Bayesian logic for formulation

$p(z_1|z_2) = N(\mu_1, \Sigma_1)$

$p(z_2|z_1) = N(\mu_2, \Sigma_2^{-1})$

$p(z_1|z_2; z_2) = N(\mu_1, \Lambda)$

$p(z_1|z_2; z_2) = N(\mu, \Lambda)$

$\hat{\mu} = \hat{\mu}_{z_1|z_2} = (\Sigma_2^{-1} + \Sigma_1^{-1})(\Sigma_1^{-1} \mu_1 + \Sigma_2^{-1} \mu_2)$

$\Lambda = \Sigma_2^{-1}(\Sigma_2^{-1} \Lambda + \Sigma_1^{-1})^{-1}$

...

$\hat{x}_{t,t-1} = \Phi_{t,t-1} \hat{x}_{t-1,t-1}$

Posterior: $p(x_t | H \cdots z_t) = N(\mu, \delta \Lambda)$

$\hat{x}_{t,t} = \Lambda = (H_t^\top \Sigma_{y_t}^{-1} H_t + \Sigma_{x_{t,t-1}}^{-1})^{-1}(H_t^\top \Sigma_{y_t}^{-1} y_t + \Sigma_{x_{t,t-1}}^{-1} \hat{x}_{t,t-1})$

$\Sigma_{x_{t,t}} = \Lambda = (H_t^\top \Sigma_{y_t}^{-1} H_t + \Sigma_{x_{t,t-1}}^{-1})^{-1}$

$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$

$D^{-1}C(A - BD^{-1}C)^{-1} = (D - CA^{-1}B)^{-1}CA^{-1}$

① $(H^\top \Sigma_y^{-1} H + \Sigma_x^{-1})^{-1} = \Sigma_x - \Sigma_x H^\top (H \Sigma_x H^\top + \Sigma_y) H \Sigma_x$

② $(H^\top \Sigma_y^{-1} H + \Sigma_x^{-1})^{-1} H^\top \Sigma_y^{-1} = \Sigma_x H^\top (H \Sigma_x H^\top + \Sigma_y)^{-1}$

$\hat{x}_{t,t} = (H_t^\top \Sigma_{y_t}^{-1} H_t + \Sigma_{x_{t,t-1}}^{-1})^{-1}(H_t^\top \Sigma_{y_t}^{-1} y_t + \Sigma_{x_{t,t-1}}^{-1} \hat{x}_{t,t-1})$

$= (H_t^\top \Sigma_y^{-1} H + \Sigma_{x_{t,t-1}}^{-1})^{-1} H_t^\top \Sigma_y^{-1} y_t + (H_t^\top \Sigma_y^{-1} H + \Sigma_{x_{t,t-1}}^{-1})^{-1} \Sigma_{x_{t,t-1}}^{-1} \hat{x}_{t,t-1}$

$= \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1} y_t + (\Sigma_{x_{t,t-1}} - \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_y)^{-1} H_t \Sigma_{x_{t,t-1}}) \Sigma_{x_{t,t-1}}^{-1} \hat{x}_{t,t-1}$

$\quad + \hat{x}_{t,t-1} - \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_y)^{-1} H_t \Sigma_{x_{t,t-1}} \Sigma_{x_{t,t-1}}^{-1} \hat{x}_{t,t-1}$

$= \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1} y_t + \hat{x}_{t,t-1} - \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1} H_t \hat{x}_{t,t-1}$

$\hat{x}_{t,t} = \hat{x}_{t,t-1} + \Sigma_{x_{t,t-1}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1}(y_t - H_t \hat{x}_{t,t-1})$

$\hat{x}_{t,t} = \hat{x}_{t,t-1} + F_t(y_t - H_t \hat{x}_{t,t-1})$

gain $\to F_t = \Sigma_{x_{t,t-1}} H^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1}$

$\Sigma_{x_{t,t}} = (I - F_t H_t) \Sigma_{t,t-1}$

time update

$$p(x_{t-1}|y_{t-1}) \to p(x_t|y_{t-1}) \to$$
$$p(x_t|y_t) \leftarrow p([x_t^\top y_t^\top]^\top | y_{t-1}) \leftarrow p(y_t)$$

measurement update

Discrete Kalman Filter

$\begin{cases} x_t = \Phi x_{t-1} + e_t \\ y_t = H_t x_t + v_t \end{cases}$

Prediction step:

$\boxed{\hat{x}_{t,t-1}} = \Phi_{t,t-1} \hat{x}_{t-1,t-1}$

$\boxed{\delta^2 \Sigma_{x_{t,t}}} = \delta^2 \Phi_{t,t-1} \Sigma_{x_{t-1,t-1}} \Phi_{t,t-1}^\top + Q_{t-1}$

Estimation step:

$\hat{x}_{t,t} = \hat{x}_{t,t-1} + F_t(y_t - H_t \hat{x}_{t,t-1})$

$F_t = \Sigma_{x_{t,t}} H_t^\top (H_t \Sigma_{x_{t,t-1}} H_t^\top + \Sigma_{y_t})^{-1}$

$\Sigma_{x_{t,t}} = (I - F_t H_t) \Sigma_{x_{t,t-1}}$

$D\{x_t | y_1 \cdots y_t\} = \delta^2 \Sigma_{x_{t,t}}$

$\hat{\delta}_t^2 = (n + 2(\hat{\delta}_{t-1}^2)^2 / V_{\delta_{t-1}^2} + 2)^{-1} \{ 2[(\hat{\delta}_{t-1}^2)^2 / V_{\delta_{t-1}^2} + 1] \hat{\delta}_{t-1}^2 + (\hat{x}_{t,t} - \hat{x}_{t,t-1})^\top \Sigma_{x_{t,t}}^{-1}(\hat{x}_{t,t} - \hat{x}_{t,t-1}) + (y_t - H_t \hat{x}_{t,t})^\top \Sigma_{y_t}^{-1}(y_t - H_t \hat{x}_{t,t}) \}$

$V_{\delta_t^2} = \dfrac{2(\hat{\delta}_t^2)^2}{n + 2(\hat{\delta}_{t-1}^2)^2 / V_{\delta_{t-1}^2}}$