$ appears in the denominator. For a typical cluster, $\log N \approx \log 10^6 \approx 14$.

Relaxation Time

The exact calculation leads to the formula

$$\Delta t_r \approx \frac{1}{12\ln(N/2)}\frac{NR}{v},$$

from which we obtain

$$\Delta t_r \approx \frac{1}{12\ln(10^6/2)}\frac{10^6 \times 5\,\mathrm{pc}}{16\,\mathrm{km/s}} \approx 2\,\mathrm{Gyr},$$

which is physically plausible, at last!

Relaxation Time

- The calculation carried out in this lecture is explained in exercise 1.14 of *Cosmology and Astrophysics through problems* (T. Padmanabhan, Cambridge University Press, 1996).

- Chapter 10.7 of volume I of *Theoretical Astrophysics* (T. Padmanabhan, Cambridge U. P., 2000) contains the full derivation of the formula in the case of a plasma (the Coulomb potential depends on r as the Newtonian one does).

Age of globular clusters

THE AGES AND DISTANCES OF GLOBULAR CLUSTERS WITH THE LUMINOSITY FUNCTION METHOD: THE CASE OF M5 AND M55

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ABSTRACT

We present new age and distance determinations for the Galactic globular clusters M55 and M5, using the luminosity function (LF) method of Padoan & Jimenez. Using the set of stellar tracks computed by Jimenez & MacDonald and taking into account a value of $[\alpha/\text{Fe}] = 0.4$, we find an age of 12.5 ± 1.0 Gyr for M55 and 10.6 ± 0.8 Gyr for M5. We also find $m - M = 14.21 \pm 0.08$ mag for M55, and $m - M = 14.55 \pm 0.10$ mag for M5, where the errors refer to the internal accuracy obtained from the method applied to the observed LFs of M5 and M55. These values of $m - M$ agree with the ones obtained using the tip of the red giant branch and with the ones given by the subdwarf fitting method with Reid's new *Hipparcos* results. If the accuracy of the LF method is further confirmed, it will mean that the period of the formation of the Galactic halo took place over a few Gyr and that halo globular clusters are not older than 14 Gyr.

Subject headings: globular clusters: individual (M5, M55) — stars: distances — stars: evolution

Age of globular clusters

The LF is a natural clock because the number of stars in a given luminosity bin decreases with time, since more massive stars evolve more rapidly than less massive ones. The fact that small differences in stellar masses correspond to large differences in evolutionary time explains the power of the LF clock…
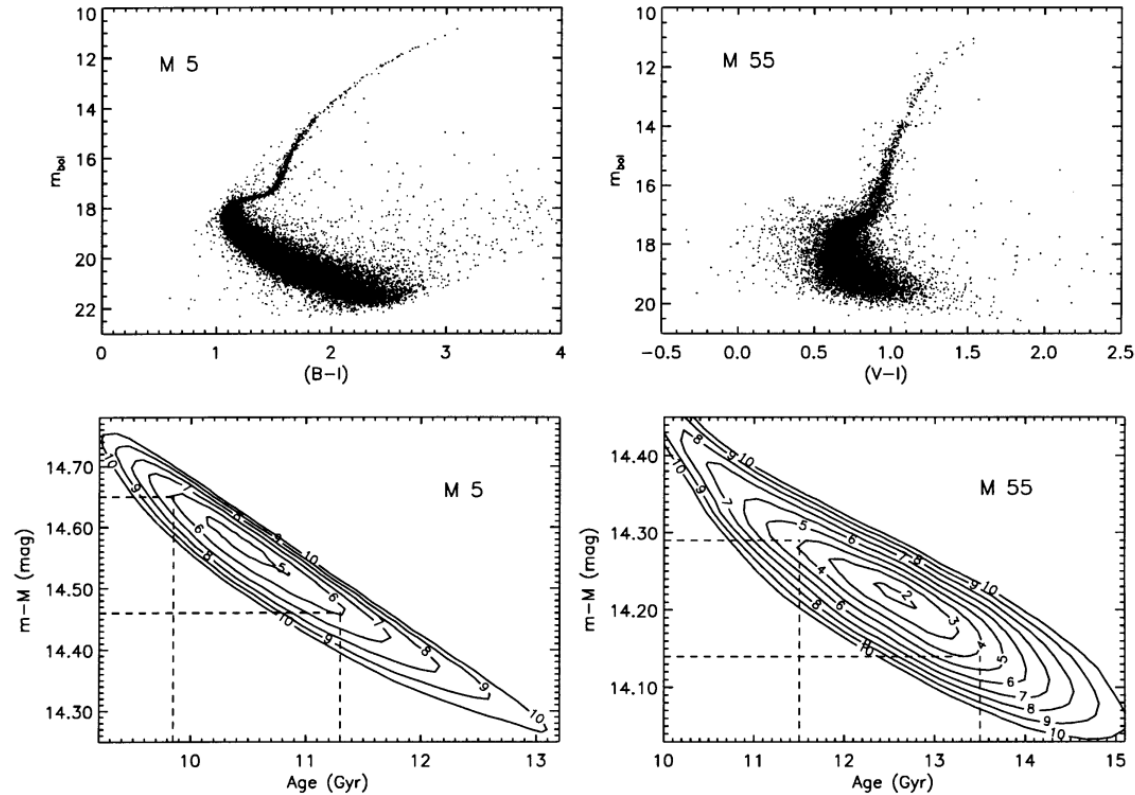
Age of globular clusters

FIG. 1.—*Upper panels*: Color-magnitude diagrams for M5 (*left*) and M55 (*right*) containing all the stars, after the elimination of the AGB and HB stars, used in this work. *Lower panels*: Contour plots of $R(t, m − M)$ (see text), the residual between theoretical and observational counts for M5 (*left*) and M55 (*right*). The contour lines are closed around a minimum value of $R(t, m − M)$, which shows that the LF method can break the age-distance degeneracy. The numbers along the contour lines are the values of $100 × R$ that are to be compared with the uncertainty in the observational stellar counts in percent. The dashed lines mark the 1σ value around the most probable age and distance modulus.

Lifetime of GCs

- The course's reference text (Kutner) claims that the lifetime of globular clusters is much longer than the age of the Universe ($\sim 200\,\mathrm{Gyr}$).

- However, this excludes the perturbations that occur when globular clusters cross the Galactic disk, which can lead to their progressive destruction.

- Furthermore, computer simulations (Zonoozi et al., 2011, Zonoozi et al., 2014) show that evaporation processes are up to an order of magnitude faster than estimated by our calculation.

Open clusters

	Open clusters	**Globular clusters**
# of stars	10^3–10^4	10^4–10^6
Size	10 pc	20–100 pc (core: 5 pc)
Gas and dust?	Yes	No
Planetary nebulae?	No	Yes
# of known clusters	10^3	~160
Dove?	Disco	Stellar halo (~1% of total mass)

M45

In Taurus. $N \sim 500$, age 10^8 yr, $R \sim 8$ ly, $D \sim 440$ ly.

Open cluster population

- To date, approximately 1,100 open clusters have been cataloged.
- Most of them are in the galactic disk
- At great distances, it is difficult to conduct a census due to dust in the galactic disk.
- Open clusters, therefore, form a much larger population than globular clusters.

Relaxation time for OCs

- Most clusters have ages less than 10^9 yr.

- From the formula

$$\Delta t_r \approx \frac{1}{12 \log \frac{N}{2}} \frac{N R^{3/2}}{\sqrt{GM_*}}$$

we obtain for open clusters that $\Delta t_r \approx 10^8 \div 10^9$ **yr**: therefore a good portion of open clusters **are not relaxed**.

Relaxation time for OCs

Star clusters under stress: why small systems cannot dynamically relax

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ABSTRACT
Utilizing a series of N-body simulations, we argue that gravitationally bound stellar clusters of modest population evolve very differently from the picture presented by classical dynamical relaxation theory. The system's most massive stars rapidly sink towards the centre and form binary systems. These binaries efficiently heat the cluster, reversing any incipient core contraction and driving a subsequent phase of global expansion. Most previous theoretical studies demonstrating deep and persistent dynamical relaxation have either conflated the process with mass segregation, ignored three-body interactions, or else adopted the artificial assumption that all cluster members are single stars of identical mass. In such a uniform-mass cluster, binary formation is greatly delayed, as we confirm here both numerically and analytically. The relative duration of core contraction and global expansion is effected by stellar evolution, which causes the most massive stars to die out before they form binaries. In clusters of higher N, the epoch of dynamical relaxation lasts for progressively longer periods. By extrapolating our results to much larger populations, we can understand, at least qualitatively, why some globular clusters reach the point of true core collapse.

Key words: binaries: general – stars: kinematics and dynamics – stars: luminosity function, mass function – open clusters and associations: general.

Relaxation time for OCs

…[the] evolutionary status [of open clusters] is much less clear. Half of open clusters disintegrate within $2 \times 10^8 \, \mathrm{yr}$ after birth [], a span corresponding to at most a few initial relaxation times. Not surprisingly, there is little observational signature that relaxation has occurred.

(Converse and Stahler, MNRAS 410, 2011)

Age of OCs

Why do open clusters have such short lifespans? Some reasons:

1. In the galactic disk, there are many massive objects (giant molecular clouds) whose gravitational force can disrupt clusters;
2. Their mass is not large enough to prevent "evaporation";
3. Due to the differential rotation of the galactic plane (which we will see later), clusters are subject to tidal forces.

Stellar populations

Population I	Population II
Open clusters (and more)	Globular clusters (and more)
Galactic plane	Spherical halo
Gas and dust	Neither gas nor dust (but planetary n.)
High metallicity	Low metallicity