Solutions: Bose-Einstein Condensate (BEC)

(a) Microstates

There are $N + 1$ microstates. The system can have k particles at $E = 0$ and $N - k$ particles at $E = \epsilon$, where $k = 0, 1, \ldots, N$.

(b) Classical Partition Function under the Canonical Ensemble

The classical partition function is given by:

$$Z_C = \sum_{k=0}^{N} \binom{N}{k} e^{-\beta k \epsilon} = (1 + e^{-\beta\epsilon})^N.$$

The probability of finding a specific energy E is:

$$P(E) = \frac{\binom{N}{E/\epsilon} e^{-\beta E}}{Z_C}, \quad \text{(assuming } E \text{ is a multiple of } \epsilon\text{)}.$$

(c) Classical Average Particle Number under the Canonical Ensemble

The expectation values for the number of particles in each state are:

$$\langle n_0 \rangle_C = \sum_{k=0}^{N} (N - k) \frac{\binom{N}{k} e^{-\beta k \epsilon}}{Z_C} = \frac{N}{1 + e^{-\beta\epsilon}}.$$

$$\langle n_\epsilon \rangle_C = \sum_{k=0}^{N} k \frac{\binom{N}{k} e^{-\beta k \epsilon}}{Z_C} = \frac{N}{1 + e^{\beta\epsilon}}.$$

(d) Quantum Partition Function under the Canonical Ensemble

The quantum partition function is given by:

$$Z = \sum_{k=0}^{N} e^{-\beta k \epsilon} = \frac{1 - e^{-(N+1)\beta\epsilon}}{1 - e^{-\beta\epsilon}}.$$

The probability of finding a specific energy E is:

$$P(E) = \frac{e^{-\beta E}}{Z}.$$

(e) Quantum Average Particle Number under the Canonical Ensemble

The expected number of particles in the ground and excited states are:

$$\langle n_0 \rangle = N - \frac{1}{e^{\beta\epsilon} - 1} + \frac{N+1}{e^{(N+1)\beta\epsilon} - 1}.$$

$$\langle n_\epsilon \rangle = \frac{1}{e^{\beta\epsilon} - 1} - \frac{N+1}{e^{(N+1)\beta\epsilon} - 1}.$$

(f) Grand Canonical Partition Function

The grand canonical partition function is:

$$\Omega_G = \sum_{N=0}^{\infty} e^{\beta\mu N} Z = \frac{1}{(e^{\beta(\mu-\epsilon)} - 1)(e^{\beta\mu} - 1)}.$$

For convergence, we require $\mu < \epsilon$.

(g) Average Particle Number in the Grand Canonical Ensemble

The total average particle number is given by:

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial\mu} \ln \Omega_G.$$

Evaluating the derivative,

$$\langle N \rangle = \frac{e^{\beta(\epsilon-\mu)}(1 + e^{\beta\epsilon} - 2e^{\beta\mu})}{(e^{\beta(\mu-\epsilon)} - 1)^2 (e^{\beta\mu} - 1)^2}.$$

Alternatively, for large N:

$$\langle N \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} + \frac{1}{e^{\beta\mu} - 1}.$$

(h) Ground State Occupation in the Grand Canonical Ensemble

The expected number of particles in the ground state follows:

$$\langle n_0 \rangle = \frac{1}{\Omega_G} \sum_{N=0}^{\infty} \sum_{k=0}^{N} (N-k) e^{\beta \mu N} e^{-\beta k \epsilon}.$$

Simplifying,

$$\langle n_0 \rangle = \frac{1}{e^{-\beta \mu} - 1}.$$

(i) Numerical Solution Approach

To determine μ for a given N and β, we solve:

$$N = \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}.$$

To find $\frac{d\langle n_0 \rangle}{d\beta}$, we differentiate both sides with respect to β:

$$\frac{d\mu}{d\beta} = \text{computed numerically.}$$