Task 1: Poisson Distribution

In astrophysics, let us consider an example where stars are randomly dispersed in space with a uniform density n. The probability of the nearest star being at a distance R from us can be determined using Poisson statistics.

The probability that exactly k stars are found in a volume V follows the Poisson distribution:

$$P(k, V) = \frac{(nV)^k e^{-nV}}{k!}. \tag{1}$$

To find the probability that the closest star is within a radius R, we compute the complement of the probability that there are no stars inside a sphere of radius R:

$$1 - P(0, V) = 1 - e^{-4\pi n R^3/3}. \tag{2}$$

This gives the cumulative probability of finding at least one star within distance R. A random variable taking a particular value in a continuous distribution is zero.

Task 2: Lorentzian Response

The behavior of a system under resonance is essential in understanding energy dissipation and absorption. We analyze a driven, damped harmonic oscillator in both the time and frequency domains.

Equation of Motion

The equation governing a driven damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = Fe^{i\omega_f t}. \tag{3}$$

Taking the Fourier transform, we obtain the frequency-domain equation:

$$(-\omega^2 + i\gamma\omega + \omega_0^2)\tilde{x} = F\delta(\omega - \omega_f). \tag{4}$$

Solving for $\tilde{x}(\omega)$,

$$\tilde{x}(\omega) = \frac{F\delta(\omega - \omega_f)}{-\omega^2 + i\gamma\omega + \omega_0^2}. \tag{5}$$

Energy Absorption and Lorentzian Form

To determine the energy absorbed per cycle, we compute:

$$x(t) = \int d\omega\, \tilde{x}(\omega)e^{i\omega t} = \frac{Fe^{i\omega_f t}}{-\omega_f^2 + i\gamma\omega_f + \omega_0^2}. \tag{6}$$

The energy absorbed per cycle is then given by:

$$E = F\pi \frac{\gamma\omega_f}{(\omega_0^2 - \omega_f^2)^2 + \gamma^2\omega_f^2}. \tag{7}$$

This takes the well-known Lorentzian form, showing how energy is absorbed by the system at resonance.

Hint

Using the small-angle approximation, we can also note:

$$\sin(\Sigma) = -A\gamma\omega_f. \tag{8}$$