

Mathematica code to solve a certain degree 10 Diophantine equation in 3 variables under some conditions - Ziqing Xiang

This is the Mathematica code for the joint work with Eiichi Bannai, Etsuko Bannai, Wei-Hsuan Yu and Yan Zhu entitled "Classification of spherical 2-distance {4, 2, 1}-design".

Section 3

Definition of Gegenbauer polynomial $Q_{n,4}(\xi)$.

```
In[1]:= Q[n_, 4, ξ_] := n (n + 6) / 24 ( (n^2 + 6 n + 8) ξ^4 - 6 (n + 2) ξ^2 + 3 ) ;
```

Definition of F . Eq. (3.3).

```
In[2]:= F = Factor[Q[n, 4, 1] + k Q[n, 4, y / k] + (v - k - 1) Q[n, 4, (-y - 1) / (v - k - 1)] / .  
v → 1 / μ (k - x) (k - y) / . k → μ - x y / . n → (μ - x y) (μ - x y - x) (x + 1) / μ / (x - y) ]
```

```
Out[2]= ((x + x y - μ) (y + x y - μ))^2  
(x^2 y + x^3 y + x^2 y^2 + x^3 y^2 + 5 x μ - x^2 μ - 6 y μ - 2 x y μ - 2 x^2 y μ + μ^2 + x μ^2)  
(x^2 y + x^3 y + x^2 y^2 + x^3 y^2 + 3 x μ - x^2 μ - 4 y μ - 2 x y μ - 2 x^2 y μ + μ^2 + x μ^2)  
(x^5 y^2 + 2 x^6 y^2 + x^7 y^2 - x^2 y^3 - 2 x^3 y^3 - x^4 y^3 + x^5 y^3 + 2 x^6 y^3 + x^7 y^3 - x^2 y^4 - 2 x^3 y^4 -  
x^4 y^4 - 5 x^4 y μ - 7 x^5 y μ - 2 x^6 y μ - x y^2 μ - 2 x^2 y^2 μ + 3 x^3 y^2 μ - x^4 y^2 μ - 8 x^5 y^2 μ -  
3 x^6 y^2 μ + 2 y^3 μ + 4 x y^3 μ - x^2 y^3 μ - 3 x^3 y^3 μ + 6 x^3 μ^2 + 5 x^4 μ^2 + x^5 μ^2 - 2 x y μ^2 -  
7 x^2 y μ^2 + 6 x^3 y μ^2 + 10 x^4 y μ^2 + 3 x^5 y μ^2 + 3 y^2 μ^2 + 8 x y^2 μ^2 + 3 x^2 y^2 μ^2 - 3 x μ^3 -  
6 x^2 μ^3 - 4 x^3 μ^3 - x^4 μ^3 + y μ^3 + x y μ^3) / (24 (1 + x)^2 (x - y)^4 (x y - μ)^2 μ^4)
```

The four factors F_0, F_1, F_2, F_3 of F . Eqs. (3.4) - (3.7).

```

In[3]:= F0 = (μ - x - x y) (μ - y - x y)2 / (24 μ4 (1 + x)2 (x - y)4 (μ - x y)2);
F0
F1 = μ2 + 5 μ x + μ2 x - μ x2 - 6 μ y - 2 μ x y + x2 y - 2 μ x2 y + x3 y + x2 y2 + x3 y2;
Collect[F1, {μ, y}, Factor]
F2 = μ2 + 3 μ x + μ2 x - μ x2 - 4 μ y - 2 μ x y + x2 y - 2 μ x2 y + x3 y + x2 y2 + x3 y2;
Collect[F2, {μ, y}, Factor]
F3 = 3 μ3 x + 6 μ3 x2 - 6 μ2 x3 + 4 μ3 x3 - 5 μ2 x4 + μ3 x4 - μ2 x5 - μ3 y +
    2 μ2 x y - μ3 x y + 7 μ2 x2 y - 6 μ2 x3 y + 5 μ x4 y - 10 μ2 x4 y + 7 μ x5 y - 3 μ2 x5 y +
    2 μ x6 y - 3 μ2 y2 + μ x y2 - 8 μ2 x y2 + 2 μ x2 y2 - 3 μ2 x2 y2 - 3 μ x3 y2 + μ x4 y2 -
    x5 y2 + 8 μ x5 y2 - 2 x6 y2 + 3 μ x6 y2 - x7 y2 - 2 μ y3 - 4 μ x y3 + x2 y3 + μ x2 y3 +
    2 x3 y3 + 3 μ x3 y3 + x4 y3 - x5 y3 - 2 x6 y3 - x7 y3 + x2 y4 + 2 x3 y4 + x4 y4;
Collect[F3, {μ, y}, Factor]

Out[4]= ((-x - x y + μ) (-y - x y + μ)2) / (24 (1 + x)2 (x - y)4 μ4 (-x y + μ)2)

Out[6]= x2 (1 + x) y + x2 (1 + x) y2 + (-(-5 + x) x - 2 (3 + x + x2) y) μ + (1 + x) μ2

Out[8]= x2 (1 + x) y + x2 (1 + x) y2 + (-(-3 + x) x - 2 (2 + x + x2) y) μ + (1 + x) μ2

Out[10]= -x5 (1 + x)2 y2 - (-1 + x) x2 (1 + x)2 (1 + x + x2) y3 + x2 (1 + x)2 y4 +
    (x4 (1 + x) (5 + 2 x) y + x (1 + x) (1 + x - 4 x2 + 5 x3 + 3 x4) y2 + (1 + x) (-2 - 2 x + 3 x2) y3) μ +
    (-x3 (2 + x) (3 + x) - x (-2 - 7 x + 6 x2 + 10 x3 + 3 x4) y + (-3 - 8 x - 3 x2) y2) μ2 +
    (x (1 + x) (3 + 3 x + x2) + (-1 - x) y) μ3

```

Verify the factorization.

```

In[11]:= Simplify[F0 F1 F2 F3 - F]
Out[11]= 0

```

Section 5.2

Alternative definition of n and v . Eqs. (5.1) and (5.2).

```

In[12]:= n = (x + 1) (μ - x y) (μ - x y - x) / (μ (-y + x));
v = n + 1 - (y n + μ - x y) / x;

```

Step 1

Assumption on y in Step 1. Eq. (5.3).

```

In[14]:= yassum = y ≤ -(2 x^3 + 3 x^2 + 3 x + 2) || -(2 x^3 + 3 x^2 - 3 x - 3) ≤ y ≤ -1;

```

The computer proof for Step 1.

```

In[15]:= Simplify[v > n (n + 3) / 2 || F3 > 0, x ≥ 1 && μ ≥ 1 && yassum]
Out[15]= True

```

Step 2

Definition of a . Eq. (5.4).

In[16]:= **usub** = $\mu \rightarrow -(x + a) y$;

Definition of G_1 .

In[17]:= **G1** = **Factor**[(**F3** /. **usub**) / **y**²];
Collect[**G1**, {**y**, **x**}, **Factor**]

Out[18]= $-6 a^2 x^3 - a (17 + 5 a) x^4 + (-12 - 17 a - a^2) x^5 - 2 (7 + 2 a) x^6 - 4 x^7 +$
 $(-a (1 - 2 a + 3 a^2) x - 2 a (-1 + a + 3 a^2) x^2 - 2 (-1 - 4 a + 12 a^2 + 2 a^3) x^3 +$
 $(8 - 31 a - 22 a^2 - a^3) x^4 - 2 (7 + 20 a + 3 a^2) x^5 - 12 (2 + a) x^6 - 8 x^7) y +$
 $((-2 + a) (-1 + a) a + (2 - 2 a - 5 a^2 + a^3) x - 2 (-1 + 7 a) x^2 - 6 (1 + a) x^3 - 4 x^4) y^2$

The computer proof for Step 2(a).

In[19]:= **Simplify**[(**G1** /. **y** \rightarrow 0) < 0, 2 ≤ **x** && -**x** < **a**]

Out[19]= True

The computer proof for Step 2(b).

In[20]:= **Simplify**[(**G1** /. **y** \rightarrow -1) > 0, 2 ≤ **x** && -**x** < **a**]

Out[20]= True

The computer proof for Step 2(c).

In[21]:= **Simplify**[(**G1** /. **y** \rightarrow -(2 **x**³ + 3 **x**² + 3 **x** + 2)) > 0, 2 ≤ **x** && (-**x** < **a** ≤ -1 || 3 ≤ **a**)]

Out[21]= True

Step 3

Definition of b . Eq. (5.11).

In[22]:= **ysub** = **y** \rightarrow - $\left(2 x^3 + 3 x^2 + \frac{3}{2} (-1 + a) a x - \frac{3}{2} (-1 + a)^2 a + \frac{1}{4 x} 3 (-1 + a) a (2 - 4 a + 3 a^2) - \right.$
 $\left. \frac{1}{4 x^2} 3 (-1 + a) a^2 (3 - 6 a + 4 a^2) + \frac{1}{8 x^3} 3 (-1 + a) a (5 - 9 a + 16 a^2 - 20 a^3 + 11 a^4) + \frac{b}{x^4} \right)$;

Definition of G_2 .

```
In[23]:= G2 = Factor[G1 /. ysub];
Collect[G2, x, Factor]
```

$$\begin{aligned} \text{Out[24]} = & \frac{1}{8} \left(-150 a + 336 a^2 - 330 a^3 + 738 a^4 - 1854 a^5 + 3291 a^6 - 4305 a^7 + 3603 a^8 - 1563 a^9 + \right. \\ & \left. 234 a^{10} + 96 b - 248 a b - 176 a^2 b - 8 a^3 b \right) + \frac{(-2+a)(-1+a) a b^2}{x^8} + \frac{1}{4 x^7} b \\ & \left(-30 a^2 + 129 a^3 - 291 a^4 + 483 a^5 - 585 a^6 + 453 a^7 - 192 a^8 + 33 a^9 + 8 b - 8 a b - 20 a^2 b + 4 a^3 b \right) + \\ & \frac{1}{64 x^6} \left(450 a^3 - 3195 a^4 + 12033 a^5 - 32382 a^6 + 67608 a^7 - 112005 a^8 + 149265 a^9 - 160236 a^{10} + \right. \\ & \left. 135054 a^{11} - 84645 a^{12} + 36369 a^{13} - 9405 a^{14} + 1089 a^{15} - 480 a b + 1824 a^2 b - 1968 a^3 b - \right. \\ & \left. 336 a^4 b + 5040 a^5 b - 10320 a^6 b + 10224 a^7 b - 4512 a^8 b + 528 a^9 b + 128 b^2 - 896 a b^2 \right) + \\ & \frac{1}{64 x^5} 3 \left(150 a^2 - 990 a^3 + 2781 a^4 - 4233 a^5 + 1572 a^6 + 10824 a^7 - 34845 a^8 + 62355 a^9 - \right. \\ & \left. 78174 a^{10} + 71370 a^{11} - 45939 a^{12} + 19155 a^{13} - 4389 a^{14} + 363 a^{15} - 160 a b + 1632 a^2 b - \right. \\ & \left. 4128 a^3 b + 6400 a^4 b - 7360 a^5 b + 5152 a^6 b - 1408 a^7 b - 128 a^8 b - 128 b^2 - 128 a b^2 \right) + \\ & \frac{1}{32 x^4} \left(225 a^2 - 3015 a^3 + 13878 a^4 - 38736 a^5 + 78291 a^6 - 119925 a^7 + 140382 a^8 - \right. \\ & \left. 125874 a^9 + 84177 a^{10} - 37521 a^{11} + 7479 a^{12} + 1431 a^{13} - 792 a^{14} + 528 a b - \right. \\ & \left. 432 a^2 b - 1392 a^3 b + 3984 a^4 b - 5664 a^5 b + 4416 a^6 b - 1440 a^7 b - 128 b^2 \right) - \\ & \frac{1}{32 x^3} a \left(315 a - 477 a^2 - 3546 a^3 + 19206 a^4 - 51723 a^5 + 95265 a^6 - 129942 a^7 + \right. \\ & \left. 133326 a^8 - 100431 a^9 + 52119 a^{10} - 16497 a^{11} + 2385 a^{12} - 128 b - 32 a b + 96 a^2 b \right) + \\ & \frac{1}{16 x^2} a \left(105 a - 1098 a^2 + 4506 a^3 - 12120 a^4 + 24231 a^5 - 36246 a^6 + 40053 a^7 - \right. \\ & \left. 31977 a^8 + 17478 a^9 - 5805 a^{10} + 873 a^{11} + 256 b - 64 a b - 96 a^2 b \right) + \\ & \frac{1}{16 x} \left(-408 a^2 + 708 a^3 + 1134 a^4 - 6750 a^5 + 15636 a^6 - 22548 a^7 + 20841 a^8 - \right. \\ & \left. 12051 a^9 + 4077 a^{10} - 639 a^{11} + 160 b + 192 a b - 384 a^2 b - 64 a^3 b \right) + \\ & \frac{1}{8} \left(-180 a + 1089 a^2 - 2388 a^3 + 3714 a^4 - 4506 a^5 + 3417 a^6 - \right. \\ & \left. 1404 a^7 + 462 a^8 - 204 a^9 - 112 b - 320 a b - 48 a^2 b \right) x - \\ & \frac{3}{4} \left(-15 a - 82 a^2 + 343 a^3 - 648 a^4 + 814 a^5 - 598 a^6 + 178 a^7 + 8 a^8 + 32 b + 16 a b \right) x^2 + \\ & \frac{1}{2} \left(30 a - 27 a^2 - 84 a^3 + 249 a^4 - 354 a^5 + 276 a^6 - 90 a^7 - 16 b \right) x^3 \end{aligned}$$

The computer proof for Step 3(a).

```
In[25]:= Simplify[(G2 /. b -> -3994) > 0, 90 <= x && -1 <= a <= 3]
```

```
Out[25]= True
```

The computer proof for Step 3(b).

```
In[26]:= Simplify[(G2 /. b -> 64) < 0, 90 <= x && -1 <= a <= 3]
```

```
Out[26]= True
```

Step 4

Definition of m^2 . Eq. (5.12).

```
In[27]:= mmsub = mm → n - (4 x^2 + 4 x - 2) ;
```

Definition of G_3 .

```
In[28]:= G3 = Factor[mm /. mmsub /. usub /. ysub] ;
G3
```

```
Out[29]= (16 a b + 8 a^2 b - 30 a^2 x + 69 a^3 x - 108 a^4 x + 141 a^5 x - 78 a^6 x - 27 a^7 x + 33 a^8 x + 16 b x +
8 a^2 b x - 30 a x^2 + 84 a^2 x^2 - 129 a^3 x^2 + 168 a^4 x^2 - 195 a^5 x^2 + 186 a^6 x^2 - 117 a^7 x^2 +
33 a^8 x^2 + 12 a^2 x^3 - 48 a^3 x^3 + 90 a^4 x^3 - 108 a^5 x^3 + 78 a^6 x^3 - 24 a^7 x^3 - 24 a x^4 +
48 a^2 x^4 - 60 a^3 x^4 + 72 a^4 x^4 - 54 a^5 x^4 + 18 a^6 x^4 - 16 a x^5 + 24 a^2 x^5 - 24 a^3 x^5 +
36 a^4 x^5 - 12 a^5 x^5 - 16 a x^6 + 48 a^2 x^6 - 12 a^3 x^6 + 12 a^4 x^6 + 40 a^2 x^7 + 16 a^2 x^8) /
((a + x) (8 b - 15 a x + 42 a^2 x - 75 a^3 x + 108 a^4 x - 93 a^5 x + 33 a^6 x + 18 a^2 x^2 -
54 a^3 x^2 + 60 a^4 x^2 - 24 a^5 x^2 - 12 a x^3 + 36 a^2 x^3 - 42 a^3 x^3 + 18 a^4 x^3 -
12 a x^4 + 24 a^2 x^4 - 12 a^3 x^4 + 8 x^5 - 12 a x^5 + 12 a^2 x^5 + 24 x^6 + 16 x^7))
```

Definition of \tilde{m}^2 . Eq. (5.12).

```
In[30]:= mmtsub = mmt → a^2 - \frac{(-1+a) a^2}{x} + \frac{(-1+a) a (1+a^2)}{x^2} -
\frac{1}{2 x^3} (-1+a) a (1+2 a+2 a^3) + \frac{1}{4 x^4} (-1+a) a (7-a+4 a^2+4 a^4) + \frac{c}{x^5} ;
```

Definition of G_4 .

```
In[31]:= G4 = Factor[mmt /. mmtsub] ;
Collect[G4, x, Factor]
```

```
Out[32]= a^2 + \frac{c}{x^5} + \frac{1}{4 x^4} (-1+a) a (7-a+4 a^2+4 a^4) - \frac{1}{2 x^3}
(-1+a) a (1+2 a+2 a^3) + \frac{(-1+a) a (1+a^2)}{x^2} - \frac{(-1+a) a^2}{x}
```

The computer proof for Step 4(a).

```
In[33]:= Simplify[G3 > (G4 /. c → -1620), 90 ≤ x && -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64]
```

```
Out[33]= True
```

The computer proof for Step 4(b).

```
In[34]:= Simplify[G3 < (G4 /. c → 3), 90 ≤ x && -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64]
```

```
Out[34]= True
```

The computer proof for Step 4(c).

```
In[35]:= Simplify[G4 < 9, 90 ≤ x && -1 ≤ a ≤ 3 && -1620 ≤ c ≤ 3]
```

```
Out[35]= True
```

Step 5

Definition of \tilde{n} . Eq. (5.16).

```
In[36]:= nt = mmt + (4 x^2 + 4 x - 2);
```

```
Collect[nt /. usub /. ysub /. mmtsub, x, Factor]
```

$$\text{Out[37]} = -2 + a^2 + \frac{c}{x^5} + \frac{1}{4 x^4} (-1 + a) a (7 - a + 4 a^2 + 4 a^4) - \frac{1}{2 x^3} \\ (-1 + a) a (1 + 2 a + 2 a^3) + \frac{(-1 + a) a (1 + a^2)}{x^2} - \frac{(-1 + a) a^2}{x} + 4 x + 4 x^2$$

Definition of \tilde{v} . Eq. (5.16).

```
In[38]:= vt = nt + 1 - (y nt + μ - x y) / x;
```

```
Collect[vt /. usub /. ysub /. mmtsub, x, Factor]
```

$$\text{Out[39]} = \frac{1}{2} (-1 + a) (2 + 18 a - 39 a^2 + 25 a^3) + \frac{b c}{x^{10}} + \frac{1}{8 x^9} \\ (-1 + a) a (14 b - 2 a b + 8 a^2 b + 8 a^4 b + 15 c - 27 a c + 48 a^2 c - 60 a^3 c + 33 a^4 c) + \\ \frac{1}{32 x^8} (-1 + a) a (-105 a + 309 a^2 - 627 a^3 + 999 a^4 - 1119 a^5 + 924 a^6 - 705 a^7 + \\ 564 a^8 - 372 a^9 + 132 a^{10} - 16 b - 32 a b - 32 a^3 b - 72 a c + 144 a^2 c - 96 a^3 c) - \\ \frac{1}{16 x^7} (-1 + a) a (-15 a - 51 a^2 + 207 a^3 - 345 a^4 + 429 a^5 - 471 a^6 + 438 a^7 - \\ 306 a^8 + 114 a^9 - 16 b - 16 a^2 b - 24 c + 48 a c - 36 a^2 c) + \frac{1}{16 x^6} (-1 + a) a \\ (-72 a + 198 a^2 - 351 a^3 + 504 a^4 - 585 a^5 + 546 a^6 - 390 a^7 + 150 a^8 - 16 a b + 24 c - 24 a c) + \\ \frac{1}{8 x^5} (15 a^2 - 87 a^3 + 222 a^4 - 375 a^5 + 489 a^6 - 483 a^7 + 306 a^8 - \\ 87 a^9 - 16 b - 8 a b + 8 a^2 b + 8 c - 12 a c + 12 a^2 c) + \frac{1}{8 x^4} \\ (16 a - 26 a^2 - 4 a^3 + 74 a^4 - 254 a^5 + 425 a^6 - 330 a^7 + 99 a^8 + 16 b + 24 c) + \frac{1}{4 x^3} \\ (-34 a + 53 a^2 - 73 a^3 + 157 a^4 - 217 a^5 + 147 a^6 - 33 a^7 + 16 b + 8 c) + \\ \frac{1}{4 x^2} (-1 + a) a (30 - 74 a + 156 a^2 - 195 a^3 + 95 a^4) - \\ \frac{1}{2 x} (-1 + a) a (-4 + 33 a - 60 a^2 + 34 a^3) - \\ 2 (1 + 6 a - 10 a^2 + 4 a^3) x + 2 (3 - 4 a + 4 a^2) x^2 + 16 x^3 + 8 x^4$$

Definition of \tilde{z} . Eq. (5.15).

```
In[40]:= zt = 144 mmt - (3 vt + (8 y + 4 x^3 + 6 x^2 + 3) (2 x + 1) - 3 / 2 mmt (mmt - 7)) ^ 2;
```

Definition of G_5 .

```
In[41]:= G5 = Factor[zt /. usub /. ysub /. mmtsub];
```

```
Collect[G5, x, Factor]
```

$$\begin{aligned}
\text{Out[42]} = & -\frac{9(2b-c)^2c^2}{4x^{20}} - \frac{1}{8x^{19}} \\
& 9(-1+a)a(2b-c)c(14b-2ab+8a^2b+8a^4b+c-25ac+40a^2c-60a^3c+25a^4c) - \\
& \frac{1}{64x^{18}}9(-1+a)a(-196ab^2+252a^2b^2-284a^3b^2+260a^4b^2-320a^5b^2+320a^6b^2- \\
& 160a^7b^2+128a^8b^2-64a^9b^2+64a^{10}b^2-252abc+1716a^2bc-4164a^3bc+ \\
& 7212a^4bc-7992a^5bc+6432a^6bc-5160a^7bc+4128a^8bc-2784a^9bc+ \\
& 864a^{10}bc-64b^2c-128ab^2c-128a^3b^2c+111a^2c^2-441a^2c^2+357a^3c^2+957a^4c^2- \\
& 4692a^5c^2+9012a^6c^2-10320a^7c^2+7536a^8c^2-2913a^9c^2+393a^{10}c^2+96b^2c^2- \\
& 96abc^2+576a^2bc^2-192a^3bc^2-32c^3+80ac^3-288a^2c^3+128a^3c^3) - \frac{1}{128x^{17}} \\
& 9(-1+a)a(784a^2b-4340a^3b+12280a^4b-24880a^5b+37466a^6b-44644a^7b+ \\
& 47092a^8b-46760a^9b+42682a^{10}b-33024a^{11}b+21744a^{12}b-13584a^{13}b+8032a^{14}b- \\
& 3776a^{15}b+928a^{16}b+224ab^2+192a^2b^2-352a^3b^2+640a^4b^2-640a^5b^2+448a^6b^2- \\
& 512a^7b^2+256a^8b^2-256a^9b^2+56a^2c-1702a^3c+10592a^4c-36980a^5c+90823a^6c- \\
& 166478a^7c+236954a^8c-269872a^9c+253478a^{10}c-208167a^{11}c+156981a^{12}c- \\
& 104985a^{13}c+55160a^{14}c-18760a^{15}c+2900a^{16}c-192abc+1056a^2bc-5568a^3bc+ \\
& 9120a^4bc-11808a^5bc+13728a^6bc-12480a^7bc+9024a^8bc-2880a^9bc+256b^2c+ \\
& 256a^2b^2c-24a^2c^2+144a^2c^2-744a^3c^2+5568a^4c^2-13272a^5c^2+18816a^6c^2- \\
& 16392a^7c^2+6720a^8c^2-816a^9c^2-768abc^2+192a^2bc^2-64c^3+384ac^3-160a^2c^3) - \\
& \frac{1}{1024x^{16}}9(-1+a)a(-3136a^3+30688a^4-150340a^5+500948a^6-1263192a^7+ \\
& 2529356a^8-4161776a^9+5783596a^{10}-6963353a^{11}+7472989a^{12}-7316706a^{13}+ \\
& 6562942a^{14}-5318713a^{15}+3846233a^{16}-2501848a^{17}+1485376a^{18}-783224a^{19}+ \\
& 332752a^{20}-96048a^{21}+13456a^{22}-2016a^2b-2400a^3b+44928a^4b-129024a^5b+ \\
& 251232a^6b-393504a^7b+501888a^8b-536832a^9b+475392a^{10}b-368256a^{11}b+ \\
& 264576a^{12}b-165888a^{13}b+79872a^{14}b-19968a^{15}b-3840ab^2+3328a^2b^2-6144a^3b^2+ \\
& 6144a^4b^2-5632a^5b^2+6144a^6b^2-3072a^7b^2+3072a^8b^2+1776a^2c-14352a^3c+ \\
& 84096a^4c-326784a^5c+838512a^6c-1584624a^7c+2308416a^8c-2699328a^9c+ \\
& 2704080a^{10}c-2374080a^{11}c+1745712a^{12}c-972480a^{13}c+343104a^{14}c-54048a^{15}c- \\
& 6912abc+40704a^2bc-71424a^3bc+110592a^4bc-132864a^5bc+121344a^6bc- \\
& 90624a^7bc+29184a^8bc-2048ab^2c+2304a^2c^2-5376a^2c^2-16704a^3c^2+61632a^4c^2- \\
& 113280a^5c^2+112896a^6c^2-45504a^7c^2+4032a^8c^2+3072b^2c^2-1536c^3+512ac^3) + \\
& \frac{1}{256x^{15}}9(56a^4+1826a^5-23218a^6+124074a^7-421139a^8+1056011a^9-2097493a^{10}+ \\
& 3441405a^{11}-4827044a^{12}+5935014a^{13}-6470053a^{14}+6260053a^{15}-5372693a^{16}+ \\
& 4113699a^{17}-2830258a^{18}+1727606a^{19}-888846a^{20}+350896a^{21}-91264a^{22}+ \\
& 11368a^{23}+3264a^3b-19776a^4b+57528a^5b-122136a^6b+209760a^7b- \\
& 295968a^8b+355560a^9b-367752a^{10}b+331104a^{11}b-266304a^{12}b+187872a^{13}b- \\
& 108864a^{14}b+44736a^{15}b-9024a^{16}b+256a^2b^2+896a^3b^2-2432a^4b^2+2688a^5b^2- \\
& 2944a^6b^2+2560a^7b^2-2048a^8b^2+1024a^9b^2+600a^3c-11328a^4c+67740a^5c- \\
& 230532a^6c+553704a^7c-1006416a^8c+1442844a^9c-1715388a^{10}c+ \\
& 1740072a^{11}c-1485864a^{12}c+1014888a^{13}c-503712a^{14}c+154584a^{15}c- \\
& 21192a^{16}c-2688a^2bc+8448a^3bc-21120a^4bc+39936a^5bc-53760a^6bc+ \\
& 54144a^7bc-33024a^8bc+8064a^9bc+1024b^2c+512ab^2c-512a^2b^2c+
\end{aligned}$$

$$\begin{aligned}
& 384 a^2 c^2 + 1632 a^3 c^2 - 9408 a^4 c^2 + 24192 a^5 c^2 - 34656 a^6 c^2 + 23712 a^7 c^2 - \\
& 5664 a^8 c^2 - 192 a^9 c^2 - 2816 b c^2 + 512 a b c^2 + 1152 c^3 - 384 a c^3 + 128 a^2 c^3) - \\
& \frac{1}{256 x^{14}} 3 (7059 a^4 - 73776 a^5 + 377724 a^6 - 1308618 a^7 + 3461262 a^8 - 7342962 a^9 + \\
& 12926181 a^{10} - 19465224 a^{11} + 25597959 a^{12} - 29696976 a^{13} + 30509715 a^{14} - \\
& 27817716 a^{15} + 22604118 a^{16} - 16375854 a^{17} + 10375986 a^{18} - 5467650 a^{19} + \\
& 2193888 a^{20} - 577776 a^{21} + 72660 a^{22} - 4272 a^3 b + 29808 a^4 b - 128160 a^5 b + 339744 a^6 b - \\
& 646128 a^7 b + 970128 a^8 b - 1164480 a^9 b + 1163520 a^{10} b - 1018368 a^{11} b + 770784 a^{12} b - \\
& 472320 a^{13} b + 200448 a^{14} b - 40704 a^{15} b + 5376 a b^2 - 2688 a^2 b^2 - 2688 a^3 b^2 + \\
& 4224 a^4 b^2 - 5760 a^5 b^2 + 5376 a^6 b^2 - 7680 a^7 b^2 + 3840 a^8 b^2 - 6456 a^3 c + 59544 a^4 c - \\
& 272736 a^5 c + 848544 a^6 c - 1896840 a^7 c + 3242952 a^8 c - 4522656 a^9 c + 5244480 a^{10} c - \\
& 4957632 a^{11} c + 3638136 a^{12} c - 1887336 a^{13} c + 591336 a^{14} c - 81336 a^{15} c - \\
& 18048 a b c + 12672 a^2 b c - 5376 a^3 b c + 24192 a^4 b c - 87936 a^5 b c + 154752 a^6 b c - \\
& 106752 a^7 b c + 26496 a^8 b c - 1024 b^2 c + 2304 a c^2 + 21600 a^2 c^2 - 69984 a^3 c^2 + \\
& 132192 a^4 c^2 - 142848 a^5 c^2 + 65232 a^6 c^2 - 6048 a^7 c^2 - 2448 a^8 c^2 + 5120 b c^2 - 2304 c^3) + \\
& \frac{1}{128 x^{13}} 3 (-3066 a^4 + 33624 a^5 - 193959 a^6 + 740637 a^7 - 2062356 a^8 + 4508097 a^9 - \\
& 8120880 a^{10} + 12374793 a^{11} - 16192152 a^{12} + 18363036 a^{13} - 18183174 a^{14} + \\
& 15854712 a^{15} - 12192060 a^{16} + 8102841 a^{17} - 4425729 a^{18} + 1821084 a^{19} - 487200 a^{20} + \\
& 61752 a^{21} - 2856 a^2 b - 1344 a^3 b + 7776 a^4 b + 11088 a^5 b - 109464 a^6 b + 299040 a^7 b - \\
& 461568 a^8 b + 526176 a^9 b - 514080 a^{10} b + 438864 a^{11} b - 302112 a^{12} b + 136416 a^{13} b - \\
& 27936 a^{14} b - 128 a b^2 + 2176 a^2 b^2 - 1408 a^3 b^2 - 640 a^4 b^2 + 1024 a^5 b^2 - 2560 a^6 b^2 + \\
& 1536 a^7 b^2 - 6444 a^2 c + 47088 a^3 c - 153792 a^4 c + 367992 a^5 c - 684540 a^6 c + \\
& 1084284 a^7 c - 1618848 a^8 c + 2177064 a^9 c - 2400336 a^{10} c + 1979136 a^{11} c - \\
& 1093644 a^{12} c + 350676 a^{13} c - 48636 a^{14} c + 2816 a b c - 3712 a^2 b c - 4352 a^3 b c - \\
& 5888 a^4 b c + 35456 a^5 b c - 32384 a^6 b c + 8064 a^7 b c + 1024 b^2 c + 1344 a c^2 - 14304 a^2 c^2 + \\
& 47040 a^3 c^2 - 67392 a^4 c^2 + 41280 a^5 c^2 - 6816 a^6 c^2 - 1152 a^7 c^2 - 2048 b c^2 + 768 c^3) - \\
& \frac{1}{256 x^{12}} 3 (-1 + a) a (11088 a^2 - 62508 a^3 + 212172 a^4 - 610116 a^5 + 1519122 a^6 - \\
& 3418839 a^7 + 6790191 a^8 - 11473770 a^9 + 16275804 a^{10} - 19314393 a^{11} + \\
& 19499715 a^{12} - 17168052 a^{13} + 13151772 a^{14} - 8382894 a^{15} + 4054380 a^{16} - \\
& 1269780 a^{17} + 186108 a^{18} + 64 a b - 4608 a^2 b + 35648 a^3 b + 7360 a^4 b - 120832 a^5 b + \\
& 236992 a^6 b - 347264 a^7 b + 386560 a^8 b - 359552 a^9 b + 225536 a^{10} b - 59904 a^{11} b - \\
& 6144 b^2 - 3584 a^2 b^2 - 1280 a^3 b^2 + 256 a^4 b^2 - 11136 a c + 79008 a^2 c - 270624 a^3 c + \\
& 495648 a^4 c - 743424 a^5 c + 1125264 a^6 c - 1620384 a^7 c + 1875504 a^8 c - 1398768 a^9 c + \\
& 557040 a^{10} c - 88128 a^{11} c + 9216 b c + 16896 a b c - 4096 a^2 b c + 512 a^3 b c + \\
& 512 a^4 b c + 5376 c^2 - 43200 a c^2 + 56832 a^2 c^2 - 38592 a^3 c^2 + 10560 a^4 c^2) + \frac{1}{64 x^{11}} \\
& 3 (-1 + a) a (-1077 a^2 + 12015 a^3 - 52404 a^4 + 143055 a^5 - 347646 a^6 + 789300 a^7 - \\
& 1590750 a^8 + 2684457 a^9 - 3678690 a^{10} + 4192239 a^{11} - 4107744 a^{12} + 3455538 a^{13} - \\
& 2383425 a^{14} + 1222608 a^{15} - 396240 a^{16} + 58764 a^{17} - 976 a b + 13008 a^2 b - 22080 a^3 b + \\
& 27600 a^4 b - 14160 a^5 b - 19920 a^6 b + 43824 a^7 b - 57360 a^8 b + 40928 a^9 b - 10864 a^{10} b - \\
& 1280 a b^2 - 128 a^2 b^2 - 128 a^3 b^2 + 6984 a c - 37560 a^2 c + 76944 a^3 c - 114576 a^4 c + \\
& 148680 a^5 c - 208440 a^6 c + 275112 a^7 c - 226368 a^8 c + 95064 a^9 c - 15840 a^{10} c + \\
& 1536 b c + 1280 a b c + 128 a^2 b c + 128 a^3 b c - 3936 c^2 + 6144 a c^2 - 5472 a^2 c^2 + 1632 a^3 c^2) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{64 x^{10}} 3 \left(-4788 a^3 + 27717 a^4 - 73035 a^5 + 138462 a^6 - 267096 a^7 + 645810 a^8 - \right. \\
& 1546947 a^9 + 2963988 a^{10} - 4497708 a^{11} + 5622327 a^{12} - 5877696 a^{13} + 5045019 a^{14} - \\
& 3366954 a^{15} + 1590660 a^{16} - 459792 a^{17} + 60033 a^{18} - 3328 a^2 b + 9920 a^3 b - \\
& 27008 a^4 b + 48128 a^5 b - 48096 a^6 b + 22368 a^7 b + 24384 a^8 b - 51456 a^9 b + \\
& 31648 a^{10} b - 6560 a^{11} b + 768 b^2 + 1280 a b^2 - 1344 a^2 b^2 + 384 a^3 b^2 - 320 a^4 b^2 + \\
& 9888 a^2 c - 39168 a^3 c + 87600 a^4 c - 142224 a^5 c + 211680 a^6 c - 308400 a^7 c + \\
& 331704 a^8 c - 215832 a^9 c + 76248 a^{10} c - 11496 a^{11} c - 3456 b c + 832 a b c + 1536 a^2 b c - \\
& 384 a^3 b c + 320 a^4 b c + 3888 c^2 - 5472 a c^2 + 6288 a^2 c^2 - 4128 a^3 c^2 + 1008 a^4 c^2 \left. \right) + \\
& \frac{1}{32 x^9} 3 \left(2079 a^3 - 9921 a^4 + 23244 a^5 - 32946 a^6 + 60867 a^7 - 187125 a^8 + 474075 a^9 - \right. \\
& 920229 a^{10} + 1437156 a^{11} - 1814976 a^{12} + 1808772 a^{13} - 1341669 a^{14} + 675564 a^{15} - \\
& 201462 a^{16} + 26571 a^{17} - 1584 a b + 2392 a^2 b - 6056 a^3 b + 16184 a^4 b - 31608 a^5 b + \\
& 39600 a^6 b - 19752 a^7 b - 3800 a^8 b + 5752 a^9 b - 1128 a^{10} b - 256 b^2 - 128 a b^2 + \\
& 384 a^2 b^2 - 256 a^3 b^2 + 3564 a c - 6684 a^2 c + 11832 a^3 c - 21084 a^4 c + 48012 a^5 c - \\
& 98148 a^6 c + 116472 a^7 c - 77580 a^8 c + 28248 a^9 c - 4632 a^{10} c + 1728 b c - \\
& 384 a^2 b c + 256 a^3 b c - 2592 c^2 + 1440 a c^2 - 1056 a^2 c^2 + 192 a^3 c^2 \left. \right) + \frac{1}{64 x^8} \\
& \left(-9801 a^2 + 15336 a^3 + 7326 a^4 - 112140 a^5 + 280539 a^6 - 375750 a^7 + 310986 a^8 + \right. \\
& 241902 a^9 - 1774008 a^{10} + 4112478 a^{11} - 5827635 a^{12} + 5280966 a^{13} - \\
& 2944206 a^{14} + 916344 a^{15} - 122337 a^{16} - 7200 a b + 6432 a^2 b - 9984 a^3 b + \\
& 68640 a^4 b - 165120 a^5 b + 163968 a^6 b - 76512 a^7 b + 25920 a^8 b - 6144 a^9 b - \\
& 3328 b^2 - 1536 a b^2 + 1536 a^2 b^2 + 23328 a c - 23184 a^2 c - 9360 a^3 c - \\
& 33120 a^4 c + 262080 a^5 c - 448128 a^6 c + 342000 a^7 c - 137808 a^8 c + \\
& 24192 a^9 c + 13824 b c - 1536 a^2 b c - 15552 c^2 + 3456 a c^2 - 1152 a^2 c^2 \left. \right) + \frac{1}{16 x^7} \\
& \left(-2079 a^2 + 2070 a^3 + 14598 a^4 - 44649 a^5 + 72693 a^6 - 132210 a^7 + 239499 a^8 - \right. \\
& 259065 a^9 - 3789 a^{10} + 442440 a^{11} - 621027 a^{12} + 408600 a^{13} - 135495 a^{14} + \\
& 18414 a^{15} - 2496 a b - 288 a^2 b + 1584 a^3 b + 8976 a^4 b - 21240 a^5 b + 25032 a^6 b - \\
& 15288 a^7 b + 3720 a^8 b - 256 b^2 + 5184 a c + 4104 a^2 c - 19944 a^3 c + 4356 a^4 c + \\
& 34884 a^5 c - 49896 a^6 c + 27036 a^7 c - 5724 a^8 c + 1536 b c - 1728 c^2 \left. \right) + \frac{1}{16 x^6} \\
& \left(-1629 a^2 - 8298 a^3 + 36738 a^4 - 62541 a^5 + 87417 a^6 - 160704 a^7 + 284427 a^8 - \right. \\
& 291303 a^9 + 85284 a^{10} + 111888 a^{11} - 119997 a^{12} + 45054 a^{13} - 6336 a^{14} + 288 a b - \\
& 1056 a^2 b + 672 a^3 b + 2928 a^4 b - 9216 a^5 b + 8784 a^6 b - 2400 a^7 b - 256 b^2 - 720 a c + \\
& 8496 a^2 c - 12528 a^3 c + 360 a^4 c + 14112 a^5 c - 13320 a^6 c + 3600 a^7 c + 768 b c - 576 c^2 \left. \right) + \\
& \frac{1}{8 x^5} 3 \left(312 a^2 - 1494 a^3 + 3384 a^4 - 5115 a^5 + 10212 a^6 - 23784 a^7 + 33483 a^8 - 23694 a^9 + \right. \\
& 6888 a^{10} + 156 a^{11} - 423 a^{12} + 75 a^{13} + 384 a b - 160 a^2 b + 320 a^3 b - 144 a^4 b - 32 a^5 b + \\
& 16 a^6 b + 384 c - 1056 a c + 1008 a^2 c - 864 a^3 c + 216 a^4 c + 48 a^5 c - 24 a^6 c \left. \right) - \\
& \frac{1}{16 x^4} 3 a \left(1344 - 3408 a + 3168 a^2 - 1044 a^3 + 1176 a^4 - 13764 a^5 + 27408 a^6 - \right. \\
& 20121 a^7 + 5292 a^8 - 42 a^9 - 12 a^{10} + 3 a^{11} - 512 b + 256 a b + 1536 c - 384 a c \left. \right) + \\
& \frac{1}{x^3} 3 a \left(24 + 78 a - 30 a^2 - 192 a^3 - 27 a^4 + 549 a^5 - 555 a^6 + 153 a^7 + 32 b - 48 c \right) -
\end{aligned}$$

$$\frac{1}{x^2} 18 (-1 + a)^2 a (8 + 6 a - 10 a^2 + a^4)$$

The computer proof for Step 5(a). We use maximum value as an upper bound when $-5 \leq i \leq -2$, and use the maximum absolute sum as an upper bound when $-20 \leq i \leq -6$.

```
In[43]:= coeff = Table[If[i ≥ -5,
    Maximize[{Abs[Coefficient[G5, x^i]],
      -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64 && -1620 ≤ c ≤ 3}, {x, a, b, c}][[1]],
    FromCoefficientRules[Map[Abs, CoefficientRules[Coefficient[G5, x^i],
      {x, a, b, c}], {2}], {x, a, b, c}] /. a → 3 /. b → 3994 /. c → 1620
  ],
  {i,
    -20,
    -2}]

Out[43]= {545 102 954 553 600, 1 801 659 993 602 400,  $\frac{5\,886\,126\,860\,798\,565}{2}$ , 3 236 151 695 060 880,
   $\frac{19\,524\,150\,631\,163\,025}{8}$ ,  $\frac{23\,280\,405\,487\,750\,863}{16}$ , 741 299 023 799 055, 334 643 780 679 111,
  131 234 047 150 977, 44 282 400 488 163, 13 051 286 303 076, 3 426 985 781 691,
   $\frac{26\,026\,504\,017\,233}{32}$ ,  $\frac{639\,099\,553\,189}{4}$ ,  $\frac{37\,583\,702\,399}{2}$ , 17 107 740, 3 629 457, 1 222 632, 3672}
```

The computer proof for Step 5(b).

```
In[44]:= Simplify[Total[coeff x^Range[-20, -2]] < 1, x ≥ 120]

Out[44]= True
```

Step 6

Definition of G_6 .

```
In[45]:= G6 = Factor[μ (y - x) (mm - m0^2) /. mmsub];
Collect[G6, μ, Factor]

Out[46]= -x^2 (1 + x) y (1 + y) + (-x + m0^2 x + 5 x^2 + 4 x^3 + 2 y - m0^2 y - 2 x y - 2 x^2 y) μ + (-1 - x) μ^2
```

Definition of F_4 . Eq. (5.18)

```
In[47]:= F4 = Denominator[Factor[PolynomialExtendedGCD[F3, G6, μ]][[2]]][[1]] /
```

$$(x^2 (1+x)^2 (x-y) y^2 (1+y));$$

```
Collect[
```

```
  F4,
```

```
  y,
```

```
  Factor]
```

```
Out[48]= x^5 (-4 + m0^2 + 4 x + 4 x^2) (-3 + m0^2 + 4 x + 4 x^2)
(-3 + 3 m0^2 + 6 x + 3 m0^2 x + 21 x^2 + m0^2 x^2 + 16 x^3 + 4 x^4) -
x^2 (3 m0^2 - 6 m0^4 + 3 m0^6 + 12 x - 51 m0^2 x + 36 m0^4 x + 3 m0^6 x - 204 x^2 + 204 m0^2 x^2 +
37 m0^4 x^2 + 5 m0^6 x^2 + 204 x^3 + 305 m0^2 x^3 + 60 m0^4 x^3 + 3 m0^6 x^3 + 1048 x^4 +
179 m0^2 x^4 + 80 m0^4 x^4 + m0^6 x^4 + 688 x^5 + 352 m0^2 x^5 + 48 m0^4 x^5 + 252 x^6 +
448 m0^2 x^6 + 12 m0^4 x^6 + 704 x^7 + 240 m0^2 x^7 + 832 x^8 + 48 m0^2 x^8 + 384 x^9 + 64 x^10) y +
x (12 m0^2 - 16 m0^4 + 4 m0^6 + 48 x - 110 m0^2 x + 21 m0^4 x + 3 m0^6 x - 244 x^2 -
56 m0^2 x^2 + 34 m0^4 x^2 + 2 m0^6 x^2 - 448 x^3 + 38 m0^2 x^3 + 30 m0^4 x^3 -
180 x^4 + 8 m0^2 x^4 + 12 m0^4 x^4 - 248 x^5 - 448 x^6 - 288 x^7 - 64 x^8) y^2 +
(-12 m0^2 + 7 m0^4 - m0^6 - 48 x + 20 m0^2 x - 44 x^2 + 32 m0^2 x^2 - 8 x^3 +
24 m0^2 x^3 - 44 x^4 + 12 m0^2 x^4 - 48 x^5 - 16 x^6) y^3
```

Step 7

Definition of $y^{(1)}$, $y^{(2)}$, $y^{(3)}$.

```
In[49]:= y1 = - (2 x^3 + 3 x^2 + 3/2 m0 (m0 + 1) x + 3/4 m0 (m0 + 1));
```

```
y2 = - (2 x^3 + 3 x^2 + 3/2 m0 (m0 - 1) x + 3/4 m0 (m0 - 1));
```

```
y3 = x;
```

The computer proof for Step 7(a).

```
In[52]:= Simplify[(F4 /. y -> y1) == 0, m0 == 0]
```

```
Out[52]= True
```

The computer proof for Step 7(b).

```
In[53]:= Simplify[(F4 /. y -> y1 - 1/2) > 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```

```
Out[53]= True
```

The computer proof for Step 7(c).

```
In[54]:= Simplify[(F4 /. y -> y1 + 1/2) < 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```

```
Out[54]= True
```

The computer proof for Step 7(d).

```
In[55]:= Simplify[(F4 /. y -> y2 + 1/x) < 0, x ≥ 90 && m0 == 0]
```

```
Out[55]= True
```

The computer proof for Step 7(e).

```
In[56]:= Simplify[(F4 /. y → y^2 - 1/2) < 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```

```
Out[56]:= True
```

The computer proof for Step 7(f).

```
In[57]:= Simplify[(F4 /. y → y^2 + 1/2) > 0, x ≥ 90 && 0 ≤ m0 ≤ 2]
```

```
Out[57]:= True
```

The computer proof for Step 7(g).

```
In[58]:= Simplify[(F4 /. y → y^3 - 1) > 0, x ≥ 1 && 0 ≤ m0 ≤ 2]
```

```
Out[58]:= True
```

The computer proof for Step 7(h).

```
In[59]:= Simplify[(F4 /. y → y^3 + 1) < 0, x ≥ 1 && 0 ≤ m0 ≤ 2]
```

```
Out[59]:= True
```

Step 8

Speed up Step 8 using multiple kernels.

```
In[60]:= LaunchKernels[8];
```

The computer proof for Step 8.

```
In[61]:= Select[DeleteDuplicates[Flatten[ParallelTable[
  If[(μ /. #) > 0 && y0 ≤ -1 && y0 ≠ -(2 x0^3 + 3 x0^2), {x0, y0, μ /. #}, {}] & /@
  Solve[(F3 /. x → x0 /. y → y0) == 0, μ, Integers]
  , {x0, 1, 120}, {y0, -(2 x0^3 + 3 x0^2 + 3 x0 + 2), -(2 x0^3 + 3 x0^2 - 3 x0 - 3)}], 2]],
  Length[#] > 0 &]
```

```
Out[61]:= {{1, -1, 1}}
```