OPTIMIZING THE NUMERICAL SOLUTIONS OF BLACK-SCHOLES PDES IN OPTIONS PRICING

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ABSTRACT

In this paper we present an elegant way of solving the Black-Scholes partial differential equation using numerical methods. We begin by posing the simplest problem of pricing a vanilla European option, whose PDE is explicitly solvable. We apply central, forward, and backward differences covering different orders of precision and sparse matrix TDMA techniques to arrive at the solution. We compare the error for different step-sizes and choose metaparameters that implement the most numerically stable solution. We then proceed to modify the Black-Scholes equation to price early-exercise (American) options, a model whose partial differential equations need not have explicit solutions and use the numerically stable algorithms proposed to solve them.

1 INTRODUCTION

The world of finance, particularly the world of financial derivatives, has developed over the years to be one of the most complex systems of abstract tradeable instruments. In order for someone to successfully trade them, they must not only understand the instrument itself, but generate a model for its fair value (Samuelson, 1965; Black & M., 1973). Options, the focus of this paper, are examples of such complicated instruments.

The models to price the tradeable instruments can be very complex and are generally given by high order and high dimensional partial differential equations. Most of these models have no analytic solutions and require a lot of computational resources to approximate accurately (Higham (2004)).

In the modern competitive world, in order to have an edge in the markets, accuracy and precision must be optimized at all costs. Thus, one must devote a serious amount of time to make sense of the numerical methods being used to solve the models such that they are optimal, produce minimal errors, and overall produce a great approximation of the exact solution.

In this paper we aim to find optimal metaparameters such as the type of space discretization scheme for a finite difference method such that the errors are minimal and the solution is numerically stable. We do so by focusing on the pricing of financial options and comparing numerical approximations of the Black-Scholes model for a European option on a non dividend-paying asset with its explicit anlytical solution.

Once the optimal metaparameters have been found, we perform the same numerical analysis to solve modifications of the Black-Scholes PDE to price more complicated instruments, instruments that based on the model do not have any known explicit solution.

2 BACKGROUND

The theory of options pricing is very nicely derived and summarized in two books, Hull (2006) and Bjork (1998). In this section we attempt to illustrate some of the derivations of the pricing of options.

An option is a financial derivative contract between two counterparties in which one of the parties is given the right but not the obligation to execute a trade at most on some future date T known as the maturity date or expiration data for some price K known as the strike price (Merton, 1973).

In an options contract, one of the parties will take a long position and the other one will take a short position. The party in the long position agrees to buy an asset from the party in the short position, who sells the asset. A put option gives that right to the seller while a call option gives it to the buyer.

The date at which an option matures is another element that goes into the creation of an options contract. An option that can only be exercised at the maturity date, a date agreed on the contract, is called a European option. An option that can be exercised at any point until the maturity date is called an American option.

For example, if you purchased an American call option with a strike price of 120 and an expiration of 30 days from now, owning this call option will allow you to purchase stock at 120 per share anytime within the next 30 days no matter where the stock price is at. Even the current stock value becomes much higher, it's within your right to purchase the stock at the lower strike price. Similarly, a put option will allow you to sell the stock at the strike price regardless of the current market value.

Let S_{t^*} be the price of the asset at time t^*, the date of exercise. The payoff from a long position in a call option is $\max(S_{t^*} - K, 0)$ and $-\max(S_{t^*} - K, 0) = \min(K - S_{t^*}, 0)$ for the short position.

On the other hand, payoff from a long position in a put option is $\max(K - S_{t^*}, 0)$ and $-\max(K - S_{t^*}, 0) = \min(S_{t^*} - K, 0)$ on the short position. The payoff for each position and for either a put or call option is summarized by the graph below

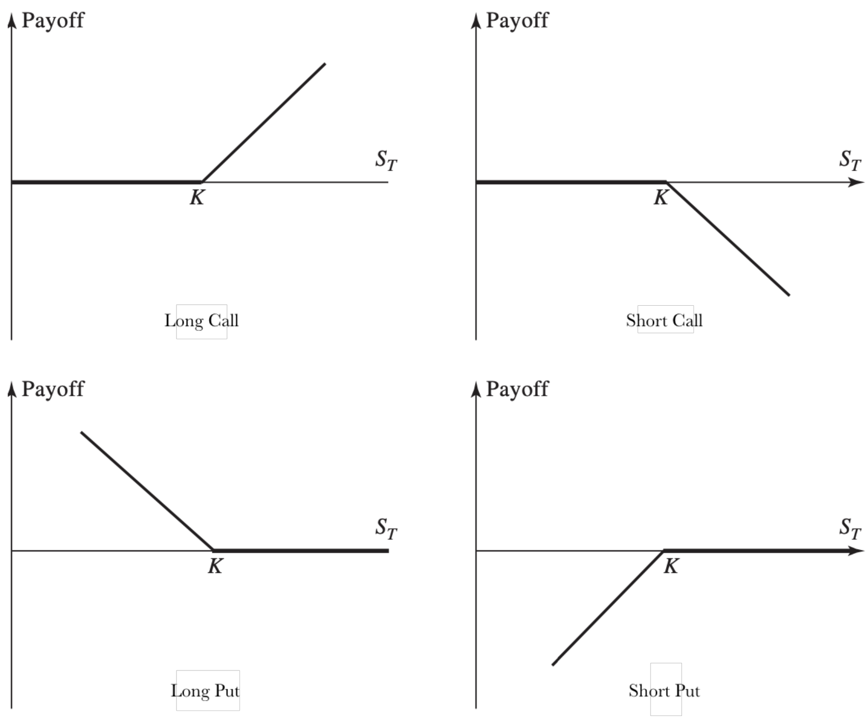

Figure 1: Payoff of options. On the x-axis we have the price of the underlying asset and on the y-axis we have the payoff. Figure taken from Hull (2006).

The question to ask now is what should be the fair value for the option, the small width between the line of slope zero and the x-axis? Pricing an option is difficult because there are many parameters at play, such as the behavior of the asset to be traded, the maturity date, and the option of early exercise.

For the sake of generality, throughout the paper we assume that there are no transaction costs, all trading profits or losses are subject to the same tax rate, there are no arbitrage opportunities, and borrowing and lending are possible at some risk-less interest rate, r.

In general, for a call option, an increase in the underlying asset's price, an increase in the volatility of the asset, and the increase of interest rates increase the price of the option. On the other hand, an increase in the agreed trade price (strike price), and an increase in future dividends will decrease the price of the option.

For a put option, an increase in the underlying asset's price and increasing interest rates decrease the price of the option while an increase in the strike price, volatility, and amount of future dividends produce a decrease in the option's price.

Before even thinking about the mechanisms of how to price an option, it makes sense to think about the upper and lower bounds of an option's price.

Let us talk about upper bounds first. For the case of a call option, the holder is being given the right to buy an asset, so the price of the option cannot be worth more than the asset's price at the date of the agreement. Otherwise, an arbitrage opportunity would arise, one could buy the asset and long the option and risklessly lock profit. For the case of a put option, the holder is being given the right to sell an asset for a price K.

Thus, the option cannot be worth more than K, otherwise one could take the short position and lock riskless profit since even if the value of the asset goes to zero, the holder would receive at most K for selling the asset, making them always produce losses. For the case of a European option in which the exercise date is known, we can even give a stricter bound. The upper bound would be the discounted strike price Ke^{-rT} since the person on the long side could invest the price of the option at the risk free rate and lock even more profit.

Now, let us examine the lower bounds. The way to arrive to the lower bound is via two portfolio arguments.

Consider a portfolio with one European put option and the underlying asset. Call it portfolio A. Now consider portfolio B, a portfolio with K dollars, the value of the strike, that can only be redeemed at time T.

If at time T, the value of the asset is less than K, then portfolio A's owner would exercise the option and sell the asset at K. If $S_T > K$, then owner doesn't exercise the option and the portfolio is worth S_T. Thus, the value of portfolio A is $\max(S_T, K)$ at time T. For the latter, portfolio B is worth K at time T, so portfolio A is worth at least the value of portfolio B at time T. Because of the no arbitrage principle, at time 0, portfolio A must be at least as much as the value of portfolio B. Thus,

$$\begin{gathered}\text{Portfolio A today} \geq \text{Portfolio B today} \\ \text{Put} + S_0 \geq K\end{gathered} \tag{1}$$

Which implies that the value of the put option is at least $K - S_0$. Obviously, it cannot have negative value, so the lower bound for a put option is $\max(K - S_0, 0)$. For the European option case, again, we can find a stricter bound by discounting the value of K to the present day, so in the bound we would replace K with Ke^{-rT}.

For the lower bound of the call option, we arrive at the conclusion using the put call parity of European options.

Consider a portfolio that goes long on put option and goes short on a call option, both options on the same underlying asset, with the same strike K and with the same maturity date T. Then the payoff diagram would be the difference of the payoff diagram for the call minus the put, which would be linear just like having the underlying asset.

The payoff at time T of this portfolio would be Call $-$ Put $= S_T - K$, but discounting to the present would give us the price at Call $-$ Put $= S_0 - Ke^{-rT}$. Solving for the put option and applying the lower bound yields a lower bound for the call option of $\max(S_0 - Ke^{-rT}, 0)$.

The following charts summarize and visualize the lower and upper bounds for options on non dividend-paying assets.

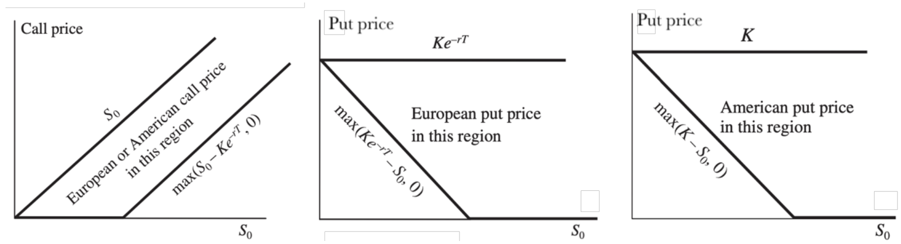

Figure 2: Upper and lower bounds for the fair present value of American and European put and call options on non dividend-paying assets. Figure taken from Hull (2006).

With a clear picture of the problem and bounds for it, we proceed to model an exact price by mathematically valuing the simplest option of all, a European option on a non-dividend paying asset.

Then we will modify the obtained solution to model the pricing of an American option on a non-dividend paying asset as well.

3 Mathematical modeling

Let the price of a stock with no dividends at time t, S_t follow a continuous time stochastic process that assumes that for a small interval of time, the return on a stock is equal to the expected return μ over that interval of time plus some Gaussian noise with magnitude as a functi on of σ^2, the variance of the stock price, and the time t. Mathematically,

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{2}$$

where $\epsilon \sim \mathcal{N}(0,1)$. In the continuum limit, this is precisely the stochastic differential equation for a process known as geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \tag{3}$$

where dZ_t is an increment in the value of the random variable Z_t in a time interval $[t, t+dt]$. It follows that $dZ_t \sim \mathcal{N}(0, dt)$. Note that this implies that S_t is lognormally distributed.

Let $V(S_t, t)$ be the function for the fair value of a European call option (Fadugba & Nwozo (2012)). By Ito's lemma, any function of S_t and t follows

$$\begin{aligned} \Delta V &= (\frac{\partial V}{\partial t} + S_t \mu \frac{\partial V}{\partial S_t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 V}{\partial Z_t^2}) \Delta t + \frac{\partial V}{\partial S_t} \sigma S_t \Delta Z_t \\ \partial V &= (\frac{\partial V}{\partial t} + S_t \mu \frac{\partial V}{\partial S_t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 V}{\partial Z_t^2}) dt + \frac{\partial V}{\partial S_t} \sigma S_t dZ_t \end{aligned} \tag{4}$$

Suppose you construct a portfolio consisting of shorting one option and longing an amount $\frac{\partial V}{\partial S_t}$ of the underlying stock. The value and the change in value are given by

$$\begin{aligned} P &= -V + \frac{\partial V}{\partial S_t} S_t \\ \Delta P &= -\Delta V + \frac{\partial V}{\partial S_t} \Delta S_t \end{aligned} \tag{5}$$

Substituting for ΔV from Eq.(4) and for ΔS_t from Eq.(2), we obtain

$$\Delta P = (-\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}) \Delta t \tag{6}$$

Since the equation above does not have a term ΔZ_t, the portfolio must be riskless and must earn the same rate of return as the risk-free rate, r.

Therefore $\Delta P = rP\Delta t$. Substituting for P and ΔP, we obtain the famous Black-Scholes differential equation.

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} = rV \tag{7}$$

Solving the differential equation implies solving a boundary value problem where the boundary is the payoff of the option. For a European call, $V = \max(S_T - K, 0)$ and for a European put, it is $V = \max(K - S_T, 0)$ as we saw previously.

It turns out that this differential equation has an explicit solution, which can be derived by risk-neutral pricing a call option. In theory, in a risk-neutral world, the value of a call option is the discounted expected payoff of the option.

$$V(S_t, t) = e^{-rT}\mathbb{E}[\max(S_T - K, 0)] \tag{8}$$

Since S_T is lognormally distributed, $\ln S_T$ is normally distributed with mean $\nu = \ln\mathbb{E}[S_T] - \frac{1}{2}\omega^2$ and variance ω^2 by the lognormal properties, we can compute the expectation by transforming it into a standard normal random variable and solving for S_T. Let $Z = \frac{\ln S_T - \nu}{\omega} \sim \mathcal{N}(0, 1)$. Then,

$$\begin{aligned}\mathbb{E}[\max(S_T - K, 0)] &= \int_K^\infty (s_T - K)f_{S_T}(s)ds_T \\ &= \int_{\frac{\ln K - \nu}{\omega}}^\infty (e^{z\omega+\nu} - K)\phi(z)dz\end{aligned} \tag{9}$$

where ϕ is the PDF of a standard normal random variable. Separating the integral and applying the normal-normal conjugacy, we obtain

$$\mathbb{E}[\max(S_T - K, 0)] = e^{\nu+\frac{1}{2}\omega^2}\int_{\frac{\ln K - \nu}{\omega}}^\infty \phi(z-\nu)dz - K\int_{\frac{\ln K - \nu}{\omega}}^\infty \phi(z)dz \tag{10}$$

If we pattern-match the integrals to the standard normal CDF, $\Phi(\cdot)$, and substitute for ν and ω we get

$$\begin{aligned}\mathbb{E}[\max(S_T - K, 0)] &= e^{\nu+\frac{1}{2}\omega^2}\Big(1 - \Phi\Big(\frac{\ln K - \nu}{\omega}\Big)\Big) - K\Big(1 - \Phi\Big(\frac{\ln K - \nu}{\omega}\Big)\Big) \\ &= \mathbb{E}[S_T]\Phi(d_1) - K\Phi(d_2)\end{aligned} \tag{11}$$

where $d_1 = \frac{\ln\frac{S_0}{K} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\frac{S_0}{K} + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$. Letting $\mathbb{E}[S_T] = S_0e^{rT}$ and multiplying by e^{-rT}, we get the solution for V,

$$V(S_t, t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{12}$$

For American options, the problem becomes slightly more complicated because a holder can exercise the option at any time τ on or before the maturity date. The value of the option will then be determined by the solution to an optimal stopping problem formulated as the following:

$$V(S_t, t) = \sup_{\tau\in[t,T]}\mathbb{E}\Big[e^{-r(\tau-t)}V(S_\tau, T)\Big|S_t\Big] \tag{13}$$

These kinds of problems are difficult to solve explicitly and lead to free boundary value problems instead of the corresponding parabolic PDEs for the European options. The mathematics to arrive at the PDE that describes this problem is beyond the scope of this paper, however it is nicely derived by Peskir (2006) to result in

$$\max\left\{\frac{\partial V(S_t,t)}{\partial t} + r\,S_t\frac{\partial V(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V(S_t,t)}{\partial S_t^2} - rV(S_t,t),\; V(S_t,T) - V(S_t,t)\right\} = 0 \tag{14}$$

This formula is a variation from that of a European option precisely because in this case the option can be exercised at any time. The problem now is to find the best time τ that maximizes the expectation, hence the profit.

It's worth noticing that the time τ is a random variable. It can be different for numerous approaches of the stock process. Therefore, at each time we need to determine whether it is optimal to exercise the option, depending on the current market value of the underlying stock.

As we know, when an American put option is not early-exercised, the premium will be equal to its European counterpart. The holder of an American put option should take an optimal early exercise strategy to get the maximum option premium.

We take a similar approach to derive a closed form solution proposed by Wang (2007) to price an early American put option on a non-dividend paying stock and obtain the following equations:

$$V_A(S_t,t) = V_E(S_0,Ke^{rT},r,T,\sigma)\Phi(-d_4) + \max\Big[(K-S_0), V_E(S_0,K,r,T,\sigma)\Big]\Phi(d_4) \tag{15}$$

where

$$V_E(S_0,K,r,T,\sigma) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$
$$V_E(S_0,Ke^{rT},r,T,\sigma) = K\Phi(-d_4) - S_0\Phi(-d_3)$$

and

$$d_1 = \frac{ln\frac{S_0}{K} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{ln\frac{S_0}{K} + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_3 = \frac{ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_4 = \frac{ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

In the next sections we will solve the Black-Scholes partial differential equation numerically and compare to the exact solution such that we find optimal hyperparameters to solve variations of the studied PDE whose explicit solutions are unknown, such as that of the American options.

4 Methods

In this section we apply some of the finite difference methods and sparse matrix algorithms proposed by Wilmott (1993) to solve the Black-Scholes partial differential equation after performing some simplifying transformations.

We begin by applying a change of variables to make the computations less expensive, $X_t = \ln S_t$. Thus, $\frac{\partial V}{\partial S_t} = \frac{\partial V}{\partial X_t}\frac{\partial X_t}{\partial S_t} = \frac{1}{S_t}\frac{\partial V}{\partial X_t}$ and $\frac{\partial^2 V}{\partial S_t^2} = \frac{\partial}{\partial S_t}\left(\frac{\partial V}{\partial S_t}\right) = \frac{\partial}{\partial S_t}\left(\frac{1}{S_t}\frac{\partial V}{\partial X_t}\right) = \frac{1}{S_t^2}(\frac{\partial^2 V}{\partial X_t^2} - \frac{\partial V}{\partial X_t})$. The PDE reduces to

$$\frac{\partial V}{\partial t} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial X_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X_t^2} = rV \tag{16}$$

With a boundary condition $V(X_T, t) = \max(e_T^X - K, 0)$. Now, we apply finite difference methods to achieve an approximation.

Taylor expanding $V(X_t, t+\Delta t), V(X_t, t-\Delta t)$ up to the second order, yields the backward, central, and forward difference approximations for $\frac{\partial V}{\partial t}$.

$$\begin{aligned}
\frac{\partial V(X_t,t)}{\partial t} &\approx \frac{V(X_t,t) - V(X_t, t-\Delta t)}{\Delta t} + \mathcal{O}(\Delta t) \\
\frac{\partial V(X_t,t)}{\partial t} &\approx \frac{V(X_t,t+\Delta t) - V(X_t, t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2). \\
\frac{\partial V(X_t,t)}{\partial t} &\approx \frac{V(X_t,t+\Delta t) - V(X_t, t)}{\Delta t} + \mathcal{O}(\Delta t)
\end{aligned} \tag{17}$$

We also Taylor expand the value formula but for a small increase and decrease in X_t, ΔX_t to obtain the symmetric central difference approximation for $\frac{\partial^2 V}{\partial X_t^2}$.

$$\frac{\partial^2 V(X_t,t)}{\partial X_t^2} \approx \frac{V(X_t + \Delta X_t, t) + V(X_t - \Delta X_t, t) - 2V(X_t, t)}{\Delta X_t^2} + \mathcal{O}(\Delta X_t^2). \tag{18}$$

Now, we discretize the domain to a finite region $[X_0, X_T] \times [t_0, T]$. For M log price steps and N time steps, we define each step as $\frac{X_T - X_0}{M}$ and $\Delta t = \frac{T - t_0}{N}$ such that $t_n = t_0 + n\Delta t$ and $X_m = X_0 + m\Delta X_t$. We let $V_m^n = V(X_0 + m\Delta X_t, t_0 + n\Delta t)$.

We can write down the backwards discretization for the time first order derivative, central discretization for the log price first order derivative, and symmetric central for the log price second order derivative.

The discretization becomes

$$\frac{V_m^{n+1} - V_m^n}{\Delta t} + (r - \frac{1}{2}\sigma^2)\frac{V_{m+1}^n - V_{m-1}^n}{2\Delta X_t} + \frac{1}{2}\sigma^2 \frac{V_{m+1}^n + V_{m-1}^n - 2V_m^n}{\Delta X_t^2} = rV_m^n. \tag{19}$$

If we rearrange the terms, we obtain an equation of the form:

$$V_m^{n+1} = aV_{m-1}^n + bV_m^n + cV_{m+1}^n \tag{20}$$

with $a = (r - \frac{1}{2}\sigma^2)\frac{\Delta t}{2\Delta X_t} - \frac{1}{2}\sigma^2 \frac{\Delta t}{\Delta X_t^2}$, $b = 1 + r\Delta t + \sigma^2 \frac{\Delta t}{\Delta X_t^2}$, and $c = -(r - \frac{1}{2}\sigma^2)\frac{\Delta t}{2\Delta X_t} - \frac{1}{2}\sigma^2 \frac{\Delta t}{\Delta X_t^2}$.

Derivations for other finite difference methods follow similarly.

If we change the space discretization to forward difference, this becomes $a = -\frac{\Delta t}{2\Delta x^2}\sigma^2, b = 1 + (r - \frac{1}{2}\sigma^2)\frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x^2}\sigma^2 + r\Delta t$, and $c = -(r - \frac{1}{2}\sigma^2)\frac{\Delta t}{\Delta x} - \frac{\Delta t}{2\Delta x^2}\sigma^2$.

For backward difference, this becomes $a = (r - \frac{1}{2}\sigma^2)\frac{\Delta t}{\Delta x} - \frac{\Delta t}{2\Delta x^2}\sigma^2, b = 1 - (r - \frac{1}{2}\sigma^2)\frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x^2}\sigma^2 + r\Delta t$, and $c = -\frac{\Delta t}{2\Delta x^2}\sigma^2$.

We will compare their accuracy in the results section.

This system of equations of multiple variables can be written in matrix form as

$$\begin{pmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ V_{M-2}^{n+1} \\ V_{M-1}^{n+1} \end{pmatrix} = \begin{pmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & a & b & c \\ 0 & 0 & 0 & a & b \end{pmatrix} \cdot \begin{pmatrix} V_1^n \\ V_2^n \\ \vdots \\ V_{M-2}^n \\ V_{M-1}^n \end{pmatrix} + \begin{pmatrix} aV_0^n \\ 0 \\ \vdots \\ 0 \\ cV_M^n \end{pmatrix} \tag{21}$$

Now, the system can be simplified as

$$V^{n+1} = DV^n + B \tag{22}$$

where D is a sparse matrix, namely a tridiagonal matrix, which can be encoded in a sparse matrix format such as compressed sparse sparse row (CSR), compressed sparse column (CSC). Then, the tridiagonal matrix algorithm (TDMA) can be applied to solve the system. The advantage is that the TDMA algorithm runs in $\mathcal{O}(n)$ time while Gaussian elimination would run in $\mathcal{O}(n^3)$ time. A tridiagonal system is solved in linear time by performing a forward pass and a backward pass.

Algorithm 1 Application of the tridiagonal matrix algorithm (TDMA) to solve the system of a european option in linear time complexity.

Forward Pass:
for $m \in 2, ..., M-1$ **do**
  $b_m \leftarrow b_m - \frac{a_m c_{m-1}}{b_{m-1}}$
  $V_m^{n+1} \leftarrow V_m^{n+1} - \frac{a_m}{b_{m-1}} V_{m-1}^{n+1}$
Backward Pass:
$V_{M-1}^n \leftarrow \frac{1}{b_{M-1}} V_{M-1}^{n+1}$
for $m \in M-1, ..., 1$ **do**
  $V_m^n \leftarrow \frac{1}{b_m}(V_m^{n+1} - c_m V_{m+1}^n)$

So far we've only discussed the numerical solution for a European option. The numerical algorithm for solving the PDE of an American option is almost identical to the algorithm for the Black-Scholes PDE as it involves numerically discretizing the space of the PDE.

The only difference for the discretization is the inclusion of a maximum and the inclusion of the payoff at the maturity date. Thus, the American option pricing algorithm only differs from the European option pricing algorithm in the last line in the backward pass where the fair value of the option at each iteration is compared to the payoff.

Algorithm 2 Application of the tridiagonal matrix algorithm (TDMA) to solve the system of an American option in linear time complexity.

Forward Pass:
for $m \in 2, ..., M-1$ **do**
  $b_m \leftarrow b_m - \frac{a_m c_{m-1}}{b_{m-1}}$
  $V_m^{n+1} \leftarrow V_m^{n+1} - \frac{a_m}{b_{m-1}} V_{m-1}^{n+1}$
Backward Pass:
$V_{M-1}^n \leftarrow \frac{1}{b_{M-1}} V_{M-1}^{n+1}$
for $m \in M-1, ..., 1$ **do**
  $V_m^n \leftarrow max\{\frac{1}{b_m}(V_m^{n+1} - c_m V_{m+1}^n), V_{m+1}^N - V_{m+1}^{n+1}\}$

In the next section we solve the PDEs using the methods discussed above and analyze the accuracy and perform a robust error analysis of different finite difference methods used.

5 Evaluation and Results

We consider options with strike K, maturity T. The stock price S_0 is not relevant for the algorithm. We will use it in the end to compute the value of the option for S_0.

Table 1: Hyperparameters for our method

Hyperparameter	**Value**
r	0.1
σ	0.2
K	100
M space steps	3000
N time steps	2000
S_{max}	300
S_{min}	33.3

We choose our hyperparameters according to Table 1. Note that a common practice is to choose the computational region between 3K and K/3 . Then we have $X_0 = \log K/3$ and $X_T = \log 3K$.

As a result, we can solve the Black-Scholes PDE numerically, and this gives a Black-Scholes curve that represent the fair value of a European option as a function of the price S. Figures 3, 4, and 5 show the evaluation of our method at t = 1999, 1000, and 0, respectively.

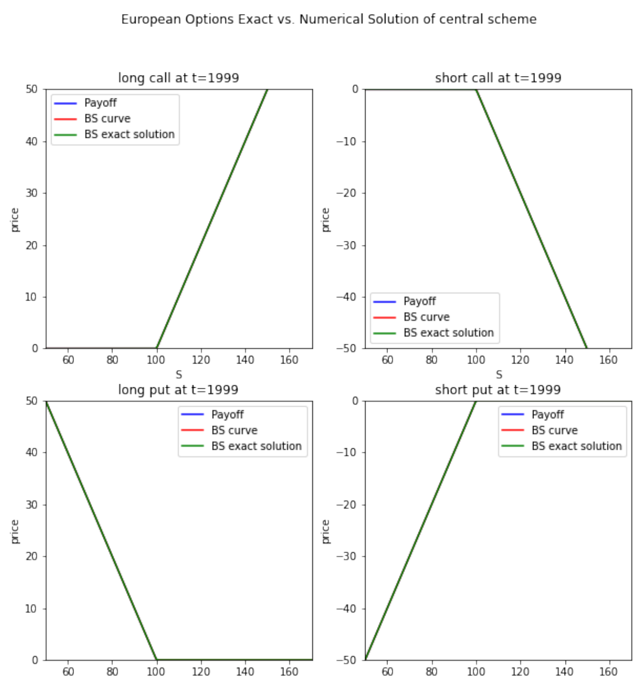

Figure 3: Central space discretization v.s. exact solution of a European option at t = 1999

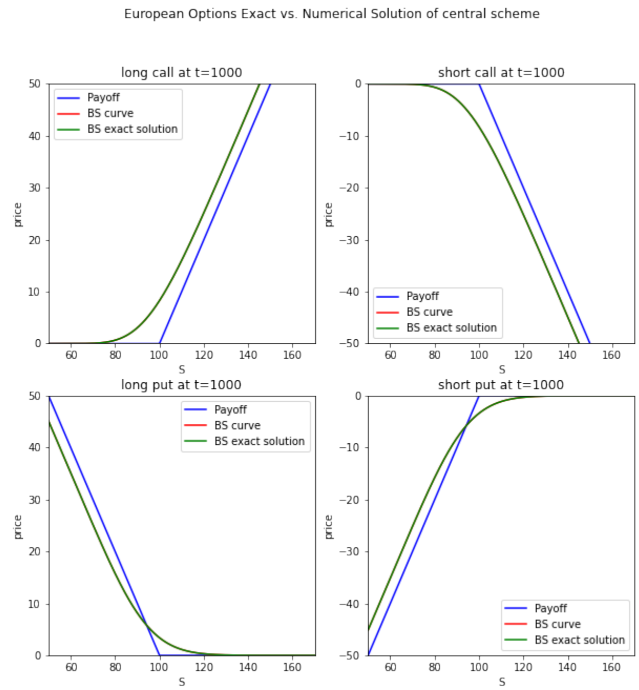

Figure 4: Central space discretization v.s. exact solution of a European option at t = 1000

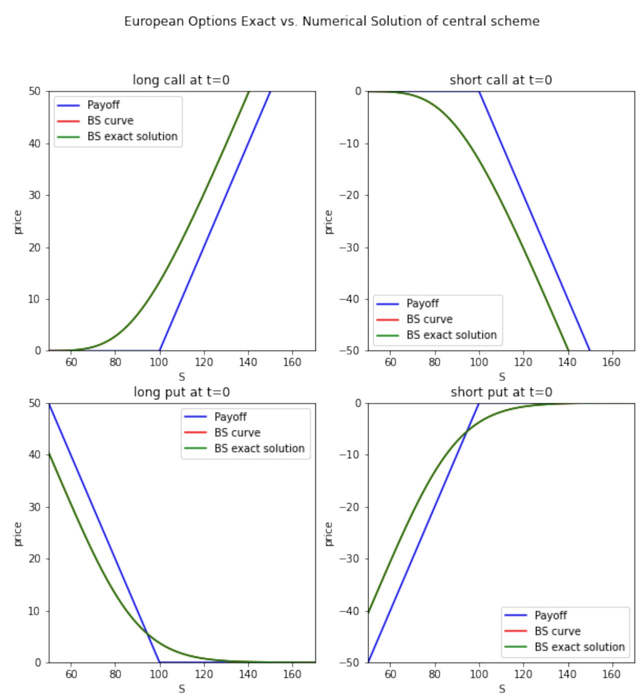

Figure 5: Central space discretization v.s. exact solution of a European option at t = 0

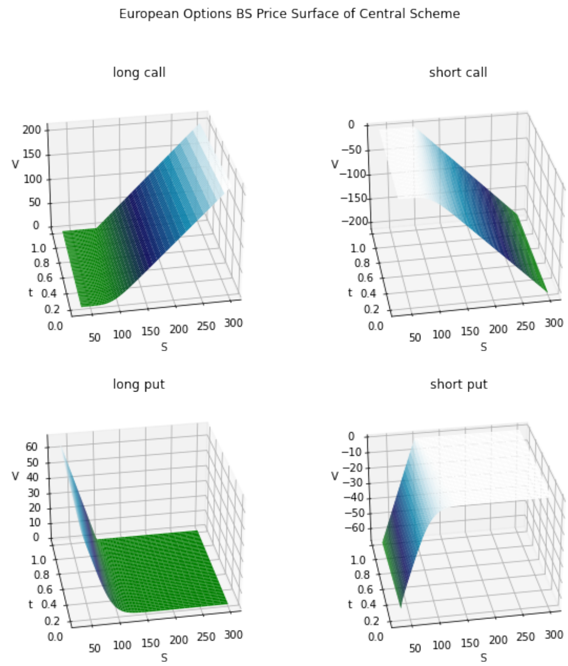

Figure 6: Black-Scholes 3D surface of Central Scheme for a European option

For better visualization, figure 6 shows the Black-Scholes curve of a European option in 3-dimension.

Because we solve the PDE backward, the BS curve at the biggest time step (t=1999) should be the most similar to the payoff. We can see from figure 3 that the solved BS curves are indeed the same as their corresponding payoffs, which makes sense. Moreover, in all the cases, our numerical solution looks exactly the same as the exact solution on the plots; this implies that our method is a good approximation of the exact solution. To further show our method's effectiveness, we include the mean square error (MSE) between exact solution and numerical method for all options in Table 2.

Table 2: MSEs of a European option with central scheme at different times

Time	**MSE (call)**	**MSE (put)**
t = 0	1.0113273086402072e-07	0.21148645593720758
t = 1000	3.951392258050094e-06	0.04226666991143163
t = 1999	5.9214910671433004e-05	5.4675179612994014e-05

The errors of both call and put options are on the scale of 10^{-5} at the end of the iterations, indicating that the accuracy of our numerical solution is high.

To further explore, we proceed to implement different schemes for space discretiazation, as discussed in section 4.

Figures 7 and 8 visualize the forward and backward space discretizations compared to the exact solution at $t = 0$. From the plots, we see that they both produce similar results as the central difference method. To have a better sense, we again calculate the MSEs between numerical methods using different discretization schemes and the exact solution, which is shown in table 3.

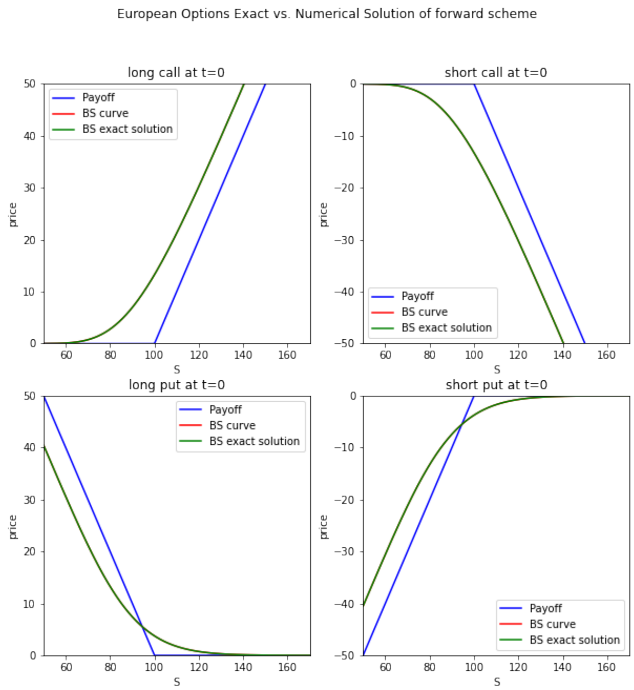

Figure 7: Forward space discretization v.s. exact solution of a European option at t = 0

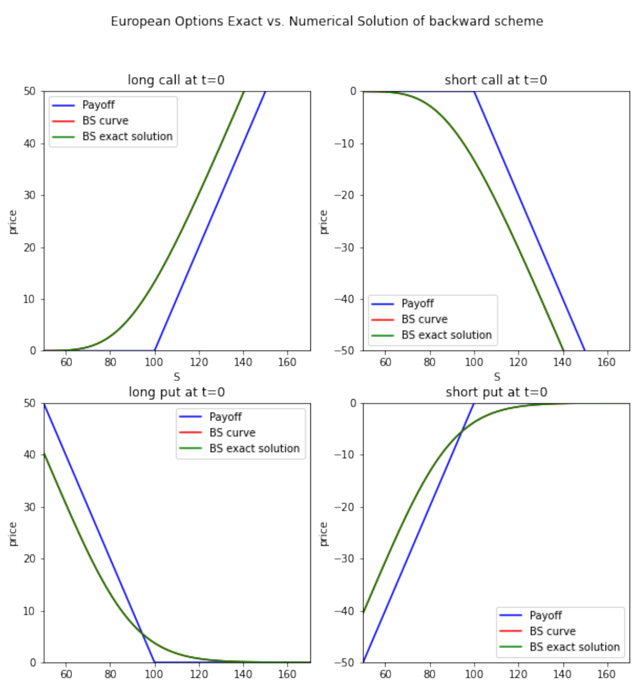

Figure 8: Backward space discretization v.s. exact solution of a European option at t = 0

Table 3: MSEs of a European option at t = 0 with different schemes

Scheme	MSE (call)	MSE (put)
central	1.0113273086402072e-07	0.21148645593720758
forward	1.748007132601301e-05	0.211535800361691
backward	1.991173785618039e-05	0.21144258121428852

Comparing the mean squared errors given by different space dscretization schemes, we can see that for a European long or short put option, different discretization schemes produce similar performances. However, for call options, the central difference scheme is approximately two orders of magnitude more accurate than both forward and backward schemes.

Hence we conclude that central difference method is the best discretization scheme for our options pricing numerical approximations.

Choosing central difference method as our optimal space discretization scheme and using the same hyperparameters in Table 1, we can also solve the slightly varied Black-Scholes PDE numerically to obtain the fair value of an early exercise American option as a function of the price S.

Figures 9, 10, and 11 show the evaluations of our method at t = 1999 1000, and 0, respectively. Similarly, we plotted the Black-Scholes curve of an American option in 3-dimension for better visualization.

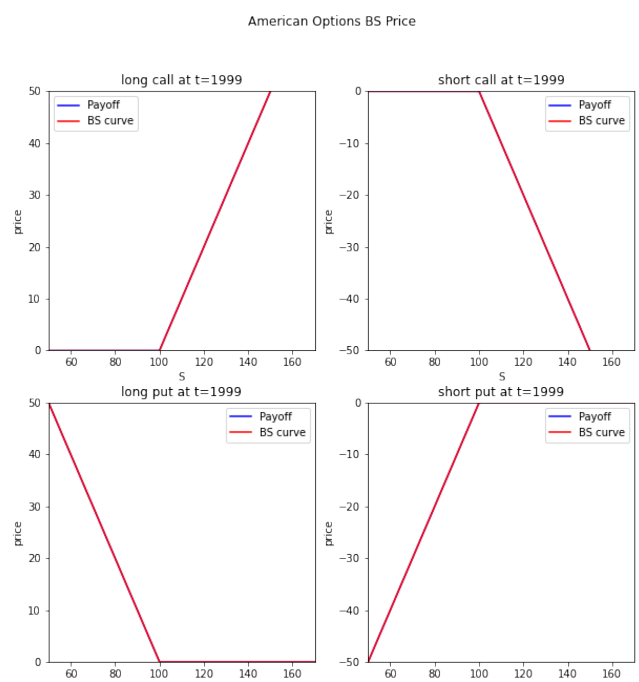

Figure 9: Black-Scholes curve v.s. Payoff for American options at t = 1999

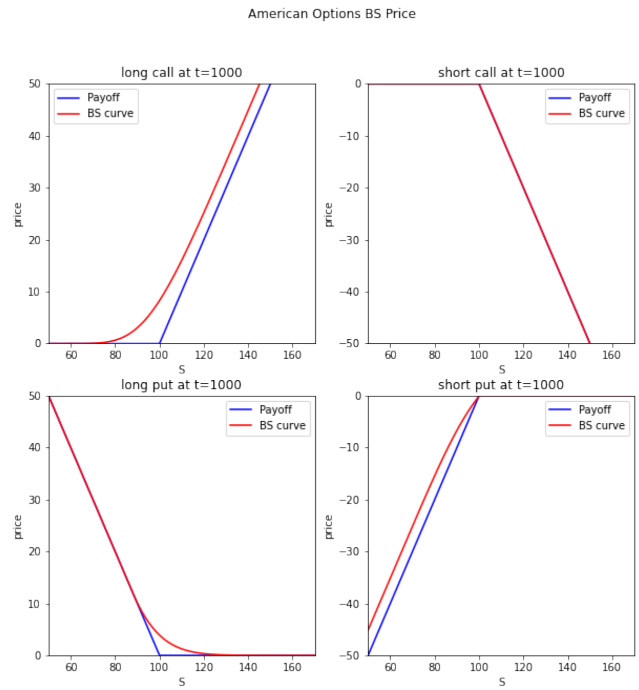

Figure 10: Black-Scholes curve v.s. Payoff for American options at $t = 1000$

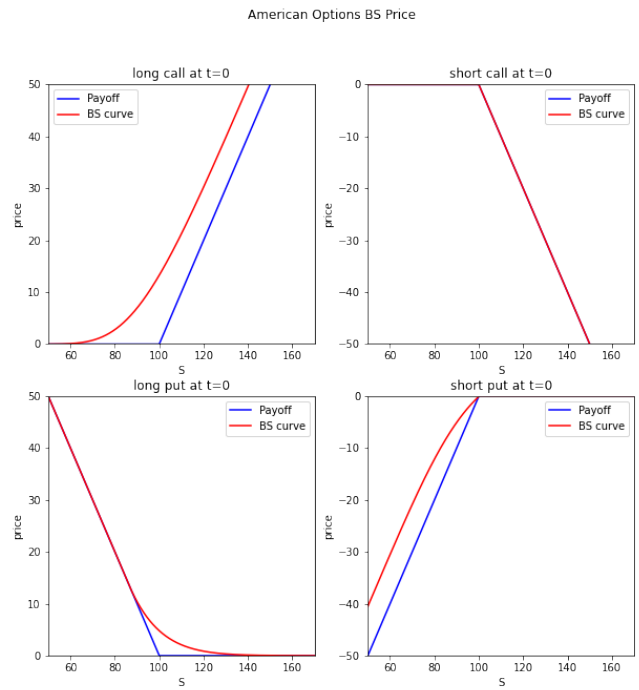

Figure 11: Black-Scholes curve v.s. Payoff for American options at $t = 0$

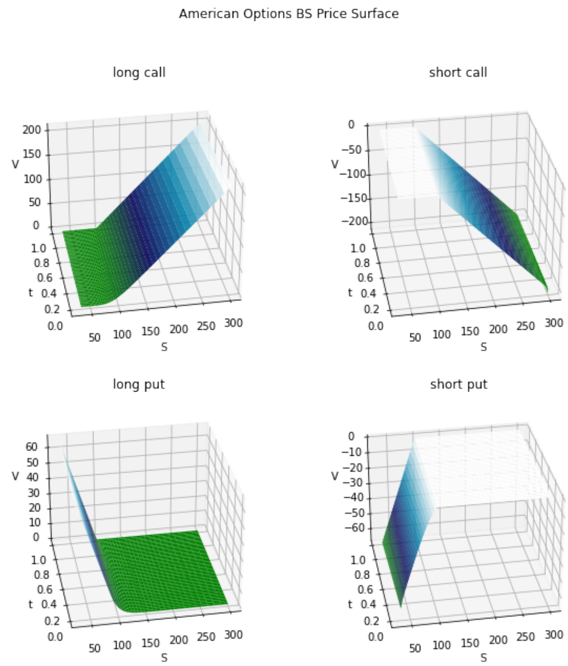

Figure 12: American options Black-Scholes 3D surface

As shown in the figures above, the Black-Scholes curve of an American option converges to the payoff as the time step progresses, which exactly corresponds to our backward iteration method.

Because we were only able to derive a closed form solution for the case of an optimally early-exercised American put option, we compared our numerical approximations to the exact solutions of an American put option at different times to determine the accuracy of our numerical solutions.

Figures 13, 14 and 15 show the results.

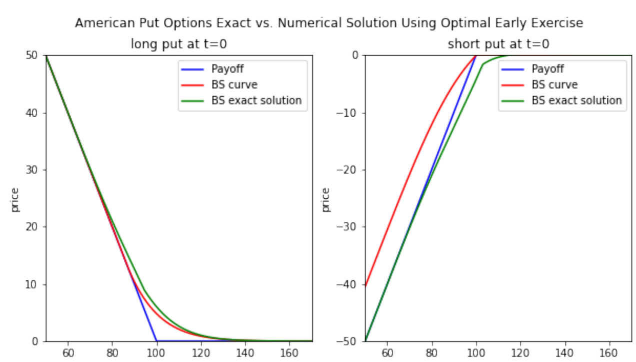

Figure 13: Exact vs. numerical solution of an early exercised American put option at t = 0

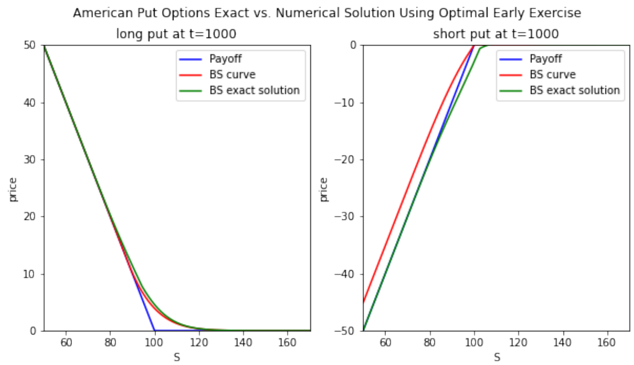

Figure 14: Exact vs. numerical solution of an early exercised American put option at t = 1000

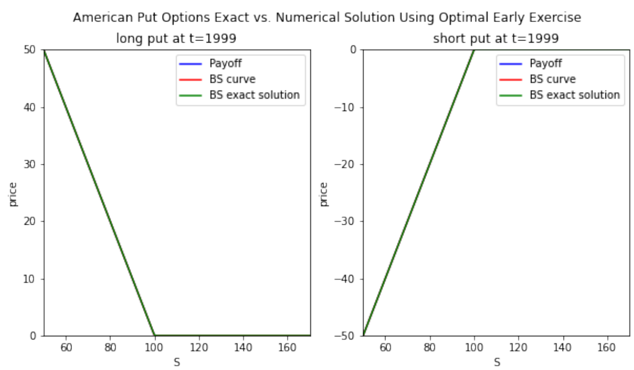

Figure 15: Exact vs. numerical solution of an early exercised American put option at t = 1999

We also computed the mean squared errors.

Table 4: MSEs of an optimally early-exercised American put option at different times

Time	MSE (long)	MSE (short)
t = 0	0.3361013819656262	38.3938235207546
t = 1000	0.07840086639662198	10.728448019176101
t = 1999	0.0043019246648554246	0.00430102876013778

The modified Black-Scholes curve of an American put option converges nicely to the analytical solution. And the eventual MSE scores of both long and short positions are on the scale of 10^{-3}, which proves that our numerical approximation of an American put option is highly accurate as well.

Hence, we conclude that the our numerical method is effective in pricing of both European and American options on non-dividend paying assets.

6 Conclusions

In conclusion, we found that central difference schemes yielded the lowest error when pricing options against the exact solutions. The goal of this paper was to provide a framework of numerical methods to use to solve the complicated problem of options pricing. More than providing theory for the pricing of options, we aimed to provide tools to analyze methods which will most accurately and precisely price options. One limitation of our method is that we only implemented TDMA as our solver for the finite difference regime. There exist more efficient solvers (Wilmott (1993)) to compute the numerical solution, such as the successive over-relaxation method.

This paper allows for extensions to try different finite difference discretization methods that are higher order accurate yet more computationally expensive, to implement different solvers that are more efficient, to price both European and American options on dividend-paying assets, or to explore numerical methods to solve options with no maturity date, known as perpetual options.

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