Dimensions of Specht modules

Ziqing Xiang

University of Georgia

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Specht module

For each partition $\lambda \vdash n$, we have a Specht module S^λ.

Specht module

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For a fixed n, they form a complete set of nonisomorphic simple $k\Sigma_n$-modules when characteristic is 0.

Standard Young tableau

A standard Young tableau of shape $(4, 2, 1)$:

1	4	5	7
2	6		
3			

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$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

$\dim S^\lambda =$ number of standard Young tableaux of shape λ.

Standard Young tableaux of shape $(3, 2)$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Standard Young tableaux of shape $(3,2)$

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$\dim S^{(3,2)} = 5.$

Standard Young tableaux of shape $(3, 2, 1)$

$$\begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array}\quad\begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array}\quad\begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array}\quad\begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array}$$

$$\begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array}\quad\begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array}\quad\begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array}\quad\begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array}$$

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$$\begin{array}{ccc}1&3&6\\2&4&\\5&&\end{array}\quad\begin{array}{ccc}1&3&6\\2&5&\\4&&\end{array}\quad\begin{array}{ccc}1&4&5\\2&6&\\3&&\end{array}\quad\begin{array}{ccc}1&4&6\\2&5&\\3&&\end{array}$$

Standard Young tableaux of shape $(3, 2, 1)$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

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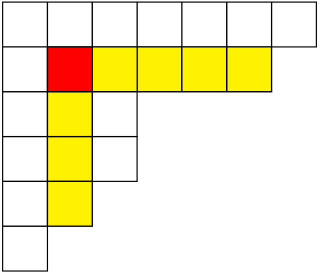
$\dim S^{(3,2,1)} = 16.$

Hook-length formula

A hook of length 8: 

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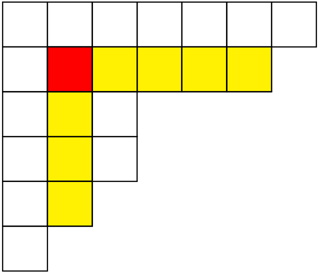


Theorem 1 (Frame-Robinson-Thrall)

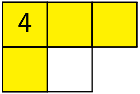
Let λ be a partition. Then,

$$\dim S^{\lambda} = \frac{\prod_{i=1}^{|\lambda|} i}{\prod_{i \in \lambda} h_i},$$

where h_i is the hook length of the hook i.

Hook-length formula

Example: $(3, 2)$ and $(3, 2, 1)$



Hook-length formula

Example: $(3,2)$ and $(3,2,1)$

4	3	

Hook-length formula

Example: $(3,2)$ and $(3,2,1)$

4	3	1

Hook-length formula

Example: $(3, 2)$ and $(3, 2, 1)$

4	3	1
2		

Hook-length formula

Example: $(3, 2)$ and $(3, 2, 1)$

4	3	1
2	1	

Hook-length formula

Example: $(3,2)$ and $(3,2,1)$

4	3	1
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$$\dim S^{(3,2)} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 1\cdot 2\cdot 1} = 5.$$

Hook-length formula

Example: $(3, 2)$ and $(3, 2, 1)$

4	3	1
2	1	

$$\dim S^{(3,2)} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 1\cdot 2\cdot 1} = 5.$$

5	3	1
3	1	
1		

$$\dim S^{(3,2,1)} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 3\cdot 1\cdot 3\cdot 1\cdot 1} = 16.$$

A question

What is the integer factorization of

$$\dim S^\lambda = \frac{\prod_{i=1}^{|\lambda|} i}{\prod_{i\in\lambda} h_i}?$$

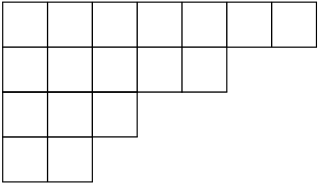
Core of a partition

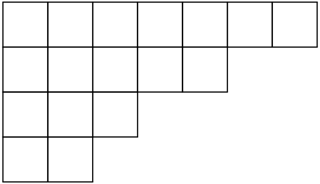
Definition 2

Let l be a natural number. The l-core of a partition λ is obtained by repeatedly removing l-hooks from λ.

Core of a partition

Example: $(7,5,3,2) \vdash 17$ and $l = 4$





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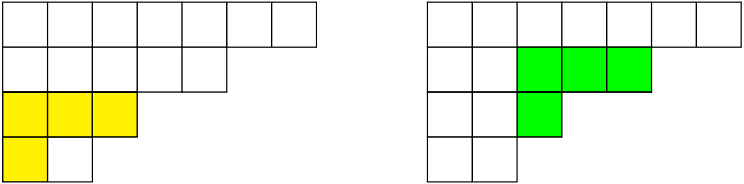

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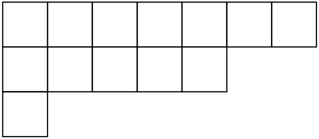

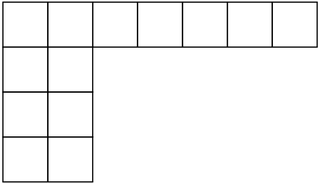

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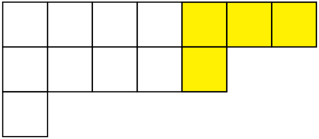

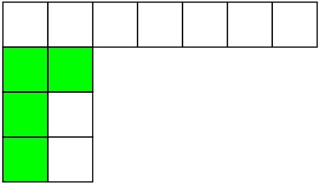

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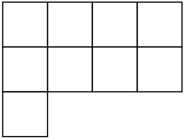

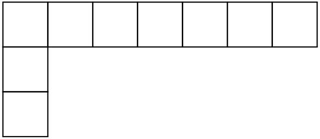

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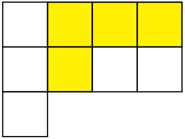

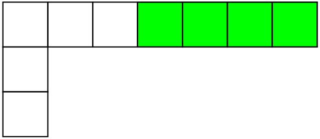

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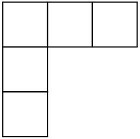

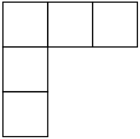

$$\mathrm{core}_4(7, 5, 3, 2) = (3, 1, 1)$$

q-integer

$$[n]_q := \frac{1 - q^n}{1 - q}.$$

q-integer

$$[n]_q := \frac{1-q^n}{1-q}.$$

$$[n]_1 = n.$$

Graded dimension

Definition 3

For a partition λ, let

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where h_i is the hook length of the hook i.

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$$\dim_1 S^\lambda = \dim S^\lambda.$$

Factorization of graded dimension

Theorem 4 (Nakano-X.)

Let λ be a partition. Then,

$$\dim_q S^\lambda = \prod_l \Phi_l(q)^{\mathrm{wt}_l |\mathrm{core}_l \lambda|},$$

where Φ_l is the l-th cyclotomic polynomial and $\mathrm{wt}_l n := \lfloor n/l \rfloor$.

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Corollary 5

$$\dim S^\lambda = \prod_{p,r} p^{\mathrm{wt}_{p^r} |\mathrm{core}_{p^r} \lambda|}.$$

An application in representation theory

Proposition 6

Let k be an algebraically closed field of characteristic p and $G := \Sigma_p^m$. If a kG module M is projective, then

$$p^m \mid \dim M.$$

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Theorem 7 (Nakano-X.)

Let k be an algebraically closed field of characteristic p, $G := \Sigma_p^m$ and λ be a partition. If S^λ is projective as kG-module, then

$$m \leq \sum_r \mathrm{wt}_{p^r} |\mathrm{core}_{p^r} \lambda|.$$

An application in representation theory

Proposition 6

Let k be an algebraically closed field of characteristic p and $G := \Sigma_p^m$. If a kG module M is projective, then

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Theorem 7 (Nakano-X.)

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Thank you for your attention.