Astrofisica Generale II — 1

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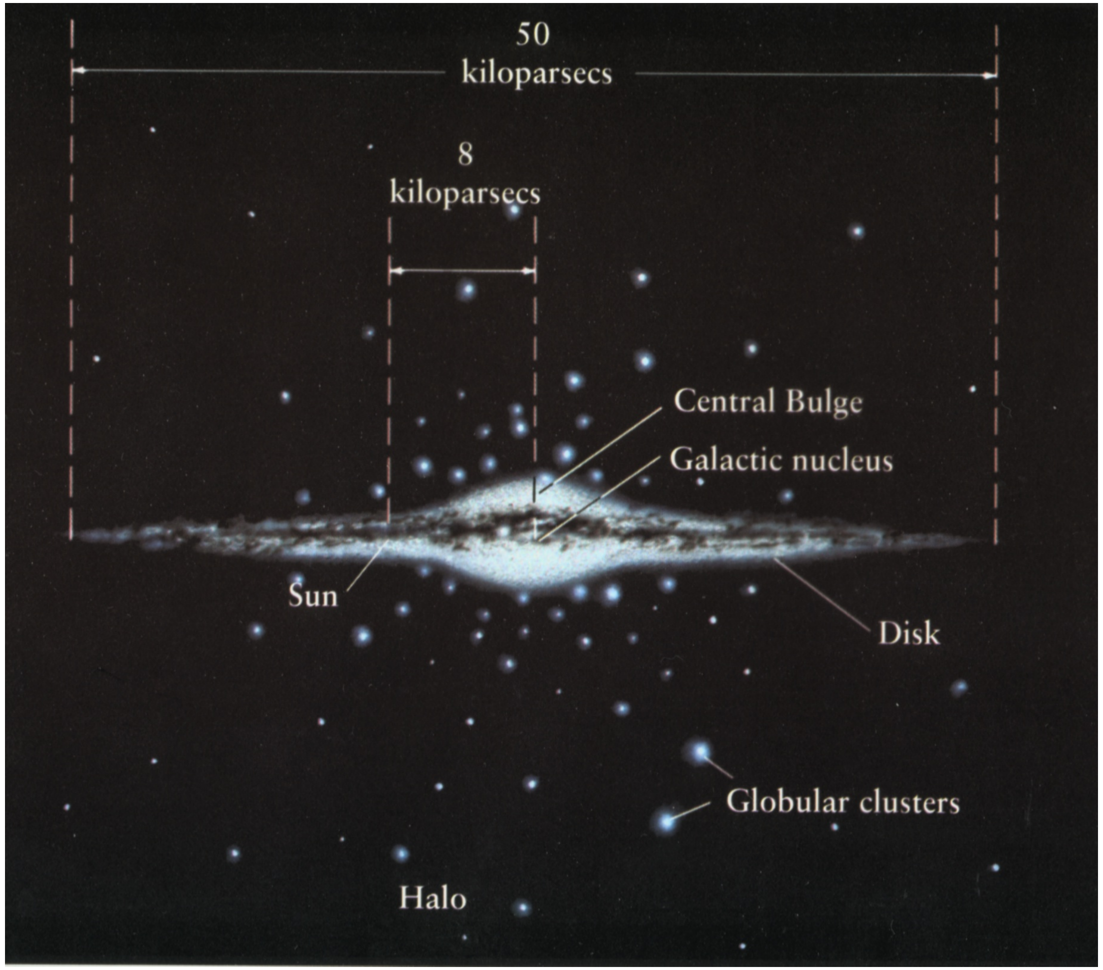
These slides

- They are available at ziotom78.github.io/lezioni-astronomia/.

- Each time I will make a PDF copy of the slides available on the website.

The Milky Way

Structure of the Milky Way

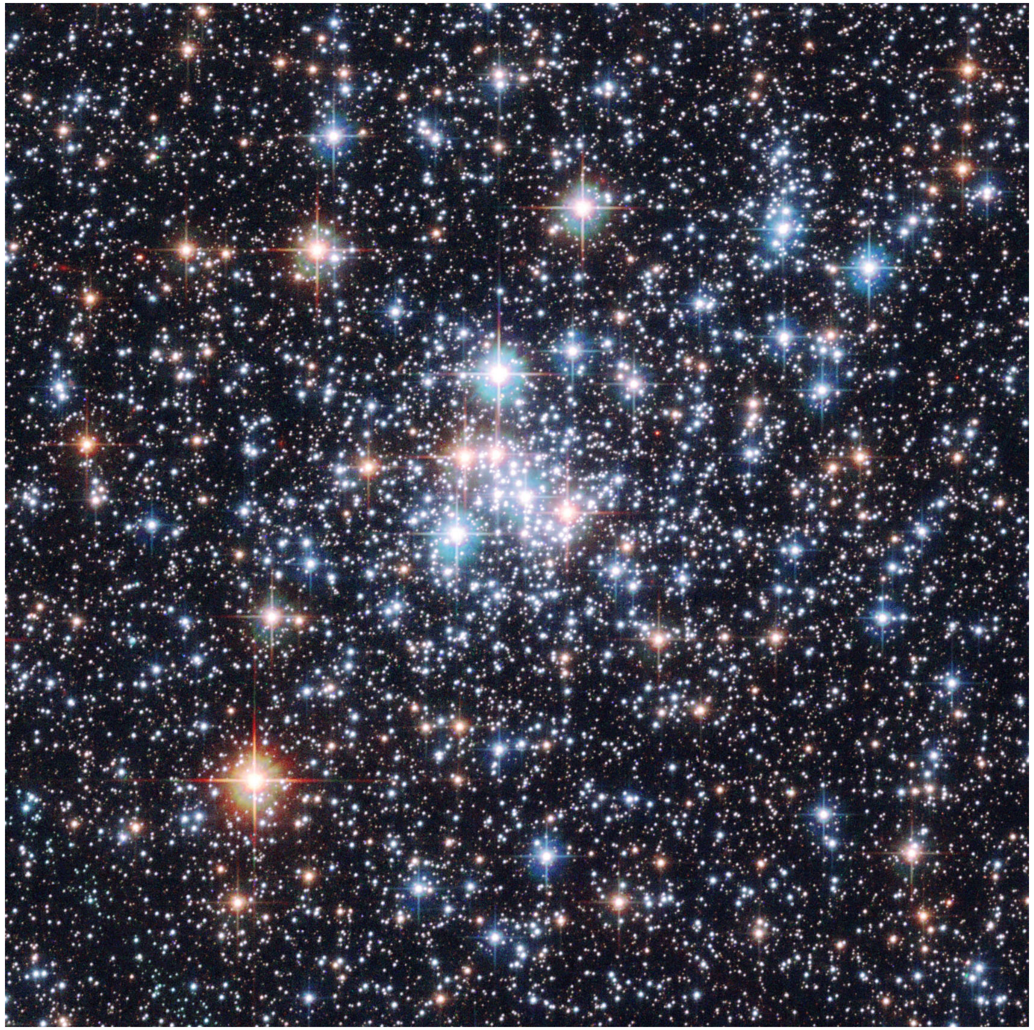


Masses and sizes

Component	Mass	Shape and size
Stellar halo	$10^9\,M_\odot$	Sphere ($r > 20\,\mathrm{kpc}$)
Disk (gas)	$10^{10}\,M_\odot$	Disk ($r = 25\,\mathrm{kpc}, h = 0.15\,\mathrm{kpc}$)
Central bulge	$10^{10}\,M_\odot$	Ellipsoid $(6 \times 2 \times 2\,\mathrm{kpc})$
Disk (stars)	$10^{11}\,M_\odot$	Disk ($r = 15\,\mathrm{kpc}, h = 1\,\mathrm{kpc}$)
Dark matter halo	$10^{12}\,M_\odot$	Sphere ($r > 60\,\mathrm{kpc}$?)

Stellar clusters

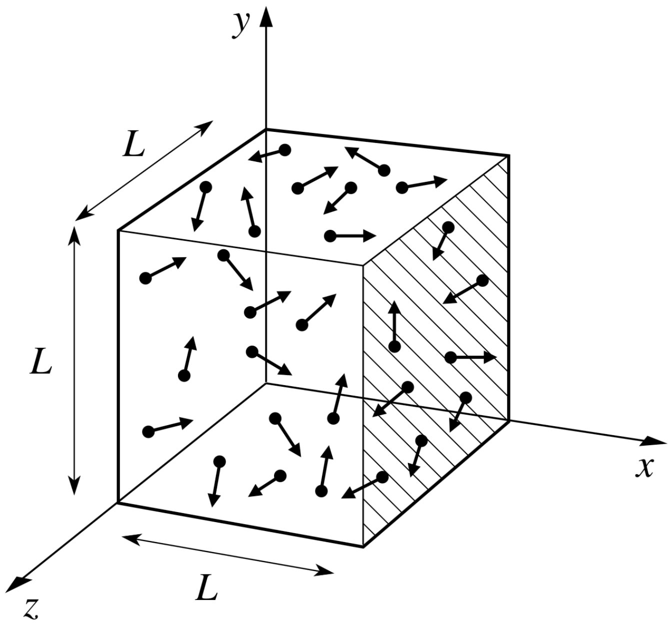
NGC 290 (open cluster)



M22 (globular cluster)

	Open clusters	Globular clusters
		
# of stars	10^3–10^4	10^4–10^6
Size	10 pc	20–100 pc (core: 5 pc)
Gas and dust?	Yes	No
Planetary nebulae?	No	Yes
# of known clusters	10^3	~160
Where?	Disk	Stellar halo (~1% of total mass)

Thermodynamics and astrophysics



Being systems composed of many particles, can we think of using classical thermodynamics to describe them?

Thermodynamics and Astrophysics

- **NO!** Ideal gas theory only works in systems without long-range forces!
- From this point of view, gravity is a problem!

> *Properties of systems with long range interactions are still poorly understood despite being of importance in most areas of physics.*

(*Dynamics and Thermodynamics of Systems with Long Range Interactions*, Springer)

Virial Theorem

- There is a suitable tool for the description of gravitationally bound systems: the *virial theorem*.
- Let's consider a physical system of N particles confined in a volume V by internal forces.
- Each particle is located at point P_i, the resulting force on it is $\vec{F}_i$, and K_i is its kinetic energy.

Time Averages

- The quantities $P_i, \vec{F}_i$ and K_i vary over time.
- However, we are more interested in their **average value** than in their instantaneous evolution.
- Given a time-dependent quantity $f = f(t)$, the value of

$$\langle f \rangle_t = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t)\,\mathrm{d}t$$

is the time average of f.

Definition of «Virial»

- Given an origin O of the reference frame, the quantity

$$G \equiv \sum_{i=1}^{N} (P_i - O) \cdot \vec{p}_i = \sum_{i=1}^{N} \vec{r}_i \cdot \vec{p}_i,$$

  where $\vec{r}_i = P_i - O$ is the vector pointing towards the i-th particle and $\vec{p}_i$ the momentum, is called **virial**.

- If the particles are located in a limited volume V, then

  1. G is a limited quantity;
  2. After a certain time, G tends to become constant.

Upper bounds for the virial

- If the system is confined in a volume V and its energy is finite, then there exist upper bounds P and R for the momentum p_i and r_i.
- Consequently,

$$|G| = \left|\sum_{i=1}^{N} \vec{r}_i \cdot \vec{p}_i\right| \leq \sum_{i=1}^{N} |\vec{r}_i| \cdot |\vec{p}_i| \leq NRP,$$

and the hypothesis is proven.

Time Variation of the Virial

- The time variation of the virial has zero average:

$$\left|\left\langle \dot{G} \right\rangle_t\right| = \left|\lim_{t\to\infty} \frac{1}{\tau} \int_0^\tau \dot{G}(t)\,\mathrm{d}t\right|$$
$$= \left|\lim_{\tau\to\infty} \frac{G(\tau) - G(0)}{\tau}\right|$$
$$\leq \lim_{\tau\to\infty} \frac{2NRP}{\tau} = 0.$$

- After a certain *relaxation time*, G becomes approximately constant.

Virial Theorem

The virial theorem states that in a system confined in a volume V, after the relaxation time, the following equality holds:

$$2\left\langle K\right\rangle_t = -\left\langle\sum_{i=1}^{N}\vec{r}_i\cdot\vec{F}_i\right\rangle_t,$$

where $K=\sum_{i=1}^{N}K_i$ is the total kinetic energy of the system.

Proof of the theorem

Using the property $\left\langle \dot{G} \right\rangle_t = 0$ we immediately obtain the thesis:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \vec{r}_i \cdot \vec{p}_i \right\rangle_t = 0,$$

$$\left\langle 2 \sum_{i=1}^{N} K_i + \sum_{i=1}^{N} \vec{r}_i \cdot \vec{F}_i \right\rangle_t = 0,$$

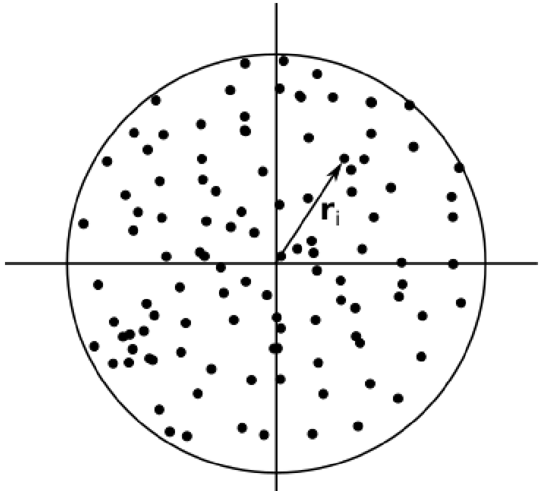
$$2 \left\langle \sum_{i=1}^{N} K_i \right\rangle_t = - \left\langle \sum_{i=1}^{N} \vec{r}_i \cdot \vec{F}_i \right\rangle_t .$$

Case of central forces

We now show that for forces with potential $U_i = k r_i^{\alpha}$ («central forces») in spherical systems the virial theorem reduces to:

$$\alpha \left\langle U \right\rangle_t = 2 \left\langle K \right\rangle_t ,$$

where $U = \sum_{i=1}^{N} U_i$ is the total potential energy.



$$\vec{F}_i = -\vec{\nabla} U_i(r_i) = -\partial_r U_i(r_i)\,\hat{e}_r = -\alpha\,k\,r_i^{\alpha-1}\hat{e}_r,$$

implies that

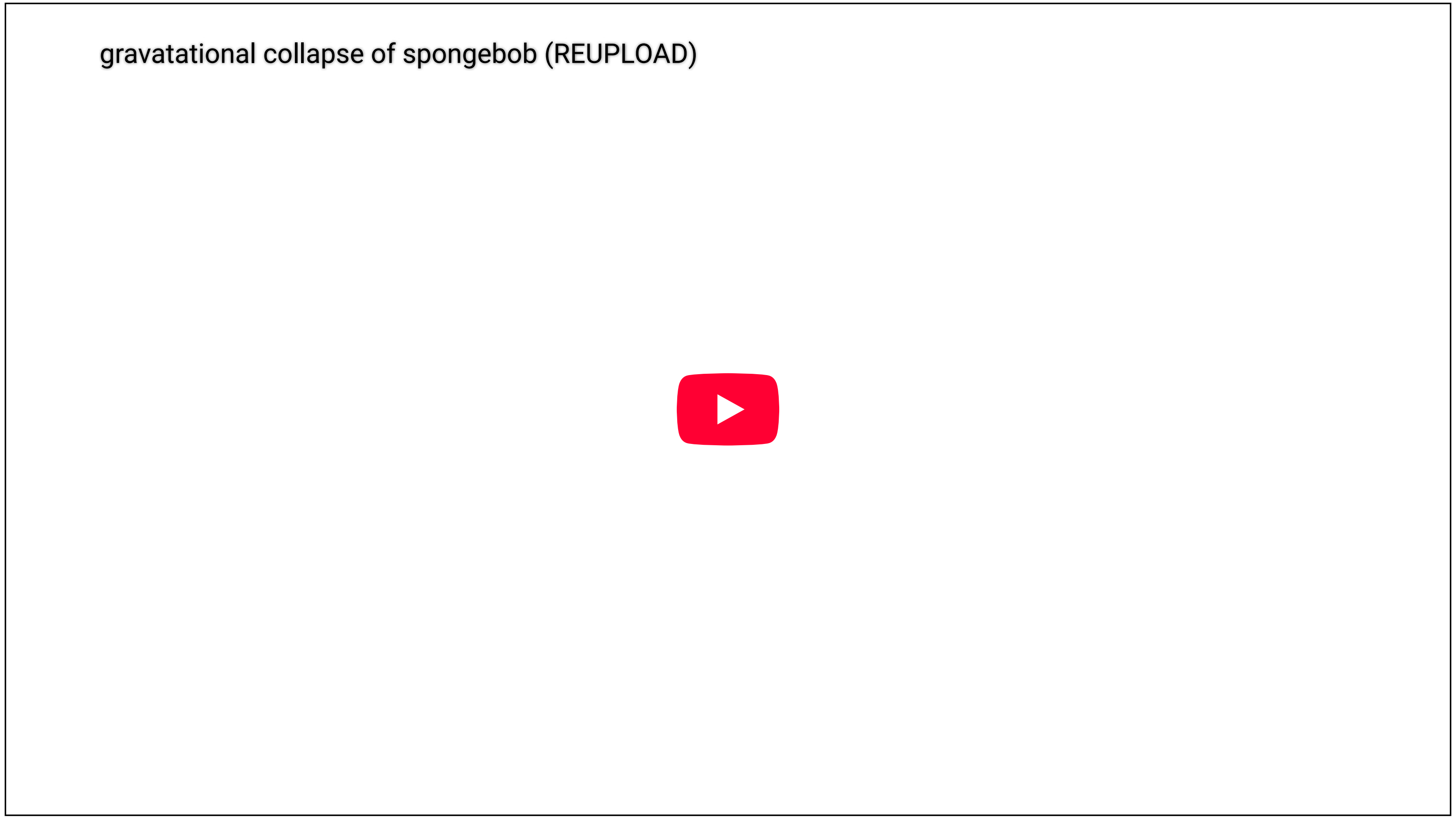
$$\begin{aligned}
2\left\langle K\right\rangle_t = -\left\langle\sum_{i=1}^N \vec{r}_i\cdot\vec{F}_i\right\rangle_t &= \\
= \left\langle\sum_{i=1}^N \vec{r}_i\cdot\vec{\nabla} U_i(r_i)\right\rangle_t &= \\
= \left\langle\sum_{i=1}^N \alpha\,k\,r_i^\alpha\right\rangle_t &= \alpha\left\langle U_{\mathrm{tot}}\right\rangle_t.
\end{aligned}$$

The Virial Theorem in Gravitational Systems

- In a system of bodies of mass m where the only force is gravitational, $U = kr^{-1}$ (with $k = -Gm^2$), and therefore $\alpha = -1$:

$$\left\langle U\right\rangle_t = -2\left\langle K\right\rangle_t.$$

- In a virialized system dominated by gravity, the potential energy is *twice* (in absolute value) the kinetic energy.
- (Actually, the relationship $U \propto r^{-1}$ is valid only far from the center, where instead $U \propto M(r)/r \propto r^2$ and the motion is like that of a spring).



Gravitational collapse of Spongebob

“Virialized” Systems

- A system for which the virial theorem holds is called “virialized”
- Virialized systems exhibit considerable symmetry, because the kinetic energy of their components is statistically distributed
- It is a condition similar to that of thermodynamic equilibrium

Potential Energy Level

- Remember that potential energy is defined up to an additive constant (it derives from an indefinite integral).
- The virial theorem, however, assumes a very specific constant for U: since we assumed that $U = kr^{-1}$, it means that we assume that the potential energy of i and j tends to zero if the two particles are moved infinitely far apart.

Applications (1/2)

- As an example, let’s estimate the average temperature of the Sun using the virial theorem.
- The Sun is a bounded spherical volume system, certainly relaxed, so the theorem is applicable.

Applications (1/2)

The gravitational potential energy of the Sun (sphere of radius R) is

$$U = \frac{3}{5} G \frac{M^2}{R},$$

while the total kinetic energy is

$$K = \sum_{i=1}^{N} \frac{3}{2} kT$$

(we assume that the temperature is constant inside).

Applications (1/2)

Using the virial theorem

$$2\left\langle K\right\rangle_t = -\left\langle U\right\rangle_t$$

we obtain that the **virial temperature** is

$$T = \frac{1}{5}\frac{GM_\odot^2}{NkR_\odot} \sim 10^6 \div 10^7\,\mathrm{K}.$$

It roughly corresponds to the core temperature.

Applications (2/2)

- Let’s now calculate the average binding energy per nucleon in an atomic nucleus.
- Also in this case we have a system of particles obviously relaxed and confined in a limited volume, but it is **not classical**: let’s try to apply the virial theorem anyway.

Applications (2/2)

- An atomic nucleus has a radius $R \sim 10^{-15}\,\mathrm{m}$.
- The average classical kinetic energy $p^2/(2m)$ can be estimated from the uncertainty principle:

$$\Delta p_x \Delta x \sim \frac{\hbar}{2} \qquad \Rightarrow \qquad p_x \approx \frac{\hbar}{2R}.$$

- Since $p^2 = p_x^2 + p_y^2 + p_z^2 \approx 3p_x^2 \approx 3\hbar^2/4R^2$, then

$$K \approx A\frac{p^2}{2m_p} \approx A\frac{3\hbar^2}{8R^2 m_p} \sim A\frac{\hbar^2}{R^2 m_p}.$$

Applications (2/2)

Under the hypothesis that $U \propto r^\alpha$, and that $|\alpha|$ is not too far from unity, from the virial theorem it holds that $K \sim U$ (same order of magnitude), i.e.

$$A \frac{\hbar^2}{R^2 m_p} \sim U$$

We are interested in the binding energy **per nucleon**, i.e., U/A:

$$U/A \sim \frac{\hbar^2}{R^2 m_p} \sim 10\,\mathrm{MeV}/\mathrm{nucleon}.$$

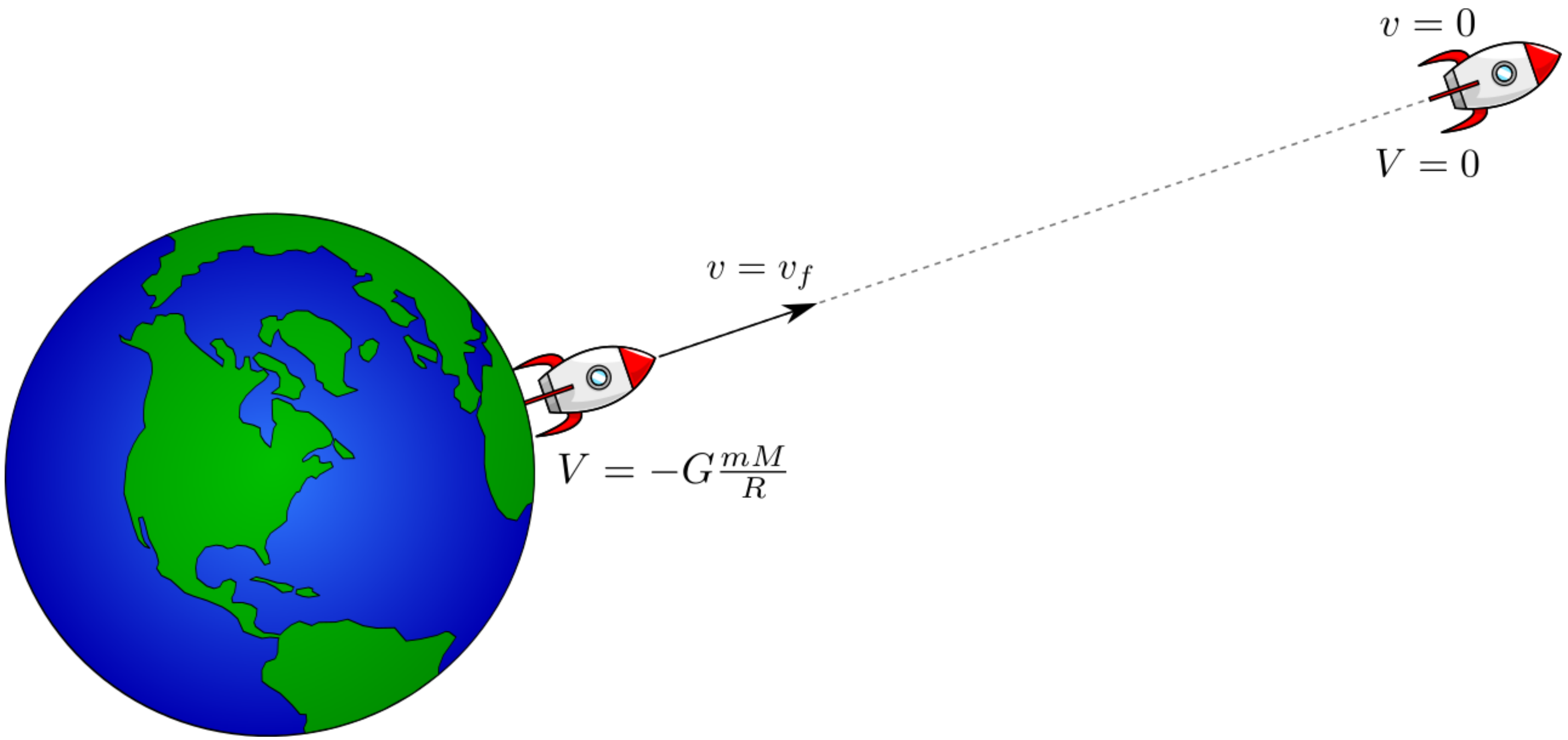
Globular cluster dynamics

- Globular clusters are spherically symmetric, therefore virialized.
- Using the virial theorem, we calculate the following quantities for a typical cluster:
  1. Escape velocity;
  2. Root mean square velocity;
  3. Mass.

Escape Velocity

- Is it reasonable to assume that a globular cluster is bound? To answer this, we need to estimate the escape velocity.
- If the average velocity of the stars were greater than the escape velocity, then the cluster could not be bound: it would «evaporate» letting its stars escape into space.

Escape Velocity



To estimate the escape velocity v_f we impose the conservation of energy between the two instants shown in the figure.

Escape Velocity

- In the case of a star initially located at a distance R from the center of mass of the cluster, the energy conservation equation becomes:

$$\frac{1}{2} M_* v_f^2 - G \frac{M_*\, M_{\rm GC}}{R} = 0,$$

- If $M_{\rm GC} \sim 10^6\, M_\odot$ and $R \sim 10\,{\rm pc}$, we have that

$$v_f = \sqrt{\frac{2 G M_{\rm GC}}{R}} \sim 30\,{\rm km/s}.$$

- Note that for an escaping particle the total energy is **zero**.

Root Mean Square Velocity

We want to calculate the root mean square velocity of the stars in a globular cluster. This quantity is related to the kinetic energy K:

$$
\begin{aligned}
K &= \sum_{i=1}^{N} \frac{1}{2} M_* v_i^2 = \frac{1}{2} M_* N \frac{1}{N} \sum_{i=1}^{N} v_i^2 \\
&= \frac{1}{2} M_{\mathrm{GC}} v_{\mathrm{rms}}^2
\end{aligned}
$$

Root Mean Square Velocity

Consequently, from the virial theorem

$$2\left\langle K\right\rangle_t = -\left\langle U\right\rangle_t = -\left\langle \frac{3}{5}\frac{GM_{\mathrm{GC}}^2}{R}\right\rangle_t$$

we have that

$$v_{\mathrm{rms}} = \sqrt{\frac{3GM_{\mathrm{GC}}}{5R}} \sim 15\,\mathrm{km/s}.$$

Escape Velocity and Root Mean Square Velocity

Since

$$\left(v_{\mathrm{rms}} = \sqrt{\frac{3GM_{\mathrm{GC}}}{5R}}\right) < \left(v_f = \sqrt{\frac{2GM_{\mathrm{GC}}}{R}}\right),$$

this confirms the hypothesis that the globular cluster (and in general any virialized gravitational system) is a bound system.

Virial mass of a GC

- Let's now calculate the total mass of a globular cluster from observational parameters.
- The potential and kinetic energy of the cluster is

$$K = \frac{1}{2} M_{\mathrm{GC}} \left\langle v^2 \right\rangle_t , \quad U = -\frac{3}{5} \frac{G M_{\mathrm{GC}}^2}{R} .$$

- From the fact that $2 \langle K \rangle_t + \langle U \rangle_t = 0$ we have that

$$M_{\mathrm{GC}} = \frac{5}{3G} \left\langle v^2 \right\rangle_t R.$$

Virial Mass of a GC

For our typical cluster with $R = 5\,\mathrm{pc}$ and $v = 15\,\mathrm{km/s}$ we have that

$$M \sim 10^{39}\,\mathrm{g} \approx 5 \times 10^{5}\,M_{\odot}.$$

This value of the mass is called **virial mass**.