$Ax + Bv - y = 0$

$x = [z_1 \dots z_p \; z_1 \dots z_n]^T$

m: number of observations
p: number of unknown points
n: number of columns in Ax

$y = \underset{m \times 1}{A} \underset{p \times 1}{x} + \underset{m \times 1}{z}$

$z = S' + e$

Stochastic signal + noise

$\hat{x} = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} y = G y$

$\hat{z} = Q_{zy} Q_{yy}^{-1} (y - A\hat{x}) = L y$

$\boxed{Q_{\hat{x}\hat{x}}} \quad \boxed{Q_{\hat{z}\hat{z}}}$

$Q_{yy} = Q_{zz} + D$

$\hat{x} = x$

$e_{\hat{x}} = x - \hat{x}$

$e_{\hat{x}} = x - G y = x - G(Ax + z)$
$= x - GAx - Gz$
$= x - x - Gz$
$e_{\hat{x}} = -Gz$

$Q_{\hat{x}\hat{x}} = E\{e_{\hat{x}} e_{\hat{x}}^T\} = E\{G z z^T G^T\} = G \underbrace{E\{z z^T\}}_{Q_{yy}} G^T = G Q_{yy} G^T$

$\boxed{Q_{\hat{x}\hat{x}} = (A^T Q_{yy}^{-1} A)^{-1}}$

$\hat{S} = Q_{sy} Q_{yy}^{-1}(y - A\hat{x})$

$\hat{S} = L y = H(I - AG)y$

$\boxed{L = H(I - AG)}$

$\boxed{L = Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}]}$

$L A = H(I - AG)A = H(A - AGA) = H(A - A) = 0$

$A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} A = A$

$\boxed{e_{\hat{S}}} = S - \hat{S}$
$= S - L(Ax + z)$
$= S - LAx - Lz = S - Lz = S - H(I - AG)z$

$Q_{\hat{S}} = E\{e_{\hat{S}} e_{\hat{S}}^T\} = E\{(S - Lz)(S^T - z^T L^T)\}$
$= E\{S S^T - L z S^T - S z^T L^T + L z z^T L^T\}$

$Q_{\hat{S}} = Q_{ss} - L Q_{zs} - Q_{sy} L^T + L Q_{yy} L^T$

$L Q_{ys} = Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}] Q_{ys}$

$L Q_{ys} = Q_{sy} L^T$

$L Q_{yy} L^T = Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}]Q_{yy}Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}]Q_{ys}$

$= Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}]^2 Q_{ys}$

$= Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}]Q_{ys}$

$L Q_{yy} L^T = Q_{sy} L^T$

$Q_{\hat{S}} = Q_{ss} - L Q_{ys} - Q_{sy} L^T + L Q_{yy} L^T$
$= Q_{ss} - L Q_{ys} - Q_{sy}L^T + Q_{sy}L^T$

$\boxed{Q_{\hat{S}} = Q_{ss} - L Q_{ys}}$

$\boxed{Q_{\hat{S}} = Q_{ss} - Q_{sy} Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}] Q_{ys}}$

$\boxed{= Q_{ss} - Q_{sy} Q_{yy}^{-1} Q_{ys} + H A Q_{\hat{x}} A^T H^T}$

$h^T = (I - AG)^T H^T$
$= H^T - G^T A^T H^T$
$= Q_{yy}^{-1} Q_{ys} - Q_{yy}^{-1} A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} Q_{ys}$
$= Q_{yy}^{-1}[Q_{ys} - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} Q_{ys}]$
$= Q_{yy}^{-1}[I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}] Q_{ys}$

$\boxed{Q_{\hat{x}}} \quad \boxed{Q_{\hat{S}}} \quad \boxed{Q_{SS}}$

Application to interpolation

$F(u) = f(u) + S(u)$

$y_i = F(u) + e_i$

$s_i' = S(u_i)$

$y_i = F(u_i) + S_i + e_i$

$f(u) = b_1 + b_2 u + b_3 u^2 + \dots + b_n u^{n-1}$

n: unknowns

$x = [b_1 \; b_2 \dots b_n]^T$

$A = \begin{bmatrix} 1 & u_1 & u_1^2 & \dots \\ 1 & u_2 & u_2^2 & \dots \\ \vdots & & & \\ 1 & u_m & u_m^2 & \dots \end{bmatrix}$ $m \times n$

$\hat{x} = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} y$

$\boxed{Q_{yy}} = \boxed{Q_{SS}} + \boxed{Q_{ee}}$

$\begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} + \begin{bmatrix} \sigma_1^2 \\ & \sigma_2^2 \\ & & \ddots \end{bmatrix}$

Q_{ii}

$F(u_i)$
$S(u_i)$

$C_s = \sigma^2 I$
$C_s = \sigma^2 S_s$

$Q_{sy} \quad P \times M$
U, V

$Q_{s's} = \begin{bmatrix} u_1 & \dots & u_m \\ v_1 & & \\ \vdots & & \\ v_p & & \end{bmatrix}$ $P \times M$

$Q_{s's}$

$E\{SS'\} = C(r) = \frac{1}{2(a-r)}\int_{a-r}^{a} S(u)S(u+r)\,du$

$\boxed{C(r) = C_0 e^{-\frac{r^2}{d^2}}}$

Q_{ss}
Q_{sy}

$Cov(S_i', S_j') = C(r_{ij}) \quad r_{ij} = |u_i - y_j|$

$\boxed{Q_{yy}}$ $Cov(S_i, S_j) = C(r_{ij}')$
$r_{ij}' = |u_i - v_j|$

$\boxed{Q_{yy}} \quad \boxed{Q_{sy}} \quad \boxed{Q_{ee}}$

Q_{sy}
Q_{yy}

$\hat{S} = Q_{sy} Q_{yy}^{-1}(y - A\hat{x})$

$Q_{\hat{S}} = Q_{ss} - Q_{sy} Q_{yy}^{-1} Q_{ys} + H A Q_{\hat{x}} A^T H^T$