University of Stuttgart
Institute of Geodesy

Physical Geodesy

Gauss's divergence theorem: quantification of a disturbing mass

GIS

Potential theory

Gauss's divergence theorem

- Divergence of a vector field (from vector calculus)

$$\operatorname{div} \boldsymbol{a} = \nabla \cdot \boldsymbol{a} = \lim_{V \to 0} \frac{\iint_S \ \boldsymbol{a} \cdot d\boldsymbol{S}}{V}$$

- Vector flow through a surface

$$\iint_S \boldsymbol{a} \cdot d\boldsymbol{S} = -4\pi G \iiint_V \rho \mathrm{d}V$$

$$\Downarrow$$

$$\operatorname{div} \boldsymbol{a} = \nabla \cdot \boldsymbol{a} = \lim_{V \to 0} \frac{\iint_S \ \boldsymbol{a} \cdot d\boldsymbol{S}}{V} = \lim_{V \to 0} \frac{-4\pi G \iiint_V \ \rho \mathrm{d}V}{V} = \begin{cases} -4\pi G \rho & \text{Poisson} \\ 0 & \text{Laplace} \end{cases}$$

- Gauss's divergence identity

$$\iiint_V \operatorname{div} \boldsymbol{a} \mathrm{d}V = \iint_S \boldsymbol{a} \cdot d\boldsymbol{S}$$

Potential theory

Gauss's divergence theorem for Earth's gravitational potential field

- Gradient field $\boldsymbol{a}$

$$\boldsymbol{a} = \nabla\Phi$$

- Poisson equation

$$\operatorname{div}\boldsymbol{a} = \nabla\cdot\boldsymbol{a} = \nabla\cdot\nabla\Phi = \operatorname{div}\operatorname{grad}\Phi = \Delta\Phi = -4\pi G\rho$$

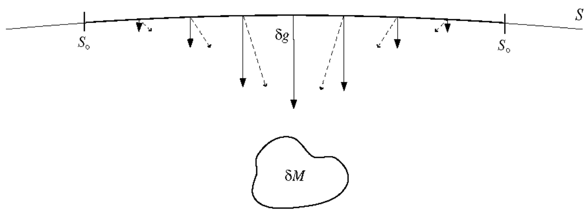
- Gauss's divergence identity

$$\iiint\limits_V \nabla\cdot\boldsymbol{a}\mathrm{d}V = \iint\limits_S \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} \Leftrightarrow \iiint\limits_V \Delta\Phi\mathrm{d}V = \iint\limits_S \nabla\Phi\cdot\boldsymbol{n}dS$$

- Total mass

$$-4\pi G\iiint\limits_V \rho\mathrm{d}V = \iint\limits_S \frac{\partial\Phi}{\partial n}\mathrm{d}S \Longrightarrow -4\pi GM = \iint\limits_S -g\mathrm{d}S \Longrightarrow M = \frac{1}{4\pi G}\iint\limits_S gdS$$

- Disturbance mass



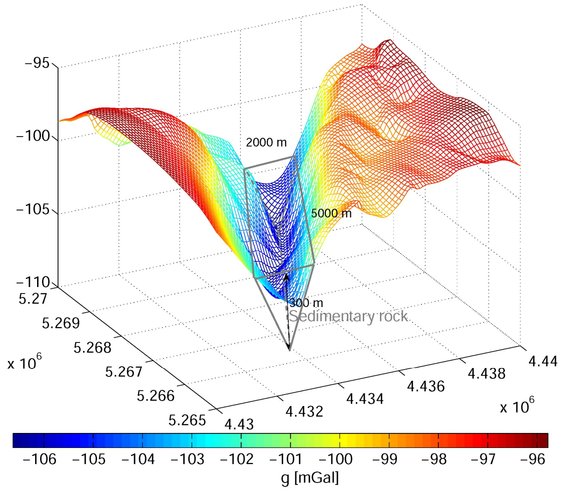
$$\delta M = \frac{1}{4\pi G}\iint\limits_{S_0} \delta g dS$$

Potential theory

Gauss's divergence theorem: an application

- Task 1: Derive equation to quantify a disturbing mass from surface gravity measurements

$$\delta M = \frac{1}{4\pi G} \iint_{S_0} \delta g dS$$



- Task 2: Compute disturbing mass of a sedimentary rock using Gauss's divergence theorem

- Task 3: Compute the disturbing mass using its geometry & density contrast

- Task 4: Compare outcomes of Task 2 & Task 3

Potential theory

Gauss's divergence theorem for Earth's gravitational potential field

- Task 2: disturbance mass from Gauss's divergence theorem

$$\delta M = \frac{1}{4\pi G}\iint_{S_0} \delta g dS = \frac{1}{4\pi G}\sum_{i=1}^{i_{max}}\sum_{j=1}^{j_{max}} \delta g_{ij}\Delta x \Delta y = \frac{\Delta x \Delta y}{4\pi G}\sum_{i=1}^{i_{max}}\sum_{j=1}^{j_{max}} \delta g_{ij}$$

$$\delta g = g - g_0$$

$$\delta g_{ij} = g_{ij} - g_0$$

- Task 3: Disturbance mass from its geometry (parallelogram) & density

$$\delta M = V\delta\rho = \frac{\text{depth} \times \text{surface area}}{2}\delta\rho$$

- ✓ δg: disturbing gravity
- ✓ g_0: background (reference) gravity
- ✓ $\{\Delta x, \Delta y\}$: grid spacing
- ✓ $\{i_{max}, i_{max}\}$: number of data in x & y directions
- ✓ $\delta\rho$: density contrast
- ✓ V: volume of disturbance mass