Nonexistence of nontrivial tight 8-designs

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Abstract

Tight t-designs are t-designs which achieve the Fisher type lower bound. Nontrivial tight t-designs with $t \geq 3$ are rare. There do not exist such designs for odd t. In the case of even t with $t \geq 4$, there are only two known examples, Witt 4-$(23, 7, 1)$ and Witt 4-$(23, 16, 52)$. This paper shows that there do not exist nontrivial tight 8-designs.

Keywords: Intersection number, product of consecutive integers.

1 Introduction

Let t, v, k and λ be positive integers. A t-(v, k, λ) *design*, or simply a t-*design*, is a set $\mathcal{B} \subseteq \binom{[v]}{k}$ satisfying

$$\#\{b \in \mathcal{B} : T \subseteq b\} = \lambda$$

for all $T \in \binom{[v]}{t}$, where $[v] = \{1, \ldots, v\}$. A design is *trivial* if it consists of all k-element subsets.

D. Ray-Chaudhuri and R. Wilson [10] showed the *Fisher type lower bound* on the size a design. For a $2e$-design $\mathcal{B}$, if $v \geq k + e$, then $|\mathcal{B}| \geq \binom{v}{e}$. And for a $(2e-1)$-design $\mathcal{B}$, if $v \geq k + e$, then $|\mathcal{B}| \geq 2\binom{v-1}{e-1}$. A t-design is *tight* provided that the size of the design achieves the lower bound. Note that, for a nontrivial tight t-design, we always have $v > k + t > 2t$.

The *complementary design* of a t-(v, k, λ) design $\mathcal{B}$ with $v \geq k + t$ consists of blocks $\{[v] \setminus b : b \in \mathcal{B}\}$, and it is a t-$(v, v-k, \lambda')$ design for some positive integer λ'. In this article, we prove the following theorem.

Theorem 1.1. *There do not exist nontrivial tight* 8*-designs.*

The proof of this theorem can be found in Proof 6.2. Table 1.2 gives some works on the classification of nontrivial tight designs till now. It is conjectured that there do not exist nontrivial tight $2e$-designs for every $e \geq 3$.

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t	Result	Reference
$2e-1$ ($e \geq 2$)	Nonexistence	D. Ray-Chaudhuri, R. Wilson [10]
4	Finitely many	H. Enomoto, N. Ito, R. Noda [6]
4	4-$(23, 7, 1)$, 4-$(23, 16, 52)$	A. Bremner [3], R. Stroeker [12]
6	Nonexistence	C. Peterson [9]
8	Finitely many	E. Bannai, T. Ito [2]
8	Nonexistence	Theorem 1.1
$2e$ ($e \geq 5$)	Finitely many for each e	E. Bannai [1]
$10, 12, \ldots, 18$	Nonexistence	P. Dukes, J. Short-Gershman [5]

Table 1.2: Results on non-isomorphic nontrivial tight t-designs

We will use the following notation. For nonnegative integer n, the *n-th rising factorial* of x is

$$x^{\overline{n}} = x(x+1)\ldots(x+n-1),$$

and the *n-th falling factorial* of x is

$$x^{\underline{n}} = x(x-1)\ldots(x-n+1).$$

As usual, $x^{\overline{0}} = x^{\underline{0}} = 1$. Note that, $n!\binom{x}{n} = x^{\underline{n}}$.

2 Intersection numbers

Given a t-(v, k, λ) design $\mathcal{B}$, the integers in the set

$$\left\{ |b \cap b'| : \ \{b, b'\} \in \binom{\mathcal{B}}{2} \right\}$$

are called the *intersection numbers* of the design. The following proposition shows that if $\mathcal{B}$ is a tight $2e$-design, then we can determine all intersection numbers of $\mathcal{B}$ just using v, k and e.

Proposition 2.1 ([4, implicit], [10, p. 743], [9, implicit], [1, Proposition 1], [5, Proposition 1.1])**.** *For a tight $2e$-(v, k, λ) design, the zeros of the polynomial $\Phi_e \in \mathbb{Q}[x]$ are distinct and the zeros coincide with the intersection numbers of the design, where*

$$\Phi_e(x) := \sum_{i=0}^{e} (-1)^{e-i} \frac{\binom{v-e}{i}\binom{k-i}{e-i}\binom{k-i-1}{e-i}}{\binom{e}{i}} \binom{x}{i}.$$

For each $0 \leq i \leq e$, let

$$p_i := \binom{e}{i} \frac{(k-e)^{\overline{i}}(k-e+1)^{\overline{i}}}{(v-2e+1)^{\overline{i}}} \in \mathbb{Q}(v, k). \tag{2.2}$$

Corollary 2.3. *For a tight $2e$-(v, k, λ) design and each $0 \leq i \leq e$, the value of p_i at the point (v, k) is an integer.*

Proof. The leading coefficient of Φ_e is $\frac{\binom{v-e}{e}}{e!}$. By Lemma 2.1, all roots of $\frac{e!}{\binom{v-e}{e}}\Phi_e$ are intersection numbers, which are integers, hence the monic polynomial $\frac{e!}{\binom{v-e}{e}}\Phi_e$ has integral coefficients. We proceed with the following calculation:

$$\begin{aligned}\frac{e!}{\binom{v-e}{e}}\Phi_e(x) &= \sum_{i=0}^{e}(-1)^{e-i}\binom{e}{e-i}\frac{(k-e)^{\overline{e-i}}(k-e+1)^{\overline{e-i}}}{(v-2e+1)^{\overline{e-i}}} \cdot x^{\underline{i}} \\ &= \sum_{i=0}^{e}(-1)^{i}p_i(v,k) \cdot x^{\underline{e-i}}.\end{aligned}$$

For each i, the polynomial $x^{\underline{e-i}} \in \mathbb{Z}[x]$ is a monic polynomial of degree $e - i$. Therefore, all $p_i(v, k)$ are integers. □

3 Some ideals

In this section, we encounter some ideals I and try to find an explicit positive integer n in the intersection of I and $\mathbb{Z}$. Our key result in this section is Corollary 3.6 used in the proof of Lemma 4.2 and Theorem 4.5. Theorem 4.5 describes a necessary condition for the existence of nontrivial tight designs. The smaller n is, the stronger the necessary condition is.

Lemma 3.1. *Let d be a nonnegative integer and x_0, x_1 be integers. Let $f_d := (x + x_0)^{\overline{d}}$ and $g_d := (x + x_1)^{\underline{d}}$ be polynomials in $\mathbb{Z}[x]$. The following statements are equivalent.*

- *The ideal $I_{d,x_0,x_1} := \langle f_d, g_d\rangle$ in $\mathbb{Z}[x]$ intersects $\mathbb{Z}$ trivially.*
- *The polynomials f_d and g_d have a common factor of positive degree.*
- $x_1 - x_0 \in [0, 2d - 2]$, *where* $[0, 2d - 2] = \emptyset$ *when* $d = 0$.

Proof. Straightforward. □

We do not know much about the nonnegative generator of $I_{d,x_0,x_1} \cap \mathbb{Z}$. A computer experiment suggests that there should be some patterns. It seems that the generator is very close to $(x_1 - x_0)^{\underline{2d-1}}$, which is much smaller than the resultant of f_d and g_d.

Conjecture 3.2. Let d be a nonnegative number, u_{d,x_0,x_1} be the nonnegative generator of the ideal I_{d,x_0,x_1}, and

$$v_{d,x_0,x_1} := \begin{cases} \frac{(x_1-x_0)^{\underline{2d-1}}}{u_{d,x_0,x_1}}, & u_{d,x_0,x_1} \neq 0, \\ 0, & u_{d,x_0,x_1} = 0. \end{cases}$$

The following statements hold:

- v_{d,x_0,x_1} is an integer.
- $v_{d,1,0}$ divides A246466$(d-2)$, which is the $(d-2)$-th term of the sequence A246466 in the OEIS [8].
- $v_{d,1,0}$ divides v_{d,x_0,x_1}.
- The sequence $\{v_{d,x_0,0}\}_{x_0\geq 1}$ is periodic in the sense that for all $x_0 \geq 0$, we have $v_{d,x_0+p,0} = v_{d,x_0,0}$ for some positive integer p. Moreover, the period is $1, 2, 6, 10, 70, 126, 154, 286, 4290$ when d is $1, 2, 3, 4, 5, 6, 7, 8, 9$, respectively.

The number v_{d,x_0,x_1} also seems to be related to OEIS [8] sequences A007947, A099985 and A204455. The following lemma studies the two variable version of Lemma 3.1.

Lemma 3.3. *Let d be a nonnegative integer and x_0, y_0, x_1, y_1 be integers. For each $0 \leq i \leq d$, let $f_{d,i} := (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i}}$ and $g_{d,i} := (x+x_1)^{\underline{i}}(y+y_1)^{\underline{d-i}}$ be polynomials in $\mathbb{Z}[x,y]$. Consider the ideal*

$$I_{d,x_0,y_0,x_1,y_1} := \langle f_{d,i}, g_{d,i} :\ 0 \leq i \leq d\rangle$$

in $\mathbb{Z}[x,y]$. It holds that $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z} \supseteq (I_{d,x_0,x_1} \cap \mathbb{Z}) \cup (I_{d,y_0,y_1} \cap \mathbb{Z}) \cup (I_{d,x_0+y_0,x_1+y_1} \cap \mathbb{Z})$, where I_{d,x_0,x_1}, I_{d,y_0,y_1} and I_{d,x_0+y_0,x_1+y_1} are the ideals defined in Lemma 3.1.

Proof. The result holds trivially when $d = 0$. Assume that $d \geq 1$. Since $f_{d,d} = (x+x_0)^{\overline{d}}$ and $g_{d,d} = (x+x_1)^{\underline{d}}$, we have $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z} \supseteq \langle f_{d,d}, g_{d,d}\rangle \cap \mathbb{Z} = I_{d,x_0,x_1} \cap \mathbb{Z}$. Similarly, since $f_{d,0} = (y+y_0)^{\overline{d}}$ and $g_{d,0} = (y+y_1)^{\underline{d}}$, we have $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z} \supseteq \langle f_{d,0}, g_{d,0}\rangle \cap \mathbb{Z} = I_{d,y_0,y_1} \cap \mathbb{Z}$. The sum of $f_{d,i}$ and $f_{d,i+1}$ is in the ideal I_{d,x_0,y_0,x_1,y_1}:

$$\begin{aligned}
I_{d,x_0,y_0,x_1,y_1} \ni & f_{d,i} + f_{d,i+1} \\
= & (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i-1}}((x+x_0+i) + (y+y_0+d-i-1)) \\
= & (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i-1}}(x+y+x_0+y_0+d-1) \\
= & (x+y+x_0+y_0+d-1) f_{d-1,i}.
\end{aligned}$$

Using induction on the equation above, we obtain

$$\begin{aligned}
(x+y+x_0+y_0)^{\overline{d}} = & (x+y+x_0+y_0+d-1)^{\underline{d}} f_{0,0} \\
= & \sum_{i=0}^{d} \binom{d}{i} f_{d,i} \in I_{d,x_0,y_0,x_1,y_1}.
\end{aligned} \tag{3.4}$$

We do the same thing for g's and get

$$\begin{aligned}
I_{d,x_0,y_0,x_1,y_1} \ni & g_{d,i} + g_{d,i+1} \\
= & (x+x_1)^{\underline{i}}(y+y_1)^{\underline{d-i-1}}((x+x_1-i) + (y+y_1-d+i+1)) \\
= & (x+x_1)^{\underline{i}}(y+y_1)^{\underline{d-i-1}}(x+y+x_1+y_1-d+1) \\
= & (x+y+x_1+y_1-d+1) g_{d,i}.
\end{aligned}$$

Again using induction, we have

$$
\begin{aligned}
(x+y+x_1+y_1)^{\underline{d}} =& (x+y+x_1+y_1-d+1)^{\overline{d}} g_{0,0} \\
=& \sum_{i=0}^{d} \binom{d}{i} g_{d,i} \in I_{d,x_0,y_0,x_1,y_1}.
\end{aligned}
\tag{3.5}
$$

Therefore, combing Eq. (3.4) and Eq. (3.5), it follows that $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z} \supseteq \langle (x+y+x_0+y_0)^{\overline{d}}, (x+y+x_1+y_1)^{\underline{d}} \rangle \cap \mathbb{Z} = I_{d,x_0+y_0,x_1+y_1} \cap \mathbb{Z}$ $\square$

Corollary 3.6. *Let d be a nonnegative integer and x_0, y_0, x_1, y_1 be integers. For each $0 \leq i \leq d$, let*

$$f_{d,i} := (x+x_0)^{\overline{i}} (y+y_0)^{\overline{d-i}} \text{ and } g_{d,i} := (x+x_1)^{\underline{i}} (y+y_1)^{\underline{d-i}}$$

be polynomials in $\mathbb{Q}[x,y]$. The set of polynomials $\{f_{d,i}, g_{d,i} : 0 \leq i \leq d\}$ generates the whole ring $\mathbb{Q}[x,y]$ provided that either $x_1 - x_0 < 0$, or $y_1 - y_0 < 0$, or $x_1 - x_0 + y_1 - y_0 > 2d - 2$.

Proof. Since either $x_1 - x_0 < 0$, or $y_1 - y_0 < 0$, or $x_1 - x_0 + y_1 - y_0 > 2d - 2$ holds, by Lemma 3.1, at least one of I_{d,x_0,x_1}, I_{d,y_0,y_1} and I_{d,x_0+y_0,x_1+y_1} intersects $\mathbb{Z}$ nontrivially. According to Lemma 3.3, there exists a nonzero integer which is a $\mathbb{Z}[x,y]$-linear combination of $\{f_{d,i}, g_{d,i} : 0 \leq i \leq d\}$, then the result follows. $\square$

$y_1 - y_0$ \ $x_1 - x_0$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
7	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520
6		2520	1080	360	720	1080	360	360	*	360	360	1080	720	360	1080	2520
5			2520	360	240	720	360	120	*	*	120	360	720	240	360	2520
4				2520	720	720	144	72	*	*	*	72	144	720	720	2520
3					2520	1080	360	72	*	*	*	*	72	360	1080	2520
2						2520	360	120	*	*	*	*	*	120	360	2520
1							2520	360	*	*	*	*	*	*	360	2520
0								2520	*	*	*	*	*	*	*	2520
-1									2520	360	120	72	72	120	360	2520
-2										2520	360	360	144	360	360	2520
-3											2520	1080	720	720	1080	2520
-4												2520	720	240	720	2520
-5													2520	360	360	2520
-6														2520	1080	2520
-7															2520	2520
-8																2520

Table 3.7: The nonnegative generator of intersection of the ideal I_{4,x_0,y_0,x_1,y_1} and $\mathbb{Z}$. An asterisk stands for the intersection being trivial.

We list the intersection of the ideal I_4 and $\mathbb{Z}$ in Table 3.7. We can read some patterns from the table. For instance, we see some constant lines, and the table is symmetric with respect to the three medians of the triangle made by asterisks. Computation supports the following conjecture.

Conjecture 3.8. Let I_{d,x_0,y_0,x_1,y_1} be the ideal defined in Lemma 3.3. The following statements hold.

- $I_{d,0,0,x_1,y_1} \cap \mathbb{Z} = I_{d,0,0,y_1,x_1} \cap \mathbb{Z} = I_{d,0,0,x_1,2d-2-x_1-y_1} \cap \mathbb{Z} = I_{d,0,0,2d-2-x_1-y_1,y_1} \cap \mathbb{Z}$.

- If $x_1 = 2d-1$, or $y_1 = 2d-1$, or $x_1 + y_1 = -1$, then $I_{d,0,0,x_1,y_1} \cap \mathbb{Z} = I_{d,1,0} \cap \mathbb{Z}$.
- The generator of the intersection $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z}$ is a multiple of $d!$.

Lemma 3.1 is the case of one variable, and Lemma 3.3 is the case of two variables. Maybe, a similar behaviour occurs with more variables. For an n-part partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a nonnegative integer d and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, the *rising factorial* of $\mathbf{x}$ with respect to λ is

$$\mathbf{x}^{\overline{\lambda}} := \prod_{i=1}^{n} x_i^{\overline{\lambda_i}},$$

and the *falling factorial* of $\mathbf{x}$ with respect to λ is

$$\mathbf{x}^{\underline{\lambda}} := \prod_{i=1}^{n} x_i^{\underline{\lambda_i}}.$$

Conjecture 3.9. Let d be a nonnegative integer and $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{Z}^n$. For each n-part partition λ of d, let $f_\lambda := (\mathbf{x}+\mathbf{x}_0)^{\overline{\lambda}}$ and $g_\lambda := (\mathbf{x}+\mathbf{x}_1)^{\underline{\lambda}}$ be polynomials in $\mathbb{Q}[\mathbf{x}]$. The ideal generated by f_λ and g_λ, where λ runs over all n-part partitions of d, is not the whole ring if and only if $\mathbf{x}_1 - \mathbf{x}_0 \geq \mathbf{0}$ and $\|\mathbf{x}_1 - \mathbf{x}_0\|_1 \leq 2d-2$.

4 A necessary condition

We establish in this section further necessary conditions for the existence of tight designs. For each $e \leq i \leq 2e$, let

$$\lambda_i := \frac{1}{e!} \frac{k^{\underline{i}}}{(v-e)^{\underline{i-e}}} \in \mathbb{Q}(v,k). \tag{4.1}$$

It is well known that a t-design $\mathcal{B}$ is also an i-design for each $1 \leq i \leq t$. In particular, in the case that $\mathcal{B}$ is a tight $2e$-(v,k,λ)-design, it is also an i-$(v,k,\lambda_i(v,k))$ design.

Lemma 4.2. *Let e be an nonnegative integer. For every $0 \leq i \leq e$,*

$$h_i := \frac{(k-e)^{\overline{i+1}}}{(v-2e+1)^{\overline{i}}} \in \mathbb{Q}(v,k) \tag{4.3}$$

is a $\mathbb{Q}[v,k]$-linear combination of $p_1, \ldots, p_e$ defined in Eq. (2.2) *and $\lambda_{e+1}, \ldots, \lambda_{2e}$ defined in Eq.* (4.1).

Proof. First, $h_0 = k - e \in \mathbb{Q}[v,k]$, so we only need to consider $h_1, \ldots, h_e$. For each $1 \leq i \leq e$, let

$$f_i := (-1)^{i-1} e! \lambda_{e+i} = (-k+e+1)^{\overline{i-1}} (v-2e+1)^{\overline{e-i}} \cdot h_e,$$

and

$$g_i := (-1)^{i-1}\binom{e}{i}^{-1}(k)_{\underline{e-i}}p_i = (-k+e-1)_{\underline{i-1}}(v-e)_{\underline{e-i}} \cdot h_e.$$

By Lemma 3.3 applied with the variables $x=-k$, $y=v$ and the parameters $x_0 = e+1$, $y_0 = -2e+1$, $x_1 = e-1$, $y_1 = -e$ and $d = e-1$, the ideal in $\mathbb{Q}[v,k]$

$$\begin{aligned}&\langle f_i/h_e, g_i/h_e :\ 1 \le i \le e\rangle\\ =&\langle(-k+e+1)^{\overline{i-1}}(v-2e+1)^{\overline{e-i}}, (-k+e-1)_{\underline{i-1}}(v-e)_{\underline{e-i}} :\ 1 \le i \le e\rangle\end{aligned}$$

contains 1. Therefore, h_e is a $\mathbb{Q}[v,k]$-linear combination of $f_1,\dots,f_e$ and $g_1,\dots,g_e$, hence a $\mathbb{Q}[v,k]$-linear combination of $p_1,\dots,p_e$ and $\lambda_{e+1},\dots,\lambda_{2e}$.

Let i be an arbitrary integer in $[1,e-1]$. In the quotient ring $\mathbb{Z}[k]/(k-e+i+1)$, we have $(k-e+1)^{\overline{i-1}} = (-i)^{\overline{i-1}} = (-1)^{i-1}i!$. So, we can express $(k-e+1)^{\overline{i-1}}$ as $(k-e+i+1)u_i + (-1)^{i-1}i!$ for some $u_i \in \mathbb{Z}[k]$. Then,

$$\begin{aligned}&(v-2e+i+1)u_i \cdot h_{i+1}\\ =&(k-e+i+1)u_i \cdot h_i\\ =&((k-e+1)^{\overline{i-1}} + (-1)^i i!) \cdot h_i\\ =&(k-e+1)^{\overline{i-1}} \cdot h_i + (-1)^i i! \cdot h_i\\ =&\binom{e}{i}^{-1} \cdot p_i + (-1)^i i! \cdot h_i,\end{aligned}$$

which shows that h_i is a $\mathbb{Q}[v,k]$-linear combination of h_{i+1} and p_i. The result follows by an induction. □

Remark 4.4. When $e=4$, we give explicit $\mathbb{Z}[v,k]$-linear combinations for some multiples of h_i:

$$\begin{aligned}q_4 := 24h_4 \quad&=288(v-7)(k-5)\lambda_6 - 96((v-15)(k-13)-64)\lambda_7\\ &\quad - 24(38v+3k-262)\lambda_8 + 6(k-5)k(k-1)(k-2)p_1\\ &\quad - 2((v-6)(k-5)+1)k(k-1)p_2\\ &\quad + ((v-9)(k-4)+2)kp_3 + (2v+3k-22)p_4,\\ q_3 := 144h_3 \quad&=6p_3 - (v-4)(k-5)q_4,\\ q_2 := 288h_2 \quad&=-24p_2 + (v-5)q_3,\\ q_1 := 4h_1 \quad&=p_1,\\ q_0 := h_0 \quad&=k-4.\end{aligned}$$

The one for q_4 is obtained by calculating the Gröbner basis for a certain ideal. And the ones for q_3 and q_2 follow from the proof of Lemma 4.2.

Theorem 4.5. *For each positive integer e, there exists a nonzero constant c_e such that if there exists a nontrivial tight $2e$-(v,k,λ) design, then for every $0 \le i \le e$, the rational function h_i in Eq. (4.3) takes value in $c_e^{-1}\mathbb{Z}$ at the point (v,k).*

Proof. By Corollary 2.3, p_i takes integer value at (v,k). A tight $2e$-design is also an i-(v,k,λ_i) design for each $i \le 2e$, so λ_i in Eq. (4.1) takes integer value at (v,k) as well. Due to Lemma 4.2, for every $0 \le i \le e$, the rational function h_i is a $\mathbb{Q}[v,k]$-linear combination of $p_1,\dots,p_e$ and $\lambda_{e+1},\dots,\lambda_{2e}$. Let c_e to be the least common multiple of denominators of all nonzero coefficient that appeared in the linear combinations. Then, $c_e h_i$ is a $\mathbb{Z}[v,k]$-linear combination of $p_1,\dots,p_e$ and $\lambda_{e+1},\dots,\lambda_{2e}$, therefore, it takes integer value at (v,k). □

For use in the proof of Theorem 1.1 in Proof 6.2, we strengthen Theorem 4.5 in the case $e = 4$. Proposition 4.6 can be obtained by similar arguments as in Theorem 4.5. We omit the proof here.

Proposition 4.6. *If there exists a tight* 8-(v,k,λ) *design, then for every* $0 \le i \le 4$, *the rational function* q_i *in Remark 4.4 takes integer value at* (v,k). □

If there exists infinitely many nontrivial tight $2e$-designs for a fixed e, then by Theorem 4.5, the h_i takes bounded denominator value infinitely many times in the region $v > k + 2e$. It seems to be possible to reprove the finiteness of nontrivial tight $2e$-designs by analyzing h_i. If the following conjecture holds for $n = e$, then there exist only finitely many tight $2e$-designs, furthermore, we may expect an efficient bound for v and k.

Conjecture 4.7. Let n be an integer at least 3. For every non-zero integer c, there are only finitely many pairs (x,y) of positive integers satisfying $y \ge x+2$ and such that for every $1 \le i \le n$,

$$\frac{x^{\overline{i+1}}}{y^{\overline{i}}} \in \frac{1}{c}\mathbb{Z}\,.$$

(In other words, $\frac{x(x+1)}{y}$, $\frac{x(x+1)(x+2)}{y(y+1)}$, …, have denominators that divide c.)

Note that, the smaller n is, the stronger Conjecture 4.7 is. In the case where $n = 3$ and $c = 1$, there do not exist any such pairs for x up to 5 billions.

5 Asymptotic behaviour of f_4

Let $f_4 \in \mathbb{Z}[v,k]$ be the polynomial of degree 13 given in §7. The polynomial f_4 is first found by E. Bannai and T. Ito in an unpublished work [2], and it is shown that for all but finitely many nontrivial tight 8-designs, f_4 takes integral value at (v,k). The polynomial is rediscovered by P. Dukes and J. Short-Gershman in [5] and they strengthen the result as follows.

Proposition 5.1 ([5, §4])**.** *If there exists a nontrivial tight* 8-(v_0,k_0,λ) *design, then* $f_4(v_0,k_0) = 0$.

The polynomial f_4 satisfies Runge's condition, so by Runge's theorem [11], it has finitely many integral solutions. Using this approach, E. Bannai and T. Ito showed in [2] that there are only finitely many tight 8-designs.

Quantitative versions of Runge's theorem have been established, and using the results in [7] and [13], we can obtain the bounds $e^{e^{8600}}$ and $e^{e^{22}}$, respectively, for the size $\max(|v|,|k|)$ of an integral solution (v,k). The bounds are too large for any computer search to terminate.

The curve defined by the polynomial f_4 has an involution $(v,k) \mapsto (v, v-k)$, which corresponds to the construction of complementary designs. The geometric genus of the curve is 20, so by Faltings' theorem, the curve has only finitely many rational points. However, Faltings' theorem is not effective.

There are 32 known rational zeros of $f_4(v,k)$. They are $(-1,-3)$, $(-1,-2)$, $(-1,1)$, $(-1,2)$, $(11/5,1)$, $(11/5,6/5)$, $(4,0)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(13/3,2)$, $(13/3,7/3)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(27/5,2)$, $(27/5,13/5)$, $(27/5,14/5)$, $(27/5,17/5)$, $(6,2)$, $(6,11/4)$, $(6,3)$, $(6,13/4)$, $(6,4)$, $(125/19,54/19)$, $(125/19,71/19)$, $(7,3)$, $(7,4)$, $(15,2)$, $(15,13)$. However, we are unable to show that they are the only rational zeros. None of the known zeros could be realized by nontrivial tight 8-designs, since for nontrivial tight 8-(v,k,λ), we have $v > k+8 > 16$.

The main result of this section is Proposition 5.5, which describes the zeros of f_4 in the region $k \geq 100000$ and $2k \leq v \leq 0.8k^2$. In the remaining part of this section, the parameters v and k are assumed to be real numbers.

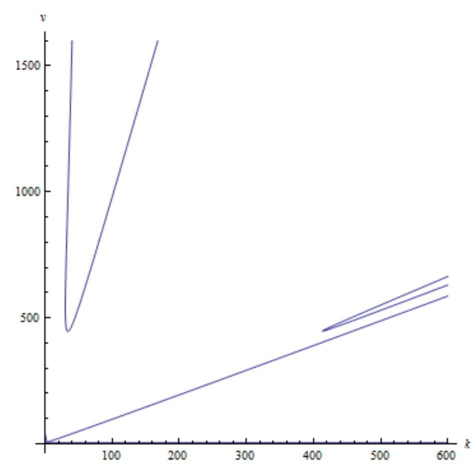

Figure 5.2: Real zeros of f_4 in the first quadrant.

We plot the reals zeros of f_4 in Figure 5.2. In the first quadrant, it seems that there are six branches going to infinity (one branch is so close to the x-axis that it is undistinguishable from it in Figure 5.2). Some necessary conditions for the existence of tight designs suggest that we focus on the branches which have growth rate of the form $v = ak + b + o(1)$ with $a \geq 2$. Figure 5.2 indicates that it is likely that there is only one such branch but we do not need this fact here.

Lemma 5.3. *If there exists a function* $\widetilde{v}(k) : \mathbb{R} \to \mathbb{R}$ *such* $f_4(\widetilde{v}(k), k) = 0$ *and*

$\widetilde{v}(k) \geq 2k$ *for all* $k \in \mathbb{R}$*, and*

$$\widetilde{v}(k) = ak + b + o(1)$$

as $k \to +\infty$*, then*

$$\begin{cases} a = \frac{2}{1-\sqrt[4]{\frac{3}{8}}} = \frac{2}{5}(8 + 2\sqrt{6} + \sqrt{48 + 22\sqrt{6}}) \approx 9.1971905725, \\ b = \frac{23}{500}\left(249 + 86\sqrt{6} + \sqrt{171312 + 70918\sqrt{6}}\right) \approx 48.1640392521. \end{cases} \tag{5.4}$$

Proof. Since $\widetilde{v}(k) \geq 2k$, we have $a \geq 2$. Substituting $\widetilde{v}(k) = ak + b + o(1)$ into $f_4(\widetilde{v}(k), k)$, we obtain

$$\begin{aligned} 0 = f_4(\widetilde{v}(k), k) = &- 128a(1-a)^4(128 - 256a + 192a^2 - 64a^3 + 5a^4)k^{13} \\ &\Big(16(1-a)^2(4096 - 16384a + 12288a^2 + 20480a^3 \\ &- 40232a^4 + 27216a^5 - 7976a^6 + 512a^7 + 21a^8) \\ &- 128(1-a)^3(128 - 1152a + 2112a^2 - 1600a^3 \\ &+ 537a^4 - 45a^5)b\Big)k^{12} \\ &+ o(k^{12}) \end{aligned}$$

as $k \to +\infty$. Therefore, the coefficients of k^{13} and k^{12} on the right hand side must vanish, which gives us the unique solution in Eq. (5.4). $\square$

Proposition 5.5. *The real zeros* (v, k) *of the polynomial* f_4 *in the region* $k \geq 10^5$ *and* $2k \leq v \leq 0.8k^2$*, satisfy*

$$ak + b \leq v \leq ak + b + \frac{1}{100},$$

where a *and* b *are the numbers in Eq.* (5.4)*.*

Proof. Combine Lemmas 5.6, 5.7, 5.8 and 5.9. $\square$

Lemma 5.6. *If* $v \in [2k, 9k]$ *and* $k \geq 20000$*, then* $f_4(v, k) > 0$*.*

Proof. Let

$$\begin{aligned} g(v,k) :=& 65535k^{12} - 16404k^{12}v + 98304k^{11}v^2 - 253969k^{10}v^3 + \\ & 368584k^9v^4 - 328320k^8v^5 + 182701k^7v^6 - 61184k^6v^7 + \\ & 10744k^5v^8 - 640k^4v^9 + 335k^2v^{10} + 45v^{11}. \end{aligned}$$

We can check that, viewing $f(v, k)$ and $g(v, k)$ as polynomials in v with coefficient in $\mathbb{Z}[k]$ and using the fact that $k \geq 20000$, for every $i \geq 0$, the coefficient of v^i in $f(v, k)$ is no smaller than that in $g(v, k)$. Since $f(v, k) \neq g(v, k)$, we know $f(v, k) > g(v, k)$.

Let $t := v/k$, hence $t \in [2, 9]$. We have

$$\begin{aligned}
&f(v,k) > g(v,k) = g(tk,k)\\
=&\Big(-16404t + 98304t^2 - 253969t^3 + 368584t^4 - 328320t^5\\
&+ 182701t^6 - 61184t^7 + 10744t^8 - 640t^9\Big)k^{13}\\
&+ \big(65535 + 335t^{10}\big)k^{12} + 45t^{11}k^{11}.
\end{aligned}$$

The coefficients of k^{13}, k^{12} and k^{11} in the equation above are positive when $t \in [2, 9]$, then the result follows. □

Lemma 5.7. *If $v \in [9k, ak + b]$ and $k \geq 100$, then $f_4(v,k) > 0$.*

Proof. Let $t := ak + b - v$, hence $t \in [0, 0.9k]$. Let

$$\begin{aligned}
g(t,k) :=& 13700000000tk^{12} - 13400000000t^2k^{11} - 140000000t^3k^{10}\\
&- 1380000000t^4k^9 - 6000000t^5k^8 - 21000000t^6k^7\\
&- 10000t^7k^6 - 73000t^8k^5 - 40000t^9k^3 - 7t^9k^4\\
&- 50000t^{10}k - 200t^{10}k^2 - 312000000000t^{11}.
\end{aligned}$$

We can check that, viewing $f(ak + b - t, k)$ and $g(t, k)$ as polynomials in k with coefficient in $\mathbb{R}[t]$ and using the fact that $t \geq 0$, for every $i \geq 0$, the coefficient of k^i in $f(ak + b - t, k)$ is no smaller than that in $g(t, k)$. Since $f(ak + b - t, k) \neq g(t, k)$, we know $f(ak + b - t) > g(t, k)$.

Let $s := t/k$, hence $s \in [0, 0.9]$. We have

$$\begin{aligned}
&f(v,k) = f(ak+b-t) > g(t,k) = g(sk,k)\\
=&\Big(13700000000s - 13400000000s^2 - 140000000s^3\\
&- 1380000000s^4 - 6000000s^5 - 21000000s^6 - 10000s^7\\
&- 73000s^8 - 7s^9\Big)k^{13}\\
&- \big(40000s^9 + 200s^{10}\big)k^{12}\\
&- \big(50000s^{10} + 312000000000s^{11}\big)k^{11}\\
\geq&\Big(13700000000s - 13400000000s^2 - 140000000s^3 \qquad (k \geq 100)\\
&- 1380000000s^4 - 6000000s^5 - 21000000s^6 - 10000s^7\\
&- 73000s^8 - 407s^9 - 7s^{10} - 31200000s^{11}\Big)k^{13}.
\end{aligned}$$

The result follows from the fact that the coefficient of k^{13} is positive when $s \in [0, 0.9]$. □

Lemma 5.8. *If $ak + b + \frac{1}{100} \le v \le 10k$ and $k \ge 100000$, then $f_4(v,k) < 0$.*

Proof. Let $t := v - ak - b$, hence $t \in [\frac{1}{100}, k]$. Let

$$\begin{aligned} g(t,k) := &- 137892000k^{12} + 13318642886180k^{11} + 713748202323829k^{10} \\ &+ 4883767326152566 8k^9 + 5098485316801241991k^8 \\ &+ 980132640412645508268k^7 + 48891070993530297659493 4k^6 \\ &+ 130500990628997779567527 7621k^5 \\ &+ 15141857225127491774321097 1453210k^4 \\ &+ 64000t^9k^3 + 2000t^{10}k^2 + 7000t^{10}k + 127x^{11}. \end{aligned}$$

We can check that, viewing $f(ak + b + t, k)$ and $g(t, k)$ as polynomials in k with coefficient in $\mathbb{R}[t]$ and using the fact that $t \ge \frac{1}{100}$, for every $i \ge 0$, the coefficient of k^i in $f(ak + b + t, k)$ is no greater than that in $g(t, k)$. Since $f(ak + b + t, k) \ne g(t, k)$, we know $f(ak + b + t) < g(t, k)$. It follows from $t \le k$ that $g(t, k) < 0$. □

Lemma 5.9. *If $v \in [9.24k, 0.8k^2]$ and $k \ge 10^5$, then $f_4(v,k) < 0$.*

Proof. Let

$$\begin{aligned} g(v,k) :=&65536k^{12} - 16384k^{12}v + 98312k^{11}v^2 - 253952k^{10}v^3 \\ &+ 368640k^9v^4 - 328299k^8v^5 + 182784k^7v^6 - 61177k^6v^7 \\ &+ 10752k^5v^8 - 639k^4v^9 + 336k^2v^{10} + 45v^{11}. \end{aligned}$$

We can check that, viewing $f(v, k)$ and $g(v, k)$ as polynomials in v with coefficient in $\mathbb{Z}[k]$ and using the fact that $k \ge 10^5$, for every $i \ge 0$, the coefficient of v^i in $f(v, k)$ is no smaller than that in $g(v, k)$. Since $f(v, k) \ne g(v, k)$, we know $f(v, k) < g(v, k)$.

Let $t := v/k$, hence $t \in [9.24, 0.8k]$. Let

$$\begin{aligned} h(x) :=& - 1 + 16384x - 98312x^2 + 253952x^3 - 368640x^4 \\ &+ 328299x^5 - 182784x^6 + 61177x^7 - 10752x^8 + 639x^9. \end{aligned}$$

We have

$$\begin{aligned} &f(v,k) < g(v,k) = g(tk,k) \\ =& - \big(k - 65536\big)k^{12} - \big(h(t) - 298t^9\big)k^{13} \\ &- \big(298 - 336(t/k) - 45(t/k)^2\big)t^9k^{13} \qquad (5.10) \\ =& - \big(k - 65536\big)k^{12} - \big(h(t) - 0.1t^9\big)k^{13} \\ &- \big(0.1 - 336(t/k) - 45(t/k)^2\big)t^9k^{13}. \qquad (5.11) \end{aligned}$$

Since $k \ge 10^5$, we have either $v \le \frac{1}{3800}k^2$ or $v \ge 26k$.

Case *1*: $v \in [26k, 0.8k^2]$.

Consider Eq. (5.10). The result follows from the following facts:

- $k - 65536 \geq 0$;
- $h(x) - 298x^9 \geq 0$ when $x \geq 26$;
- $298 - 336x - 45x^2 \geq 0$ when $x \in [0, 0.8]$.

Case *2*: $v \in [9.24k, \frac{1}{3800}k^2]$.

Consider Eq. (5.11). The result follows from the following facts:

- $k - 65536 \geq 0$;
- $h(x) - 0.1x^9 \geq 0$ when $x \geq 9.24$;
- $0.1 - 336x - 45x^2 \geq 0$ when $x \in [0, \frac{1}{3800}]$. □

6 Putting everything together

Here is a heuristic argument for the nonexistence of nontrivial tight 8-designs with large coefficients (v, k). Assume that there exists a nontrivial tight 8-(v, k, λ) design with $v \geq 2k$. When k and v are big, by Lemma 5.3, we may expect $v = ak + O(1)$. According to Theorem 4.5, h_i takes a rational value with bounded denominator at (v, k), and when k and v are big, h_i is roughly ka^{-i}. So, for every polynomial $f = \sum_{i=0}^{e} c_i x^i \in \mathbb{Q}[x]$, as $k \to \infty$,

$$\begin{aligned} X_f(v,k) := \sum_{i=0}^{e} c_i h_i(v,k) &= \sum_{i=0}^{e} c_i \frac{(k-e)^{\overline{i+1}}}{(v-2e+1)^{\overline{i}}} \\ &= \sum_{i=0}^{e} c_i (ka^{-i} + O(1)) = kf(a^{-1}) + O(1). \end{aligned}$$

In the degenerate case where $f(a^{-1}) = 0$, we get that $X_f = O(1)$, and we hope that $\lim_{k\to\infty} X_f(v,k)$ becomes an irrational number. On the other hand, X_f only takes rational values with bounded denominator at (v, k), so we get a contradiction.

To make this idea precise, we choose

$$X := 48q_4 - 16q_3 + 6q_2 - 144q_1 + 45q_0 \in \mathbb{Q}(v,k), \tag{6.1}$$

and

$$\begin{aligned} &\lim_{k\to\infty} X(ak+b,k) \\ =&\frac{9}{100}\left(6522 + 2808\sqrt{6} - \sqrt{56993328 + 24204417\sqrt{6}}\right) \approx 235.5086988785. \end{aligned}$$

Proof 6.2 (Proof of Theorem 1.1). Assume that there exists a nontrivial tight 8-design. By the construction of complementary designs, we know that there must exist a nontrivial tight 8-(v, k, λ) design with parameters $v \geq 2k$ and $k \geq 8$. By Proposition 5.1, $f_4(v, k) = 0$.

A computer check shows that the polynomial $f_4(v,k)$ has no integer zero in the region $8 \le k \le 10^5$ and $v \ge 2k$. So, $k \ge 10^5$.

According to Proposition 4.6, we know that $q_1(v,k)$ is a positive integer. If $q_1(v,k) \in \{1,2,3,4\}$, then we substitute $v = \frac{4(k-4)(k-3)}{q_1(v,k)}$ into $f_4(v,k) = 0$ and get $k \in \{2,3,4\}$, which is too small. So, $q_1(v,k) \ge 5$, hence $v \le 0.8k^2$. Then, Proposition 5.5 gives a bound for v, namely $ak+b \le v \le ak+b+\epsilon$, where $\epsilon := \frac{1}{100}$.

Again, by Proposition 4.6, $q_0(v,k), \dots, q_4(v,k)$ are all integers. According to Eq. (6.1), $X(v,k)$ is an integral linear combination of $q_0(v,k), \dots, q_4(v,k)$, hence an integer as well. Expanding the expression of $X(v,k)$, we obtain

$$\begin{aligned} X(v,k) =& \frac{9(k-4)}{(v-7)^4}\Big(\big(128k^4 + 256k^3 - 896k^2 - 1024k - 1944\big) \\ & - \big(256k^3 + 192k^2 - 1088k - 2186\big)v \\ & + \big(192k^2 - 833\big)v^2 - \big(64k - 82\big)v^3 + 5v^4\Big), \end{aligned}$$

in which $256k^3 + 192k^2 - 1088k - 2186 \ge 0$, $192k^2 - 833 \ge 0$ and $64k - 82 \ge 0$. Therefore,

$$\begin{aligned} & X(v,k) \\ \ge & \frac{9(k-4)}{(ak+b+\epsilon-7)^4}\Big(\big(128k^4 + 256k^3 - 896k^2 - 1024k - 1944\big) \\ & - \big(256k^3 + 192k^2 - 1088k - 2186\big)(ak+b+\epsilon) \\ & + \big(192k^2 - 833\big)(ak+b)^2 - \big(64k - 82\big)(ak+b+\epsilon)^3 + 5(ak+b)^4\Big) \\ \ge & 235 + \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} & X(v,k) \\ \le & \frac{9(k-4)}{(ak+b-7)^4}\Big(\big(128k^4 + 256k^3 - 896k^2 - 1024k - 1944\big) \\ & - \big(256k^3 + 192k^2 - 1088k - 2186\big)(ak+b) \\ & + \big(192k^2 - 833\big)(ak+b+\epsilon)^2 - \big(64k - 82\big)(ak+b)^3 + 5(ak+b+\epsilon)^4\Big) \\ \le & 235 + \frac{3}{4}, \end{aligned}$$

which contradicts the fact that $X(v,k)$ is an integer. $\square$

7 The thirteen degree polynomial

$f_4(v,k) := -16384k^{12}v + 65536k^{12} + 98304k^{11}v^2 - 393216k^{11}v - 253952k^{10}v^3 + 786432k^{10}v^2 + 1744896k^{10}v - 3309568k^{10} + 368640k^9v^4 - 327680k^9v^3 - 8724480k^9v^2 + 16547840k^9v - 328320k^8v^5 - 1102464k^8v^4 + 17194752k^8v^3 - 21567744k^8v^2 - 49810560k^8v + 62323584k^8 + 182784k^7v^6 + 2050560k^7v^5 - 16432128k^7v^4 - 13016064k^7v^3 + 199242240k^7v^2 - 249294336k^7v - 61184k^6v^7 - 1642240k^6v^6 + 6536960k^6v^5 + 58253568k^6v^4 - 293538048k^6v^3 + 209662720k^6v^2 + 511604992k^6v - 488998144k^6 + 10752k^5v^8 + 698880k^5v^7 + 1258752k^5v^6 - 59703552k^5v^5 + 183266304k^5v^4 + 243542016k^5v^3 - 1534814976k^5v^2 + 1466994432k^5v - 640k^4v^9 - 143664k^4v^8 - 2296192k^4v^7 + 27050224k^4v^6 - 7038496k^4v^5 - 582955856k^4v^4 + 1856597696k^4v^3 - 1428764528k^4v^2 - 1015706784k^4v + 974873344k^4 + 7520k^3v^9 + 772608k^3v^8 - 2875616k^3v^7 - 58917568k^3v^6 + 469164960k^3v^5 - 1155170432k^3v^4 + 412538336k^3v^3 + 2031413568k^3v^2 - 1949746688k^3v + 336k^2v^{10} - 52816k^2v^9 - 1582560k^2v^8 + 27560816k^2v^7 - 127930016k^2v^6 + 28759472k^2v^5 + 1497511456k^2v^4 - 4944873072k^2v^3 + 6922441360k^2v^2 - 4733985888k^2v + 1506333312k^2 - 2352kv^{10} + 203472kv^9 - 764688kv^8 - 24513072kv^7 + 293023248kv^6 - 1459281552kv^5 + 3929166288kv^4 - 5947568016kv^3 + 4733985888kv^2 - 1506333312kv + 45v^{11} + 972v^{10} - 191952v^9 + 2961396v^8 - 14780538v^7 - 18769932v^6 + 544096980v^5 - 2755473732v^4 + 7281931941v^3 - 11097146016v^2 + 9310949028v - 3408102864.$

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