Isomorphism theorem between q-Schur algebras of type B and type A

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Joint work with Chun-Ju Lai and Daniel K. Nakano.

Weyl groups and Iwahori-Hecke algebras of classical type

Let $G := \mathrm{Aut}(\{\pm 1, \ldots, \pm d\})$.

1. Type A: $W^{\mathrm{A}} \subset G$. $s_i := (i, i+1)(-i, -i-1)$.
2. Type B: $W^{\mathrm{B}} \subset G$. $s_0^{\mathrm{B}} := (1, -1)$.
3. Type D: $W^{\mathrm{D}} \subset G$. $s_0^{\mathrm{D}} := (1, -1)(2, -2) = s_0^{\mathrm{B}} s_1 s_0^{\mathrm{B}}$.

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$$W^{\mathrm{A}} \subseteq W^{\mathrm{D}} \subseteq W^{\mathrm{B}} \subseteq G.$$

$$\mathcal{H}_q^{\mathrm{A}} \subseteq \mathcal{H}_q^{\mathrm{D}} \subseteq \mathcal{H}_{Q=1,q}^{\mathrm{B}}.$$

Tensor space

For $n = 2r + \epsilon$, let

$$V := V_{>0} \oplus V_0 \oplus V_{<0},$$

where

$$V_{>0} \cong k^r, \quad V_0 \cong k^\epsilon, \quad \text{and} \quad V_{<0} \cong k^r.$$

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The tensor space

$$V^{\otimes d}$$

admits an action of the type B Weyl group $\langle s_0^{\mathrm{B}}, s_1, \ldots, s_{d-1} \rangle$, hence actions of W^Φ for $\Phi \in \{A_{d-1}, B_d, D_d\}$.

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The tensor space $V^{\otimes d}$ is a $\mathcal{H}_{\mathbf{q}}^\Phi$-module, for $\Phi \in \{A_{d-1}, B_d, D_d\}$.

q-Schur algebra of classical type

Definition 1

The q-Schur algebra of type Φ is

$$S^{\Phi}_{\mathbf{q}}(n,d) := \mathrm{End}_{\mathcal{H}^{\Phi}_{\mathbf{q}}}(V^{\otimes d}).$$

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$$\begin{tikzcd}
S_q^{\mathrm{A}} \arrow[dr, bend left] & & \mathcal{H}_q^{\mathrm{A}} \arrow[dl, bend right] \arrow[d, hook] \\
S_q^{\mathrm{D}} \arrow[u, hook] \arrow[r, bend left] & V^{\otimes d} & \mathcal{H}_q^{\mathrm{D}} \arrow[l, bend right] \arrow[d, hook] \\
S_{Q=1,q}^{\mathrm{B}} \arrow[u, hook] \arrow[ur, bend left] & & \mathcal{H}_{Q=1,q}^{\mathrm{B}} \arrow[ul, bend right]
\end{tikzcd}$$

Morita equivalences between Iwahori-Hecke algebras of classical type

From now on, assume that $f_d(Q,q) := \prod_{i=1-d}^{d-1}(Q^{-2}+q^{2i})$ is invertible in k.

Theorem 2 (Dipper-James '92, Pallikaros '94, Hu '02)

$$\mathcal{H}_{Q,q}^{\mathrm{B}_d} \sim_{\textit{Morita}} \bigoplus_{i=0}^{d} \mathcal{H}_q^{\mathrm{A}_{i-1}} \otimes \mathcal{H}_q^{\mathrm{A}_{d-i-1}}.$$

$$\mathcal{H}_q^{\mathrm{D}_d} \sim_{\textit{Morita}} \mathcal{A}_{d/2} \oplus \bigoplus_{i=0}^{\lceil d/2-1 \rceil} \mathcal{H}_q^{\mathrm{A}_{i-1}} \otimes \mathcal{H}_q^{\mathrm{A}_{d-i-1}}.$$

where $\mathcal{A}_{d/2} = 0$ if d is odd, and
$\mathcal{A}_{d/2}//\left(\mathcal{H}_q^{\mathrm{A}_{d/2-1}} \otimes \mathcal{H}_q^{\mathrm{A}_{d/2-1}}\right) \cong k\mathbb{Z}_2$ if d is even.

Isomorphism theorem for q-Schur algebras of type B

Theorem 3 (Lai-Nakano-X. '19)

There exists an algebra isomorphism

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_{q}(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_{q}(\lfloor n/2 \rfloor, d-i).$$

Sketch of the proof of the isomorphism theorem for type B

For $e = e_{i,d-i}$ such that

$$e\,\mathcal{H}^{\mathrm{B}}\,e \cong \mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{A}}\,.$$

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Calculation shows that

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where $V_{\geq 0} := V_{>0} \oplus V_0$.

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$$\mathrm{End}_{e\,\mathcal{H}^{\mathrm{B}}_{Q,q}\,e}(V^{\otimes d} e) \cong \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q}(V_{\geq 0}^{\otimes i}) \otimes \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q}(V_{<0}^{\otimes d-i}).$$

Structure of q-Schur algebras of type D

Conjecture 4

$$S_q^{\mathrm{D}}(n,d) \sim_{\mathsf{Morita}} M_2\left(S_q^{\mathrm{A}}(\lceil n/2\rceil, d/2) \otimes S_q^{\mathrm{A}}(\lfloor n/2\rfloor, d/2)\right)^{\mathbb{Z}_2} \oplus \bigoplus_{i=0}^{\lceil d/2-1\rceil} S_q^{\mathrm{A}}(\lceil n/2\rceil, i) \otimes S_q^{\mathrm{A}}(\lfloor n/2\rfloor, d-i).$$

Corollary: Simple modules of q-Schur algebras

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i).$$

The simple modules of $S^{\mathrm{A}}_q(n,d)$ are indexed by partitions

$$\Lambda^{\mathrm{A}}(n,d) := \{\lambda \vdash d : \ \ell(\lambda) \leq n\}.$$

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$$\Lambda^{\mathrm{B}}(n,d) := \{(\lambda,\mu) \vdash d : \ \ell(\lambda) \le \lceil n/2\rceil, \ell(\mu) \le \lfloor n/2\rfloor)\}.$$

Corollary: Cellular structure

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i).$$

Theorem 5

The algebra $S^{\mathrm{A}}_q(n,d)$ is cellular with poset $\Lambda^{\mathrm{A}}(n,d)$ ordered by dominance order.

Corollary: Cellular structure

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i).$$

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Theorem 6 (Lai-Nakano-X. '19)

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Corollary: Representation type of q-Schur algebras

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_{q}(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_{q}(\lfloor n/2 \rfloor, d-i).$$

Let l be the order of q^2. Assume that $l \geq 2$ and $n \geq 5$.

Theorem 7 (Erdmann-Nakano '01)

The algebra $S^{\mathrm{A}}_{q}(n,d)$ is semisimple, finite but not semisimple, and wild if and only if $d < l$, $l \leq d < 2l$, and $2l \leq d$, respectively.

Theorem 8 (Lai-Nakano-X. '19)

The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is semisimple, finite but not semisimple, and wild if and only if $d < l$, $l \leq d < 2l$, and $2l \leq d$, respectively.

Corollary: Quasi-hereditary cover

Definition 9 (Rouquier '08)

A quasi-hereditary cover $F\colon A\text{-mod} \to B\text{-mod}$ is 1-faithful if

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_B(FM,FN) \text{ and } \mathrm{Ext}^1_A(M,N) \cong \mathrm{Ext}^1_B(FM,FN)$$

for M and N admitting Δ-filtration.

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Theorem 10 (Hemmer-Nakano '04)

For $n \geq d$, the algebra $S_q^{\mathrm{A}}(n,d)$ is the quasi-hereditary 1-cover of $\mathcal{H}_q(\Sigma_d)$.

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Theorem 11 (Lai-Nakano-X. '19)

For $n \geq 2d$, the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is the quasi-hereditary 1-cover of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$.

Comparison between different type B Schur algebras

	QHA/ cellular	Schur functor	double cent'r prop	coord const'n	can'l bases
Orthogonal q-Schur algebra			✓	✓	
Sakamoto-Shoji's q-Schur algebra			✓		
Coideal q-Schur algebra	new	new	✓	new	✓
Cyclotomic q-Schur algebra	✓				
q-Schur^2 algebra	✓				
Rouquier's q-Schur algebra		✓			

Thanks for your attention.