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The fast spinning motion of a rigid body in the presence of a gyrostatic momentum ℓ_3

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Summary. In this paper, the rotational motion of a rigid body about a fixed point in the Newtonian force field [1] with a gyrostatic momentum ℓ_3 about the z-axis is considered. The equations of motion and their first integrals are obtained and reduced to a quasi-linear autonomous system with two degrees of freedom with one first integral. Poincaré's small parameter method [2] is applied to investigate the analytical periodic solutions of the equations of motion of the body with one point fixed, rapidly spinning about one of the principal axes of the ellipsoid of inertia. A geometric interpretation of motion is given by using Euler's angles [3] to describe the orientation of the body at any instant of time.

1 Equations of motion and change of variables

Consider a rigid body of mass M, with one fixed point O; its ellipsoid of inertia is arbitrary and acted upon by a central Newtonian force field arising from an attracting centre O_1 being located on a downward fixed axis (OZ) passing through the fixed point with a gyrostatic momentum ℓ_3 about the z-axis. Consider an element at a point $p(x, y, z)$ and located at a distance ρ from O and r from O_1 (see Fig. 1).
The general differential equations of motion and their first integrals are

$$\frac{dp}{dt} + A_1 qr + qA^{-1}\ell_3 = MgA^{-1}(y_0\gamma'' - z_0\gamma') + NA_1\gamma'\gamma'', \tag{1.1}$$

$$\frac{dq}{dt} + B_1 pr - pB^{-1}\ell_3 = MgB^{-1}(z_0\gamma - x_0\gamma'') + NB_1\gamma''\gamma, \tag{1.2}$$

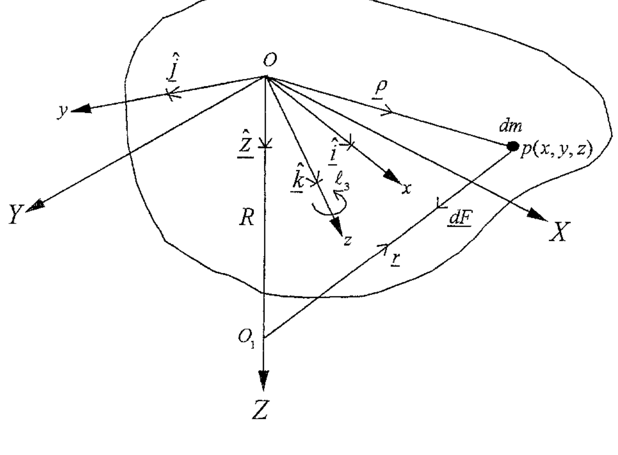

Fig. 1. The force component

$$\frac{dr}{dt} + C_1 pq = MgC^{-1}(x_0\gamma' - y_0\gamma) + NC_1\gamma\gamma', \tag{1.3}$$

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma', \tag{1.4–6}$$

$$\left(A_1 = \frac{C-B}{A}, \qquad B_1 = \frac{A-C}{B}, \qquad C_1 = \frac{B-A}{C}, \qquad N = \frac{3g}{R}, \qquad g = \frac{\lambda}{R^2}\right)$$

and

$$\begin{aligned} &Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma + y_0\gamma' + z_0\gamma'') + N(A\gamma^2 + B\gamma'^2 + C\gamma''^2) \\ &= A{p_0}^2 + B{q_0}^2 + C{r_0}^2 - 2Mg(x_0\gamma_0 + y_0\gamma_o' + z_0\gamma_0'') + N(A{\gamma_0}^2 + B\gamma_0'^2 + C{\gamma_0''}^2), \\ &Ap\gamma + Bq\gamma' + (Cr + \ell_3)\,\gamma'' = Ap_0\gamma_0 + Bq_0\gamma_0' + (Cr_0 + \ell_3)\,\gamma_0'', \qquad \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \end{aligned} \tag{2}$$

where A, B and C are the principal moments of inertia of the body under consideration; x_0, y_0 and z_0 are the coordinates of the centre of mass in the moving coordinate system $(Oxyz)$; γ, γ' and γ'' are the direction cosines of the downwards fixed Z-axis of the fixed frame in the space $(OXYZ)$; p, q and r are the projections of the angular velocity vector of the body on the principal axes of inertia; R is the distance from the fixed point O to the centre of attraction O_1; λ is the coefficient of attraction of such a centre; and p_0, q_0, r_0, γ_0, γ_0' and γ_0'' are the initial values of the corresponding variables. The gyrostatic momentum ℓ_3 is obtained by a rotor the moments of inertia of which are negligeable when compared to the housekeeping but which rotates with high angular velocity to achieve finite momentum values.

When $\ell_3 = 0$ and $\ell_3 = N = 0$, one obtains the equations of motion in [4], [5] and [6], respectively. It is taken into consideration that at the initial time the body rotates about the z-axis with a high angular velocity r_0, and that this axis makes an angle $\theta \neq n\pi/2$ $(n = 0, 1, 2, \ldots)$ with the Z-axis. Without loss of generality, we select the positive branches of the z-axis and of the x-axis in a way to avoid an obtuse angle with the direction of the Z-axis. According to the restriction on θ_0 and the selection of the coordinate system, one obtains

$$\gamma_0 \geq 0, \qquad 0 < \gamma_0'' < 1. \tag{3}$$

Consider the following parameters:

$$\begin{aligned} &a = \frac{A}{C}, \qquad b = \frac{B}{C}, \qquad c^2 = \frac{Mg\ell}{C}, \qquad \varepsilon = \frac{c\sqrt{\gamma_0''}}{r_0}, \\ &x_0 = \ell x_0', \qquad y_0 = \ell y_0', \qquad z_0 = \ell z_0', \qquad \ell^2 = {x_0}^2 + {y_0}^2 + {z_0}^2, \end{aligned} \tag{4}$$

where ε is a small parameter, that is r_0 is large. We introduce the following new variables:

$$\begin{aligned} &p = c\sqrt{\gamma_0''}\,p_1, \qquad q = c\sqrt{\gamma_0''}\,q_1, \qquad r = r_0 r_1, \qquad k = N/c^2, \\ &\gamma = \gamma_0''\gamma_1, \qquad \gamma' = \gamma_0''\gamma_1', \qquad \gamma'' = \gamma_0''\gamma_1'', \qquad t = \tau/r_0. \end{aligned} \tag{5}$$

Substituting (5) into Eqs. (1) and their integrals (2), one gets:

$$\dot{p}_1 + A_1 q_1 r_1 + A^{-1}{r_0}^{-1} q_1 \ell_3 = \varepsilon a^{-1}(y_0'\gamma_1'' - z_0'\gamma_1' + kaA_1\gamma_1'\gamma_1''), \tag{6.1}$$

$$\dot{q}_1 + B_1 p_1 r_1 - B^{-1}{r_0}^{-1} p_1 \ell_3 = \varepsilon b^{-1}(z_0'\gamma_1 - x_0'\gamma_1'' + kbB_1\gamma_1\gamma_1''), \tag{6.2}$$

$$\dot{r}_1 = \varepsilon^2(-C_1 p_1 q_1 + x_0'\gamma_1' - y_0'\gamma_1 + kC_1\gamma_1\gamma_1'), \tag{6.3}$$

$$\dot{\gamma}_1 = r_1\gamma_1' - \varepsilon q_1\gamma_1'', \quad \dot{\gamma}_1' = \varepsilon p_1\gamma_1'' - r_1\gamma_1, \quad \dot{\gamma}_1'' = \varepsilon(q_1\gamma_1 - p_1\gamma_1') \qquad (\cdot \equiv d/d\tau), \tag{6.4–6}$$

and

$$r_1{}^2 = 1 + \varepsilon^2 S_1\,, \qquad r_1\gamma_1{}'' = 1 + \varepsilon S_2\,, \qquad \gamma_1{}^2 + \gamma_1{}'^2 + \gamma_1{}''^2 = (\gamma_0{}'')^{-2}\,, \tag{7.1–3}$$

where

$$S_1 = a(p_{10}^2 - p_1{}^2) + b(q_{10}^2 - q_1{}^2) - 2[x_0{}'(\gamma_{10} - \gamma_1) + y_0{}'(\gamma_{10}' - \gamma_1{}') + z_0{}'(1 - \gamma_1{}'')]$$
$$+k[a(\gamma_{10}^2 - \gamma_1{}^2) + b(\gamma_{10}'^2 - \gamma_1{}'^2) + (1 - \gamma_1{}''^2)]\,,$$
$$S_2 = a(p_{10}\gamma_{10} - p_1\gamma_1) + b(q_{10}\gamma_{10}' - q_1\gamma_1{}') + Y(1 - \gamma_1{}'')\,, \qquad Y = \ell_3/(Cc\sqrt{\gamma_0{}''})\,.$$

2 Reduction of the equations of motion to a quasi-linear autonomous system

From Eqs. (7.1, 2) one can express the variables r_1 and $\gamma_1{}''$ in the following form:

$$\begin{aligned} r_1 &= 1 + \frac{1}{2}\varepsilon^2[S_1 + 2z_0{}'(1 - \gamma_1{}'') - k(1 - \gamma_1{}''^2)] + \dots\,, \\ \gamma_1{}'' &= 1 + \varepsilon S_2 - \frac{1}{2}\varepsilon^2[S_1 + 2z_0{}'(1 - \gamma_1{}'') - k(1 - \gamma_1{}''^2)] + \dots\,. \end{aligned} \tag{8}$$

We differentiate Eqs. (6.1) and (6.4), and use (8) to reduce the four remaining equations to the following two second-order differential equations:

$$\begin{aligned} \ddot{p}_1 + \omega'^2 p_1 = {} & \varepsilon\{z_0{}'(a^{-1} - A_1 b^{-1})\,\gamma_1 + A_1 b^{-1} x_0{}' + k(\omega^2 - A_1)\,\gamma_1 + [b^{-1}(x_0{}' - z_0{}'\gamma_1) \\ & - kB_1\gamma_1]\,A^{-1} r_0{}^{-1}\ell_3\} + \varepsilon^2\Big\{[-\omega^2 p_1 S_1 + A_1 b^{-1} x_0{}' S_2 + A_1 C_1 p_1 q_1{}^2 - A_1 q_1 \\ & \times (x_0{}'\gamma_1{}' - y_0{}'\gamma_1) + a^{-1} y_0{}'(q_1\gamma_1 - p_1\gamma_1{}') - a^{-1} z_0{}' p_1] + A_1 k[p_1(1 - \gamma_1{}'^2) \\ & + q_1(1 - C_1)\,\gamma_1\gamma_1{}' - S_2(1 + B_1)\,\gamma_1] + \frac{1}{2} r_0{}^{-1}\ell_3 p_1 (A^{-1}B_1 - A_1 B^{-1}) \\ & [S_1 + 2z_0{}'(1 - \gamma_1{}'') - k(1 - \gamma_1{}''^2)] + A^{-1} r_0^{-1}\ell_3(b^{-1}x_0{}' - kb_1\gamma_1)\,S_2\Big\} \\ & + \varepsilon^3\Big\{\frac{1}{2} z_0{}'(a^{-1} - A_1 b^{-1})\,\gamma_1[S_1 + 2z_0{}'(1 - \gamma_1{}'') - k(1 - \gamma_1{}''^2)] \\ & + \frac{1}{2} A^{-1} r_0{}^{-1}\ell_3(kB_1\gamma_1 - b^{-1}x_0{}')\,[S_1 + 2z_0{}'(1 - \gamma_1{}'') - k(1 - \gamma_1{}''^2)] \\ & + (2kA_1 - a^{-1}z_0{}')\,p_1 S_2\Big\} + \cdots\,, \end{aligned} \tag{9}$$

$$\begin{aligned} \ddot{\gamma}_1 + \gamma_1 = {} & \varepsilon[(1 + B_1) - B^{-1} r_0{}^{-1}\ell_3]\,p_1 + \varepsilon^2[-S_1\gamma_1 + (1 + B_1)\,p_1 S_2 + (1 - C_1)\,p_1 q_1 \gamma_1{}' \\ & + x_0{}'\gamma_1{}'^2 + x_0{}' b^{-1} - \gamma_1(y_0{}'\gamma_1{}' + z_0{}' b^{-1} + q_1{}^2) + k(C_1\gamma_1{}'^2 - B_1)\,\gamma_1{}'] \\ & + \varepsilon^3[2b^{-1}x_0{}' - \gamma_1(b^{-1}z_0{}' + 2kB_1)]\,S_2 + \cdots\,, \end{aligned} \tag{10}$$

where

$$\omega^2 = -A_1B_1 = (A-C)(B-C)/AB = (a-1)(b-1)/ab\,,$$
$$\omega'^2 = \omega^2 - (A^{-1}B_1 - A_1B^{-1})\,{r_0}^{-1}\ell_3\,. \tag{11}$$

Here r_0 is large, so ${r_0}^{-2}$, ${r_0}^{-3}$, $\cdots$ are neglected. By solving Eqs. (6.1) and (6.4), we obtain q_1 and γ_1' in the form

$$q_1 = {A_1}^{-1}{r_1}^{-1}(1 - A^{-1}{A_1}^{-1}{r_0}^{-1}\ell_3{r_1}^{-1} + \cdots)$$
$$[-\dot{p}_1 + \varepsilon a^{-1}(y_0'\gamma_1'' - z_0'\gamma_1' + kaA_1\gamma_1'\gamma_1'')]\,, \tag{12}$$
$$\gamma_1' = {r_1}^{-1}(\dot{\gamma}_1 + \varepsilon q_1\gamma_1'')\,,$$

in which r_1 and γ_1'' are replaced by (8). Substituting (12) into (9) and (10), one obtains a quasilinear autonomous system with two degrees of freedom depending on p_1, $\dot{p}_1$, γ_1, $\dot{\gamma}_1$, p_{10}, $\dot{p}_{10}$, γ_{10} and $\dot{\gamma}_{10}$.

We introduce new variables p_2 and γ_2 such that

$$p_2 = p_1 - \varepsilon\chi - \varepsilon\chi_1\gamma_2\,, \qquad \gamma_2 = \gamma_1 - \varepsilon\nu p_2\,, \tag{13}$$

where

$$\chi = x_0'(b\omega'^2)^{-1}(A_1 + A^{-1}{r_0}^{-1}\ell_3)\,, \qquad \nu = (1-\omega'^2)^{-1}[1 + B_1 - B^{-1}{r_0}^{-1}\ell_3]\,,$$
$$\chi_1 = (1-\omega'^2)^{-1}[-z_0'(a^{-1} - A_1b^{-1}) + k(A_1 - \omega^2) + (b^{-1}z_0' + kB_1)\,A^{-1}{r_0}^{-1}\ell_3]\,. \tag{14}$$

Making use of (13), (8) and (12), one gets the following expressions for S_1 and S_2 in terms of power series in ε:

$$S_i = S_{i1} + 2^{2-i}\varepsilon S_{i2} + \cdots\,, \tag{15}$$

where

$$S_{11} = a(p_{20}^2 - {p_2}^2) + bX^2(\dot{p}_{20}^2 - {\dot{p}_2}^2) - 2x_0'(\gamma_{20} - \gamma_2) - 2y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2)$$
$$+ k[a(\gamma_{20}^2 - {\gamma_2}^2) + b(\dot{\gamma}_{20}^2 - {\dot{\gamma}_2}^2)]\,,$$
$$S_{12} = a[\chi(p_{20} - p_2) + \chi_1(p_{20}\gamma_{20} - p_2\gamma_2)] - bX^2[a^{-1}y_0'(\dot{p}_{20} - \dot{p}_2)$$
$$- \chi_2(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma})] - \nu x_0'(p_{20} - p_2) - y_0'\nu_2(\dot{p}_{20} - \dot{p}_2)$$
$$+ (z_0' - k)\,S_{21} + k[\nu a(p_{20}\gamma_{20} - p_2\gamma_2) + \nu_2 b(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)]\,, \tag{16}$$
$$S_{21} = a(p_{20}\gamma_{20} - p_2\gamma_2) - bX(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)\,,$$
$$S_{22} = a[\nu(p_{20}^2 - {p_2}^2) + \chi(\gamma_{20} - \gamma_2) + \chi_1(\gamma_{20}^2 - {\gamma_2}^2)] + bX[-\nu_2(\dot{p}_{20}^2 - {\dot{p}_2}^2)$$
$$+ a^{-1}y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2) - \chi_2(\dot{\gamma}_{20}^2 - {\dot{\gamma}_2}^2)] - YS_{21}\,,$$

with

$$X = {A_1}^{-1}(1 - A^{-1}{A_1}^{-1}r_0^{-1}\ell_3)\,, \qquad \chi_2 = \chi_1 + a^{-1}z_0' - kA_1\,, \qquad \nu_2 = \nu - X\,. \tag{17}$$

Formulas (8) and (15) lead to

$$\begin{aligned} r_1 &= 1 + \frac{1}{2}\,\varepsilon^2 S_{11} + \varepsilon^3 (S_{12} - z_0' S_{21} + k S_{21}) + \cdots\,, \\ \gamma_1'' &= 1 + \varepsilon S_{21} + \varepsilon^2 \left(S_{22} - \frac{1}{2}\, S_{11} \right) - \varepsilon^3 (S_{12} - z_0' S_{21} + k S_{21}) + \cdots\,. \end{aligned} \tag{18}$$

In terms of the new variables p_2 and γ_2, the variables q_1 and γ_1' have the forms:

$$\begin{aligned} q_1 &= -X\dot{p}_2 + \varepsilon X(a^{-1}y_0' - \chi_2\dot{\gamma}_2) + \varepsilon^2 [X(kA_1 - a^{-1}z_0')\,\nu_2\dot{p}_2 \\ &\quad + \left(X - \frac{1}{2}A_1^{-1} \right) S_{11}\dot{p}_2 + X(kA_1\dot{\gamma}_2 + a^{-1}y_0')\,S_{21}] + \cdots\,, \\ \gamma_1' &= \dot{\gamma}_2 + \varepsilon\nu_2\dot{p}_2 + \varepsilon^2 \left[X(a^{-1}y_0' - \chi_2\dot{\gamma}_2 - S_{21}\dot{p}_2) - \frac{1}{2} S_{11}\dot{\gamma}_2 \right] + \cdots\,. \end{aligned} \tag{19}$$

Substituting (13), (15), (16), (18) and (19) into (9) and (10), we obtain the following quasi-linear autonomous system of two degrees of freedom:

$$\ddot{p}_2 + \omega'^2 p_2 = \varepsilon^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon)\,, \qquad \ddot{\gamma}_2 + \gamma_2 = \varepsilon^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon)\,, \tag{20}$$

where

$$\begin{aligned} F &= F_2 + \varepsilon F_3 + \cdots\,, \qquad \Phi = \Phi_2 + \varepsilon\Phi_3 + \cdots\,, \\ F_2 &= f_2 - \nu\chi_1(1 - \omega'^2)\,p_2\,, \qquad \Phi_2 = \varphi_2 + \nu(1 - \omega'^2)\,(\chi + \chi_1\gamma_2)\,, \\ F_3 &= f_3 - \chi_1\varphi_2 - \nu\chi_1(1 - \omega'^2)\,(\chi + \chi_1\gamma_2)\,, \qquad \Phi_3 = \varphi_3 - \nu f_2 + \nu^2\chi_1(1 - \omega'^2)\,p_2\,, \\ f_2 &= -\,\omega^2 S_{11} p_2 + A_1 x_0'(b^{-1}S_{21} + X\dot{\gamma}_2\dot{p}_2) + A_1 C_1 X^2 p_2 \dot{p}_2^{\,2} - y_0' X\gamma_2\dot{p}_2(A_1 + a^{-1}) \\ &\quad - a^{-1}p_2(z_0' + y_0'\dot{\gamma}_2) + A_1 k(1 - \dot{\gamma}_2^{\,2})\,p_2 + (C_1 - 1)\,X\gamma_2\dot{\gamma}_2\dot{p}_2 - (1 + B_1)S_{21}\gamma_2 \\ &\quad + \frac{1}{2}\,r_0^{-1}\ell_3[2p_2(A^{-1}B_1 - A_1B^{-1})\,S_{11} + A^{-1}(b^{-1}x_0' - kB_1\gamma_2)\,S_{21}]\,, \\ f_3 &= -\,2\omega^2 p_2 S_{12} + (\chi + \chi_1\gamma_2)\,\{-\omega^2 S_{11} - a^{-1}(z_0' + y_0'\dot{\gamma}_2) + A_1[C_1 X^2\dot{p}_2^{\,2} \\ &\quad + k(1 - \dot{\gamma}_2^{\,2})]\} + A_1 b^{-1}x_0' S_{22} + A_1 X\dot{p}_2(x_0'\nu_2\dot{p}_2 - y_0'\nu p_2) - p_2\dot{p}_2[a^{-1}y_0'(\nu_2 + \nu\chi) \\ &\quad + 2A_1 k\nu_2\dot{\gamma}_2] + X\dot{p}_2(\nu_2\gamma_2\dot{p}_2 + \nu\dot{\gamma}_2 p_2)\,(C_1 - 1) - (1 + B_1)\,(\nu S_{21}p_2 + S_{22}\gamma_2) \\ &\quad + \frac{1}{2}\,z_0'(a^{-1} - A_1b^{-1})\,\gamma_2 S_{11} + (2kA_1 - a^{-1}z_0')\,p_2 S_{21} + X(a^{-1}y_0' - \chi_2\dot{\gamma}_2) \\ &\quad \times [-A_1(2C_1 X p_2\dot{p}_2 + x_0'\dot{\gamma}_2) + y_0'\gamma_2(A_1 + a^{-1}) + \gamma_2\dot{\gamma}_2^{\,2}(1 - C_1)] \\ &\quad + \frac{1}{2}\,r_0^{-1}\ell_3\{(A^{-1}B_1 - A_1B^{-1})\,[2P_2(S_{12} - z_0'S_{21} + kS_{21}) + (\chi + \chi_1\gamma_2)\,S_{11}] \\ &\quad + 2A^{-1}[(b^{-1}x_0' - kB_1\gamma_2)\,S_{22} - kB_1\nu S_{21}p_2] + A^{-1}(kB_1\gamma_2 - b^{-1}x_0')\,S_{11}]\}\,, \\ \varphi_2 &= [(1 + B_1)\,S_{21} - (1 - C_1)\,X\dot{\gamma}_2\dot{p}_2]\,p_2 + x_0'(b^{-1} + \dot{\gamma}_2^{\,2}) + [k(C_1\dot{\gamma}_2^{\,2} - B_1) \\ &\quad - y_0'\dot{\gamma}_2 - z_0'b^{-1} - X^2\dot{p}_2^{\,2} - S_{11}]\,\gamma_2\,, \\ \varphi_3 &= (1 + B_1)\,[p_2 S_{22} + (\chi + \chi_1\gamma_2)\,S_{21}] + X(1 - C_1)\,\{(a^{-1}y_0' - \chi_2\dot{\gamma}_2)\,\dot{\gamma}_2 p_2 \\ &\quad - \dot{p}_2[\nu_2 p_2\dot{p}_2 + (\chi + \chi_1\gamma_2)\,\dot{\gamma}_2]\} - 2\gamma_2 S_{12} - \nu p_2 S_{11} + 2x_0'\nu_2\dot{\gamma}_2\dot{p}_2 \\ &\quad - y_0'(\nu_2\gamma_2\dot{p}_2 + \nu\dot{\gamma}_2 p_2) - \nu p_2(b^{-1}z_0' + X^2\dot{p}_2^{\,2}) + 2X^2(a^{-1}y_0' - \chi_2\dot{\gamma}_2)\,\gamma_2\dot{p}_2 \\ &\quad + k[2C_1\nu_2\gamma_2\dot{\gamma}_2\dot{p}_2 + \nu(C_1\dot{\gamma}_2^{\,2} - B_1)\,p_2] + [2b^{-1}x_0' - (b^{-1}z_0' + 2kB_1)\,\gamma_2]\,S_{21}. \end{aligned} \tag{21}$$

System (20) has the following first integral obtained from (7) in the form:

$$\gamma_2{}^2 + \dot{\gamma}_2{}^2 + 2\varepsilon(\nu\gamma_2 p_2 + \nu_2\dot{\gamma}_2\dot{p}_2 + S_{21}) + \varepsilon^2[\nu^2 p_2{}^2 + 2X(a^{-1}y_0{}' - \chi_2\dot{\gamma}_2 - S_{21}\dot{p}_2)\,\dot{\gamma}_2 \\ - (1+\dot{\gamma}_2{}^2)\,S_{11} + 2S_{22}] + \cdots = (\gamma_0{}'')^{-2} - 1\,. \tag{22}$$

Our aim is to find the periodic solutions of this system under the conditions $A > B > C$ or $A < B < C$ (ω'^2 is positive). In the first case, the body is set in a fast initial spin r_0 with respect to the major axis of the ellipsoid of inertia while in the second case the body is set in a fast initial spin r_0 with respect to the minor axis of the ellipsoid of inertia.

3 Formal construction of the periodic solutions

Since the system (20) is autonomous, the following conditions:

$$p_2(0,0) = 0\,, \qquad \dot{p}_2(0,0) = 0\,, \qquad \dot{\gamma}_2(0,\varepsilon) = 0\,, \tag{23}$$

do not affect the generality of the solutions [7]. The generating system of (20) is

$$\ddot{p}_2{}^{(0)} + \omega'^2 p_2{}^{(0)} = 0\,, \qquad \ddot{\gamma}_2{}^{(0)} + \gamma_2{}^{(0)} = 0\,, \tag{24}$$

which admits periodic solutions in the form:

$$p_2{}^{(0)} = M_1\cos\omega'\tau + M_2\sin\omega'\tau\,, \qquad \gamma_2{}^{(0)} = M_3\cos\tau\,, \tag{25}$$

with period $T_0 = 2\pi n$. M_i with $i = (1,2,3)$ are constants to be determined. We suppose the required periodic solutions of the initial autonomous system in the form:

$$p_2(\tau,\varepsilon) = (M_1+\beta_1)\,\cos\omega'\tau + (M_2+\beta_2)\,\sin\omega'\tau + \sum_{k=2}^{\infty}\varepsilon^k G_k(\tau)\,, \\ \gamma_2(\tau,\varepsilon) = (M_3+\beta_3)\,\cos\tau + \sum_{k=2}^{\infty}\varepsilon^k H_k(\tau)\,, \tag{26}$$

with period $T(\varepsilon) = T_0 + \alpha(\varepsilon)$. The quantities β_1, $\omega'\beta_2$ and β_3 represent the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of the system (20) from their values of the system (24); these deviations are functions of ε and vanish when $\varepsilon = 0$. We express the initial conditions of (26) by the relations:

$$p_2(0,\varepsilon) = M_1+\beta_1\,, \qquad \dot{p}_2(0,\varepsilon) = \omega'(M_2+\beta_2)\,, \\ \gamma_2(0,\varepsilon) = M_3+\beta_3\,, \qquad \dot{\gamma}_2(0,\varepsilon) = 0\,. \tag{27}$$

Let us define the functions $G_k(\tau)$ and $H_k(\tau)$ $(k = 2,3\cdots)$ by the operator [8]

$$U = u + \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 + \frac{1}{2}\frac{\partial^2 u}{\partial M_1{}^2}\,\beta_1{}^2 + \cdots\,, \qquad \begin{pmatrix} U = G_k, H_k \\ u = g_k, h_k \end{pmatrix}, \tag{28}$$

where the functions $g_k(\tau)$ and $h_k(\tau)$ take the forms:

$$g_k(\tau) = \frac{1}{\omega'}\int_0^\tau F_k{}'(t_1)\,\sin\omega'(\tau - t_1)\,dt_1\,, \qquad h_k(\tau) = \int_0^\tau \Phi_k{}'(t_1)\,\sin(\tau - t_1)\,dt_1\,, \qquad (k = 2,3)$$

with

$$F_k{}'(\tau) = \frac{1}{(k-2)!}\left(\frac{d^{k-2}F}{d\varepsilon^{k-2}}\right)_{\beta=\varepsilon=0}, \qquad \Phi_k{}'(\tau) = \frac{1}{(k-2)!}\left(\frac{d^{k-2}\Phi}{d\varepsilon^{k-2}}\right)_{\beta=\varepsilon=0}.$$

We notice that the right-hand sides of the system (20) begin from a term of order ε^2, and therefore we have

$$F_k'(\tau) = F_k({p_2}^{(0)}, {\dot{p}_2}^{(0)}, {\gamma_2}^{(0)}, {\dot{\gamma}_2}^{(0)}) \equiv {F_k}^{(0)} ,$$

$$\Phi_k'(\tau) = \Phi_k({p_2}^{(0)}, {\dot{p}_2}^{(0)}, {\gamma_2}^{(0)}, {\dot{\gamma}_2}^{(0)}) \equiv {\Phi_k}^{(0)} , \qquad (k = 2, 3) .$$

Now, we try to find the expressions of the functions ${F_2}^{(0)}$ and ${\Phi_2}^{(0)}$. The periodic solutions (25) can be rewritten in the forms:

$$p_2{}^{(0)} = E \cos(\omega'\tau - \eta) , \qquad \gamma_2{}^{(0)} = M_3 \cos\tau , \tag{29}$$

where

$E = \sqrt{{M_1}^2 + {M_2}^2}$ and $\eta = \tan^{-1}\dfrac{M_2}{M_1}$. Making use of (29) and (16), one gets:

$$S_{ij}^{(0)} = S_{ij}^{(0)}({p_2}^{(0)}, {\dot{p}_2}^{(0)}, {\gamma_2}^{(0)}, {\dot{\gamma}_2}^{(0)}) \qquad (i, j = 1, 2) ,$$

$$\begin{aligned}
S_{11}^{(0)} &= E^2\left[a\left(\cos^2\eta - \frac{1}{2}\right) + bX^2\omega'^2\left(\sin^2\eta - \frac{1}{2}\right) + \frac{1}{2}(bX^2\omega'^2 - a)\cos 2(\omega'\tau - \eta)\right] \\
&\quad - 2M_3[x_0'(1 - \cos\tau) + y_0'\sin\tau] - \frac{1}{2}kM_3{}^2C_1(1 - \cos 2\tau) , \\
S_{21}^{(0)} &= M_3E\left\{a\cos\eta + \frac{1}{2}(b\omega'X - a)\cos[(\omega' - 1)\tau - \eta]\right. \\
&\quad \left. - \frac{1}{2}(b\omega'X + a)\cos[(\omega' + 1)\tau - \eta]\right\} , \\
S_{12}^{(0)} &= aE\{X[\cos\eta - \cos(\omega'\tau - \eta)] + \chi_1M_3[\cos\eta - \cos\tau\cos(\omega'\tau - \eta)]\} \\
&\quad - bX^2E\omega'\{a^{-1}y_0'[\sin\eta + \sin(\omega'\tau - \eta)] + \chi_2M_3\sin\tau\sin(\omega'\tau - \eta)\} \\
&\quad - \nu x_0'E[\cos\eta - \cos(\omega'\tau - \eta)] + kEM_3\{\nu a[\cos\eta - \cos\tau\cos(\omega'\tau - \eta)] \\
&\quad - \nu_2b\sin\tau\sin(\omega'\tau - \eta)\} + (z_0' - k)\, S_{21}^{(0)} , \\
S_{22}^{(0)} &= a\{\nu E^2[\cos^2\eta - \cos^2(\omega'\tau - \eta)] + \chi M_3(1 - \cos\tau) + \chi_1M_3{}^2\sin^2\tau\} \\
&\quad + bX\{a^{-1}y_0'M_3\sin\tau - \nu_2E^2\omega'^2[\sin^2\eta - \sin^2(\omega'\tau - \eta)] + \chi_2M_3{}^2\sin^2\tau\} - YS_{21}^{(0)} .
\end{aligned} \tag{30}$$

Substituting (29) and (30) into formulas (21) we obtain

$$\begin{aligned}
F_2{}^{(0)} &= M_1L(\omega')\cos\omega'\tau + M_2L(\omega')\sin\omega'\tau + \cdots , \\
\Phi_2{}^{(0)} &= M_3N(\omega')\cos\tau + \cdots ,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
L(\omega') &= \omega^2[-(aM_1{}^2 + b\omega'^2X^2M_2{}^2) + (M_1{}^2 + M_2{}^2)\, b\omega'^2X^2] + A_1C_1\omega'^2X^2 \\
&\quad \times (M_1{}^2 + M_2{}^2) + 2M_3x_0'\omega^2 + k\left(A_1 + \frac{1}{2}M_3{}^2\omega^2C_1\right) - [z_0'a^{-1} + \nu\chi_1(1 - \omega'^2)] \\
&\quad + \frac{1}{2}r_0{}^{-1}\ell_3(A^{-1}B_1 - A_1B^{-1})\,[aM_1{}^2 + b\omega'^2X^2M_2{}^2 - b\omega'^2X^2(M_1{}^2 + M_2{}^2) \\
&\quad - 2M_3x_0' - \frac{1}{2}kM_3{}^2C_1] ,
\end{aligned} \tag{32}$$

$$N(\omega') = -(aM_1^2 + b\omega'^2 X^2 M_2^2) - (M_1^2 + M_2^2)[aB_1 + \omega'^2 X^2(1-b)] + 2M_3 x_0' - [z_0' b^{-1} - \nu\chi_1(1-\omega'^2)] + k(M_3^2 C_1 - B_1)] . \tag{32}$$

From (28), (31) and (32), the following results are obtained:

$$g_2(T_0) = -\pi n(\omega')^{-1} M_2 L(\omega') , \qquad \dot g_2(T_0) = \pi n M_1 L(\omega') ,$$
$$h_2(T_0) = 0 , \qquad \dot h_2(T_0) = \pi n M_3 N(\omega') . \tag{33}$$

The constants M_1, $\omega' M_2$ and M_3, which represent the initial conditions of the generating solution (25), the deviations $\beta_1(\varepsilon)$, $\omega'\beta_2(\varepsilon)$, and $\beta_3(\varepsilon)$, and the correction for the period α must be found from the conditions of periodicity of the solutions $p_2(\tau,\varepsilon)$, $\gamma_2(\tau,\varepsilon)$, and their first derivatives. These conditions can be written in a form:

$$\psi_1 = p_2(T_0+\alpha,\varepsilon) - p_2(0,\varepsilon) = 0 , \qquad \psi_2 = \dot p_2(T_0+\alpha,\varepsilon) - \dot p_2(0,\varepsilon) = 0 ,$$
$$\psi_3 = \gamma_2(T_0+\alpha,\varepsilon) - \gamma_2(0,\varepsilon) = 0 , \qquad \psi_4 = \dot\gamma_2(T_0+\alpha,\varepsilon) - \dot\gamma_2(0,\varepsilon) = 0 . \tag{34}$$

However, on the strength of the existence of the first integral (22) of system (20), the condition for periodicity $\psi_3 = 0$ is not independent [7]. Writing the integral (22) in the form

$$\begin{aligned}
&\gamma_2^2(T_0+\alpha,\varepsilon) + \dot\gamma_2^2(T_0+\alpha,\varepsilon) + 2\varepsilon[\nu\gamma_2(T_0+\alpha,\varepsilon)\,p_2(T_0+\alpha,\varepsilon) + \nu_2\dot\gamma_2(T_0+\alpha,\varepsilon)\,\dot p_2(T_0+\alpha,\varepsilon)\\
&+ S_{21}(T_0+\alpha,\varepsilon)] + \varepsilon^2\{\nu^2 p_2^2(T_0+\alpha,\varepsilon) + 2X[a^{-1}y_0' - \chi_2\dot\gamma_2(T_0+\alpha,\varepsilon)\\
&- S_{21}(T_0+\alpha,\varepsilon)\,\dot p_2(T_0+\alpha,\varepsilon)]\,\dot\gamma_2(T_0+\alpha,\varepsilon) - [1+\dot\gamma_2^2(T_0+\alpha,\varepsilon)]\,S_{11}(T_0+\alpha,\varepsilon)\\
&+ 2S_{22}(T_0+\alpha,\varepsilon)\} + \cdots = \gamma_2^2(0,\varepsilon) + \dot\gamma_2^2(0,\varepsilon) + 2\varepsilon[\nu\gamma_2(0,\varepsilon)\,p_2(0,\varepsilon) + \nu_2\dot\gamma_2(0,\varepsilon)\,\dot p_2(0,\varepsilon)\\
&+ S_{21}(0,\varepsilon)] + \varepsilon^2\{\nu^2 p_2^2(0,\varepsilon) + 2X[a^{-1}y_0' - \chi_2\dot\gamma_2(0,\varepsilon) - S_{21}(0,\varepsilon)\,\dot p_2(0,\varepsilon)]\,\dot\gamma_2(0,\varepsilon)\\
&- [1+\dot\gamma_2^2(0,\varepsilon)]\,S_{11}(0,\varepsilon) + 2S_{22}(0,\varepsilon)\} + \cdots ,
\end{aligned} \tag{35}$$

and using the condition (27), we get from (34)

$$2(M_3+\beta_3)\,\psi_3 + \psi_3^2 + \varepsilon\varphi_1(\psi_1,\psi_2,\psi_3,\psi_4,\varepsilon) = 0 .$$

Here φ_1 is an entire function of all its variables, and $\varphi_1(0,0,0,\varepsilon) = 0$. If $M_3 \neq 0$, it follows from (35) that

$$\psi_3 = f_1(\psi_1,\psi_2,\psi_3,\psi_4,\varepsilon) ,$$

where f_1 is an entire function of all its arguments, and $f_1(0,0,0,\varepsilon) = 0$. Then it follows immediately that the condition $\psi_3 = 0$ holds in (34), which is a consequence of the other ones,

$$\psi_1 = \psi_2 = \psi_4 = 0 . \tag{36}$$

Substituting the initial conditions (27) into the integral (22) for $\tau = 0$, the following equation is obtained:

$$M_3^2 + 2M_3\beta_3 + \beta_3^2 + 2\varepsilon\nu M_3(M_1+\beta_1) + \cdots = (\gamma_0'')^{-2} - 1 .$$

Supposing that γ_0'' is independent of ε, we get

$$M_3^2 = (\gamma_0'')^{-2} - 1 , \qquad \beta_3^2 + 2M_3\beta_3 + 2\varepsilon\nu M_3(M_1+\beta_1) + \cdots = 0 . \tag{37}$$

One obtains M_3 and β_3 from Eqs. (37) and condition (3) in the form:

$$M_3 = (1 - \gamma_0''^2)^{\frac{1}{2}} (\gamma_0'')^{-1} \qquad 0 < M_3 < \infty ,$$
$$\beta_3 = -\varepsilon\nu(M_1 + \beta_1) + \cdots , \tag{38}$$

because γ_0'' is an arbitrary parameter, and M_3 is an arbitrary positive constant. This means that the periodic solutions (26) depend on an arbitrary constant M_3 and a fuction $\beta_3(\varepsilon)$, and vanish when ε tends to zero. This property does not depend on the form of α. Expanding the independent conditions of periodicity (34) in a power series of α and retaining only the linear terms (neglecting even the terms $\varepsilon^2\alpha$), it yields

$$p_2(T_0, \varepsilon) + \alpha\dot{p}_2(T_0, \varepsilon) + \cdots - p_2(0, \varepsilon) = 0 ,$$
$$\dot{p}_2(T_0, \varepsilon) + \alpha\ddot{p}_2(T_0, \varepsilon) + \cdots - \dot{p}_2(0, \varepsilon) = 0 ,$$
$$\dot{\gamma}_2(T_0, \varepsilon) + \alpha\ddot{\gamma}_2(T_0, \varepsilon) + \cdots - \dot{\gamma}_2(0, \varepsilon) = 0 .$$

Using the initial values (27) in the above relations, we obtain the independent conditions for the periodicity of (36),

$$p_2(T_0, \varepsilon) + \alpha\omega'(M_2 + \beta_2) - (M_1 + \beta_1) = 0 , \tag{39.1}$$

$$\dot{p}_2(T_0, \varepsilon) - \omega'(M_2 + \beta_2) - \alpha\omega'^2(M_1 + \beta_1) = 0 , \tag{39.2}$$

$$\dot{\gamma}_2(T_0, \varepsilon) - \alpha(M_3 + \beta_3) = 0 . \tag{39.3}$$

Making use of (26), (38) and Eq. (39.3), the function $\alpha(\varepsilon)$ takes the form

$$\alpha(\varepsilon) = \varepsilon^2(M_3 + \beta_3)^{-1}[\dot{H}_2(T_0) + \varepsilon\dot{H}_3(T_0) + \cdots] . \tag{40}$$

It follows then that, by neglecting terms of order α^2 and $\varepsilon^2\alpha$ in (39), we also omit the terms of order ε^4. Making use of (23) and (27), we shall investigate those periodic solutions when the basic amplitudes vanish, i.e.

$$M_1 = M_2 = 0 . \tag{41}$$

Applying (40), (41) and (26) to Eqs. (39.1, 2), one gets the system determining β_1 and β_2 in the form

$$G_2(T_0) + \varepsilon G_3(T_0) + \omega'\beta_2(M_3 + \beta_3)^{-1}[\dot{H}_2(T_0) + \varepsilon\dot{H}_3(T_0) + \cdots] + \varepsilon^2(\cdots) = 0 ,$$
$$\dot{G}_2(T_0) + \varepsilon\dot{G}_3(T_0) - \omega'^2\beta_1(M_3 + \beta_3)^{-1}[\dot{H}_2(T_0) + \varepsilon\dot{H}_3(T_0) + \cdots] + \varepsilon^2(\cdots) = 0 .$$

By virtue of (33), the above system is transformed into

$$-\pi n\beta_2(\omega')^{-1}[L_1(\omega') - \omega'^2 N_1(\omega')] + \varepsilon[G_3(T_0) + \cdots] = 0 ,$$
$$\pi n\beta_1[L_1(\omega') - \omega'^2 N_1(\omega')] + \varepsilon[\dot{G}_3(T_0) + \cdots] = 0 , \tag{42}$$

where $L_1(\omega')$ and $N_1(\omega')$ can be obtained from (32) by replacing M_1, M_2 and M_3 by β_1, β_2, and $M_3 + \beta_3$. By (11), (14), (17) and (32), one has

$$L_1(\omega') - \omega'^2 N_1(\omega') = (\beta_1{}^2 + \beta_2{}^2)\, W_1(\omega') + z_0' W_2(\omega') + kW_3(\omega') + W_4(\omega') ,$$

where

$$W_1(\omega') = d_1 + (d_2 + d_3)\, r_0{}^{-1}\ell_3 ,$$

$$W_2(\omega') = (d_4 - d_5 d_6 d_7) + r_0{}^{-1}\ell_3[d_5 d_6(d_8 + d_9) + B^{-1}d_7 - b^{-1}d_{10}(1 + a^{-1}d_6 d_7)]\,,$$

$$W_3(\omega') = (d_5 d_6 d_{11} + d_{12}) + r_0{}^{-1}\ell_3\{d_5[d_6(d_{13} - d_{14}) - B^{-1}d_{11}] + b^{-1}d_{10}(a^{-1}d_6 d_{11} + d_{15})\}\, r_0{}^{-1}\ell_3,$$

$$W_4(\omega') = -\frac{a}{2}d_{10}\left[\beta_1{}^2 + \left(\frac{a-1}{b-1}\right)\beta_2{}^2\right] r_0{}^{-1}\ell_3\,, \qquad d_1 = b^{-1}(a-1)\,(2a-b-1)\,,$$

$$d_2 = b^{-2}[b(a-b) + (a-1)]\,[aA^{-1}(a-1)\,(1-b)^{-1} + bB^{-1}]\,,$$

$$d_3 = \frac{1}{2}A^{-1}(1-a)\,[ab^{-1}(1-a)\,(1-b)^{-1} + AB^{-1}]\,, \qquad d_4 = a^{-1}[1 - b^{-2}(a-1)\,(b-1)]\,,$$

$$d_5 = (ab)^{-2}[ab + (a-1)\,(b-1)]\,, \qquad d_6 = b^{-1}(a+b-1)\,,$$

$$d_7 = (ab)^{-1}(2b-1)\,[ab + (a-1)\,(b-1)]\,, \qquad d_8 = (Ab)^{-1}[ab + (a-1)\,(b-1)]\,,$$

$$d_9 = (ab)^{-1}(2b-1)\,[A^{-1}a(a-1) + B^{-1}b(b-1)]\,,$$

$$d_{10} = (Ab)^{-1}(a-1) + (aB)^{-1}(b-1)\,,$$

$$d_{11} = (ab)^{-1}(1-b)\,(a+b-1)\,[ab + (a-1)\,(b-1)]\,,$$

$$d_{12} = (ab^2)^{-1}(1-b)\left[b^2 - (a-1)^2 + \frac{1}{2}\,b(a-1)\,(b-a)\,M_3{}^2\right],$$

$$d_{13} = (Ab)^{-1}(a-1)\,[ab + (a-1)\,(b-1)]\,,$$

$$d_{14} = (ab)^{-1}(1-b)\,(a+b-1)\,[aA^{-1}(a-1) + bB^{-1}(b-1)]\,,$$

$$d_{15} = \frac{3}{4}\,b(b-a)\,M_3{}^2 - (a-1)\,.$$

From the conditions that the z-axis has to be directed along the major or the minor axis of the ellipsoid of inertia of the body, it follows that $W_1(\omega') > 0$ for all ω' under consideration. We assume that

$$z_0{}'W_2(\omega') + kW_3(\omega') + W_4(\omega') \neq 0\,.$$

By use of (42), the expressions for β_1 and β_2 are obtained in the form of a power series of integral powers of ε. These expansions begin with terms of order higher than ε^2. Consequently, the first terms in the expansions of the periodic solutions and the quantity $\alpha(\varepsilon)$ are expressed in the following forms:

$$p_1 = \varepsilon\{-x_0{}'(a-1)^{-1}[1 + bB^{-1}(a-1)^{-1}r_0{}^{-1}\ell_3] + \chi_1 M_3\,\cos\tau\} + \cdots\,, \tag{43.1}$$

$$q_1 = \varepsilon a(1-b)^{-1}\{y_0{}'a^{-1} + \chi_2 M_3\,\sin\tau - A^{-1}(1-b)^{-1}r_0{}^{-1}\ell_3[y_0{}' + (z_0{}' - kaA_1) \times M_3\sin\tau + ad_5[kb(1-b)\,d_6 - z_0{}'(2b-1)]]\} + \cdots\,, \tag{43.2}$$

$$r_1 = 1 - \varepsilon^2 M_3\left[x_0{}'(1-\cos\tau) + y_0{}'\sin\tau + \frac{1}{4}kM_3C_1(1-\cos 2\tau)\right] + \cdots\,, \tag{43.3}$$

$$\gamma_1 = M_3\cos\tau + \cdots\,, \qquad \gamma_1{}' = -M_3\sin\tau + \cdots\,, \tag{43.4, 5}$$

$$\gamma_1'' = 1 + \varepsilon^2 \Big\{ (1-b)^{-1} M_3 y_0' \sin\tau + (1-a)^{-1} M_3 x_0' (1-\cos\tau)$$
$$- \frac{1}{2} b^{-1}(1-b)^{-1} d_7 M_3^2 z_0'(1-\cos 2\tau) + \frac{1}{4} M_3^2 k(2abd_5 d_6 + C_1)(1-\cos 2\tau)$$
$$+ r_0^{-1}\ell_3[-abA^{-1}(1-b)^{-2} M_3 y_0' \sin\tau + abB^{-1}(a-1)^{-2} M_3 x_0'(1-\cos\tau)$$
$$+ \frac{1}{2} b^{-1}(1-b)^{-1} z_0' M_3^2 (1-\cos 2\tau)[A^{-1}a^2 b d_5 (1-b)^{-1}(2b^2-2b+1) + d_9]$$
$$+ \frac{1}{2} k(1-b)^{-1} M_3^2 (1-\cos 2\tau)[b^{-1} d_{13} - aA^{-1} d_{11}(1-b)^{-1}$$
$$- (1-b)(2b-1)^{-1} d_6 d_9]] \Big\} + \cdots, \tag{43.6}$$

$$\alpha(\varepsilon) = \varepsilon^2 \pi n \{2M_3 x_0' - z_0' b^{-1} + (ab)^{-1}(kd_{11} - z_0' d_7) d_6 + k(M_3^2 C_1 - B_1)$$
$$+ (ab)^{-1} r_0^{-1} \ell_3 [[(d_8 + d_9) d_6 + d_7 B^{-1}] z_0' + [(d_{13} - d_{14}) d_6 - d_{11} B^{-1}] k]\} + \cdots. \tag{43.7}$$

Our solutions (43) are considered as a general case of [1] and [7] and having no singularity points at all, i.e., the obtained solutions are valid for all rational values of ω'.

Now we investigate the deviations between our solutions and the Newtonian and classical ones, which were obtained in [1] and [7]. The deviations can be expressed in the form

$$\Delta p_1 = \varepsilon\{x_0' b^{-1}[B_1^{-1}(1-\omega^2\omega'^{-2}) + \omega'^{-2} A^{-1} r_0^{-1} \ell_3] + (\chi - \chi_1^*) M_3 \cos\tau\} + \cdots,$$
$$\Delta q_1 = \varepsilon\{-y_0'(aAA_1^2)^{-1} r_0^{-1} \ell_3 + A_1^{-1} M_3 \sin\tau[\chi_1 - \chi_1^* - \chi_2 A^{-1} A_1^{-1} r_0^{-1} \ell_3]\} + \cdots,$$
$$\Delta r_1 = \varepsilon^3[0] + \cdots, \qquad \Delta\gamma_1 = \varepsilon[0] + \cdots, \qquad \Delta\gamma_1' = \varepsilon[0] + \cdots,$$
$$\Delta\gamma_1'' = \varepsilon^2\{aM_3(\chi - \chi^*)(1-\cos\tau) - bM_3 y_0'(aAA_1^2)^{-1} r_0^{-1} \ell_3 \sin\tau$$
$$+ \frac{1}{2} M_3^2 (1-\cos 2\tau)[a(1-b)^{-1}(\chi_1 - \chi_1^*) - \chi_2 b(AA_1^2)^{-1} r_0^{-1} \ell_3]\} + \cdots,$$
$$\Delta\alpha(\varepsilon) = \varepsilon^2 \pi n\{(1-B_1)(\chi_1 - \chi_1^*) - B^{-1}\chi_1 r_0^{-1} \ell_3\} + \cdots$$

and

$$\Delta p_{11} = \Delta p_1 + \varepsilon(\chi_1^* - \chi_1^{**}) M_3 \cos\tau + \cdots,$$
$$\Delta q_{11} = \Delta q_1 + \varepsilon A_1^{-1} M_3 (\chi_1^* - \chi_1^{**} - kA_1) \sin\tau + \cdots,$$
$$\Delta r_{11} = -\frac{1}{4} \varepsilon^2 M_3^2 C_1 (1-\cos 2\tau) + \cdots$$
$$\Delta\gamma_{11} = \varepsilon[0] + \cdots, \qquad \Delta\gamma_{11}' = \varepsilon[0] + \cdots,$$
$$\Delta\gamma_{11}'' = \Delta\gamma_1'' + \varepsilon^2 \Big\{ \frac{k}{4} M_3^2 C_1 (1-\cos 2\tau) +$$
$$\frac{1}{2} M_3^2 (1-\cos\tau)[a(1-b)^{-1}(\chi_1^* - \chi_1^{**}) - kb] \Big\} + \cdots,$$
$$\Delta\alpha_1(\varepsilon) = \Delta\alpha + \varepsilon^2 \pi n[z_0'(2-b)^{-1} + k(M_3^2 C_1 - B_1) + (1+B_1)\chi_1^*] + \cdots,$$

where

$$\chi^* = (b\omega^2)^{-1} A_1 x_0', \qquad \chi_1^* = (1-\omega^2)^{-1}[k(A_1 - \omega^2) - z_0'(a^{-1} - A_1 b^{-1})],$$
$$\chi_1^{**} = -z_0'(1-\omega^2)^{-1}(a^{-1} - A_1 b^{-1}).$$

4 Geometric interpretation of motion

In this Section, the motion of the rigid body is investigated by introducing Euler's angles θ, ψ, and φ, which can be determined through the obtained periodic solutions (see Fig. 2).
Since the initial system is autonomous, the periodic solutions are still periodic if t is replaced by $(t + t_0)$, where t_0 is an arbitrary interval of time. Euler's angles, in terms of time t, take the forms [3]

$$\begin{aligned} &\cos\theta = \gamma'' , && \frac{d\psi}{dt} = \frac{p\gamma + q\gamma'}{1-\gamma''^2} , \\ &\tan\varphi_0 = \frac{\gamma_0}{\gamma_0'} , && \frac{d\varphi}{dt} = r - \frac{d\psi}{dt}\cos\theta . \end{aligned} \tag{44}$$

Substituting (43) into (44), in which t has been replaced by $t + t_0$, and using relations (5), the following expressions for the angles θ, ψ, and φ are obtained:

$$\varphi_0 = (\pi/2) + r_0 t_0 + \cdots , \qquad \theta_0 = \tan^{-1} M_3 ,$$

$$\theta = \theta_0 - \varepsilon^2[\theta_1(t + t_0) - \theta_1(t_0)] ,$$

$$\psi = \psi_0 + \varepsilon c \operatorname{cosec}\theta_0\sqrt{\cos\,\theta_0}\,[\psi_1(t+t_0) - \psi_1(t_0)] ,$$

$$\begin{aligned} \varphi = \varphi_0 + r_0 t &- \varepsilon c\cot\theta_0\sqrt{\cos\,\theta_0}\,[\varphi_1(t+t_0) - \varphi_1(t_0)] \\ &- \varepsilon^2\tan\,\theta_0[\varphi_2(t+t_0) - \varphi_2(t_0)] , \end{aligned}$$

where

$$\theta_1(t) = a_1\sin\,r_0 t - a_2\cos\,r_0 t - a_3\tan\,\theta_0\cos\,2r_0 t ,$$

$$\begin{aligned} \psi_1(t) = a_4 {r_0}^{-1}\sin\,r_0 t + a_5 {r_0}^{-1}\cos\,r_0 t &+ \frac{1}{2}(\chi_1 - a_6)\,t\tan\,\theta_0 \\ &+ \frac{1}{4}{r_0}^{-1}(\chi_1 - a_6)\tan\,\theta_0\sin\,2r_0 t , \end{aligned}$$

$$\varphi_1(t) = \psi_1(t) ,$$

$$\varphi_2(t) = a_7 r_0 t - {x_0}'\sin\,r_0 t - {y_0}'\cos\,r_0 t - \frac{1}{8}kC_1\tan\,\theta_0\sin\,2r_0 t ,$$

$$a_1 = (1-b)^{-1}{y_0}'[1 - abA^{-1}(1-b)^{-1}{r_0}^{-1}\ell_3] ,$$

$$a_2 = (1-a)^{-1}{x_0}'[1 + abB^{-1}(1-a)^{-1}{r_0}^{-1}\ell_3] ,$$

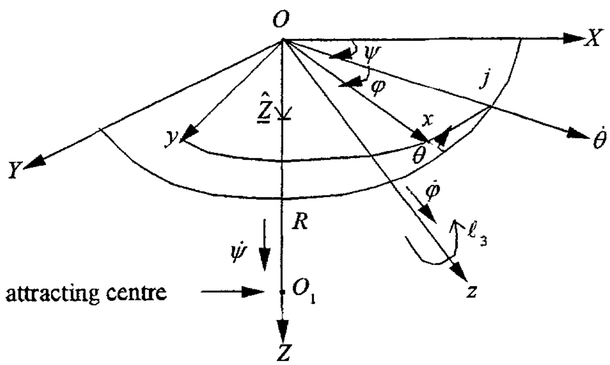

Fig. 2. Representation of Euler's angles. OX, OY, OZ are the fixed axes in space, Ox, Oy, Oz are the body moving principal axes, θ, φ, ψ are Euler's angles.

$$a_3 = \frac{1}{2} z_0' b^{-1}(1-b)^{-1}\{r_0{}^{-1}\ell_3[a^2 b d_5 A^{-1}(1-b)^{-1}(2b^2-2b+1)+d_9]-d_7\}$$
$$+\frac{1}{4}k\{(2abd_5d_6+C_1)+2(1-b)^{-1}r_0{}^{-1}\ell_3[b^{-1}d_{13}-aA^{-1}(1-b)^{-1}d_{11}$$
$$+(b-1)(2b-1)^{-1}d_6d_9]\}\,,$$
$$a_4 = -x_0'(a-1)^{-1}[1+bB^{-1}(a-1)^{-1}r_0{}^{-1}\ell_3]\,,$$
$$a_5 = (1-b)^{-1}y_0' - aA^{-1}(1-b)^{-2}r_0{}^{-1}\ell_3\{y_0' + ad_5[kb(1-b)\,d_6 - z_0'(2b-1)]\}\,,$$
$$a_6 = a(1-b)^{-1}[\chi_2 - aA^{-1}(1-b)^{-1}r_0{}^{-1}\ell_3(\chi_2-\chi_1)]\,,$$
$$a_7 = x_0' + \frac{k}{4}\,C_1 \tan\theta_0\,.$$

The expressions for the Eulerian angles θ, ψ and φ depend on some arbitrary constants θ_0, ψ_0, φ_0 and r_0 (r_0 is large).

5 Discussion of the solutions

In [1], [7], [9], and [10], there are singularities in the obtained solutions when $\omega = 1, 2, 3, 1/2, 1/3, \cdots$. The solutions for these singularities are obtained separately, see [3], [7], [8], [10], and [11]. In our problem when we used the frequency ω' instead of ω, there are no singular points at all. The obtained solutions are valid for all rational values of ω'. From Sect. 4, we conclude for $\varepsilon = 0$ that $\dot\theta = 0$, $\dot\psi = 0$ and $\dot\varphi = r_0$. This permits permanent rotations of the body with spin r_0 (sufficiently large) about the z-axis.

5.1 Numerical discussions

In this section we investigate the numerical results by computer programs for the aforementioned problem.

Let us consider two cases:

1. $A < B < C$

For this case, the following parameters for the motion of the body are determined:

$$A = 8.53\,\mathrm{kg.mm}^2\,,\quad B = 19.6\,\mathrm{kg.mm}^2\,,\quad C = 26.27\,\mathrm{kg.mm}^2\,,\quad r_0 = 1\,000\,\mathrm{mm}\,,$$
$$R = (1\,000, 1\,500, 2\,000)\,\mathrm{mm}\,,\quad \lambda = 0.6\,,\quad M = 300\,\mathrm{kg}\,,\quad x_0 = 1\,\mathrm{mm}\,,\quad y_0 = 2\,\mathrm{mm}\,,$$
$$z_0 = -1\,\mathrm{mm}\,,\quad \ell_3 = (0, 50, 100, 150)\,\mathrm{kg.mm}^2.\mathrm{s}^{-1}\,,\quad \gamma_0'' = 0.352\,,\quad T = 12.566\,371\,.$$

p_2 and γ_2 denote the analytical solutions in this case. Figure 3.1 shows that in the absence of a gyrostatic momentum about the z-axis ($\ell_3 = 0$) the position of the centre of attraction is independent of the behavior of the solution, i.e., the solutions p_2 are the same when $\ell_3 = 0$ with different distances of the centre of attraction, $R = (1\,000, 1\,500, 2\,000)$. We also note that when ℓ_3 and R increase, the amplitude of the oscillations decrases and the number of oscillations increases, see Figs. 3.2–3.4. Figure 3.5 shows the behavior of γ_2 via t for different values of ℓ_3.

2. $A > B > C$

Let us suppose:

$$A = 35.21\,\text{kg.mm}^2\,,\qquad B = 21.49\,\text{kg.mm}^2\,,\qquad C = 17.6\,\text{kg.mm}^2\,,\qquad r_0 = 1\,000\,\text{mm}\,,$$

$$R = (1\,000, 1\,500, 2\,000)\,\text{mm}\,,\qquad \lambda = 0.6\,,\qquad M = 300\,\text{kg}\,,\qquad x_0 = 1\,\text{mm}\,,\qquad y_0 = 2\,\text{mm}\,,$$

$$z_0 = -1\,\text{mm}\,,\qquad \ell_3 = (0, 50, 100, 150)\,\text{kg.mm}^2.\text{s}^{-1}\,,\qquad \gamma_0{''} = 0.352\,,\qquad T = 12.566\,371\,.$$

Figure 3.6 shows that there is no variation of the amplitude and the number of oscillations when R increases. One can see from Figs. 3.7–3.9 that when ℓ_3 increases for the same values of R and when R increases for the same values of ℓ_3, the amplitude of the wave decreases and the number of oscillations increases. Figure 3.10 shows the variation of γ_2 via t when ℓ_3 takes different values.

We conclude from the previous cases that when the minor axis of the ellipsoid of inertia of the body coincides with the z-axis ($A < B < C$), the number of oscillations increases and the amplitude of the waves decreases. When the major axis of the ellipsoid of inertia of the body coincides with the z-axis ($A > B > C$), the number of oscillations increases to some extent and the amplitude of the waves decreases.

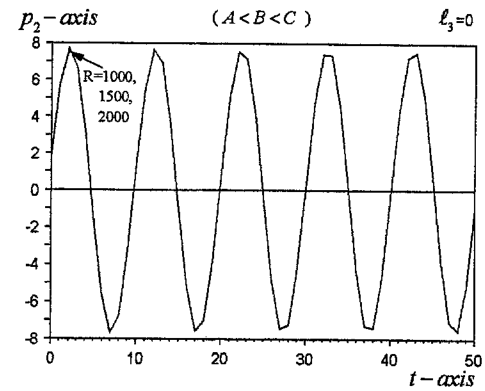

Fig. 3.1.

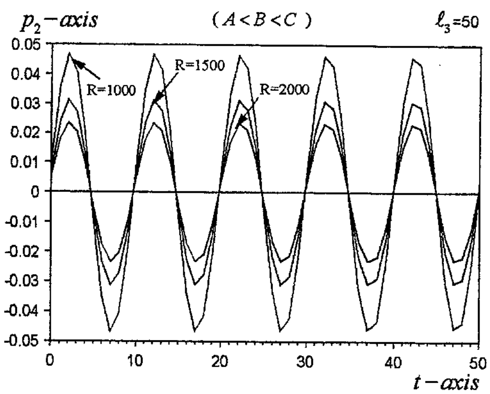

Fig. 3.2.

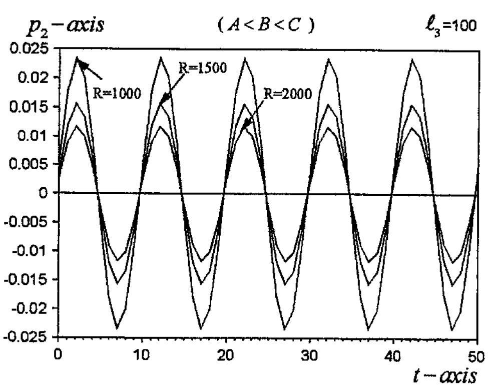

Fig. 3.3.

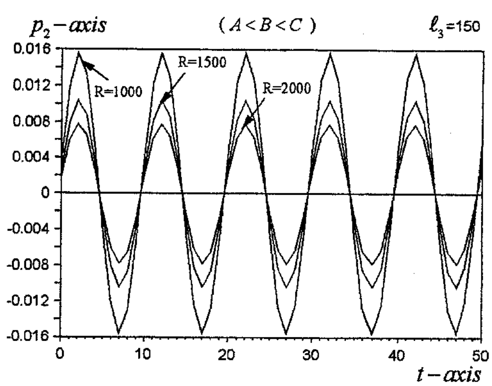

Fig. 3.4.

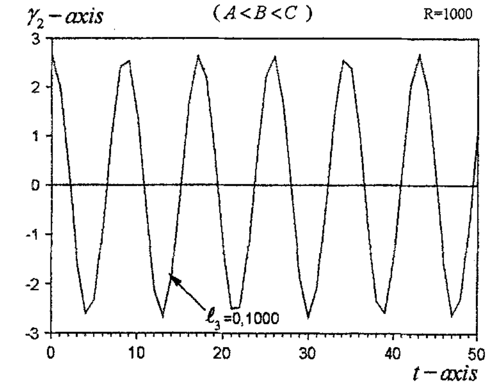

Fig. 3.5.

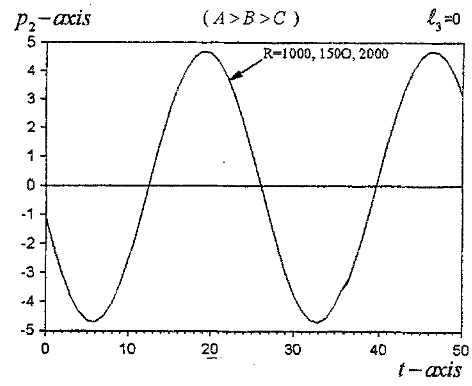

Fig. 3.6.

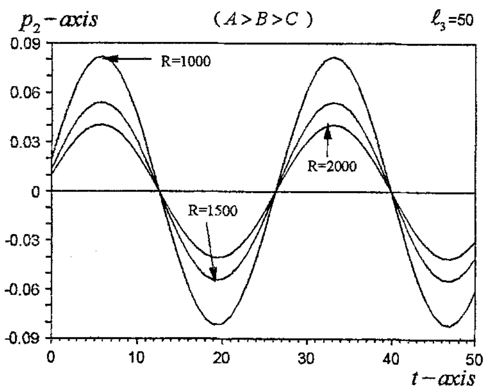

Fig. 3.7.

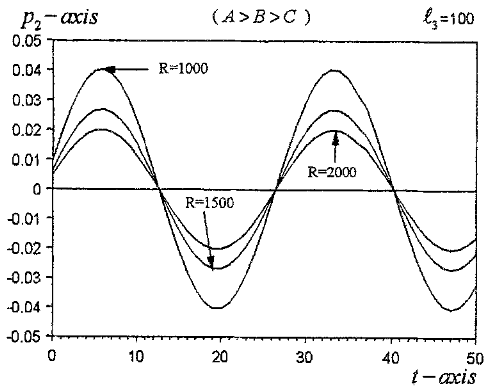

Fig. 3.8.

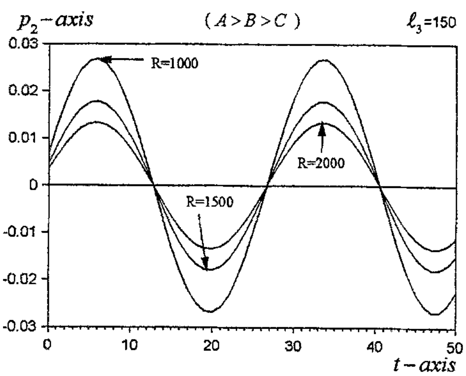

Fig. 3.9.

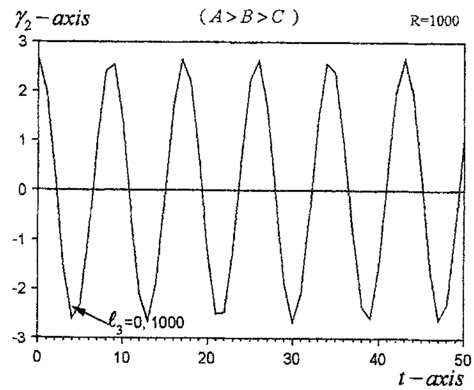

Fig. 3.10.

6 Conclusion

The problem of the three-dimensional motion of a rigid body in the Newtonian force field with a third gyrostatic momentum about one of the principal axes of the ellipsoid of inertia is investigated by reducing the six first-order nonlinear differential equations of motion and their first three integrals into a quasi-linear autonomous system with two degrees of freedom and one first integral. Poincaré's small parameter method is used to investigate the periodic solutions of the present problem up to the first-order approximation in terms of the small parameter ε. The periodic solutions (43) are considered as a generalization of those by Arkhangel'skii [7] (in the case of the uniform force field) and El-Barki et al. [1] (in the case of the Newtonian force field). The solutions and the correction of the period for the latter two problems can be deduced from our solutions as limiting cases by reducing the Newtonian terms and the third gyrostatic momentum. The introduction of an alternative frequency ω' instead of ω avoids the singularities traditionally appearing in the solutions of other treatments. When the minor axis of the ellipsoid of inertia of the body coincides with the z-axis $(A < B < C)$, the number of oscillations increases, and the amplitude of the waves decreases. Also, when the major axis of the ellipsoid of inertia of the body coincides with the z-axis $(A > B > C)$, the number of oscillations increases to some extent, and the amplitude of the waves decreases. In the case without gyrostatic momentum about the z-axis ($\ell_3 = 0$), the position of the centre of attraction is independent irrespective of the behavior of the solutions. The analytical solutions are analysed geometrically using Euler's angles to describe the orientation of the body at any instant of time. These solutions are performed by computer programs to get their graphical representations. A great effect of the third gyrostatic momentum (ℓ_3) is shown obviously from the graphical representations.

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