Representation theory for Iwahori-Hecke algebras and Schur algebras of classical type

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Part I: Support varieties for Iwahori-Hecke algebras

Based on joint work with Daniel K. Nakanao.

Coxeter group

k: an algebraically closed field of characteristic 0.

(W, S): a Coxeter system.

Example 1

Let $G := \mathrm{Aut}(\{\pm 1, \ldots, \pm d\})$.

1. Type A: $W^{\mathrm{A}} \subset G$. $s_i := (i, i+1)(-i, -i-1)$.
2. Type B: $W^{\mathrm{B}} \subset G$. $s_0^{\mathrm{B}} := (1, -1)$.
3. Type D: $W^{\mathrm{D}} \subset G$. $s_0^{\mathrm{D}} := (1, -1)(2, -2) = s_0^{\mathrm{B}} s_1 s_0^{\mathrm{B}}$.

$$W^{\mathrm{A}} \subseteq W^{\mathrm{D}} \subseteq W^{\mathrm{B}} \subseteq G.$$

Iwahori-Hecke algebra

$\mathbf{q} := (q_s)_{s\in S}$ such that $q_s = q_t$ if s and t are conjugate.

Definition 2

The *Iwahori-Hecke algebra* of (W, S) with parameter $\mathbf{q}$, denoted by $\mathcal{H}(W)$, is the free k-module with basis

$$\{T_w : \ w \in W\}$$

and with multiplication defined by

$$T_w T_s := \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ q_s T_{ws} + (q_s - 1) T_w, & \text{otherwise,} \end{cases}$$

for $w \in W$ and $s \in S$, where $\ell : \ W \to \mathbb{N}$ is the length function of (W, S).

Two simple modules

For $w \in W$, let

$$q_w := q_{s_1} \dots q_{s_{\ell(w)}}$$

for a reduced expression $w = s_1 \dots s_{\ell(w)}$.

The trivial module k, where T_w acts as q_w.

The alternating module sgn, where T_w acts as $(-1)^{\ell(w)}$.

Cohomology

M: a $\mathcal{H}(W)$-module.

Definition 3

The cohomology of $\mathcal{H}(W)$ with coefficient M is

$$\mathrm{H}^\bullet(\mathcal{H}(W), M) := \mathrm{Ext}_{\mathcal{H}(W)}(k, M).$$

The cohomology ring of $\mathcal{H}(W)$ is

$$\mathrm{H}^\bullet(\mathcal{H}(W), k) := \mathrm{Ext}_{\mathcal{H}(W)}(k, k).$$

Yoneda product gives

$$\mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, k) \times \mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, k) \to \mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, k),$$

and

$$\mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, k) \times \mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, M) \to \mathrm{Ext}^\bullet_{\mathcal{H}(W)}(k, M).$$

Cohomology ring for type A

Let $R_\lambda := \mathrm{H}^\bullet(\mathcal{H}(\Sigma_\lambda), k)$, and $R_d := R_{(d)}$.

Theorem 4

Suppose that q is an lth root of unity for $l > 2$.

1. $R_l \cong k[x] \otimes \Lambda[y]$ *with* $\deg x = 2l - 2$ *and* $\deg y = 2l - 3$.

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2. $R_{(l^a)} \cong k[x_1, \ldots, x_a] \otimes \Lambda[y_1, \ldots, y_a]$.
3. *(Benson-Erdmann-Mikaelian '10) Let $a := \lfloor d/l \rfloor$.*

$$R_d \cong R_{la} \cong R_{(l^a)}^{\Sigma_a} \cong k[x_1, \ldots, x_a]^{\Sigma_a} \otimes \Lambda[y_1, \ldots, y_a]^{\Sigma_a}.$$

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4. *(Nakano-X. '18)* $\mathrm{H}^\bullet(\mathcal{H}(\Sigma_d), M)$ *is finitely generated over* $\mathrm{H}^\bullet(\mathcal{H}(\Sigma_d), k)$.

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4. *(Nakano-X. '18)* $\mathrm{H}^\bullet(\mathcal{H}(\Sigma_d), M)$ *is finitely generated over* $\mathrm{H}^\bullet(\mathcal{H}(\Sigma_d), k)$.

Let $\widetilde{R}_d := k[x_1, \ldots, x_a]^{\Sigma_a}$. Then,

$$\operatorname{maxSpec} \widetilde{R}_d = \mathbb{A}^{\lfloor d/l \rfloor}.$$

Complexity

For a sequence of integers $a_\bullet$, let

$$r(a_\bullet) := \text{smallest natural number } c \text{ such that } a_n \ll n^{c-1}.$$

For an A-module M, the complexity of M is

$$c_A(M) := r(\dim P^\bullet),$$

where $P^\bullet$ is the minimal projective resolution of the A-module M.

Complexity of $\mathcal{H}(\Sigma_d)$ modules

Let M be a $\mathcal{H}(\Sigma_d)$-module.

1. $c(M) = 0$ if and only if M is projective.
2. $c(M) = r(\mathrm{Ext}^{\bullet}_{\mathcal{H}(\Sigma_d)}(M, M))$.
3. $c(k) = r(R_d) = r(k[x_1, \ldots, x_a]^{\Sigma_a} \otimes \Lambda[y_1, \ldots, y_a]^{\Sigma_a}) = \lfloor d/l \rfloor$.

Support variety for the group algebra of symmetric groups

Let M be a $k\Sigma_d$-module. Its support variety is

$$V(M) := \operatorname{maxSpec} \widetilde{\operatorname{Ext}}^{\bullet}_{k\Sigma_d}(k,k)/J(M),$$

where $J(M)$ is the annihilator of the action of $\widetilde{\operatorname{Ext}}^{\bullet}_{k\Sigma_d}(k,k)$ on $\operatorname{Ext}^{\bullet}_{k\Sigma_d}(M,M)$.

$$\dim V(M) = r(\operatorname{Ext}^{\bullet}_{k\Sigma_d}(M,M)) = c(M).$$

Branching result on complexity

Theorem 5 (Hemmer-Nakano '02, Nakano-X. '18)

Let A_λ be $k\Sigma_\lambda$ or $\mathcal{H}(\Sigma_\lambda)$, and M an A_d-module.

$$
\begin{aligned}
c(M) &= \max_\lambda r(\mathrm{Ext}^\bullet_{A_d}(D^\lambda, M)) \\
&= \max_\lambda r(\mathrm{Ext}^\bullet_{A_d}(S^\lambda, M)) \\
&= \max_\lambda r(\mathrm{Ext}^\bullet_{A_d}(Y^\lambda, M)) \\
&= \max_\lambda r(\mathrm{Ext}^\bullet_{A_d}(M^\lambda, M)) \\
&= \max_\lambda r(\mathrm{Ext}^\bullet_{A_\lambda}(k, \mathrm{res}_{A_\lambda} M))
\end{aligned}
$$

Support variety

The relative support of M is

$$W_\lambda(M) := \mathrm{maxSpec}\widetilde{\mathrm{Ext}}_{\mathcal{H}(\Sigma_\lambda)}(k,k)/J_\lambda(M),$$

where $J_\lambda(M)$ is the annihilator of the action of $\widetilde{\mathrm{Ext}}^\bullet_{\mathcal{H}(\Sigma_\lambda)}(k,k)$ on $\mathrm{Ext}^\bullet_{\mathcal{H}(\Sigma_\lambda)}(k,M)$.

The support variety of M is

$$V_d(M) := \bigcup_{\lambda \vDash d} \mathrm{res}^*_{d,\lambda} W_\lambda(M).$$

Properties of support varieties

Let M and N be finite dimensional $\mathcal{H}(\Sigma_d)$-modules.

1. $\dim V(M) = c(M)$.
2. For short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$,
$$V(M_{i_1}) \subseteq V(M_{i_2}) \cup V(M_{i_3}).$$
3. $V(M_1 \oplus M_2) = V(M_1) \cup V(M_2)$.
4. $V_d(M) = \bigcup_{\lambda \vDash d} \operatorname{res}^*_{d,\lambda} W_\lambda(M)$, where λ runs over l-parabolic compositions.
5. If $M \mid \operatorname{ind}_{\mathcal{H}(\Sigma_\lambda)}^{\mathcal{H}(\Sigma_d)} N$ and $N \mid \operatorname{res}_{\mathcal{H}(\Sigma_\lambda)}^{\mathcal{H}(\Sigma_d)} M$, then
$$V_d(M) = \operatorname{res}^*_{d,\lambda} V_\lambda(N).$$
6. $V(k) = \mathbb{A}^{\lfloor d/l \rfloor}$.

Vertex theory

Definition 6 (Higman's criterion)

A $\mathcal{H}(\Sigma_d)$-module M is $\mathcal{H}(\Sigma_\lambda)$-projective if and only if

$$M \mid \operatorname{ind}_{\mathcal{H}(\Sigma_\lambda)}^{\mathcal{H}(\Sigma_d)} N$$

for some $\mathcal{H}(\Sigma_d)$-module N.

Definition 7

The vertex $\mathrm{vx}(M)$ of M is the minimal partition λ such that M is $\mathcal{H}(\Sigma_\lambda)$-projective. An N such that $M \mid \operatorname{ind}_{\mathcal{H}(\Sigma_\lambda)}^{\mathcal{H}(\Sigma_d)} N$ is called a source of M.

Corollary 8

Let N be a source of M. Then,

$$V_d(M) = \operatorname{res}^*_{d,\mathrm{vx}(M)} V_{\mathrm{vx}(M)}(N).$$

Support variety of some modules

Example 9 (Permutation modules)

Let $\lambda = \lambda_0 l + \lambda_1$ with $\lambda_1 < l$. Then,

$$V_d(M^\lambda) = \mathbb{A}^{|\lambda_0|}.$$

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Example 9 (Permutation modules)

Let $\lambda = \lambda_0 l + \lambda_1$ with $\lambda_1 < l$. Then,

$$V_d(M^\lambda) = \mathbb{A}^{|\lambda_0|}.$$

Example 10 (Young modules)

Let $\lambda = \lambda_0 l + \lambda_1$ with λ_1 being l-restricted. Then,

$$V_d(Y^\lambda) = \mathbb{A}^{|\lambda_0|}.$$

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Example 11 (Blocks)

Let $\mathrm{core}\ \lambda$ be the core of λ and $\mathrm{wt}\ \lambda$ the weight of λ. Then,

$$V(\mathbb{B}^{\mathrm{core}\ \lambda}) = \mathbb{A}^{\mathrm{wt}\ \lambda}.$$

Morita equivalence

Theorem 12 (Dipper-James '92, Pallikaros '94, Hu '02)

1. *If $f_d^{\mathrm{B}}(Q,q)$ is invertible in k, then*

$$\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \sim_{\mathrm{Morita}} \bigoplus_{i=0}^{d} \mathcal{H}^{\mathrm{A}}_q(i-1) \otimes \mathcal{H}^{\mathrm{A}}_q(d-i-1).$$

2. *If $f_d^{\mathrm{D}}(q)$ is invertible in k, then*

$$\mathcal{H}^{\mathrm{D}}_{q}(d) \sim_{\mathrm{Morita}} \mathcal{A}(d/2) \oplus \bigoplus_{i=0}^{\lceil d/2-1 \rceil} \mathcal{H}^{\mathrm{A}}_q(i-1) \otimes \mathcal{H}^{\mathrm{A}}_q(d-i-1),$$

   where $\mathcal{A}(d/2) = 0$ if d is odd, and
   $\mathcal{H}^{\mathrm{A}}_q(d/2-1)^{\otimes 2} /\!/ \mathcal{A}(d/2) \cong k\mathbb{Z}_2$ if d is even.

Part II: Coordinate construction of Schur algebras

Based on joint work with Chun-Ju Lai and Daniel K. Nakanao.

Tensor space

For $n = 2r + \epsilon$, let

$$V := V_{>0} \oplus V_0 \oplus V_{<0},$$

where

$$V_{>0} \cong k^r, \quad V_0 \cong k^\epsilon, \quad \text{and} \quad V_{<0} \cong k^r.$$

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The tensor space

$$V^{\otimes d}$$

admits an action of the type B Coxeter group $\langle s_0^{\mathrm{B}}, s_1, \ldots, s_{d-1} \rangle$, hence actions of W^Φ for $\Phi \in \{A_{d-1}, B_d, D_d\}$.

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The tensor space $V^{\otimes d}$ is a $\mathcal{H}_{\mathbf{q}}^\Phi$-module, for $\Phi \in \{A_{d-1}, B_d, D_d\}$.

q-Schur algebra of classical type

Definition 13

The q-Schur algebra of type Φ is

$$S_{\mathbf{q}}^{\Phi}(n, d) := \mathrm{End}_{\mathcal{H}_{\mathbf{q}}^{\Phi}}(V^{\otimes d}).$$

Coordinate construction of q-Schur algebra of classical type

Suppose that $V = k\langle v_i : \ i \in I(n)\rangle$.

Let $k[M(n)]$ be the bialgebra

$$k\langle x_{i,j} : \ i, j \in I(n)\rangle.$$

Theorem 14 (Lai-Nakano-Xiang '19 for type B and D)

There exists a suitable coideal $J_{\mathbf{q}}^{\Phi}(n) \subseteq k[M(n)]$ *such that*

$$S_{\mathbf{q}}^{\Phi}(n, d) \cong (k[M(n)]/J_{\mathbf{q}}^{\Phi}(n))_d^*.$$

Type A relations

	$(j \sim i, m \sim l)$
$x_{li}x_{mj} - q^{-1}x_{mj}x_{li}$	$(=,<)$ or $(<,=)$
$x_{li}x_{mj} - qx_{mj}x_{li}$	$(=,>)$ or $(>,=)$
$x_{li}x_{mj} - x_{mj}x_{li}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li}x_{mj} - x_{mj}x_{li} - \widehat{q}x_{mi}x_{lj}$	$(<,<)$
$x_{li}x_{mj} - x_{mj}x_{li} + \widehat{q}x_{mi}x_{lj}$	$(>,>)$

$\widehat{q} := q^{-1} - q$

Extra type B relations

	$(i \sim -i, l \sim -l) = (i \sim 0, l \sim 0)$
$x_{li} - Q^{-1} x_{-l-i}$	$(=,<)$ or $(<,=)$
$x_{li} - Q x_{-l-i}$	$(=,>)$ or $(>,=)$
$x_{li} - x_{-l-i}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li} - x_{-l-i} - \widehat{Q} x_{-li}$	$(<,<)$
$x_{li} - x_{-l-i} + \widehat{Q} x_{-li}$	$(>,>)$

$$\widehat{Q} := Q^{-1} - Q$$

Extra type D relations

	$(j \sim -i, m \sim -l)$
$x_{li}x_{mj} - q^{-1}x_{-m-j}x_{-l-i}$	$(=,<)$ or $(<,=)$
$x_{li}x_{mj} - qx_{-m-j}x_{-l-i}$	$(=,>)$ or $(>,=)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i} - \widehat{q}x_{-mi}x_{-lj}$	$(<,<)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i} + \widehat{q}x_{-mi}x_{-lj}$	$(>,>)$

$\widehat{q} := q^{-1} - q$

Isomorphism theorem for type B

Theorem 15 (Lai-Nakano-X. '19)

If $f^{\mathrm{B}}_{Q,q}(d)$ *is invertible in* k*, then there exists an algebra isomorphism*

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_{q}(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_{q}(\lfloor n/2 \rfloor, d-i).$$

Sketch of the proof of the isomorphism theorem for type B

Theorem 16 (Dipper-James '92)

There exists a Morita equivalence

$$\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \sim \bigoplus_{i=0}^{d} e_{i,d-i}\,\mathcal{H}^{\mathrm{B}}_{Q,q}(d)\,e_{i,d-i} \cong \bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_i) \otimes \mathcal{H}_q(\Sigma_{d-i}).$$

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For $e = e_{i,d-i}$,

$$V^{\otimes d} e \cong V_{\geq 0}^{\otimes i} \otimes V_{<0}^{\otimes d-i},$$

where $V_{\geq 0} := V_{>0} \oplus V_0$.

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For $e = e_{i,d-i}$,

$$V^{\otimes d} e \cong V_{\geq 0}^{\otimes i} \otimes V_{<0}^{\otimes d-i},$$

where $V_{\geq 0} := V_{>0} \oplus V_0$.

$$\mathrm{End}_{e\mathcal{H}^{\mathrm{B}}_{Q,q}e}(V^{\otimes d} e) \cong \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q}(V_{\geq 0}^{\otimes i}) \otimes \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q}(V_{<0}^{\otimes d-i}).$$

Isomorphism conjecture for type D

Conjecture 17

If $f_q^{\mathrm{D}}(d)$ is invertible in k and d is odd, then there exists an algebra isomorphism

$$S_q^{\mathrm{D}}(n,d) \cong \bigoplus_{i=0}^{\lceil d/2-1 \rceil} S_q^{\mathrm{A}}(\lceil n/2 \rceil, i) \otimes S_q^{\mathrm{A}}(\lfloor n/2 \rfloor, d-i).$$

Conjecture 18

If $f_q^{\mathrm{D}}(d)$ is invertible in k and d, n are even, then there exists an algebra isomorphism

$$\begin{aligned} S_q^{\mathrm{D}}(n,d) \cong & \bigoplus_{i=0}^{\lceil d/2-1 \rceil} S_q^{\mathrm{A}}(n/2, i) \otimes S_q^{\mathrm{A}}(n/2, d-i) \\ & \oplus \left(S_q^{\mathrm{A}}(n/2, d/2) \otimes S_q^{\mathrm{A}}(n/2, d/2) \right)^{\mathbb{Z}_2}. \end{aligned}$$

Corollary: Simple modules of q-Schur algebras

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i).$$

The simple modules of $S^{\mathrm{A}}_q(n,d)$ are indexed by partitions

$$\Lambda^{\mathrm{A}}(n,d) := \{\lambda \vdash d : \; \ell(\lambda) \leq n\}.$$

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$$\Lambda^{\mathrm{B}}(n,d) := \{(\lambda,\mu) \vdash d : \ \ell(\lambda) \le \lceil n/2 \rceil, \ell(\mu) \le \lfloor n/2 \rfloor)\}.$$

Corollary: Cellular structure

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i).$$

Theorem 19

The algebra $S^{\mathrm{A}}_q(n,d)$ is cellular with poset $\Lambda^{\mathrm{A}}(n,d)$ ordered by dominance order.

Corollary: Cellular structure

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i).$$

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Theorem 20 (Lai-Nakano-X. '19)

The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is cellular with poset $\Lambda^{\mathrm{B}}(n,d)$ ordered by dominance order.

Corollary: Representation type of q-Schur algebras

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i).$$

Let l be the order of q^2. Assume that $l \geq 2$ and $n \geq 5$.

Theorem 21 (Erdmann-Nakano '01)

The algebra $S^{\mathrm{A}}_q(n,d)$ is semisimple, finite but not semisimple, and wild if and only if $d < l$, $l \leq d < 2l$, and $2l \leq d$, respectively.

Theorem 22 (Lai-Nakano-X. '19)

The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is semisimple, finite but not semisimple, and wild if and only if $d < l$, $l \leq d < 2l$, and $2l \leq d$, respectively.

Corollary: Quasi-hereditary cover

Definition 23 (Rouquier '08)

A quasi-hereditary cover $F\colon A\text{-mod} \to B\text{-mod}$ is 1-faithful if

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(FM, FN) \text{ and } \mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_B(FM, FN)$$

for M and N admitting Δ-filtration.

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Theorem 24 (Hemmer-Nakano '04)

For $n \geq d$, the algebra $S_q^{\mathrm{A}}(n,d)$ is the quasi-hereditary 1-cover of $\mathcal{H}_q(\Sigma_d)$.

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Theorem 25 (Lai-Nakano-X. '19)

For $n \geq 2d$, the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is the quasi-hereditary 1-cover of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$.

Comparison between different type B Schur algebras

	QHA/ cellular	Schur functor	double cent'r prop	coord const'n	can'l bases
Orthogonal q-Schur algebra			✓	✓	
Sakamoto-Shoji's q-Schur algebra			✓		
Coideal q-Schur algebra	new	new	✓	new	✓
Cyclotomic q-Schur algebra	✓				
q-Schur^2 algebra	✓				
Rouquier's q-Schur algebra		✓			

Thanks for your attention.