Explicit spherical designs

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Abstract

Since the introduction of the notion of spherical designs by Delsarte, Goethals, and Seidel in 1977, finding explicit constructions of spherical designs had been an open problem. Most existence proofs of spherical designs rely on the topology of the sphere, hence their constructive versions are only computable, but not explicit. That is to say that these constructions can only give algorithms that produce spherical designs up to arbitrary given precision, while they are not able to give any spherical designs explicitly. Inspired by recent work on rational designs, i.e. designs consisting of rational points, we generalize the known construction of spherical designs that uses interval designs with Gegenbauer weight, and give an explicit formula of spherical designs of arbitrary given strength on the real unit sphere of arbitrary given dimension.

Contents

1 Introduction

Spherical designs were introduced by Delsarte-Goethals-Seidel. The first existence proof of spherical designs of arbitrary strength on arbitrary dimensional sphere is given by Seymour-Zaslavsky [SZ84]. After that many other existence proofs were found [RB91, Wag91, BRV13, CXX17].

Once the existence of spherical designs is proved, one might ask how to construct spherical designs and how explicit the constructions are. As we will recall below, there are algorithms to produce an approximation of a spherical design up to arbitrary given precision. There also exists a very recent algorithm to produce a spherical design. The best we can hope for would be a formula that describes all coordinates of all points in the design, and this is what we focus on in this paper.

Most of the proofs of the existence of spherical designs use the topology of the sphere, more precisely, use that the spheres are complete metric space. Their constructive versions then use limits of points to construct the spherical designs. These constructions can only give *computable designs*, that is, designs such that there exists an algorithm to compute approximation of the designs up to arbitrary given precision, or equivalently, designs with coordinates being in the field of computable real numbers $\mathbb{R}^{\mathrm{com}}$. These constructions are not explicit in the sense that they cannot give the coordinates of the points in the designs explicitly. Computable designs are good for numerical analytic purposes, and some discussion about small computable designs can be found in [CFL11].

[RB91, Wag91] gave constructions of spherical designs using designs on *Gegenbauer interval*, that is interval equipped with Gegenbauer weight. Although this part of the construction is explicit, it is still an open problem to give an explicit construction of designs on Gegenbauer interval.

Recently, [CXX17] proves the existence of almost rational spherical designs, that is, spherical designs where every point has rational coordinates except possibly the first coordinate. This gives the first algorithm to construct spherical designs: enumerate all finite almost rational subsets of a sphere, and then test if they are designs or not. Since there are only countably many almost rational subsets and the existence of almost rational spherical designs are guaranteed in [CXX17], the above procedure terminates in finite time, hence is an algorithm. Note that the designs found by this algorithm have coordinates in the field $\mathbb{Q}(\sqrt{p} : \ p \text{ prime})$.

On intervals, some explicit designs were constructed by Kuperberg [Kup05]. He constructs a certain polynomial with integer coefficients and then express the points in the interval designs as some linear combinations of roots of the polynomial. This gives us interval designs over the field $\mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}$, the totally real part of the algebraic closure of $\mathbb{Q}$. Note that although this construction is explicit, it is not known whether the design constructed can be written down using radical expressions.

In this paper, our main result is Construction 1.1, an explicit construction of spherical designs on large dimensional spheres using explicit good spherical designs on smaller dimensional spheres. In Theorem 1.2, we apply this construction to some well-known explicit good spherical designs. In particular, Theorem 1.2(i) gives explicit spherical designs of arbitrary strength on arbitrary given dimensional sphere over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$, the totally real part of the abelian closure of $\mathbb{Q}$. Theorem 1.2 is explicit in the sense that it gives a formula for each coordinate of each point in the design. Morevover, the formula can be written in finitely many following symbols and the number of symbols is independent of the choice of t and d.

(i) Strength t, dimension d.

(ii) Rational numbers, real parts and imaginary parts of roots of unity ($\mathrm{Re}\zeta_n$, $\mathrm{Im}\zeta_n$).

(iii) Ceiling $\lceil \ \rceil$, floor $\lfloor \ \rfloor$, arithmetic operation (sum $+$, difference $-$, product $\cdot$, quotient $/$).

(iv) Finite sum $\sum_{i=a}^{b}$ and finite product $\prod_{i=a}^{b}$.

Let d be a natural number and consider the d-dimensional unit sphere S^d and hemisphere H^d:

$$S^d := \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} : \ x_0^2 + \cdots + x_d^2 = 1\} \quad \text{and} \quad H^d := \{(x_0, \dots, x_d) \in S^d : \ x_0 > 0\}.$$

On the sphere S^d, denote the spherical measure ν^d. On the hemisphere H^d, we will consider various measures ν_s^d indexed by a natural number s. The measure ν_s^d is a "shift" of the spherical measure ν^d, and the definition of ν_s^d is postponed to § 2.1. We equip S^d and H^d with these measures and denote the resulting measure spaces

$$\mathcal{S}^d := (S^d, \nu^d) \quad \text{and} \quad \mathcal{H}_s^d := (H^d, \nu_s^d).$$

Designs on $\mathcal{S}^d$ are subsets that approximate the sphere $\mathcal{S}^d$ nicely with respect to polynomials on $\mathcal{S}^d$. We will define in § 2.4 *semidesigns* on $\mathcal{H}_s^d$, that are subsets that approximate the hemisphere $\mathcal{H}_s^d$ nicely with respect to exactly one half of the polynomials on $\mathcal{H}_s^d$. In § 2.4, we also generalize the *antipodal* property on $\mathcal{S}^d$ and define the *semiantipodal* property on $\mathcal{H}_s^d$.

Weighted version of (semi)designs are used in this paper as well. A weighted (semi)design $\mathcal{X}$ is called *integer/rational-weighted* if all weights of $\mathcal{X}$ are integer/rational. For a field $\mathbb{F} \subseteq \mathbb{R}$, we say that $\mathcal{X}$ is *defined over* $\mathbb{F}$, if it consists of only $\mathbb{F}$-points. In particular, $\mathcal{X}$ is called *rational* if it consists of only rational points. The precise definitions for these concepts are postponed to § 2.2 and 2.3.

Construction 1.1. *Let t and a be positive integers and $\mathcal{Y}^a$ an explicit antipodal spherical t-design $\mathcal{Y}^a$ on $\mathcal{S}^a$ over a field $\mathbb{F} \subseteq \mathbb{R}$ for some natural number t.*

Step 1 *Apply Corollary 3.9 to strength $t+d-1$, and get an explicit finite semiantipodal rational-weighted rational $(t+d-1)$-semidesign on $\mathcal{H}_0^1$, denoted by $\mathcal{X}_0^1$.*

Step 2 *Apply Corollary 3.12 to strength $t+d-1$, and get an explicit finite semiantipodal rational-weighted rational $(t+d-1)$-semidesign on $\mathcal{H}_1^1$, denoted by $\mathcal{X}_1^1$.*

Step 3 *For each positive even (resp. odd) integer $s < d$, apply Corollary 4.6 to $\mathcal{X}_0^1$ (resp. $\mathcal{X}_1^1$) and $\mathcal{H}_s^1$, and get an explicit finite semiantipodal rational-weighted rational t-semidesign $\mathcal{X}_s^1$ on $\mathcal{H}_s^1$.*

Step 4 *For each positive integer $s < d$, apply Corollary 4.10 to $\mathcal{X}_s^1, \dots, \mathcal{X}_{d-1}^1$, and get an explicit finite semiantipodal rational-weighted rational t-semidesign $\mathcal{X}_s^{d-s} := \mathcal{X}_s^1 \times \cdots \times \mathcal{X}_{d-1}^1$ on $\mathcal{H}_s^{d-s}$.*

Step 5 *For each positive integer $s < d$, apply Corollary 4.13 to $\mathcal{X}_s^{d-s}$, and get an explicit semiantipodal integer-weighted rational t-semidesign $\overline{\mathcal{X}}_s^{d-s}$ on $\mathcal{H}_s^{d-s}$.*

Step 6 *Apply Corollary 4.17 to $\mathcal{Y}^a$, and get an explicit reflection $s_\alpha \in O(a+1, \mathbb{Q})$ such that $s_\alpha \mathcal{Y}^a$ is an explicit antipodal t-design on $\mathcal{S}^a$ over $\mathbb{F}$ such that every point in $s_\alpha \mathcal{Y}^a$ has nonzero first coordinate.*

Step 7 *Apply Corollary 4.25 to $\mathcal{Y}^a$ and $\overline{\mathcal{X}}_a^{d-a}$, and get an explicit antipodal t-design $\mathcal{Y}^d := (s_\alpha \mathcal{Y}^a) \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$ on $\mathcal{S}^d$ over $\mathbb{F}$, where $\rtimes_\xi$ is the twisted product defined in § 4.6, and ξ is a certain map $\mathbb{N} \to O(a+1, \mathbb{Q})$ determined by $\mathcal{Y}^a$ defined in Corollary 4.25.*

Theorem 1.2 below is an immediate corollary of Construction 1.1 applied to some well-known choices of explicit spherical designs $\mathcal{Y}^a$.

Theorem 1.2. (i) *Choose $a = 1$ and $\mathcal{Y}^1$ the set of the vertices of a regular $4(t+1)$-gon on $\mathcal{S}^1$ where one point is $(1, 0)$. Then, $\mathcal{Y}^d$ is an explicit spherical t-design on $\mathcal{S}^d$ over $\mathbb{Q}(\zeta_{4(t+1)}) \cap \mathbb{R}$, where ζ_n is a primitive n-th root of unity. Moreover, all points in $\mathcal{Y}^d$ have rational coordinates except for their first two coordinates.*

(ii) *hoose $a = 3$ and $\mathcal{Y}^3$ the rational 5-design from the vertices of 24-cell on $\mathcal{S}^3$. Then, $\mathcal{Y}^d$ is an explicit rational spherical 5-design on $\mathcal{S}^d$ for every $d \geq 3$.*

(iii) *Choose $a = 7$ and $\mathcal{Y}^7$ the rational 7-design from E_8 lattice on $\mathcal{S}^7$ as in [Ven84, CXX17]. Then, $\mathcal{Y}^d$ is an explicit rational spherical 7-design on $\mathcal{S}^d$ for every $d \geq 7$.*

(iv) *Choose $a = 23$ and $\mathcal{Y}^{23}$ the rational 11-design from Leech lattice on $\mathcal{S}^{23}$ as in [Ven84, CXX17]. Then, $\mathcal{Y}^d$ is an explicit rational spherical 11-design on $\mathcal{S}^d$ for every $d \geq 23$.*

Since compositum of all cyclotomic fields $\mathbb{Q}(\zeta_n)$ is $\mathbb{Q}^{\mathrm{ab}}$, the abelian closure of $\mathbb{Q}$, we can construct t-designs over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$ for all t, as shown in the following corollary.

Corollary 1.3. *For every positive integers t and d, Theorem 1.2(i) gives an explicit t-design on $\mathcal{S}^d$ over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$.*

Except for the initial design $\mathcal{Y}^a$, Construction 1.1 uses only rational objects to construct the final design $\mathcal{Y}^d$. Therefore, field automorphism commutes with the construction. This observation gives the following immediate corollary.

Corollary 1.4. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield and $\sigma \in \mathrm{Aut}(\mathbb{F} / \mathbb{Q})$ be a field automorphism fixing $\mathbb{Q}$. Let $\mathcal{Y}^a$ be as in Construction 1.1 and $\mathcal{Y}^d$ the resulting design obtained by Construction 1.1. If $\mathcal{Y}^a$ is stable under the action of σ, then $\mathcal{Y}^d$ is also stable under the action of σ.*

Eiichi Bannai first asked the question whether rational spherical designs exist or not. Theorem 1.2(iii) and (iv) give explicit affirmative answers when $t \leq 7$ and $d \geq 7$ or $t \leq 11$ and $d \geq 23$. It motivates us to give Conjecture 1.5.

Conjecture 1.5. For every positive integer t, there exists a rational spherical t-design on $\mathcal{S}^d$ for some positive integer d.

Note that if Conjecture 1.5 is true, applying Construction 1.1, we can get a rational t-design on $\mathcal{S}^d$ for all sufficiently large d.

The paper is organized as follow. We introduce necessary concepts in § 2. In § 3, we consider one-dimensional semicircle $\mathcal{H}_s^1$, and do *Step 1* and *Step 2* of Construction 1.1. In § 4, we consider high-dimensional hemispheres $\mathcal{H}_s^d$ and spheres $\mathcal{S}^d$, and do *Step 3* to *Step 7* of Construction 1.1. We will explain at the beginning of subsections of § 4 the motivations of the corresponding steps in Construction 1.1.

Notation. Throughout the paper, for a real interval I, the set of all integers in I is denoted by $I_{\mathbb{Z}}$. For instance, $[0, t)_{\mathbb{Z}}$ consists of all natural numbers smaller than t, and $[1, d]_{\mathbb{Z}}$ consists of all positive integers no greater than d.

2 Preliminary

2.1 Radon-Nikodym derivative

Let X be a *measurable space*, that is a set equipped with a σ-algebra consisting of subsets of X called *measurable sets*. The *Radon-Nikodym derivative* of a measure μ on X with respect to another measure ν on X, denoted by $\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}$, is a measurable function $X \to \mathbb{R}^{\geq 0}$ such that

$$\mu(E) = \int_E \frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu} \,\mathrm{d}\,\nu \tag{2.1}$$

for all measurable sets E. In probability theoretic language, the Radon-Nikodym derivative is known as the probability density function.

When we know the measure ν, by specifying the Radon-Nikodym derivative $\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}$, we get a description of the measure μ by Eq. (2.1). We use this method to construct certain measures ν_s^d on the hemisphere H^d indexed by a natural number s.

Definition 2.1. Let s be a natural number and $g_s(x_0, \ldots, x_d) := x_0^s$, regarded as a polynomial on H^d. Set ν_s^d to be the measure on H^d with Radon-Nikodym derivative with respect to the spherical measure ν^d

$$\frac{\mathrm{d}\,\nu_s^d}{\mathrm{d}\,\nu^d} = g_s.$$

The topological space H^d equipped with the measure ν_s^d is denoted by $\mathcal{H}_s^d$.

It might be worth explaining here why we use hemispheres in our approach instead of intervals and balls.

Let $\mathcal{I}_s$ be the interval $(-1, 1)$ equipped with the density function $(1 - x^2)^{s/2}$. The interval $\mathcal{I}_s$ is sometimes called an interval with Gegenbauer weight. Consider the natural projection to the second coordinate $H^1 \to (-1, 1)$. This projection induces an isomorphism of measure spaces $\mathcal{H}_s^1 \xrightarrow{\sim} \mathcal{I}_{s-1}$. In high dimensional space, we have a similar isomorphism of measure spaces $\mathcal{H}_s^d \xrightarrow{\sim} \mathcal{B}_{s-1}^d$, where $\mathcal{B}_{s-1}^d$ is the d-dimensional real open unit ball equipped with a certain measure indexed by $s - 1$.

However, a rational point in $(-1, 1)$ might not lift to a rational point on H^1 under this projection. In our Construction 1.1, for the final design $\mathcal{Y}^d$ being defined over the same field $\mathbb{F}$ as the initial design $\mathcal{Y}^a$, we found that it is necessary to work with rational points on hemispheres, and not with rational points on intervals or balls. Using hemispheres also allows us to only handle polynomials and eliminate the use of radical expression like $\sqrt{1 - x^2}$.

2.2 Levelling spaces and related constructions

In this paper, we use the notation in [CXX17]. For readers' convenience, we repeat some important definitions here.

A *levelling space* $\mathcal{X} = (X, \mu_X)$ is a nonempty Hausdorff topological space X, which is called the *support*, equipped with a measure μ_X on X such that the total measure is finite and the measures of nonempty open sets are positive.

In this paper, we use the convention that $\mathcal{X} \subseteq \mathcal{Z}$ means only X is the topological subspace of Z and we assume nothing on μ_X and μ_Z.

The *total measure* of $\mathcal{X}$ is denoted by $|\,\mathcal{X}\,|$, and it is clear that $|\,\mathcal{X}\,| = \int_X 1\,\mathrm{d}\,\mu_X$. We say that a levelling space $\mathcal{X}$ is *finite* if its cardinality of the support $|X|$ is finite.

We say that $\mathcal{X}$ is *rational-weighted* (resp. *integer-weighted*) if the image of μ_X is contained in $\mathbb{Q}$ (resp. $\mathbb{Z}$). When X is finite, the counting measure on X is denoted by $\underline{1}_X$, namely $\underline{1}_X(E) = |E|$ for all subsets $E \subseteq X$. We say that $\mathcal{X}$ is $\underline{1}$*-weighted* if the measure μ_X is the counting measure $\underline{1}_X$.

A *map* g between levelling spaces $\mathcal{X}$ and $\mathcal{Y}$ is a map that is both a continuous map of topological space and a homomorphism of measure space. The map g is a *dominant open embedding* if g is a dominant open embedding of topological spaces (i.e. g maps domain homeomorphically to the image and the image is dense in codomain) and the measure of the image equals to the measure of codomain. If we have a dominant open embedding between $\mathcal{X}$ and $\mathcal{Y}$, then we can basically think of them as the same for our purposes, as shown in Lemma 4.3.

Given two levelling space $\mathcal{X}$ and $\mathcal{Y}$, and a map $\iota : \mathcal{X} \to \mathcal{Y}$, we can define some related spaces as follow.

(i) *Scalar* $c\mathcal{X} := (X, c\mu_X)$ for positive real c. For continuous measurable function f on $\mathcal{X}$,

$$\int_X f \,\mathrm{d}\, c\mu_X = c \int_X f \,\mathrm{d}\, \mu_X. \tag{2.2}$$

(ii) *Sum* $\mathcal{X} + \mathcal{Y} := (X \cup Y, \mu_X + \mu_Y)$. Note that we only have this sum if the topologies of X and Y agree on the set intersection $X \cap Y$. For continuous measurable function f on $\mathcal{X} + \mathcal{Y}$,

$$\int_{X\cup Y} f \,\mathrm{d}(\mu_X + \mu_Y) = \left(\int_X f \,\mathrm{d}\, \mu_X\right) + \left(\int_Y f \,\mathrm{d}\, \mu_Y\right). \tag{2.3}$$

(iii) *Product* $\mathcal{X} \times \mathcal{Y} := (X \times Y, \mu_X \times \mu_Y)$. For continuous measurable function f on $\mathcal{X}$ and g on $\mathcal{Y}$,

$$\int_{X\times Y} f \otimes g \,\mathrm{d}\, \mu_X \times \mu_Y = \left(\int_X f \,\mathrm{d}\, \mu_X\right)\left(\int_Y g \,\mathrm{d}\, \mu_Y\right). \tag{2.4}$$

(iv) *Image* $g(\mathcal{X}) := (\iota(X), \iota_* \mu_X)$, where $\iota_*\mu_X$ is the pushfoward measure of μ_X. For every continuous measurable function f on $\iota(X)$,

$$\int_X \iota^* f \,\mathrm{d}\, \mu_X = \int_{\iota(X)} f \,\mathrm{d}\, \iota_* \mu_X, \quad \text{where} \quad \iota^* f := f \circ \iota. \tag{2.5}$$

2.3 Designs and polynomials

Let $\mathcal{Z} = (Z, \mu_Z)$ be a levelling space and V a real vector space of continuous integrable functions on $\mathcal{Z}$. A V*-design* on $\mathcal{Z}$ is a levelling space $\mathcal{X} = (X, \mu_X) \subseteq \mathcal{Z}$ such that

$$\frac{1}{|\,\mathcal{X}\,|} \int_X f \,\mathrm{d}\, \mu_X = \frac{1}{|\,\mathcal{Z}\,|} \int_Z f \,\mathrm{d}\, \mu_Z$$

for all $f \in V$. For a field $\mathbb{F} \subseteq \mathbb{R}$, we call $\mathcal{X}$ a V*-design over* $\mathbb{F}$ when it makes sense to talk about $\mathbb{F}$-points in $\mathcal{Z}$ and $\mathcal{X}$ consists of only $\mathbb{F}$-points in $\mathcal{Z}$. The V-designs over $\mathbb{Q}$ are also called *rational* V*-designs.*

Let $Z \subseteq \mathbb{R}^d$ be a subspace. Denote $\mathcal{P}^t[Z]$ the vector space of all polynomials on Z with degree bounded above by t. We use the convention that $\mathcal{P}^\infty[Z]$ is the vector space of all polynomials on Z. It is clear that

$$\mathcal{P}^t[S^d] \cong \mathcal{P}^t[H^d] \cong \mathbb{R}[x_0, \ldots, x_d]_{\le t}/(x_0^2 + \cdots + x_d^2 - 1)_{\le t},$$

where subscript $\le t$ means the degree $\le t$ part.

Definition 2.2. Assume that polynomials are integrable on $\mathcal{Z}$. A *weighted* t*-design on* $\mathcal{Z}$ is a $\mathcal{P}^t[\mathcal{Z}]$-design on $\mathcal{Z}$. A t*-design on* $\mathcal{Z}$ is a $\underline{1}$-weighted $\mathcal{P}^t[\mathcal{Z}]$-design on $\mathcal{Z}$.

In particular, a (weighted) t-design on $\mathcal{S}^d$ is just an ordinary (weighted) spherical t-design.

2.4 Semidesigns and antipodal maps

For tuples $\mathbf{x} = (x_0, \ldots, x_d)$ and $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_d)$, let $\mathbf{x}^{\boldsymbol{\lambda}} := \prod_{i=0}^d x_i^{\lambda_i}$. For each natural number r, let

$$\mathcal{P}^{t,r}[\mathbb{R}^d] := \mathbb{R}\langle \mathbf{x}^{\boldsymbol{\lambda}} :\ \boldsymbol{\lambda} \in \mathbb{N}^d,\ \textstyle\sum_{i=0}^r \lambda_i \text{ is even}\rangle_{\le t},$$

and set $\mathcal{P}^{t,0,r}[\mathbb{R}^d] := \mathcal{P}^{t,0}[\mathbb{R}^d] \cap \mathcal{P}^{t,r}[\mathbb{R}^d]$.

Since the defining equation of S^d and H^d, $x_0^2 + \cdots + x_d^2 - 1$, is in $\mathcal{P}^{t,r}[\mathbb{R}^d]$ for all r, we have the following well-defined quotients

$$\begin{aligned} \mathcal{P}^{t,0}[H^d] &:= \mathcal{P}^{t,0}[\mathbb{R}^d]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[H^d], \\ \mathcal{P}^{t,0,r}[H^d] &:= \mathcal{P}^{t,0,r}[\mathbb{R}^d]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[H^d], \\ \mathcal{P}^{t,r}[S^d] &:= \mathcal{P}^{t,r}[\mathbb{R}^d]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[S^d]. \end{aligned} \tag{2.6}$$

It is easy to see that the polynomials on H^d admit a direct sum decomposition

$$\mathcal{P}^t[H^d] = \mathcal{P}^{t,0}[H^d] \oplus x_0\, \mathcal{P}^{t-1,0}[H^d].$$

Definition 2.3. A levelling space $\mathcal{X} \subseteq H^d$ is a *weighted t-semidesign* on $\mathcal{H}_s^d$ provided that $\mathcal{X}$ is a $\mathcal{P}^{t,0}[H^d]$-design on $\mathcal{H}_s^d$. A weighted t-semidesign is called a *t-semidesign* if it is $\underline{1}$-weighted.

The name *semidesign* comes from the fact that $\mathcal{P}^\infty[H^d] \cong \mathcal{P}^{\infty,0}[H^d] \oplus \mathcal{P}^{\infty,0}[H^d]$.

Definition 2.4. Let a, b be natural numbers such that $a \le b \le d$. The $[a,b]$-*antipodal map* is defined to be

$$-_{[a,b]} : (x_0, \ldots, x_d) \mapsto (x_0, \ldots, x_{a-1}, -x_a, \ldots, -x_b, x_{b+1}, \ldots, x_d).$$

A set $X \subseteq H^d$ (resp. levelling space $\mathcal{X} \subseteq H^d$) is said to be $[a,b]$-*antipodal* if $-_{[a,b]}X = X$ (resp. $-_{[a,b]}\,\mathcal{X} = \mathcal{X}$). We say a set/levelling space *antipodal* (resp. *semiantipodal*) if it is $[0,d]$-antipodal (resp. $[1,d]$-antipodal).

Lemma 2.5. *Let $\mathcal{X} \subseteq H^d$ be a $[1,r]$-antipodal levelling space. Then, $\mathcal{X}$ is a weighted t-semidesign on $\mathcal{H}_s^d$ if and only if $\mathcal{X}$ is a $\mathcal{P}^{t,0,r}[H^d]$-design on $\mathcal{H}_s^d$.*

Proof. Let $\mathbf{x}^{\boldsymbol\lambda} \in \mathcal{P}^{t,0}[H^d]$ be a monomial. When $\sum_{i=1}^r \lambda_i$ is even, $\mathbf{x}^{\boldsymbol\lambda} \in \mathcal{P}^{t,0,r}[H^d]$. When $\sum_{i=1}^r \lambda_i$ is odd, since both $\mathcal{H}_s^d$ and $\mathcal{X}$ are $[1,r]$-antipodal,

$$\frac{1}{|\,\mathcal{H}_s^d\,|}\int_{\mathcal{H}^d} \mathbf{x}^{\boldsymbol\lambda}\, \mathrm{d}\,\nu_s^d = \frac{1}{|\,\mathcal{X}\,|}\int_X \mathbf{x}^{\boldsymbol\lambda}\, \mathrm{d}\,\mu_X = 0. \qquad \square$$

Lemma 2.6. *Let $\mathcal{X} \subseteq S^d$ be a $[0,r]$-antipodal levelling space. Then, $\mathcal{X}$ is a weighted t-design on $\mathcal{S}^d$ if and only if $\mathcal{X}$ is a $\mathcal{P}^{t,r}[S^d]$-design on $\mathcal{S}^d$.*

Proof. The result can be proved use similar arguments as in the proof of Lemma 2.5. $\square$

3 Designs on semicircles

In § 3.1, we show another interpretation of semidesigns on the semicircle $\mathcal{H}_s^1$. In § 3.2, with the help of that interpretation, we give a strategy to construct semidesigns on the semicircle $\mathcal{H}_s^1$. In § 3.3 and § 3.4, we apply this strategy to $\mathcal{H}_0^1$ and $\mathcal{H}_1^1$ and do *Step 1* and *Step 2* of Construction 1.1, respectively.

We always use $(t-1)$-semidesigns in this section, since most formulas would look more complicated if we used t-semidesigns.

3.1 Vandermonde matrix and semidesigns

For every finite subset $\mathbf{a} \subseteq (-1, 1)$, we associate it a finite subset $X_{\mathbf{a}}$ of the semicircle H^1 as follow:

$$X_{\mathbf{a}} := \{x_a \in H^1 : \ a \in \mathbf{a}\} \quad \text{where} \quad x_a := (\sqrt{1-a^2}, a) \in H^1 \,. \tag{3.1}$$

We will show in Lemma 3.3 that semidesigns on $\mathcal{H}^1_s$ with support $X_{\mathbf{a}}$ can be viewed as positive solutions $\mathbf{x}$ of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}_s$, where $\mathbf{A}$ is the Vandermonde matrix of $\mathbf{a}$ defined in Definition 3.1, and $\mathbf{b}_s$ is some column vector determined by $\mathcal{H}^1_s$ and defined in Lemma 3.2. Some estimate on the behavior of $\mathbf{A}$ is given in Lemma 3.5.

Definition 3.1. Let $\mathbf{a}$ be a finite set of real numbers. The *t-th Vandermonde matrix* of $\mathbf{a}$ is the matrix with rows indexed by $[0, t)_{\mathbb{Z}}$, columns indexed by $\mathbf{a}$ and entries $\mathbf{A}_{j,a} := a^j$ for $j \in [0, t)_{\mathbb{Z}}$ and $a \in \mathbf{a}$. The *Vandermonde matrix* of $\mathbf{a}$ is the $|\mathbf{a}|$-th Vandermonde matrix of $\mathbf{a}$, which is a square matrix.

Lemma 3.2. *Let s and j be natural numbers. Let*

$$b_{s,j} := \frac{1}{|\,\mathcal{H}^1_s\,|} \int_{H^1} y_1^j \,\mathrm{d}\,\nu^1_s, \tag{3.2}$$

where $y_1^j \in \mathbb{R}[y_1] \cong \mathcal{P}^{\infty,0}[H^1]$ is a polynomial on $\mathcal{H}^1_s$. Then,

$$b_{s,j} = \begin{cases} 0, & j \text{ is odd,} \\ \dfrac{\Gamma(\frac{s+2}{2})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{s+j+2}{2})} \in \mathbb{Q}, & j \text{ is even,} \end{cases}$$

where Γ is the gamma function. In particular,

$$b_{0,j} = \begin{cases} 0, & j \text{ is odd,} \\ \frac{1}{2^j}\binom{j}{\frac{j}{2}}, & j \text{ is even,} \end{cases} \quad \text{and} \quad b_{1,j} = \begin{cases} 0, & j \text{ is odd,} \\ \frac{1}{j+1}, & j \text{ is even.} \end{cases} \tag{3.3}$$

Proof. Straightforward calculation. □

Lemma 3.3. *Let s, t, n be natural numbers, and let $\mathbf{a}$ be a finite subset of $(-1, 1)$ of size n and $\mathbf{A}$ the t-th Vandermonde matrix of $\mathbf{a}$. We associate every measure χ on $X_{\mathbf{a}}$ a vector $\mathbf{x} = (\chi(x_a) : \ a \in \mathbf{a})$, and vice versa. Then, $(X_{\mathbf{a}}, \chi)$ is a weighted $(t-1)$-semidesign on $\mathcal{H}^1_s$ if and only if $\mathbf{A}\mathbf{x} = n\mathbf{b}_s$ and $\mathbf{x}$ is positive, where $\mathbf{b}_s := (b_{s,j} : \ j \in [0, t)_{\mathbb{Z}})$.*

Proof. The result follows from the definitions of semidesigns (see Definition 2.3), Vandermonde matrix $\mathbf{A}$ (see Definition 3.1) and constants $b_{s,j}$ (see Eq. (3.2)). □

The L^∞-norm of a (column) vector $\mathbf{a}$ is the maximum absolute value of coordinates of $\mathbf{a}$:

$$\|\mathbf{a}\|_\infty := \max_{j \in J} |a_j|,$$

where J is the index set for rows. It induces an L^∞-norm on a matrix $\mathbf{A}$, which is the maximum absolute row sum of $\mathbf{A}$:

$$\|\mathbf{A}\|_\infty := \max_{j \in J} \sum_{a \in \mathbf{a}} |\mathbf{A}_{j,a}|, \tag{3.4}$$

where $\mathbf{a}$ is the index set for columns.

Definition 3.4. Let X and $\widetilde{X}$ be subsets of a metric space. Fix a bijection $\sim\,: X \to \widetilde{X}$.

(i) Let $\operatorname{dist}_{\min} X$ be the infimum of distances among distinct points of X, namely

$$\operatorname{dist}_{\min} X := \inf\{\operatorname{dist}(x,y) : \ x, y \in X, x \neq y\}.$$

(ii) Let $\operatorname{dist}(\widetilde{X}, X)$ be the distance between $\widetilde{X}$ and X with respect to $\sim$, namely

$$\operatorname{dist}(\widetilde{X}, X) := \sup_{x\in X} \operatorname{dist}(\widetilde{x}, x).$$

We analyze in Lemma 3.5 the norms of Vandermonde matrix and related matrices.

Lemma 3.5. *Let* $\mathbf{a}$ *(resp.* $\widetilde{\mathbf{a}}$*) be a subset of* $(-1,1)$ *of size* t *and* $\mathbf{A}$ *(resp.* $\widetilde{\mathbf{A}}$*) the Vandermonde matrix of* $\mathbf{a}$ *(resp.* $\widetilde{\mathbf{a}}$*). Fix a bijection* $\sim\,: \mathbf{a} \to \widetilde{\mathbf{a}}$*. Then, the following statements hold.*

(i) *Let* $\delta := \operatorname{dist}_{\min} \mathbf{a}$*. Then,*

$$\|\mathbf{A}\|_\infty = t \quad \textit{and} \quad \|\mathbf{A}^{-1}\|_\infty \leq (2/\delta)^{t-1}.$$

(ii) *Let* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$*. Then,*

$$\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty \leq t(t-1)\varepsilon.$$

Proof. (i) Since $\mathbf{a}$ consists of numbers in $(-1,1)$, it is straightforward to calculate $\|\mathbf{A}\|_\infty$, and [Gau62] gives the desired estimate on $\|\mathbf{A}^{-1}\|_\infty$.
(ii) We expand $\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty$ according to Eq. (3.4) and estimate it.

$$\begin{aligned}
\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty &= \max_{j\in[0,t)_{\mathbb{Z}}} \sum_{a\in\mathbf{a}} |\widetilde{a}^j - a^j| && \text{Eq. (3.4) and Definition 3.1} \\
&= \max_{j\in[0,t)_{\mathbb{Z}}} \sum_{a\in\mathbf{a}} \left(|\widetilde{a} - a| \cdot \left| \sum_{k=0}^{j-1} \widetilde{a}^k a^{j-k-1} \right| \right) \\
&\leq \max_{j\in[0,t)_{\mathbb{Z}}} \sum_{a\in\mathbf{a}} \varepsilon j && |\widetilde{a} - a| \leq \varepsilon \text{ and } |\widetilde{a}_i|, |a_i| \in (-1,1) \\
&= t(t-1)\varepsilon. && |\mathbf{a}| = t \quad \square
\end{aligned} \tag{3.5}$$

3.2 A strategy to construct designs on $\mathcal{H}_s^1$

Here is our strategy to construct rational-weighted rational semidesigns on $\mathcal{H}_s^1$. First, we start with some subset X of H^1 which is "almost" a semidesign when every point has measure 1. Then, we choose a subset $\widetilde{X}$ of $H^1 \cap \mathbb{Q}^2$ to approximate X. At the end, we choose a subset $\widetilde{X}'$ of $\widetilde{X}$, and tweak the measure on $\widetilde{X}'$ to get a desired design. Theorem 3.6 shows the details of this strategy.

Theorem 3.6. *Let* s, t, n *be natural numbers. Let* $\mathbf{a}$ *and* $\widetilde{\mathbf{a}}$ *be subsets of* $(-1,1)$ *of size* n *and fix a bijection* $\sim\,: \mathbf{a} \to \widetilde{\mathbf{a}}$*. Let* $\mathbf{a}'$ *be a subset of* $\mathbf{a}$ *of size* t*, and let* $\widetilde{\mathbf{a}}'$ *be the image of* $\mathbf{a}'$ *in* $\widetilde{\mathbf{a}}$ *under the bijection. In other words, we assume that we have the following commutative diagram.*

$$\begin{tikzcd}
\mathbf{a} \textit{ of size } n \arrow[r, "\sim"] & \widetilde{\mathbf{a}} \textit{ of size } n \\
\mathbf{a}' \textit{ of size } t \arrow[r, "\sim"] \arrow[u, hook] & \widetilde{\mathbf{a}}' \textit{ of size } t \arrow[u, hook]
\end{tikzcd} \tag{3.6}$$

Let

$$\boldsymbol{\epsilon} := (\boldsymbol{\epsilon}_j : \ j \in [0,t)_{\mathbb{Z}}), \quad \text{where} \quad \boldsymbol{\epsilon}_j := \frac{1}{n}\sum_{a\in\mathbf{a}} a^j - b_{s,j}, \tag{3.7}$$

and

$$\widetilde{\boldsymbol{\epsilon}} := (\widetilde{\boldsymbol{\epsilon}}_j : \ j \in [0,t)_{\mathbb{Z}}), \quad \text{where} \quad \widetilde{\boldsymbol{\epsilon}}_j := \frac{1}{n}\sum_{\widetilde{a}\in\widetilde{\mathbf{a}}} \widetilde{a}^j - b_{s,j}, \tag{3.8}$$

where $b_{s,j}$ is some rational number defined in Lemma 3.2. Let $\delta' := \operatorname{dist}_{\min}\mathbf{a}'$ (see Definition 3.4(i)) and $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ (see Definition 3.4(ii)). Assume that

$$t^2\varepsilon + n\|\boldsymbol{\epsilon}\|_\infty < (\delta'/2)^{t-1}. \tag{3.9}$$

Then, $X_{\widetilde{\mathbf{a}}}$ (see Eq. (3.1)) is the support a unique weighted $(t-1)$-semidesign $\mathcal{X} = (X_{\widetilde{\mathbf{a}}}, \chi)$ on $\mathcal{H}_s^1$ such that $\chi(X_{\widetilde{\mathbf{a}}}) = n$ and $\chi(x) = 1$ for every $x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\widetilde{\mathbf{a}}'}$. Moreover, the unique measure χ is given by

$$\chi(x) = \begin{cases} 1 - n\displaystyle\sum_{j=0}^{t-1}(-1)^j \frac{e_{t-j-1}(\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\})}{\prod_{\widetilde{b}'\in\widetilde{\mathbf{a}}'\setminus\{\widetilde{a}'\}}(\widetilde{b}' - \widetilde{a}')}\widetilde{\boldsymbol{\epsilon}}_j, & x = x_{\widetilde{a}'} \in X_{\widetilde{\mathbf{a}}'} \\ 1, & x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\mathbf{a}'}, \end{cases} \tag{3.10}$$

where $e_{t-j-1}(\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\})$ is the $(t-j-1)$-th elementary polynomial in $t-1$ numbers $\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\}$.

Proof. Let $\mathbf{A}$, $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}'$ be the t-th Vandermonde matrix of $\mathbf{a}$, $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$, respectively. The definitions of $\boldsymbol{\epsilon}$ and $\widetilde{\boldsymbol{\epsilon}}$ are equivalent to

$$\mathbf{A}\mathbf{1} = n\mathbf{b}_s + n\boldsymbol{\epsilon} \quad \text{and} \quad \widetilde{\mathbf{A}}\mathbf{1} = n\mathbf{b}_s + n\widetilde{\boldsymbol{\epsilon}}. \tag{3.11}$$

Let $\widetilde{\mathbf{x}}' \in \mathbb{R}^t$ be an indeterminate vector and $\widetilde{\mathbf{x}} \in \mathbb{R}^n$ the extension of $\widetilde{\mathbf{x}}'$ by 0 according to the inclusion $\widetilde{\mathbf{a}}' \subseteq \widetilde{\mathbf{a}}$. Consider the equation

$$\widetilde{\mathbf{A}}(\mathbf{1} + \widetilde{\mathbf{x}}) = n\mathbf{b}_s. \tag{3.12}$$

By the definition of $\widetilde{\mathbf{x}}$, we have $\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{A}}'\widetilde{\mathbf{x}}'$, hence Eq. (3.12) is equivalent to

$$\widetilde{\mathbf{A}}\mathbf{1} + \widetilde{\mathbf{A}}'\widetilde{\mathbf{x}}' = n\mathbf{b}_s. \tag{3.13}$$

Since $\widetilde{\mathbf{A}}'$ is a square Vandermonde matrix, using Eq. (3.11), we see that Eq. (3.12) has a unique solution

$$\widetilde{\mathbf{x}}' = -n\widetilde{\mathbf{A}}'^{-1}\widetilde{\boldsymbol{\epsilon}}. \tag{3.14}$$

Lemma 3.3 and the constraints of the desired design implies that $X_{\widetilde{\mathbf{a}}}$ is the support a desired design if and only if Eq. (3.12) has a solution $\widetilde{\mathbf{x}}$ such that $\mathbf{1} + \widetilde{\mathbf{x}} > \mathbf{0}$. Moreover, the measure associated χ is uniquely determined by $\mathbf{1} + \widetilde{\mathbf{x}}$, and Eq. (3.10) is the expansion of $\mathbf{1} + \widetilde{\mathbf{x}}$ using Eq. (3.14) and the explicit formula for the inverse of the Vandermonde matrix $\widetilde{\mathbf{A}}'$.

Let $\widetilde{\delta}' := \operatorname{dist}_{\min}\widetilde{\mathbf{a}}'$. Clearly,

$$\widetilde{\delta}' \geq \delta' - 2\varepsilon. \tag{3.15}$$

Now, we estimate the L^∞-norm of $\widetilde{\mathbf{x}}'$.

$$
\begin{aligned}
\|\widetilde{\mathbf{x}}'\|_\infty =& \|n\widetilde{\mathbf{A}}'^{-1}\widetilde{\boldsymbol{\epsilon}}\|_\infty && \text{Eq. (3.14)}\\
=& \left\|\widetilde{\mathbf{A}}'^{-1}\left((\widetilde{\mathbf{A}}-\mathbf{A})\mathbf{1}+n\boldsymbol{\epsilon}\right)\right\|_\infty && \text{Eq. (3.11)}\\
\leq& \|\widetilde{\mathbf{A}}'^{-1}\|_\infty\left(\|\widetilde{\mathbf{A}}-\mathbf{A}\|_\infty\cdot\|\mathbf{1}\|_\infty+n\|\boldsymbol{\epsilon}\|_\infty\right) && \text{Property of norms}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\widetilde{\delta}'/2)^{t-1}} && \text{Lemma 3.5}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\delta'/2-\varepsilon)^{t-1}} && \text{Eq. (3.15)}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\delta'/2)^{t-1}-(t-1)\varepsilon} && 2\varepsilon<\delta'<2 \text{ when } t\geq 2 \text{ by Eq. (3.9)}\\
<& 1. && \text{Eq. (3.9)}
\end{aligned}
$$

Therefore, $\mathbf{1}+\widetilde{\mathbf{x}}>\mathbf{0}$, and the result follows from Lemma 3.3. □

Remark 3.7. It is clear that for the degree d elementary symmetric polynomial in $a_1,\ldots,a_n$, we have

$$e_d(a_1,\ldots,a_n)=\sum_{i_1=1}^{d}\sum_{i_2=i_1}^{d}\cdots\sum_{i_t=i_t-1}^{d}\prod_{j=1}^{t}a_{i_j}.$$

However, this expression uses d finite sums $\sum$. It is a little tricker to use only a constant number of finite sum $\sum$, finite product $\prod$ and other operations mentioned in the introduction to write down $e_d(a_1,\ldots,a_n)$, and we leave it as an exercise to the reader.

3.3 *Step 1* of Construction 1.1: Designs on $\mathcal{H}_0^1$

In § 3.3, we fix a positive integer $t\geq 2$, and let $n:=t$. Let $\mathbf{a}:=\{a_i:\ i\in[0,n)_\mathbb{Z}\}$ where

$$a_i:=\operatorname{Im}\zeta_{2n}^{-n+1+2i}=\sin\frac{-n+1+2i}{n}\frac{\pi}{2}. \tag{3.16}$$

The set $X_\mathbf{a}$ (see Eq. (3.1)) is semiantipodal (see Definition 2.4) since $a_i=a_{n-i-1}$. For each $i\in[0,n)_\mathbb{Z}$, we choose some to-be-determined approximation $\widetilde{a}_i$ of a_i so that $x_{\widetilde{a}_i}$ (see Eq. (3.1)) is a rational point on H^1 and $\widetilde{a}_i=\widetilde{a}_{n-i-1}$. We set $\widetilde{\mathbf{a}}:=\{\widetilde{a}_i:\ i\in[0,t)_\mathbb{Z}\}$, and the set $X_{\widetilde{\mathbf{a}}}$ is also semiantipodal. We fix a bijection $\sim:\ \mathbf{a}\to\widetilde{\mathbf{a}}$ where $a_i\mapsto\widetilde{a}_i$. Choose $\mathbf{a}':=\mathbf{a}$ and $\widetilde{\mathbf{a}}':=\widetilde{\mathbf{a}}$.

Then, $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ and $\widetilde{\mathbf{a}}'$ form the commutative diagram in Eq. (3.6), all of $X_\mathbf{a}$, $X_{\widetilde{\mathbf{a}}}$, $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, and $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are rational.

Lemma 3.8. *Let $\boldsymbol{\epsilon}$ be defined using $\mathbf{a}$ (see Eq. (3.18)) and $b_{0,j}$ (see Eq. (3.3)) as in Eq. (3.7). Then,*

$$\|\boldsymbol{\epsilon}\|_\infty=0.$$

Proof. It is well-known that vertices of a regular $2t$-gon give a spherical $(2t-1)$-design on $\mathcal{S}^1$, hence a $(t-1)$-design on $\mathcal{S}^1$. The set $X_\mathbf{a}$ (see Eq. (3.1)) is one half of the vertices of a regular $2t$-gon and $\mathcal{H}_0^1$ is one half of $\mathcal{S}^1$. By the symmetry, $X_\mathbf{a}$, when equipped with the counting measure, is a $(t-1)$-semidesign. The result follows immediately from Lemma 3.3. □

Corollary 3.9. *Let* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ *(see Definition 3.4(ii)). If*

$$\varepsilon < \frac{\pi^{2t}}{12 \cdot 2^t t^{2t}}, \tag{3.17}$$

then, applying Theorem 3.6 to $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ *and* $\widetilde{\mathbf{a}}'$*, we get an explicit finite semiantipodal rational-weighted rational* $(t-1)$*-semidesign on* $\mathcal{H}_0^1$*.*

Proof. Let $\delta' := \operatorname{dist}_{\min} \mathbf{a}'$ (see Definition 3.4(i)). Eq. (3.16) gives

$$\delta' = a'_{t-1} - a'_{t-2} = a_{t-1} - a_{t-2} = 2 \sin \frac{\pi}{2t} \sin \frac{\pi}{t}$$

and Eq. (3.17) gives an upper bound on ε. Lemma 3.8 gives $\|\boldsymbol{\epsilon}\|_\infty = 0$. It is easy to check that ε, δ' and $\|\boldsymbol{\epsilon}\|_\infty$ satisfy Eq. (3.9). Then, Theorem 3.6 gives an explicit weighted $(t-1)$-semidesign $\mathcal{X} = (X_{\widetilde{\mathbf{a}}}, \chi)$.

Consider the $[1,1]$-antipodal map in Definition 2.4. Since $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, the levelling space $\frac{1}{2}\left(-_{[1,1]}\mathcal{X}\right) + \frac{1}{2}\mathcal{X}$ is also a $(t-1)$-semidesign that satisfies conditions $\chi(X_{\widetilde{\mathbf{a}}}) = n$ and $\chi(x) = 1$ for $x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\widetilde{\mathbf{a}}'}$. According to the uniqueness of $\mathcal{X}$ showed by Theorem 3.6, $\mathcal{X} = \frac{1}{2}\left(-_{[1,1]}\mathcal{X}\right) + \frac{1}{2}\mathcal{X}$, hence $\mathcal{X}$ is semiantipodal.

Since every $x_{\widetilde{a}_i}$ is a rational point on H^1, $\mathcal{X}$ is a rational semidesign. Since every $\widetilde{a}_i$ is a rational number and every $b_{0,j}$ is a rational number (see Eq. (3.3)), every $\widetilde{\boldsymbol{\epsilon}}_j$ is a rational number (see Eq. (3.8)). Therefore, the explicit formula for χ (see Eq. (3.10)) shows that $\mathcal{X}$ is rational-weighted. The semidesign $\mathcal{X}$ is finite since its support $X_{\widetilde{\mathbf{a}}}$ is finite. ∎

Remark 3.10. One explicit choice for $\widetilde{a}_i$ with $i < (n-1)/2$ in Corollary 3.9 is

$$\widetilde{a}_i := \frac{2\widetilde{c}_i}{\widetilde{c}_i^2 + 1} \quad \text{where} \quad \widetilde{c}_i := \frac{\lfloor m c_i \rfloor}{m}, \quad c_i := \frac{1 - \sqrt{1 - a_i^2}}{a_i}, \quad \text{and} \quad m := 2^{t+2} t^{2t}.$$

The other $\widetilde{a}_i$'s are obtained from $\widetilde{a}_i = \widetilde{a}_{n-i-1}$.

3.4 *Step 2* of Construction 1.1: Designs on $\mathcal{H}_1^1$

In § 3.4, we fix a positive even integer t and fix a to-be-determined positive even integer n such that $n \geq t \geq 2$ and n/t is an odd integer. Let $\mathbf{a} := \{a_i : \ i \in [0, n)_{\mathbb{Z}}\}$ where

$$a_i := \frac{-n + 1 + 2i}{n}. \tag{3.18}$$

The set $X_{\mathbf{a}}$ (see Eq. (3.1)) is semiantipodal (see Definition 2.4) since $a_i = a_{n-i-1}$. For each $i \in [0, n)_{\mathbb{Z}}$, we choose some to-be-determined approximation $\widetilde{a}_i$ of a_i so that $x_{\widetilde{a}_i}$ (see Eq. (3.1)) is a rational point on H^1 and $\widetilde{a}_i = \widetilde{a}_{n-i-1}$. We set $\widetilde{\mathbf{a}} := \{\widetilde{a}_i : \ i \in [0, t)_{\mathbb{Z}}\}$, and the set $X_{\widetilde{\mathbf{a}}}$ is also semiantipodal. We fix a bijection $\sim \, : \, \mathbf{a} \to \widetilde{\mathbf{a}}$ where $a_i \mapsto \widetilde{a}_i$.

For each $i \in [0, t)_{\mathbb{Z}}$, let

$$a'_i := a_\ell = \frac{-t + 1 + 2i}{t}, \quad \text{where} \quad \ell := \frac{(2i+1)n - t}{2t} \in [0, n)_{\mathbb{Z}}, \tag{3.19}$$

and set $\mathbf{a}' := \{a'_i : \ i \in [0, t)_{\mathbb{Z}}\}$. Let $\widetilde{\mathbf{a}}'$ be the image of $\mathbf{a}$ under the bijection $\sim$. Since $a'_i = a'_{t-i-1}$ and $\widetilde{a}'_i = \widetilde{a}'_{t-i-1}$, both $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal.

Then, as in case $\mathcal{H}_0^1$, we also get the commutative diagram Eq. (3.6), all of $X_{\mathbf{a}}$, $X_{\widetilde{\mathbf{a}}}$, $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, and $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are rational.

Lemma 3.11. *Let $\boldsymbol{\epsilon}$ be defined using* $\mathbf{a}$ *(see Eq. (3.18)) and $b_{1,j}$ (see Eq. (3.3)) as in Eq. (3.7). Then,*

$$\|\boldsymbol{\epsilon}\|_\infty < \frac{t}{4n^2}.$$

Proof. When j is odd, $\frac{1}{n}\sum_{i=0}^{n-1} a_i^j = 0$ since $X_{\mathbf{a}}$ is semiantipodal, and Eq. (3.3) gives $b_{1,j} = 0$, hence $\boldsymbol{\epsilon}_j = S - b_{1,j} = 0$. From now on, assume that j is even.

The sum of powers of first n integers can be calculated using Bernoulli's formula:

$$\sum_{i=1}^{n} i^j = \frac{1}{j+1}\sum_{k=0}^{j}(-1)^k\binom{j+1}{k}B_k n^{j+1-k}. \tag{3.20}$$

Applying this formula to a_i's, we get

$$\begin{aligned}
\boldsymbol{\epsilon}_j &= \frac{2}{n}\sum_{i=n/2}^{n-1} a_i^j - \frac{1}{j+1} && \text{Eq. (3.3), } a_i = -a_{n-1-i}\\
&= \frac{2}{n^{j+1}}\left(\sum_{i=1}^{n} i^j - 2^j\sum_{i=1}^{n/2} i^j\right) - \frac{1}{j+1} && \text{Eq. (3.18)}\\
&= \frac{1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k}(-1)^k B_k(2-2^k)n^{-k} - \frac{1}{j+1} && \text{Eq. (3.20)}\\
&= -\frac{1}{j+1}\sum_{\text{even } k=2}^{j}\binom{j+1}{j}B_k(2^k-2)n^{-k}. && B_0 = 1 \text{ and } B_k = 0 \text{ for odd } k \geq 3
\end{aligned}$$

For positive even k, [AS64, 23.1.15] gives a bound on Bernoulli number:

$$|B_k| < \frac{2k!}{(2\pi)^k}\frac{1}{1-2^{1-k}}. \tag{3.21}$$

We use this bound to estimate $\boldsymbol{\epsilon}_j$.

$$\begin{aligned}
|\boldsymbol{\epsilon}_j| &\leq \frac{1}{j+1}\sum_{\text{even k}=2}^{j}\binom{j+1}{k}|B_k|(2^k-2)n^{-k}\\
&< \frac{1}{j+1}\sum_{\text{even k}=2}^{\infty}\frac{(j+1)^k}{k!}\frac{2k!}{(2\pi)^k}\frac{1}{1-2^{1-k}}(2^k-2)n^{-k} && \text{Eq. (3.21)}\\
&= \frac{2}{j+1}\sum_{\text{even } k=2}^{\infty}\left(\frac{j+1}{\pi n}\right)^k = \frac{2(j+1)}{(\pi n)^2-(j+1)^2}\\
&\leq \frac{2(j+1)}{(\pi n)^2-n^2} < \frac{j+1}{4n^2}. && j+1 \leq t \leq n
\end{aligned}$$

Therefore, $\|\boldsymbol{\epsilon}\|_\infty = \min_{j\in[0,t)_{\mathbb{Z}}}|\boldsymbol{\epsilon}_j| < \frac{t}{4n^2}$. □

Corollary 3.12. *Let $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ (see Definition 3.4(ii)). If*

$$\varepsilon < \frac{1}{t^{t+1}} - \frac{1}{4nt} \tag{3.22}$$

then, applying Theorem 3.6 to $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ *and* $\widetilde{\mathbf{a}}'$*, we get an explicit finite semiantipodal rational-weighted rational $(t-1)$-semidesign on $\mathcal{H}_1^1$.*

Proof. We follow the the strategy as in the proof of Corollary 3.9. Let $\delta' := \operatorname{dist}_{\min} \mathbf{a}'$ (see Definition 3.4(i)). Eq. (3.19) gives

$$\delta' = a'_{t-1} - a'_{t-2} = \frac{2}{t},$$

Eq. (3.22) gives an upper bound on ε and Lemma 3.11 gives an upper bound for $\|\boldsymbol{\epsilon}\|_\infty$. The result follows from similar arguments in Corollary 3.9. $\square$

Remark 3.13. One explicit choice for n is $n := (t^{t-1}+1)t$. For this particular choice of n, one explicit choice for $\widetilde{a}_i$ with $i < (n-1)/2$ in Corollary 3.12 is

$$\widetilde{a}_i := \frac{2\widetilde{c}_i}{\widetilde{c}_i^2 + 1} \quad \text{where} \quad \widetilde{c}_i := \frac{\lfloor mc_i \rfloor}{m}, \quad c_i := \frac{1 - \sqrt{1 - a_i^2}}{a_i} \quad \text{and} \quad m := 3t^{t+1}.$$

The other $\widetilde{a}_i$'s are obtained from $\widetilde{a}_i = \widetilde{a}_{n-i-1}$.

4 Designs on spheres and hemispheres

§ 4.1 studies the structure of spheres and hemispheres. Proposition 4.2 allows us to view spheres and hemispheres as the products of lower dimensional spheres and hemispheres. In each of § 4.2, § 4.3, § 4.4, § 4.5, and § 4.6, we show a different type of construction of designs in general spaces and explain why we need it in Construction 1.1. Then, we specialize these constructions to spheres and hemispheres cases and get *Step 3*, *Step 4*, *Step 5*, *Step 6* and *Step 7* of Construction 1.1, respectively.

4.1 Structure of spheres and hemispheres

Let a and b be two natural numbers. Consider the double branched cover of topological spaces

$$\begin{array}{ccccccc} \iota_{a,b}: & S^a & \times & S^b & \to & S^{a+b} \\ & (x_0,\ldots,x_a) & \times & (y_0,\ldots,y_b) & \mapsto & (x_0y_0,\ldots,x_ay_0,y_1\ldots,y_b). \end{array}$$

The map $\iota_{a,b}$ induces a dominant open embedding (see § 2.2) of topological spaces

$$\iota_{a,b}: \ S^a \times H^b \to S^{a+b} \tag{4.1}$$

and an isomorphism of topological spaces

$$\iota_{a,b}: \ H^a \times H^b \to H^{a+b}. \tag{4.2}$$

Remark 4.1. The map $\iota_{a,b}$, regarded as a product operator, is associative in the sense that we have the following commutative diagram.

$$\begin{tikzcd} H^a \times H^b \times H^c \arrow[r, "\iota_{a,b}\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times\iota_{b,c}"] & H^{a+b} \times H^c \arrow[d, "\iota_{a+b,c}"] \\ H^a \times H^{b+c} \arrow[r, "\iota_{a,b+c}"] & H^{a+b+c} \end{tikzcd}$$

By associativity, $\iota_{a,b}$ induces an isomorphism of topological spaces

$$\iota_{1^d}: \ \underbrace{H^1 \times \cdots \times H^1}_{d \text{ copies of } H^1} \to H^d, \tag{4.3}$$

which can be explicitly described as

$$((x_{i,0}, x_{i,1}) : \ i \in [1,d]_{\mathbb{Z}}) \mapsto \left(x_{i,1} \prod_{j=i+1}^{d} x_{j,0} : \ i \in [0,d]_{\mathbb{Z}} \right), \quad \text{where } x_{0,1} := x_{1,0}.$$

When we equip the spheres and hemispheres with suitable measures, we can make the map in Eqs. (4.1) to (4.3) dominant open embedding or isomorphisms of levelling spaces (see § 2.2).

Proposition 4.2. *Let a, b, s be natural numbers. The following statements hold.*

(i) *The map $\iota_{a,b}$ in Eq. (4.1) induces a dominant open embedding of levelling spaces*

$$\iota_{a,b} : \ \mathcal{S}^a \times \mathcal{H}^b_a \to \mathcal{S}^{a+b}.$$

(ii) *The map $\iota_{a,b}$ in Eq. (4.2) induces an isomorphism of levelling spaces*

$$\iota_{a,b} : \ \mathcal{H}^a_s \times \mathcal{H}^b_{a+s} \to \mathcal{H}^{a+b}_s.$$

(iii) *The map $\iota_{(1^d)}$ in Eq. (4.3) induces an isomorphism of levelling spaces*

$$\iota_{1^d} : \ \mathcal{H}^1_s \times \cdots \times \mathcal{H}^1_{s+d-1} \to \mathcal{H}^d_s.$$

Proof. (ii) Using the parametrization $(x_1, \ldots, x_d) \mapsto (x_0, \ldots, x_d)$ of the hypersurface $\mathcal{H}^d_s$, for all continuous measurable function f on $\mathcal{H}^d_s$,

$$\int_{H^d} f \,\mathrm{d}\,\nu^d_s = \int_{B^d} x_0^{s-1} f(x_0, \ldots, x_d) \,\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_d, \tag{4.4}$$

where $B^d := \{(x_1, \ldots, x_d) : \ x_1^2 + \cdots + x_d^2 < 1\}$ is the a-dimensional unit open ball. With Eq. (4.4) and separation of variables, it is easy to check that for all continuous integrable function f on $\mathcal{H}^{a+b}_s$,

$$\int_{H^{a+b}} f \,\mathrm{d}\,\nu^{a+b}_s = \int_{H^b} \left(\int_{H^a} \iota^*_{a,b} f \,\mathrm{d}\,\nu^a_s \right) \mathrm{d}\,\nu^b_{a+s}, \quad \text{where} \quad \iota^*_{a,b} : \ f \mapsto f \circ \iota_{a,b},$$

from which the result follows.
(i): This can be proved similarly as in (ii).
(iii): Since $\iota_{a,b}$, regarded as a product operator, is associative, the result follows from an induction on (ii). $\square$

Recall that the image $\iota(\mathcal{X})$ of a levelling space $\mathcal{X}$ under a map ι is defined in § 2.2.

Lemma 4.3. *Suppose that $\iota : \mathcal{W} \to \mathcal{Z}$ is a dominant open embedding of levelling spaces (see § 2.2). Let V be a real vector space of continuous integrable function on $\mathcal{Z}$. Then, for every $\iota^* V$-design $\mathcal{X}$ on $\mathcal{W}$, $\iota(\mathcal{X})$ is a V-design on $\mathcal{Z}$.*

Proof. For arbitrary $f \in V$,

$$\begin{aligned}
\frac{1}{|\iota(\mathcal{X})|} \int_{\iota(X)} f \,\mathrm{d}\,\iota_* \mu_X &= \frac{1}{|\,\mathcal{X}\,|} \int_X \iota^* f \,\mathrm{d}\,\mu_X && \text{Eq. (2.5)} \\
&= \frac{1}{|\,\mathcal{W}\,|} \int_W \iota^* f \,\mathrm{d}\,\mu_W && \iota^* f \in \iota^* V,\ \mathcal{X} \text{ is an } \iota^* V\text{-design} \\
&= \frac{1}{|\iota(\mathcal{W})|} \int_{\iota(W)} f \,\mathrm{d}\,\iota_* \mu_W && \text{Eq. (2.5)} \\
&= \frac{1}{|\,\mathcal{Z}\,|} \int_Z f \,\mathrm{d}\,\mu_Z. && \iota \text{ is a dominant open embedding}
\end{aligned}$$

Therefore, $\iota(\mathcal{X})$ is a V-design on $\mathcal{Z}$. $\square$

Remark 4.4. In the remaining sections, we regard $\mathcal{S}^a \times \mathcal{H}^b_a$ as a subspace of $\mathcal{S}^{a+b}$ using the dominant open embedding $\mathcal{S}^a \times \mathcal{H}^b_a \to \mathcal{S}^{a+b}$ in Proposition 4.2(i), and regard designs on $\mathcal{S}^a \times \mathcal{H}^b_a$ as designs on $\mathcal{S}^{a+b}$ using Lemma 4.3 without explicitly mentioning it. We do similar identifications for the dominant open embeddings $\mathcal{H}^a_s \times \mathcal{H}^b_{a+s} \to \mathcal{H}^{a+b}_s$ in Proposition 4.2(ii) and $\mathcal{H}^1_s \times \cdots \times \mathcal{H}^1_{s+d-1} \to \mathcal{H}^d_a$ in Proposition 4.2(iii).

4.2 *Step 3* of Construction 1.1: Lifts of designs

The goal of *Step 3* is to reduce the need of constructing designs on $\mathcal{H}^1_s$ for all positive integers $s < d$ down to constructing designs for only finitely many s independent with the choice of d. For our purpose, we only need construct design manually for $s = 0$, which is done in *Step 1*, and for $s = 1$, which is done in *Step 2*.

Recall that the Radon-Nikodym derivative is defined in § 2.1.

Lemma 4.5. *Let Z be a measurable space, $\mathcal{Z}_0 = (Z, \mu_{Z_0})$ and $\mathcal{Z}_1 = (Z, \mu_{Z_1})$ two levelling spaces on Z, and $\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}$ the Radon-Nikodym derivative of μ_{Z_1} with respect to μ_{Z_0}. Let V_0 (resp. V_1) be a real vector space of continuous integrable functions on $\mathcal{Z}_0$ (resp. V_1). Assume that*

$$V_0 \supseteq \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} V_1 := \left\{ \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} f :\ f \in V_1 \right\}.$$

Let $\mathcal{X}_0 = (X, \mu_{X_0})$ be a V_0-design on $\mathcal{Z}_0$. Then, $\mathcal{X}_1 := (X, \mu_{X_1})$ with

$$\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} = \frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}} \tag{4.5}$$

is a V_1-design on $\mathcal{Z}_1$.

Proof. Let f be an arbitrary function in V_1. Then,

$$\begin{aligned}
\frac{1}{|\mathcal{Z}_1|}\int_Z f\,\mathrm{d}\,\mu_{Z_1} &= \left(\int_Z f\,\mathrm{d}\,\mu_{Z_1}\right) / \left(\int_Z 1\,\mathrm{d}\,\mu_{Z_1}\right) \\
&= \left(\int_Z f\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}\,\mathrm{d}\,\mu_{Z_0}\right) / \left(\int_Z 1\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}\,\mathrm{d}\,\mu_{Z_0}\right) && \text{Eq. (2.1)} \\
&= \left(\int_X f\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}\,\mathrm{d}\,\mu_{X_0}\right) / \left(\int_X 1\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}\,\mathrm{d}\,\mu_{X_0}\right) && f\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}, 1\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \in V_0,\ \mathcal{X}_0 \text{ is a } V_0\text{-design} \\
&= \left(\int_X f\frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}}\,\mathrm{d}\,\mu_{X_0}\right) / \left(\int_X 1\frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}}\,\mathrm{d}\,\mu_{X_0}\right) && \text{Eq. (4.5)} \\
&= \left(\int_X f\,\mathrm{d}\,\mu_{X_1}\right) / \left(\int_X 1\,\mathrm{d}\,\mu_{X_1}\right) = \frac{1}{|\mathcal{X}_1|}\int_X f\,\mathrm{d}\,\mu_{X_1} && \text{Eq. (2.1)}
\end{aligned}$$

which proves that $\mathcal{X}_1$ is a V_1-design on $\mathcal{Z}_1$. ☐

Corollary 4.6. *Let s and $\widetilde{s}$ be two natural numbers such that $s - \widetilde{s}$ is an even natural number. Let $\mathcal{X}^1_{\widetilde{s}} = (X, \mu^1_{\widetilde{s}})$ be a finite semiantipodal rational-weighted rational $(t+s-\widetilde{s})$-semidesign on $\mathcal{H}^1_{\widetilde{s}}$. Then, $\mathcal{X}^1_s := (X, \mu^1_s)$, where*

$$\mu^d_s(x_0, x_1) := x_0^{s-\widetilde{s}} \mu^d_{\widetilde{s}}(x_0, x_1). \tag{4.6}$$

is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}^1_s$, *Proof.* By Definition 2.1, the Radon-Nikodym derivative $\mathrm{d}\,\nu_s^1/\,\mathrm{d}\,\nu_{\widetilde{s}}^1$ is $g_{s-\widetilde{s}}$, which is a degree $s-\widetilde{s}$ polynomial in $\mathcal{P}^{\infty,0}[H^1]$ since $s-\widetilde{s}$ is an even natural number. Eq. (4.6) shows that the Radon-Nikodym derivative $\mathrm{d}\,\mu_s^1/\,\mathrm{d}\,\mu_{\widetilde{s}}^1$ is $g_{s-\widetilde{s}}$ as well. Since

$$g_{s-\widetilde{s}}\mathcal{P}^{t,0}[H^1] \subseteq \mathcal{P}^{t+s-\widetilde{s},0}[H^1],$$

by Lemma 4.5, $\mathcal{X}_s^1$ is a weighted t-semidesign on $\mathcal{H}_s^1$. The semiantipodal and finite property of $\mathcal{X}_s^1$ inherits from the semiantipodal and finite property of $\mathcal{X}_{\widetilde{s}}^1$, and the rationality of $\mathcal{X}_s^1$ follows from the rationality of $\mathcal{X}_{\widetilde{s}}^1$ and Eq. (4.6). □

Remark 4.7. It is straightforward to generalize Corollary 4.6 to higher dimensional hemispheres $\mathcal{H}_s^d$ and $\mathcal{H}_{\widetilde{s}}^d$.

4.3 *Step 4* of Construction 1.1: Products of designs

The goal of *Step 4* is to construct designs on high dimensional hemispheres using known designs on semicircles which we constructed in *Step 3*.

Recall that the product of levelling spaces is defined in § 2.2.

Lemma 4.8. *For each $i \in \{0,1\}$, let $\mathcal{Z}_i$ be a levelling space, V_i a real vector space of continuous integrable functions on $\mathcal{Z}_i$, and $\mathcal{X}_i$ a V_i-design on $\mathcal{Z}_i$. Then, $\mathcal{X}_0 \times \mathcal{X}_1$ is a $V_0 \otimes V_1$-design on $\mathcal{Z}_0 \times \mathcal{Z}_1$.*

Proof. For arbitrary $f_0 \in V_0$ and $f_1 \in V_1$,

$$\begin{aligned}
&\frac{1}{|\,\mathcal{X}_0 \times \mathcal{X}_1\,|}\int_{X_0\times X_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{X_0} \times \mu_{X_1} \\
=&\left(\frac{1}{|\,\mathcal{X}_0\,|}\int_{X_0} f_0 \,\mathrm{d}\,\mu_{X_0}\right)\left(\frac{1}{|\,\mathcal{X}_1\,|}\int_{X_1} f_1 \,\mathrm{d}\,\mu_{X_1}\right) && \text{Eq. (2.4)}\\
=&\left(\frac{1}{|\,\mathcal{Z}_0\,|}\int_{Z_0} f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\frac{1}{|\,\mathcal{Z}_1\,|}\int_{Z_1} f_1 \,\mathrm{d}\,\mu_{Z_1}\right) && \mathcal{X}_i \text{ is a } V_i\text{-design on } \mathcal{Z}_i\\
=&\frac{1}{|\,\mathcal{Z}_0 \times \mathcal{Z}_1\,|}\int_{Z_0\times Z_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{Z_0} \times \mu_{Z_1}. && \text{Eq. (2.4)}
\end{aligned}$$

Since $V_0 \otimes V_1$ is generated by functions of the form $f_0 \otimes f_1$, $\mathcal{X}_0 \times \mathcal{X}_1$ is a $V_0 \otimes V_1$-design. □

Recall that we say a levelling space $\mathcal{X}$ is semiantipodal if it is stable under a certain antipodal map (see Definition 2.4).

Lemma 4.9. *Let $\mathcal{X}_s^a$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^a$ and let $\mathcal{X}_{a+s}^b$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_{a+s}^b$. Then, $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$, regarded as a subspace of $\mathcal{H}_s^{a+b}$ using Remark 4.4, is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^{a+b}$.*

Proof. We label the coordinates of H^a, H^b and H^{a+b} as follow

$$H^a = \{(x_0,\dots,x_a)\}, \quad H^b = \{(y_0,\dots,y_b)\}, \quad H^{a+b} = \{(z_0,\dots,z_{a+b})\}.$$

Since both $\mathcal{X}_s^a$ and $\mathcal{X}_{a+s}^b$ are semiantipodal, the product $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$, regarded as a subspace of $\mathcal{H}_s^{a+b}$ by Remark 4.4, is both semiantipodal and $[1,a]$-antipodal.

Recall that $\mathcal{P}^{t,0}$ and $\mathcal{P}^{t,0,a}$ are defined in Eq. (2.6), and $\mathcal{P}^{t,0,a}$ is a subfunctor of $\mathcal{P}^{t,0}$. Consider the dominant open embedding $\iota_{a,b}$ in Proposition 4.2(ii). Its comorphism, the pullback $\iota_{a,b}^*$, gives an inclusion

$$\begin{aligned} \iota_{a,b}^*: \mathcal{P}^{t,0,a}[H^{a+b}] &\hookrightarrow \mathcal{P}^{t,0}[H^a] \otimes \mathcal{P}^{t,0}[H^b] \\ z_i &\mapsto \begin{cases} x_i \otimes y_0, & i \in [1,a]_{\mathbb{Z}}, \\ 1 \otimes y_{i-a}, & i \in [a+1, a+b]_{\mathbb{Z}}. \end{cases} \end{aligned}$$

Note that although y_0 is not in $\mathcal{P}^{t,0}[H^b]$, any monomial in $\mathcal{P}^{t,0,a}[H^{a+b}]$ maps to some monomial with even degree in y_0, hence in $\mathcal{P}^{t,0}[H^b]$.

Since $\mathcal{X}_s^a$ is a $\mathcal{P}^{t,0}[H^a]$-design on $\mathcal{H}_s^a$ and $\mathcal{X}_s^a$ is a $\mathcal{P}^{t,0}[H^a]$-design on $\mathcal{H}_{a+s}^b$, according to Lemma 4.8, $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$ is a $\mathcal{P}^{t,0}[H^a] \otimes \mathcal{P}^{t,0}[H^a]$-design on $\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b$, hence a $\mathcal{P}^{t,0,a}[H^{a+b}]$-design on $\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b$ by the comorphism $\iota_{a,b}^*$. According to Lemma 2.5, the $[1,a]$-antipodal $\mathcal{P}^{t,0,a}[H^{a+b}]$-design $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$ is a weighted t-semidesign on $\mathcal{H}_s^{a+b}$.

Since both of $\mathcal{X}_s^a$ and $\mathcal{X}_{a+s}^b$ are finite rational-weighted rational, their product is finite rational-weighted and rational as well. $\square$

Corollary 4.10. *For each $i \in [s, s+b)_{\mathbb{Z}}$, let $\mathcal{X}_i^1$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_i^1$. Then, $\mathcal{X}_s^1 \times \cdots \times \mathcal{X}_{s+b-1}^1$, regarded as a subspace of $\mathcal{H}_s^b$ using Remark 4.4, is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^b$.*

Proof. The result follows from Proposition 4.2(iii) and Lemma 4.9. $\square$

4.4 *Step 5* of Construction 1.1: Weights of designs

The goal of *Step 5* is to view a rational-weighted design as a design with repeated points. Many known constructions of spherical design first construct designs with repeated points, and then try to apply some separation result to get designs.

Recall that scalars of a levelling space is defined in § 2.2.

Lemma 4.11. *Let $\mathcal{X} = (X, \mu_X)$ be a finite rational-weighted V-design on $\mathcal{Z}$. Then, $\overline{\mathcal{X}} := c\,\mathcal{X}$ is an integer-weighted V-design on $\mathcal{Z}$, where*

$$c := \operatorname{lcm}_{x \in X} \textit{denominator of } \mu_X(x).$$

Proof. For arbitrary $f \in V$,

$$\frac{1}{|c\,\mathcal{X}\,|} \int_X f \,\mathrm{d}\, c\mu_X = \frac{1}{|\,\mathcal{X}\,|} \int_X f \,\mathrm{d}\, \mu_X. \qquad \textit{Eq. (2.2)}$$

Therefore, $\overline{\mathcal{X}}$ is a V-design on $\mathcal{Z}$. Since for every $x \in X$, $(c\mu_X)(x)$ is an integer, $\overline{\mathcal{X}}$ is integer-weighted. $\square$

Remark 4.12. In Lemma 4.11, it suffices to choose a positive integer c that is an integer multiple of the lcm. For instance, we can choose

$$c := \prod_{x \in X} \text{denominator of } \mu_X(x).$$

Corollary 4.13. *Let $\mathcal{X}_s^d$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^d$. Then, $\overline{\mathcal{X}}_s^d$ is an semiantipodal integer-weighted rational t-semidesign on $\mathcal{H}_s^d$.*

Proof. This is immediate corollary of Lemma 4.11. □

Remark 4.14. By the definition of levelling spaces, an integer-weighted levelling space is automatically finite.

Remark 4.15. In *Step 5*, we apply Corollary 4.13 to explicit designs we constructed in *Step 4*. For those designs, we can write down an explicit formula of a multiple of denominators of $\mu_X(x)$. Therefore, the use of the operations of taking denominators of rational numbers and taking least common multiples of integers can be eliminated.

4.5 *Step 6* of Construction 1.1: Designs at generic position

The goal of *Step 6* is to put the designs to a "generic position", for our purpose, nonzero first coordinate, so that it would be easier to write down explicit formulas in *Step 7*.

Recall that the image of a levelling space is defined in § 2.2.

Lemma 4.16. *Let $x = (x_0, \ldots, x_d) \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ be a point such that $|x_i| \leq 1$. Let δ be a positive real number that is no greater than the absolute values of nonzero coordinates of x. Let ϵ be an arbitrary rational number such that $0 < \epsilon \leq \delta/3$, and consider the reflection $s_\alpha \in O(d+1, \mathbb{Q})$:*

$$s_\alpha(x) := x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha, \quad \text{where} \quad \alpha := (\epsilon^i : \ i \in [0, d]_{\mathbb{Z}}). \tag{4.7}$$

Then, $s_\alpha(x)$ has nonzero first coordinate.

Proof. Let $s_\alpha(x)_0$ be the first coordinate of $s_\alpha(x)$, and let $k \in [0, d]_{\mathbb{Z}}$ be the smallest index such that $x_k \neq 0$. Then,

$$\begin{aligned}
|s_\alpha(x)_0| =& \left| x_0 - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha_0 \right| && \text{Eq. (4.7)} \\
=& \frac{1-\epsilon^{2d+2}}{1-\epsilon^2} \left| \sum_{i=0}^{d} (2-\epsilon^i)\epsilon^i x_i \right| && \text{Eq. (4.7)} \\
\geq& \frac{1-\epsilon^{2d+2}}{1-\epsilon^2} \left((2-\epsilon^k)\epsilon^k |x_k| - \sum_{i=k+1}^{d} (2-\epsilon^i)\epsilon^i |x_i| \right) && \text{definition of } k \\
>& \frac{1-\epsilon^{2d+2}}{1-\epsilon^2} \left((2-\epsilon^k)\epsilon^k 3\epsilon - \sum_{i=k+1}^{\infty} (2-\epsilon^i)\epsilon^i \right) && |x_k| \geq \delta \geq 3\epsilon,\ |x_i| \leq 1 \\
=& \frac{(1-\epsilon^{2d+2})\epsilon^{k+1}}{(1-\epsilon^2)^2} \left((4 - 2\epsilon - 6\epsilon^2) - \epsilon^k(3 - \epsilon + 3\epsilon^2) \right) && \\
\geq& \frac{(1-\epsilon^{2d+2})\epsilon^{k+1}}{(1-\epsilon^2)^2} \left(1 + \epsilon - 9\epsilon^2 \right) && \epsilon \leq \delta/3 \leq 1/3, \epsilon^k \leq 1 \\
>& 0. && \epsilon \leq \delta/3 \leq 1/3
\end{aligned}$$

Therefore, $s_\alpha(x)$ has nonzero first coordinate. □

Corollary 4.17. *Let $\mathcal{Y}^d$ be an antipodal t-design on $\mathcal{S}^d$ over a field $\mathbb{F} \subseteq \mathbb{R}$. Let $\delta \in \mathbb{F}$ be a positive number that is no greater than the absolute values of nonzero coordinates of points in $\mathcal{Y}^d$. Let ϵ be an arbitrary rational number such that $0 < \epsilon \leq \delta/3$, and consider the reflection $s_\alpha \in O(d+1, \mathbb{Q})$ defined in Eq. (4.7). Then, $s_\alpha \mathcal{Y}^d$ is an antipodal t-design on $\mathcal{S}^d$ over $\mathbb{F}$ such that all points in the design have nonzero first coordinates.*

Proof. Orthogonal transformations over $\mathbb{Q}$ preserves antipodal spherical designs over $\mathbb{F}$, hence $s_\alpha \mathcal{Y}^d$ is a t-design on $\mathcal{S}^d$ over $\mathbb{F}$. According to Lemma 4.16, for an arbitrary point $x \in \mathcal{Y}^d$, the first coordinate of $s_\alpha(x)$ is nonzero. ☐

Remark 4.18. The finite minimum is used to define δ in Corollary 4.17. However, if we apply our construction to explicit designs, as in Theorem 1.2, the minimum operator can be avoided. For Theorem 1.2(i), (ii), (iii) and (iv), we have $\delta = \mathrm{Im}\zeta_{4(t+1)}$, $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{8}$, respectively. An explicit choice of ϵ is $\epsilon := 1/\lceil 3/\delta \rceil$.

4.6 *Step 7* of Construction 1.1: Twisted products of designs

After *Step 5*, we can view our weighted designs are designs with repeated points. Using topology to separate the repeated points, we can get existence of designs. Using constructive topology to separate the repeated points, we get computable designs. [CXX17] uses analytic number theory to separate the repeated points, which gives algorithms that finds designs. *Step 7* uses extra dimensions to separate the repeated points, which lead us to explicit designs.

Recall that in § 2.2 we define integer-weighted and $\underline{1}$-weighted levelling spaces, and we define the maps, sums, products and images of levelling spaces. For a levelling space $\mathcal{Z}$, let $\mathrm{Aut}(\mathcal{Z})$ be the set of all automorphisms of $\mathcal{Z}$.

It is easy to see that, for an integer-weighted levelling space $\mathcal{X} = (X, \mu_X)$, we have

$$\mathcal{X} = \sum_{x\in X} \sum_{i=1}^{\mu_X(x)} \boldsymbol{x}, \quad X = \bigcup_{x\in X} \bigcup_{i=1}^{\mu_X(x)} \{x\} \quad \text{and} \quad \mu_X = \sum_{x\in X} \sum_{i=1}^{\mu_X(x)} \underline{1}_x, \tag{4.8}$$

where $\boldsymbol{x}$ is understood as the levelling space with support $\{x\}$ equipped with the counting measure $\underline{1}_x$ (see § 2.2) on $\{x\}$.

Definition 4.19. Let $\mathcal{Y} \subseteq \mathcal{Z}$ be a levelling space and $\mathcal{X} = (X, \mu_X)$ an integer-weighted levelling space. Let $\xi : \mathbb{N} \to \mathrm{Aut}(\mathcal{Z})$ be a map. The *twisted product* of $\mathcal{Y}$ and $\mathcal{X}$ with respect to ξ is the levelling space

$$\mathcal{Y} \rtimes_\xi \mathcal{X} := \sum_{x\in X} \sum_{i=1}^{\mu_X(x)} \xi(i)\, \mathcal{Y} \times \boldsymbol{x}.$$

In other words, the twisted product is $\mathcal{Y} \rtimes_\xi \mathcal{X} := (Y \rtimes_\xi X, \mu_Y \rtimes_\xi \mu_X)$, where

$$Y \rtimes_\xi X := \bigcup_{x\in X} \bigcup_{i=1}^{\mu_X(x)} \xi(i) Y \times \{x\} \quad \text{and} \quad \mu_Y \rtimes_\xi \mu_X := \sum_{x\in X} \sum_{i=1}^{\mu_X(x)} \xi(i)_* \mu_Y \times \underline{1}_x.$$

Since $\mathcal{X}$ is integer-weighted and the total measure is finite, all sums are finite sums.

The total measures satisfy the equation $|\,\mathcal{Y} \rtimes_\xi \mathcal{X}\,| = |\,\mathcal{Y}\,| \cdot |\,\mathcal{X}\,|$. When ξ is the constant function with image the identity automorphism of $\mathcal{Z}$, the twisted product is just the ordinary product, namely, $\mathcal{Y} \rtimes_\xi \mathcal{X} = \mathcal{Y} \times \mathcal{X}$.

Theorem 4.20. *Let $\mathcal{Z}_0$ and $\mathcal{Z}_1$ be levelling spaces, and let V_0 and V_1 be real vector spaces of continuous integrable functions on $\mathcal{Z}_0$ and $\mathcal{Z}_1$, respectively. Let $\mathcal{X}_0$ be a V_0-design on $\mathcal{Z}_0$ and $\mathcal{X}_1$ an integer-weighted V_1-design on $\mathcal{Z}_1$. Let $\xi : \mathbb{N} \to \mathrm{Aut}(\mathcal{Z})$ be a map and assume that $\mathrm{Aut}(\mathcal{Z})$ preserves V_0. Then, the twisted product $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1$ is a $V_0 \otimes V_1$-design on $\mathcal{Z}_0 \times \mathcal{Z}_1$. Moreover, if $\mathcal{X}_0$ is $\underline{1}$-weighted and $\bigcup_{i\in\mathbb{N}} \xi(i) X_0$ is a disjoint union, then $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1$ is $\underline{1}$-weighted.*

Proof. For arbitrary $f_0 \in V_0$ and $f_1 \in V_1$,

$$\begin{aligned}
&\frac{1}{|\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1|}\int_{X_0 \rtimes_\xi X_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{X_0} \rtimes_\xi \mu_{X_1} \\
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\int_{\xi(i)X_0\times x_1} f_0\otimes f_1 \,\mathrm{d}\,\xi(i)_*\mu_{X_0}\times \underline{1}_{x_1} && \text{Definition 4.19 and Eq. (2.3)}\\
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{\xi(i)X_0} f_0 \,\mathrm{d}\,\xi(i)_*\mu_{X_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) && \text{Eq. (2.4)}\\
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{X_0} \xi(i)^*f_0 \,\mathrm{d}\,\mu_{X_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) && \text{Eq. (2.5)}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{Z_0} \xi(i)^*f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) && \xi(i)^*f_0\in V_0,\ \mathcal{X}_0 \text{ is a } V_0\text{-design}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{\xi(i)Z_0} f_0 \,\mathrm{d}\,\xi(i)_*\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) && \text{Eq. (2.5)}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{Z_0} f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) && \xi(i) \text{ is an automorphism}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\left(\int_{Z_0} f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{X_1} f_1\,\mathrm{d}\,\mu_{X_1}\right) && \text{Eqs. (2.3) and (4.8)}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{Z}_1|}\left(\int_{Z_0} f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{Z_1} f_1\,\mathrm{d}\,\mu_{Z_1}\right) && f_1\in V_1,\ \mathcal{X}_1 \text{ is a } V_1\text{-design}\\
=&\frac{1}{|\mathcal{Z}_0\times\mathcal{Z}_1|}\int_{Z_0\times Z_1} f_0\otimes f_1\,\mathrm{d}\,\mu_{Z_0}\times\mu_{Z_1}. && \text{Eq. (2.4)}
\end{aligned}$$

Therefore, the twisted product is a $V_0 \otimes V_1$-design on $\mathcal{Z}_0 \times \mathcal{Z}_1$.

When $\mathcal{X}_0$ is $\underline{1}$-weighted, the levelling space $\xi(i)\,\mathcal{X}_0 \times x_1$ is also $\underline{1}$-weighted. Since $\bigcup_{i\in\mathbb{N}}\xi(i)X_0$ is a disjoint union, $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1 = \sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\xi(i)\,\mathcal{X}_0\times x_1$ is a disjoint union as well. Therefore, $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1$ is $\underline{1}$-weighted. □

Let Z be a metric space. Denote $\mathrm{Iso}(Z)$ the set of all isometries of Z. It is easy to see that $\mathrm{Iso}(\mathcal{S}^d) = O(d+1,\mathbb{R}) \subseteq \mathrm{Aut}(\mathcal{S}^d)$.

For a real number s, let $\mathrm{Iso}_s(Z)$ be the set of all isometries g of Z such that $\mathrm{dist}(gz,z) < s$ for all $z \in Z$. Recall that, we define in Definition 3.4(i) that, for a subset $X \subseteq Z$, $\mathrm{dist}_{\min} X$ is the infimum of distances between any two distinct points of X.

Lemma 4.21. *Let X be a set in a metric space Z and $s := (\mathrm{dist}_{\min} X)/2$. Then, $\bigcup_g gX$ is a disjoint union where g runs over $\mathrm{Iso}_s(Z)$.*

Proof. Let $g_0, g_1 \in \mathrm{Iso}_s(Z)$ be two arbitrary distinct isometries. For distinct $x_0, x_1 \in X$

$$\mathrm{dist}(g_0x_0, g_1x_1) \geq \mathrm{dist}(g_0x_0, g_0x_1) - \mathrm{dist}(x_1, g_0x_1) - \mathrm{dist}(x_1, g_1x_1) > \mathrm{dist}(x_0, x_1) - \mathrm{dist}_{\min}(X) \geq 0,$$

hence $g_0x_0 \neq g_1x_1$. For $x_0 = x_1 \in X$, since $g_0 \neq g_1$, $g_0x_0 \neq g_1x_1$. Therefore, g_0X and g_1X are disjoint. □

Lemma 4.22. *Let $X \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ be a finite subset. Let $p : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathcal{S}^1$ be the projection $(x_0, x_1) \mapsto \left(\frac{x_0}{\sqrt{x_0^2+x_1^2}}, \frac{x_1}{\sqrt{x_0^2+x_1^2}} \right)$. Consider the map*

$$\begin{aligned} \xi_X^1 : \mathbb{N} &\to \mathrm{O}(2, \mathbb{Q}) \subseteq \mathrm{Aut}(\mathcal{S}^1) \\ i-1 &\mapsto \frac{1}{i^2+s^2} \begin{bmatrix} i^2-s^2 & -2is \\ 2is & i^2-s^2 \end{bmatrix}, \end{aligned}$$

where $s := \max\{\frac{1}{n+1} : \frac{1}{n+1} < (\mathrm{dist}_{\min} p(X))/2, n \in \mathbb{N}\} \in \mathbb{Q}$. Then, the $\bigcup_{i \in \mathbb{N}} \xi_X^1(i) X$ is a disjoint union.

Proof. It is easy to check that the image of ξ_X^1 is in $\mathrm{Iso}^s(\mathcal{S}^1)$, hence it is also in $\mathrm{Iso}^{(\mathrm{dist}_{\min} p(X))/2}(\mathcal{S}^1)$. Applying Lemma 4.21 to $p(X) \subseteq \mathcal{S}^1$, we know that $\bigcup_{i \in \mathbb{N}} \xi_X^1(i) p(X)$ is a disjoint union, hence $\bigcup_{i \in \mathbb{N}} \xi_X^1(i) X$ is a disjoint union as well. $\square$

Remark 4.23. Although Lemma 4.22 uses the projection p, the minimum distance $\mathrm{dist}_{\min}(\dots)$ and maximum $\max\{\dots\}$ to define ξ_X^1, all three operations can be avoided. Applying the idea of Lemma 4.22 to the unions of concentric spheres, we can avoid using p. For explicit X, for instance the X's obtained from well-known spherical designs in Theorem 1.2, the minimum distance $\mathrm{dist}_{\min}$ is known. Instead of maximum, we can use ceiling/floor to define s.

Corollary 4.24. *Let d be a positive integer. Let $X \subseteq \mathcal{S}^d$ be a finite subset such that every point in X has nonzero first coordinate. Let $p : \mathcal{S}^d \to \mathbb{R}^2$ be the projection $(x_0, \dots, x_d) \mapsto (x_0, x_1)$. Consider the map*

$$\xi_X^d := j(\xi_{p(X)}^1) : \mathbb{N} \to O(d+1, \mathbb{Q}) \subseteq \mathrm{Aut}(\mathcal{S}^d)$$

where $j : \mathrm{O}(2, \mathbb{Q}) \to \mathrm{O}(d+1, \mathbb{Q})$ is the inclusion corresponding to p. Then, the $\bigcup_{i \in \mathbb{N}} \xi_X^d(i) X$ is a disjoint union.

Proof. Since every point in X has nonzero first coordinate, $p(X) \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$. By Lemma 4.22, $\bigcup_{i \in \mathbb{N}} \xi_{p(X)}^1(i) p(X) = \bigcup_{i \in \mathbb{N}} p(\xi_X^d(i) X)$ is a disjoint union, hence $\bigcup_{i \in \mathbb{N}} \xi_X^d(i) X$ is a disjoint union. $\square$

Corollary 4.25. *Let a be a positive integer and let b be a natural number. Let $\mathcal{Y}^a$ be an antipodal t-design on $\mathcal{S}^a$ over $\mathbb{F}$ such that every point in $\mathcal{Y}^a$ has nonzero first coordinate, and let $\overline{\mathcal{X}}_a^b$ a semiantipodal integer-weighted rational t-semidesign on $\mathcal{H}_a^b$. Let $\xi := \xi_{\mathcal{Y}^a}^a$, which is defined in Corollary 4.24. Then, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^b$, regarded as a subspace of $\mathcal{S}^{a+b}$, is an antipodal t-design on $\mathcal{S}^{a+b}$ over $\mathbb{F}$.*

Proof. We use the similar ideas in Lemma 4.8. We label the coordinates of S^a, H^b and S^{a+b} as follow

$$S^a = \{(x_0, \dots, x_a)\}, \quad H^b = \{(y_0, \dots, y_b)\}, \quad S^{a+b} = \{(z_0, \dots, z_{a+b})\}.$$

Since $\mathcal{Y}^a$ is antipodal and $\overline{\mathcal{X}}_a^b$ is semiantipodal, using the fact that orthogonal transformations preserve antipodal map, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^b$, regarded as a subspace of $\mathcal{S}^{a+b}$ by Remark 4.4, is antipodal and $[0,a]$-antipodal (see Definition 2.4 for definition of antipodal).

Recall that $\mathcal{P}^{t,a}$ is defined in Eq. (2.6), and $\mathcal{P}^{t,a}$ is a subfunctor of $\mathcal{P}^t$. Consider the dominant open embedding $\iota_{a,b}$ in Proposition 4.2(i). The comorphism $\iota_{a,b}^*$ gives an inclusion

$$\begin{aligned} \iota_{a,b}^* : \mathcal{P}^{t,a}[S^{a+b}] &\hookrightarrow \mathcal{P}^t[S^a] \otimes \mathcal{P}^{t,0}[H^b] \\ z_i &\mapsto \begin{cases} x_i \otimes y_0, & i \in [1,a]_{\mathbb{Z}}, \\ 1 \otimes y_{i-a}, & i \in [a+1, a+b]_{\mathbb{Z}}. \end{cases} \end{aligned}$$

Note that although y_0 is not in $\mathcal{P}^{t,0}[H^b]$, any monomial in $\mathcal{P}^{t,a}[S^{a+b}]$ maps to some monomial with even degree in y_0, hence in $\mathcal{P}^{t,0}[H^b]$.

Since $\mathcal{Y}^a$ is a $\underline{1}$-weighted $\mathcal{P}^t[S^a]$-design on $\mathcal{S}^a$ and $\overline{\mathcal{X}}_a^{d-a}$ is an integer-weighted $\mathcal{P}^{t,0}[H^{d-a}]$-design on $\mathcal{H}_a^{d-a}$, by Theorem 4.20, $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$ is a $\underline{1}$-weighted $\mathcal{P}^t[S^a] \otimes \mathcal{P}^{t,0}[H^{d-a}]$-design on $\mathcal{S}^a \times \mathcal{H}_a^{d-a}$. According to the inclusion $\iota_{a,b}^*$, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$, regarded a subspace of $\mathcal{S}^d$, is a $\underline{1}$-weighted $\mathcal{P}^{t,a}[S^{a+b}]$-design on $\mathcal{S}^d$. Therefore, by Lemma 2.6, $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$ is a $\underline{1}$-weighted t-design on $\mathcal{S}^d$, hence a t-design on $\mathcal{S}^d$.

Since $\overline{\mathcal{X}}_a^{d-a}$ is rational and ξ has rational coefficients, the twisted product preserves the field. Thus $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$ is a design over $\mathbb{F}$. □

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