1 Overview

My research interests mainly focus on three fields: (1) representation theory, (2) combinatorics and (3) number theory. I will give an overview of my research in these three fields in sections 1.1 to 1.3, and discuss my research in details in sections 2 to 4, respectively.

1.1 Representation theory

The representation theory of the symmetric group has been studied for a long time. The theory at characteristic zero is fairly well-understood, while at positive characteristic, there are many open problems. *Hecke algebras* of type A are deformations of the group algebras of symmetric groups. Such algebras at root of unity can be viewed as some approximation of the symmetric group algebra at positive characteristic.

The calculation of the *cohomology ring* of the symmetric group has been an open problem for a long time. At this time, it is only known to be finitely generated. Recently, Benson, Erdmann and Mikaelian [BEM10] calculated the cohomology ring for Hecke algebra of type A. Linckelmann [Lin11] proved the finite generation of Hochschild cohomology of Hecke algebras of type A. In joint work with Nakano [NX18], we proved that the ordinary cohomologies over the cohomology ring are also finitely generated.

Support varieties are the support of cohomologies of modules. It can detect natural homological properties of the modules, such as the complexity of modules. Support varieties have been developed in a variety of contexts that involve a Hopf structure, for example, for finite group schemes, quantum groups and Lie superalgebras [FP86; FP87; FP05; NP98; BNPP14]. Very little is known about extracting geometric properties from Frobenius categories where there is no underlying coproduct.

In joint work with Nakano [NX18], we developed a support theory for Hecke algebras of type A which has a canonical description in an affine space where computations are tractable. The ideas involve the interplay with the computation of the cohomology ring due to Benson, Erdmann and Mikaelian [BEM10], the theory of vertices due to Dipper and Du [DD93], and branching results for cohomology by Hemmer and Nakano [HN02]. Using Morita equivalence between Hecke algebras of classical types by Dipper, James, Pallikaros and Hu [DJ92; Pal94; Hu02], our theory can be extended to Hecke algebras of other classical types.

The q-Schur algebras are quasi-hereditary covers of Hecke algebras of type A. With the Schur functor, we can relate the representation theory of q-Schur algebras and Hecke algebras of type A. With the Morita equivalence mentioned above, in joint work with Nakano and Lai [LNX18], we proved that in many cases, q-Schur algebras of type B are quasi-hereditary covers of Hecke algebras of type B. We also proved that q-Schur algebras of type B are direct sums of tensor products of Schur algebras of type A. With this structural result, we can generalize many results for type A results to type B. For example, we can derive the representation type of q-Schur algebra using the representation type of q-Schur algebra of type A obtained by Nakano and Erdmann [EN01].

1.2 Combinatorics

Many of my works in combinatorics are in a subfield called *design theory.* These works introduce many tools from other field of mathematics to design theory. Besides design theory, I also do research in graph theory, phylogenetic combinatorics and symbolic dynamics, which will not be discussed in this research statement.

Designs are good finite subsets that approximate a given space nicely. Information on designs can be used to recover the information on the given space. There are mainly two classes of designs:

(i) *combinatorial designs*, whose study goes back to Kirkman, Steiner, Fisher, Witt, Bose, Hughes, etc., such as *block designs* (also called t-(v, k, λ) designs) on Johnson association scheme;

(ii) *geometric designs*, originated from Delsarte-Goethals-Seidel [DGS77], such as *spherical t-designs* on real spheres.

Here, t is called the *strength* of the design. The higher the strength is, the more nicely the designs approximate the space.

Finding good constructions of designs on various spaces is always an important topic in design theory. Wilson proved the existence of t-(v, k, λ) designs for $t = 2$ in [Wil72a; Wil72b; Wil75] under necessary conditions, and the existence for general t was proved recently by Keevash in [Kee14]. However, it is still an open problem to construct an explicit 6-$(v, k, 1)$ design. In [SZ84], Seymour-Zaslavsky proved the existence of geometric designs on path-connected space. In joint work with Cui and Xia [CXX17], with the help of analytic number theory, we proved the existence of designs consisting of rational points on algebraically path-connected space. Inspired by such existence, in a recent paper of mine [Xia18a], I give the first explicit construction of spherical designs of arbitrary strength and dimension.

Another direction of studying designs is classifying small designs on various spaces. We usually expect to have a certain lower bound on the size of designs, called *Fisher type lower bound* [RW75; DGS77]. I proved such lower bound for relative designs on Hamming association scheme in [Xia12] using total positivity. Designs achieving this bound are rare, and it is already proved that such designs (almost) do not exist for large strength on various spaces [BD79; BD80; RW75; Ban77]. One of my result in this direction is to give a new lower bound on spaces that do not support Q-polynomial association schemes [Xia18b]. One application of [Xia18b] is a nonexistence result for tight designs on unitary group, whose proof involves generalized hook formula and Hankel determinants. For small strength, the classification of designs achieving Fisher type lower bounds are usually very difficult, since they involve classification of finite projective planes, classification of Hadamard matrices, classification of maximal equiangular lines, etc. By solving a certain explicit high degree Diophantine equation, I proved such nonexistence for block designs of strength 8 [Xia18c], and with my coauthors for a variant of spherical designs of strength ≤ 4 [BBXYZ18].

1.3 Number theory

In my research of algebraic combinatorics [Xia18c; BBXYZ18], I usually encounter some Diophantine equations. The integral solutions of the Diophantine equations correspond to feasible parameters of a certain interesting algebraic combinatorial objects. Finding all integral (rational) solutions is an important step to classify the corresponding algebraic combinatorial objects.

These Diophantine solutions are explicitly given, having large coefficients, high degree (say, degree 10) in two or three variables. Existing methods are not able to solve them. A computer proof is impossible due to the answer to Hilbert's 10-th problem. For curves, Siegel's theorem gives finiteness of integral points, and Falting's theorem gives finiteness of rational points. But, these two results are not effective. For surfaces, Bombieri-Lang conjecture gives a conjectural behaviour of rational points on a generic surface. Even if this conjecture was proved, it is still unknown how it can be applied our explicit Diophantine equations. For some curves, Runge's theorem gives finiteness of integral points. However, since coefficients and degree are large, effective versions of Runge's theorem would give very large bound, say $10^{10^{10}}$, and a computer search for small solutions would be impossible.

In [Xia18a], I develop a new method solving these Diophantine equations under some additional conditions coming from combinatorial properties. In many cases, parameters of algebraic combinatorial objects are related by rational functions. Some parameters are integers, and some parameters are positive. The positivity conditions and integral conditions of new parameters allow me to find all integral solutions. This method has been applied successfully to solve a degree-13 Diophantine equation with large coefficients in two variables [Xia18a] and a degree-10 Diophantine equation in three variables [BBXYZ18] under new integral and positivity conditions.

2 Representation theory: Hecke algebras and q-Schur algebras

2.1 Cohomology of Hecke algebra

Let Σ_d be the symmetric group on d letters, and $q \in \mathbb{C}^*$ a non-zero complex number. The Hecke algebra $\mathcal{H}_q^A(d) := \mathcal{H}_q(\Sigma_d)$ is a deformation of the group algebra $\mathbb{C}\,\Sigma_d$. The trivial $\mathcal{H}_q^A(d)$-module is denoted by $\mathbb{C}$. For an $\mathcal{H}_q^A(d)$-module M, define the *cohomology* $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), M)$ as $\mathrm{Ext}^\bullet_{\mathcal{H}_q^A(d)}(\mathbb{C}, M)$. The *cohomology ring* $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), \mathbb{C})$ was calculated recently by Benson, Erdmann and Mikaelian.

Theorem 2.1 (Benson-Erdmann-Mikaelian [BEM10])**.** *Let q be a primitive l-th root of unity with $l \geq 3$. Then,*

$$\mathrm{H}^\bullet(\mathcal{H}^A(d), \mathbb{C}) \cong k[x_1, \ldots, x_{\lfloor d/l \rfloor}] \otimes \Lambda[y_1, \ldots, y_{\lfloor d/l \rfloor}],$$

where $\deg x_i = 2l - 2$ *and* $\deg y_i = 2l - 3$.

For permutation modules and Young modules, we can calculate its cohomology using the vertices of these modules [DD93]. Like the symmetric group algebra, it is still an open problem to calculate the cohomology of Specht modules of Hecke algebras.

Problem 2.2. For an l-regular partition $\lambda \vdash d$. Calculate the cohomology $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), S^\lambda)$, where S^λ is the Specht module indexed by λ.

For group algebras, it is known that the cohomology is finitely generated over the cohomology ring, and it is still the case for Hecke algebras of type A.

Theorem 2.3 (Nakano-X. [NX18]). *For every finitely generated $\mathcal{H}_q^A(d)$-module M, the cohomology $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), M)$ is finitely generated over $\mathrm{H}^\bullet(\mathcal{H}_q(d), \mathbb{C})$.*

For a composition $\lambda = (\lambda_1, \lambda_2, \dots) \vDash d$, denote by Σ_λ the Young subgroup $\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots$ of Σ_n, and denote by R_λ the cohomology ring $\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), \mathbb{C})$ of Hecke algebra $\mathcal{H}_q^A(\lambda) := \mathcal{H}_q(\Sigma_\lambda)$. Using Künneth Theorem and formula of the cohomology ring, it is possible to write down R_λ explicitly. The ring R_λ has a commutative subring generated by preimages of x_i's in Theorem 2.1, and let $\widetilde{R}_\lambda$ denote this commutative subring. Then, we have

$$\widetilde{R}_\lambda \cong \mathbb{C}[x_1, \dots, x_{\lfloor d/l \rfloor}].$$

Given a pair of algebras $(\mathcal{H}_q^A(d), \mathcal{H}_q^A(\lambda))$, we can generalize cohomology of $\mathcal{H}_q^A(\Sigma_d)$ to cohomology of $\mathcal{H}_q^A(\Sigma_d)$ relative to $\mathcal{H}_q^A(\lambda)$. For group algebras, the relative cohomology may not be finitely generated over the relative cohomology ring. It is an interesting question to see if it is true for Hecke algebras.

Problem 2.4. Let M be a $\mathcal{H}_q^A(d)$-module. Is the relative cohomology $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), \mathcal{H}_q^A(\lambda), M)$ finitely generated over $\mathrm{H}^\bullet(\mathcal{H}_q^A(d), \mathcal{H}_q^A(\lambda), \mathbb{C})$?

2.2 Support variety

The definition of the support variety for group algebras involves tensor products of modules, which are not available for Hecke algebras. This suggests that we may find another way to define the support variety for Hecke algebras. For symmetric group algebras, Hemmer and Nakano have the following branching results.

Theorem 2.5 (Hemmer-Nakano [HN02]). *Let k be an algebraically closed filed with positive characteristic, $V_{k\Sigma_d}(M)$ the support variety of $k\Sigma_d$-module M, and $V_{k\Sigma_\lambda}(\mathbb{C}, M)$ the relative support variety of $k\Sigma_\lambda$-module M for a given composition $\lambda \vDash d$. Then,*

$$V_{k\Sigma_d}(M) = \bigcup_{\lambda \vDash d} \operatorname{res}^*_{\Sigma_d, \Sigma_\lambda} V_{k\Sigma_\lambda}(k, M).$$

Note that the definition for relative support $V_{k\Sigma_\lambda}(\mathbb{C}, M)$ does not need tensor products of modules. This indicate that we might be able to construct support variety for Hecke algebras by mimicking the result above.

In joint work with Nakano, we define the support variety for Hecke algebras as follows. Let $W_\lambda := \operatorname{MaxSpec} \widetilde{R_\lambda}$, which is an affine space. For a finitely generated $\mathcal{H}_q^A(\lambda)$-module M, its *relative support variety* $W_\lambda(M)$ is the variety of the annihilator ideal, $J_{\mathcal{H}_q^A(\lambda)}(\mathbb{C}, M)$, in $\widetilde{R}_\lambda$ for its action on $\mathrm{H}^\bullet(\mathcal{H}_q(\lambda), M)$. These relative support varieties are closed conical subvarieties of W_λ.

For each $\lambda \vDash d$, there exists a restriction map in cohomology $\operatorname{res}^*_{d,\lambda} : W_\lambda \to W_d$ which is induced by the inclusion of $\mathcal{H}_q^A(\lambda) \leq \mathcal{H}_q^A(d)$. Inspired by the branching results, we define the support variety as follows

Definition 2.6. Let M be a finitely generated $\mathcal{H}_q^A(d)$-module. The *support variety* of M is

$$V_d(M) := \bigcup_{\lambda \vDash d} \operatorname{res}^*_{d,\lambda}(W_\lambda(M)).$$

Using vertices of permutation modules and Young modules, we can calculate that the support of these modules are affine spaces. It is still an open problem to calculate the support varieties of Specht modules.

For symmetric group algebra, support varieties have another description as rank varieties. It allows us to calculate the support variety from some local data. It is an interesting open problem to find such description for support varieties for Hecke algebras.

Problem 2.7. Find a rank variety description for the support variety for Hecke algebras.

2.3 Hecke algebras of other types

For a Coxeter group of other type, we can define its associated Hecke algebra similarly. For type B_d (or C_d), the Hecke algebra involves two parameters $Q \in \mathbb{C}^*$ and $q \in \mathbb{C}^*$, and let $\mathcal{H}^B_{Q,q}(d)$ denote this Hecke algebra. For type D_d, the Hecke algebra only involves one parameter $q \in \mathbb{C}^*$, and let $\mathcal{H}_q^D(d)$ denote this Hecke algebra.

When the parameters are generic, we can relate the Hecke algebras of classical type to Hecke algebra of type A via Morita equivalences.

Theorem 2.8. (i) (Dipper-James [DJ92]) *If Q is generic, then*

$$\mathcal{H}^B_{Q,q}(d) \sim_{Morita} \bigoplus_{i=0}^{d} \mathcal{H}_q^A(i) \otimes \mathcal{H}_q^A(d-i).$$

(ii) (Pallikaros [Pal94]) *If Q is generic and d is odd, then*

$$\mathcal{H}_q^D(d) \sim_{Morita} \bigoplus_{i=0}^{(d-1)/2} \mathcal{H}_q^A(i) \otimes \mathcal{H}_q^A(d-i).$$

(iii) (Hu [Hu02]) *If q is generic and d is even, then*

$$\mathcal{H}_q^D(d) \sim_{Morita} A(d/2) \oplus \bigoplus_{i=0}^{d/2-1} \mathcal{H}_q^A(i) \otimes \mathcal{H}_q^A(d-i),$$

for some algebra $A(d/2)$.

With this Morita equivalence, we can calculate cohomology ring and define support variety at generic parameters. The details are given in [NX18].

At special parameters, i.e. some roots of unity, the above Morita equivalences fail, and the cohomology rings have not been calculated yet.

Problem 2.9. Calculate the cohomology ring for Hecke algebras of type B and D at special parameters.

2.4 Coordinate coalgebra construction of q-Schur algebras

The q-Schur algebra of type A $\mathcal{S}^A(n,d)$ is the quasi-hereditary cover of the Hecke algebra of type A $\mathcal{H}^A(d)$. These two algebras have the Schur-Weyl duality, namely double centralizer properties. The q-Schur algebra $\mathcal{S}^A(n,d)$ has a coordinate coalgebra construction, more precisely, it is the dual of the degree d part of a deformation of the matrix algebra $M_n(\mathbb{C})$. Using this description, the calculation in $S^A(n,d)$ is tractable.

In joint work with Lai and Nakano [LNX18], we generalize this description to type B, and obtain a coordinate coalgebra description for the q-Schur algebra of type B, denoted by $\mathcal{S}^B(n,d)$. We proved that the type B Schur algebra is related to type A Schur algebra as follows.

Theorem 2.10 (Nakano-Lai-X. [LNX18])**.** *Suppose that q is generic. Then,*

$$\mathcal{S}^B(n,d) \cong \bigoplus_{i=0}^{d} \mathcal{S}^A(n,i) \otimes \mathcal{S}^A(n,d-i).$$

The isomorphism above allows us to get many results on type B Schur algebra using the results on type A Schur algebras. The cellularity and quasi-hereditariness of q-Schur algebras of type B follow immediately from those of q-Schur algebras of type A. The representation type of q-Schur algebras is determined in [EN01]. With the above isomorphism theorem, we can determine the representation type of q-Schur algebras of type B.

We have an ongoing project to generalize these results to other types, especially type D.

Problem 2.11. Find a coordinate coalgebra construction of the q-Schur algebras of type D.

3 Combinatorics: Design theory

Designs are good finite subsets of a given space that they approximate the given space with respect to polynomials on the space. One well-known example is the set of minimum vectors of the Leech lattice, which approximates the 23-dimensional sphere with respect to polynomials of degree up to degree 11.

For a space $\mathcal{Z}$ studied in algebraic combinatorics, there is usually a natural measure μ_Z on $\mathcal{Z}$ and a concept of polynomials on $\mathcal{Z}$. Let $\mathcal{P}^t[\mathcal{Z}]$ the real vector space of continuous integrable degree $\leq t$ polynomials on $\mathcal{Z}$. A *t-design X* on $\mathcal{Z}$ is a finite subset such that

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|\,\mathcal{Z}\,|}\int_{\mathcal{Z}} f \,\mathrm{d}\,\mu_Z$$

for all $f \in \mathcal{P}^t[\mathcal{Z}]$. Designs are classified as *combinatorial/geometric designs* if $\mathcal{Z}$ has combinatorial/geometric structure.

For combinatorial designs, $\mathcal{Z}$ is usually a discrete topological space equipped with some additional combinatorial structure, the counting measure, and polynomials defined using the combinatorial structure such as association scheme. The most important case is *block designs* (also called *t-(v,k,λ)-design*), whose existence was proved by Wilson for $t = 2$ and Keevash for general t. Block designs can be regarded as designs on a Johnson association scheme.

Regular t-wise balanced designs [Xia12] are designs on a Hamming association scheme. For a general P-polynomial or Q-polynomial association schemes, we have *relative designs* on them [Del73; BB12; BBST15]. Another direction of the generalization is *designs* on polynomial spaces [God88].

For geometric designs, $\mathcal{Z}$ is usually a good manifold equipped with a natural measure and polynomials. The study of geometric designs originated from Delsarte-Goethals-Seidel [DGS77], where they introduced the concept of *spherical designs*, that is designs on real unit spheres. Seymour-Zaslavsky proved the existence of spherical designs in [SZ84]. Later, many other proofs are found [RB91; Wag91; BRV13; CXX17; Xia18a]. There is a large number of literature studying variations of spherical designs: interval designs [Kup05], Euclidean designs [NS88; DS89], Grassmannians designs [BCN02; BBC04], complex spherical designs [RS14], unitary designs [RS09; Xia18b], designs on homogeneous spaces, designs on polynomial spaces [God88; BBD99], etc.

There are mainly two directions on the study of designs, finding small designs and finding explicit designs.

3.1 Small designs

On a good space $\mathcal{Z}$, either combinatorial or geometric, we have an inner product structure on polynomials on $\mathcal{Z}$. With the help of inner structure, we can expect for every t-design X on $\mathcal{Z}$,

$$|X| \geq \dim \mathcal{P}^{t/2}[\mathcal{Z}].$$

This kind of lower bound is called a *Fisher type lower bound* for designs on $\mathcal{Z}$. Such lower bound has been proved for various spaces: block designs [RW75], spherical designs [DGS77], designs on subset of Euclidean space [Möl79], Euclidean designs [Möl79; DS89], unitary designs [RS09], complex spherical designs [RS14], regular t-wise balanced designs [Xia12], relative designs [BB12].

For some spaces $\mathcal{Z}$, it is not an easy task to give a simple formula for $\dim \mathcal{P}^e[\mathcal{Z}]$. I determined it for Hamming scheme in [Xia12]. It is unknown if a simple formula can be found for unitary group [RS09].

A design that achieves this Fisher type lower bound is called *tight*. A central topic in design theory is to find all tight (or almost tight) designs.

3.1.1 Classification of tight spherical designs

It is shown in [DGS77] that the Fisher type lower bound for a t-designs X on real unit sphere of dimension d with even t is

$$|X| \geq \binom{d+t/2}{d} + \binom{d+t/2-1}{d}.$$

There is a similar formula for odd t. Spherical designs achieving the equality attract many attentions, and there are many studies on these tight spherical designs.

Theorem 3.1 (Bannai-Damerell [BD79; BD80]). *If a tight spherical t-design exists on S^d, then $t \in \{1, 2, 3, 4, 5, 7, 11\}$.*

The classification of tight spherical t-designs for $t \in \{1, 2, 3, 11\}$ is known, and classification for $t \in \{4, 5, 7\}$ is still open.

(i) A tight 1-design is a pair of antipodal points on S^d.

(ii) A tight 2-design is the vertices of a regular d-simplex on S^d.

(iii) A tight 3-design is the vertices of a cross polytope on S^d.

(iv) (Bannai-Slone [BS81]) The unique tight 11-design is the minimal vectors of the Leech lattice.

3.1.2 Asymptotically the smallest geometric designs

Let $N(d, t)$ be the size of the smallest spherical t-design on a d-dimensional sphere. The Fisher type lower bound gives

$$N(d, t) \gg t^d.$$

It is proved recently by Bondarenko-Radchenko-Viazovska in [BRV13] that $N(d, t)$ is asymptotically t^d as $t \to \infty$. It remains to be in interesting problem to find the asymptotic size of the smallest design on other spaces.

3.1.3 Classification of tight combinatorial designs

In [RW75], it is shown that the Fisher type lower bound for a combinatorial t-(v, k, λ) design X is

$$|X| \geq \binom{v}{t/2}.$$

for even t. Recall that when the equality holds, such design is called a *tight* combinatorial t-design.

Tight 1-designs are basically partitions. There exists a unique tight 1-(v, k, λ) design up to isomorphism for each k, the 1-$(2k, k, 1)$ design.

Tight 2-designs are also called *symmetric designs*. The case $t = 2$ is quite different from the other cases. Finite projective planes and Hadamard matrices give two infinite families of nontrivial tight 2-designs. The classification for finite projective planes and the classification for Hadamard matrices are both very difficult open question, consequently a complete classification of tight 2-designs is not yet known.

Ray-Chaudhuri and Wilson [RW75] proved the nonexistence of nontrivial tight t-designs for odd $t \geq 3$. Bannai [Ban77] succeeded in proving that for any $t = 2e$ with $e \geq 5$, there exist only finitely many nontrivial tight t-designs. In an unpublished paper, Bannai and Ito proved the finiteness of the number of nontrivial tight 8-designs. The nonexistence of nontrivial tight 6-designs was proved by Peterson [Pet77]. Bremner [Bre79] and Stroeker [Str81] independently showed that there are only two nontrivial tight 4-designs up to isomorphism, which have parameters 4-$(23, 7, 1)$ and 4-$(23, 16, 52)$. Dukes and Short-Gershman [DS13] proved the nonexistence of nontrivial tight t-designs for $t \in \{10, 12, \ldots, 18\}$. I proved the nonexistence of nontrivial tight 8-design in [Xia18c], and I am interested in proving the following conjecture.

Conjecture 3.2. For each $e \geq 10$, there are no nontrivial tight combinatorial $2e$-design.

In [Xia18a], I gave a new necessary condition for the existence of tight designs, which involves a certain Diophantine equation. I expect that further analysis of the Diophantine equation may provide some hint on how to approach the conjecture above. Some partial result was obtained in joint work with Lorenzini [LX18].

3.2 Explicit designs

3.2.1 Explicit construction of geometric designs

When t and d are small, there are many good spherical t-designs coming from geometric constructions. The regular n-gon in $\mathbb{R}^2$ gives a spherical $(n-1)$-design. The root systems in $\mathbb{R}^{d+1}$ with large d give spherical 3-design. Regular dodecahedron and regular icosahedron in $\mathbb{R}^3$ give spherical 5-designs. The 120-cell and 600-cell in $\mathbb{R}^4$ give spherical 11-designs. The minimum vectors of the Leech lattice in $\mathbb{R}^{24}$ form a spherical 11-design. The orbit of a generic point in $\mathbb{R}^4$ under the reflection group of type H_4 gives spherical 19-design. It is not known if there exists some simple geometric construction that could give spherical 20-design for $d \geq 2$.

With the help of interval design, [RB91; Wag91; Baj92] gave a construction of spherical t-design on S^d for arbitrary t and d. However, this construction is not explicit since this construction uses designs on Gegenbauer intervals, which do not have an explicit construction yet.

Recently, I showed an explicit construction of spherical t-design on S^d for arbitrary t and d.

Theorem 3.3 (X. [Xia18a])**.** *For every t and d, spherical t-designs on S^d can be constructed explicitly.*

I am interested in generalizing this to generalize it to other good geometric spaces, say compact symmetric spaces such as unitary groups.

3.2.2 Explicit construction of combinatorial designs

The work of Wilson and Keevash showed that there exist combinatorial t-(v, k, λ) designs under necessary conditions. However, these results use probabilistic methods, and are both nonconstructive. The problem of finding an explicit construction of t-(v, k, λ) designs is very difficult. Teirlinck [Tei87] constructed explicit t-(v, k, λ) designs for all t, but the parameter λ in that construction is very large. The smallest open case is $t = 6$ and $\lambda = 1$.

Problem 3.4. Construct an explicit 6-$(v, k, 1)$ design for some v and k.

3.2.3 Rational designs

The existence of geometric designs on path-connected spaces is proved by Seymour-Zaslavsky.

Theorem 3.5 (Seymour-Zaslavsky [SZ84])**.** *For a path-connected space $\mathcal{Z}$ equipped with a good measure, there exists t-designs on $\mathcal{Z}$ of size n for arbitrary t and sufficiently large n.*

In this existence results, the points in the designs are real points. It is natural to ask, for a given subfield $\mathbb{F} \subseteq \mathbb{R}$, whether there exist *designs over* $\mathbb{F}$, that is designs consisting of $\mathbb{F}$-points.

In joint work with Cui and Xia [CXX17], we generalized the concept of path-connected space and define algebraic path-connected spaces, and we proved the following analogous result for *rational designs*, that is designs consisting of rational points.

Theorem 3.6 (Cui-Xia-X. [CXX17]). *For an algebraically path-connected space $\mathcal{Z}$ equipped with a good measure, there exists t-designs on $\mathcal{Z}$ of size n for arbitrary t and sufficiently large n.*

On the other hand, the spheres are not algebraically path-connected, hence the existence of rational spherical designs remains to be an open problem. I propose to investigate the following conjecture.

Conjecture 3.7. For every positive integer t, there exists a rational spherical t-design on S^d for some positive integer d.

If we enlarge the field $\mathbb{Q}$ to some intermediate filed $\mathbb{F}$, there might be possible to prove the above conjecture. I showed in [Xia18a] a construction over the filed $\mathbb{Q}(\sqrt{p} : \text{prime } p)$.

4 Number theory: Diophantine equations

In algebraic combinatorics, a certain object has two parameters v and k. It is known that these two parameters are zeros of the Diophantine equation $f_4(v,k) = 0$ where

$$f_4(v,k) := -16384k^{12}v + 65536k^{12} + 98304k^{11}v^2 - 393216k^{11}v - 253952k^{10}v^3 + 786432k^{10}v^2 + 1744896k^{10}v - 3309568k^{10} + 368640k^9v^4 - 327680k^9v^3 - 8724480k^9v^2 + 16547840k^9v - 328320k^8v^5 - 1102464k^8v^4 + 17194752k^8v^3 - 21567744k^8v^2 - 49810560k^8v + 62323584k^8 + 182784k^7v^6 + 2050560k^7v^5 - 16432128k^7v^4 - 13016064k^7v^3 + 199242240k^7v^2 - 249294336k^7v - 61184k^6v^7 - 1642240k^6v^6 + 6536960k^6v^5 + 58253568k^6v^4 - 293538048k^6v^3 + 209662720k^6v^2 + 511604992k^6v - 488998144k^6 + 10752k^5v^8 + 698880k^5v^7 + 1258752k^5v^6 - 59703552k^5v^5 + 183266304k^5v^4 + 243542016k^5v^3 - 1534814976k^5v^2 + 1466994432k^5v - 640k^4v^9 - 143664k^4v^8 - 2296192k^4v^7 + 27050224k^4v^6 - 7038496k^4v^5 - 582955856k^4v^4 + 1856597696k^4v^3 - 1428764528k^4v^2 - 1015706784k^4v + 974873344k^4 + 7520k^3v^9 + 772608k^3v^8 - 2875616k^3v^7 - 58917568k^3v^6 + 469164960k^3v^5 - 1155170432k^3v^4 + 412538336k^3v^3 + 2031413568k^3v^2 - 1949746688k^3v + 336k^2v^{10} - 52816k^2v^9 - 1582560k^2v^8 + 27560816k^2v^7 - 127930016k^2v^6 + 28759472k^2v^5 + 1497511456k^2v^4 - 4944873072k^2v^3 + 6922441360k^2v^2 - 4733985888k^2v + 1506333312k^2 - 2352kv^{10} + 203472kv^9 - 764688kv^8 - 24513072kv^7 + 293023248kv^6 - 1459281552kv^5 + 3929166288kv^4 - 5947568016kv^3 + 4733985888kv^2 - 1506333312kv + 45v^{11} + 972v^{10} - 191952v^9 + 2961396v^8 - 14780538v^7 - 18769932v^6 + 544096980v^5 - 2755473732v^4 + 7281931941v^3 - 11097146016v^2 + 9310949028v - 3408102864.$$

The geometric genus of the curve is 20, so by Faltings' theorem [HS00, Theorem E.0.1], the curve has only finitely many rational points. There are 32 known rational zeros of $f_4(v,k)$. They are $(-1,-3)$, $(-1,-2)$, $(-1,1)$, $(-1,2)$, $(11/5,1)$, $(11/5,6/5)$, $(4,0)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(13/3,2)$, $(13/3,7/3)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(27/5,2)$, $(27/5,13/5)$, $(27/5,14/5)$, $(27/5,17/5)$, $(6,2)$, $(6,11/4)$, $(6,3)$, $(6,13/4)$, $(6,4)$, $(125/19,54/19)$, $(125/19,71/19)$, $(7,3)$, $(7,4)$, $(15,2)$, $(15,13)$. It is still unknown if they are the only rational points.

The polynomial f_4 satisfies Runge's condition [Run87], so by Runge's theorem [Run87], it has finitely many integral solutions. Quantitative versions of Runge's theorem have been established, and using the results in [HS83] and [Wal92], we can obtain the bounds $e^{e^{8600}}$ and $e^{e^{22}}$, respectively, for the size $\max(|v|,|k|)$ of an integral solution (v_0,k_0). The bounds are too large for any computer search to terminate.

From combinatorial properties of the objects, we have more integral parameters

$$p_i = \binom{4}{i} \frac{(k-4)^{\overline{i}}(k-4+1)^{\overline{i}}}{(v-8+1)^{\overline{i}}} \quad \text{and} \quad \lambda_i = \frac{1}{4!} \frac{k^{\underline{i}}}{(v-4)^{\underline{i-4}}}.$$

With these new parameters and some positivity condition, in [Xia18a], I constructed

$$X := 48q_4 - 16q_3 + 6q_2 - 144q_1 + 45q_0$$

where

$$\begin{aligned}
q_4 :=\ & 288(v-7)(k-5)\lambda_6 - 96((v-15)(k-13)-64)\lambda_7 \\
& -24(38v+3k-262)\lambda_8 + 6(k-5)k(k-1)(k-2)p_1 \\
& -2((v-6)(k-5)+1)k(k-1)p_2 \\
& +((v-9)(k-4)+2)kp_3 + (2v+3k-22)p_4, \\
q_3 :=\ & 6p_3 - (v-4)(k-5)q_4, \\
q_2 :=\ & -24p_2 + (v-5)q_3, \\
q_1 :=\ & p_1, \\
q_0 :=\ & k-4.
\end{aligned}$$

The parameter X is an integers since q_i's and λ_i's are integers. When $k \geq 10^5$,

$$235 + \frac{1}{4} \leq X \leq 235 + \frac{3}{4}.$$

This gives a bound $k \leq 10^5$, which is small enough so that we can use computers to search for all zeros of f_4 under the new conditions.

References

[Baj92] B. Bajnok. "Construction of spherical t-designs". In: *Geometriae Dedicata* 43.2 (1992), pp. 167–179. DOI: 10.1007/BF00147866.

[Ban77] E. Bannai. "On tight designs". In: *The Quarterly Journal of Mathematics* 28.4 (1977), pp. 433–448. DOI: 10.1093/qmath/28.4.433.

[BB12] E. Bannai and E. Bannai. "Remarks on the concepts of t-designs". In: *Journal of Applied Mathematics and Computing* 40.1 (2012), pp. 195–207. DOI: 10.1007/s12190-012-0544-1.

[BBC04] C. Bachoc, E. Bannai, and R. Coulangeon. "Codes and designs in Grassmannian spaces". In: *Discrete Mathematics* 277.1 (2004), pp. 15–28. DOI: 10.1016/S0012-365X(03)00151-1.

[BBD99] P. Boyvalenkov, S. Boumova, and D. Danev. "Necessary Conditions for Existence of Some Designs in Polynomial Metric Spaces". In: *European Journal of Combinatorics* 20.3 (1999), pp. 213–225. DOI: 10.1006/eujc.1998.0278.

[BBST15] E. Bannai, E. Bannai, S. Suda, and H. Tanaka. "On relative t-designs in polynomial association schemes". In: *The Electornic Journal of Combinatorics* 22.4 (2015), P4.47. URL: http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i4p47.

[BBXYZ18] E. Bannai, E. Bannai, Z. Xiang, W.-H. Yu, and Y. Zhu. "Classification of spherical 2-distance $\{4, 2, 1\}$-designs". In: (2018).

[BCN02] C. Bachoc, R. Coulangeon, and G. Nebe. "Designs in Grassmannian Spaces and Lattices". In: *Journal of Algebraic Combinatorics* 16.1 (2002), pp. 5–19. DOI: 10.1023/A:1020826329555.

[BD79] E. Bannai and R. M. Damerell. "Tight spherical designs, I". In: *Journal of the Mathematical Society of Japan* 31.1 (1979), pp. 199–207. DOI: 10.2969/jmsj/03110199.

[BD80] E. Bannai and R. M. Damerell. "Tight spherical designs, II". In: *Journal of the London Mathematical Society* s2-21.1 (1980), pp. 13–30. DOI: 10.1112/jlms/s2-21.1.13.

[BEM10] D. Benson, K. Erdmann, and A. Mikaelian. “Cohomology of Hecke algebras”. In: *Homology, Homotopy and Applications* 12.2 (2010), pp. 353–370. URL: https://projecteuclid.org/euclid.hha/1296223887.

[BNPP14] C. P. Bendel, D. K. Nakano, B. J. Parshall, and C. Pillen. “Cohomology for quantum groups via the geometry of the nullcone”. In: *Memoirs of the American Mathematical Society* 229.1077 (2014). DOI: 10.1090/memo/1077.

[Bre79] A. Bremner. “A Diophantine equation arising from tight 4-designs”. In: *Osaka Journal of Mathematics* 16.2 (1979), pp. 353–356. URL: http://hdl.handle.net/11094/8744.

[BRV13] A. Bondarenko, D. Radchenko, and M. Viazovska. “Optimal asymptotic bounds for spherical designs”. In: *Annals of Mathematics* 178.2 (2013), pp. 443–452. DOI: 10.4007/annals.2013.178.2.2.

[BS81] E. Bannai and N. J. Sloane. “Uniqueness of certain spherical codes”. In: *Canadian Journal of Mathematics* 33.2 (1981), pp. 437–449.

[CXX17] Z. Cui, J. Xia, and Z. Xiang. “Rational designs”. In: (2017). URL: http://ziqing.org/data/Rational%20designs.pdf.

[DD93] R. Dipper and J. Du. “Trivial and alternating source modules of Hecke algebras of type A”. In: *Proceedings of the London Mathematical Society* s3-66.3 (1993), pp. 479–506.

[Del73] P. Delsarte. *An algebraic approach to the association schemes of coding theory*. Philips Research Reports Supplements 10. 1973.

[DGS77] P. Delsarte, J.-M. Goethals, and J. J. Seidel. “Spherical codes and designs”. In: *Geometriae Dedicata* 6.3 (1977), pp. 363–388. DOI: 10.1007/BF03187604.

[DJ92] R. Dipper and G. D. James. “Representations of Hecke algebras of type B_n”. In: *Journal of Algebra* 146.2 (1992), pp. 454–481. URL: https://doi.org/10.1016/0021-8693(92)90078-Z.

[DS13] P. Dukes and J. Short-Gershman. “Nonexistence results for tight block designs”. In: *Journal of Algebraic Combinatorics* 38.1 (2013), pp. 103–119. DOI: 10.1007/s10801-012-0395-8.

[DS89] P. Delsarte and J. J. Seidel. “Fisher type inequalities for Euclidean t-designs”. In: *Linear Algebra and its Applications* 114.Supplement C (1989), pp. 213–230. DOI: 10.1016/0024-3795(89)90462-X.

[EN01] K. Erdmann and D. K. Nakano. “Representation type of q-Schur algebras of type A”. In: *Transactions of the American Mathematical Society* 353.12 (2001), pp. 4729–4756.

[FP05] E. M. Friedlander and J. Pevtsova. “Representation-theoretic support spaces for finite group schemes”. In: *American Journal of Mathematics* 127.2 (2005), pp. 379–420. DOI: 10.1353/ajm.2005.0010.

[FP86] E. M. Friedlander and B. J. Parshall. “Support varieties for restricted Lie algebras”. In: *Inventiones mathematicae* 86.3 (1986), pp. 553–562. DOI: 10.1007/BF01389268.

[FP87] E. M. Friedlander and B. J. Parshall. “Geometry of p-unipotent lie algebras”. In: *Journal of Algebra* 109.1 (1987), pp. 25–45. DOI: 10.1016/0021-8693(87)90161-X.

[God88] C. D. Godsil. “Polynomial spaces”. In: *Discrete Mathematics* 73.1 (1988), pp. 71–88. DOI: 10.1016/0012-365X(88)90134-3.

[HN02] D. J. Hemmer and D. K. Nakano. “Support varieties for modules over symmetric groups”. In: *Journal of Algebra* 254.2 (2002), pp. 422–440. DOI: 10.1016/S0021-8693(02)00104-7.

[HS00] M. Hindry and J. H. Silverman. “Diophantine Geometry: An Introduction”. In: (2000).

[HS83] D. L. Hilliker and E. G. Straus. “Determination of bounds for the solutions to those binary Diophantine equations that satisfy the hypotheses of Runge’s theorem”. In: *Transactions of the American Mathematical Society* 280.2 (1983), pp. 637–657. DOI: 10.1090/S0002-9947-1983-0716842-3.

[Hu02] J. Hu. “A Morita equivalence theorem for Hecke algebra $H_q(D_n)$ when n is even”. In: *Manuscripta Mathematica* 108.4 (2002), pp. 409–430. DOI: 10.1007/s002290200272.

[Kee14] P. Keevash. “The existence of designs”. In: (2014). arXiv: 1401.3665. URL: http://arxiv.org/abs/1608.02792.

[Kup05] G. Kuperberg. “Special moments”. In: *Advances in Applied Mathematics* 34.4 (2005), pp. 853–870. DOI: 10.1016/j.aam.2004.11.005.

[Lin11] M. Linckelmann. “Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero”. In: *Bulletin of the London Mathematical Society* 43.5 (2011), pp. 871–885.

[LNX18] C.-J. Lai, D. K. Nakano, and Z. Xiang. “On q-Schur algebras corresponding to Hecke algebras of type B”. In: (2018).

[LX18] D. Lorenzini and Z. Xiang. “Integral points on variable separated curves”. In: (2018).

[Möl79] H. M. Möller. “Lower Bounds for the Number of Nodes in Cubature Formulae”. In: *Numerische Integration: Tagung im Mathematischen Forschungsinstitut Oberwolfach vom 1. bis 7. Oktober 1978*. Ed. by G. Hämmerlin. 1979, pp. 221–230. DOI: 10.1007/978-3-0348-6288-2_17.

[NP98] D. K. Nakano and J. H. Palmieri. "Support varieties for the Steenrod algebra". In: *Mathematische Zeitschrift* 227.4 (1998), pp. 663–684. DOI: 10.1007/PL00004398.

[NS88] A. Neumaier and J. J. Seidel. "Discrete measures for spherical designs, eutactic stars and lattices". In: *Indagationes Mathematicae (Proceedings)* 91.3 (1988), pp. 321–334. DOI: 10.1016/S1385-7258(88)80011-8.

[NX18] D. K. Nakano and Z. Xiang. "Support variety for Hecke algebras". In: *Homology, Homotopy and Applications* (2018).

[Pal94] C. Pallikaros. "Representations of Hecke algebras of type D_n". In: *Journal of Algebra* 169 (1994), pp. 20–48.

[Pet77] C. Peterson. "On tight 6-designs". In: *Osaka Journal of Mathematics* 14.2 (1977), pp. 417–435. URL: http://hdl.handle.net/11094/3990.

[RB91] P. Rabau and B. Bajnok. "Bounds for the number of nodes in Chebyshev type quadrature formulas". In: *Journal of Approximation Theory* 67.2 (1991), pp. 199–214. DOI: 10.1016/0021-9045(91)90018-6.

[RS09] A. Roy and A. J. Scott. "Unitary designs and codes". In: *Designs, Codes and Cryptography* 53.1 (2009), pp. 13–31. DOI: 10.1007/s10623-009-9290-2.

[RS14] A. Roy and S. Suda. "Complex spherical designs and codes". In: *Journal of Combinatorial Designs* 22.3 (2014), pp. 105–148. DOI: 10.1002/jcd.21379.

[Run87] C. Runge. "Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen". In: *Journal für die reine und angewandte Mathematik* 100 (1887), pp. 425–435. URL: http://eudml.org/doc/148675.

[RW75] D. K. Ray-Chaudhuri and R. M. Wilson. "On t-designs". In: *Osaka Journal of Mathematics* 12.3 (1975), pp. 737–744. URL: http://hdl.handle.net/11094/7296.

[Str81] R. J. Stroeker. "On the diophantine equation $(2y^2 - 3)^2 = x^2(3x^2 - 2)$ in connection with the existence of non-trivial tight 4-designs". In: *Indagationes Mathematicae* 84.3 (1981), pp. 353–358.

[SZ84] P. D. Seymour and T. Zaslavsky. "Averaging sets: A generalization of mean values and spherical designs". In: *Advances in Mathematics* 52.3 (1984), pp. 213–240. DOI: 10.1016/0001-8708(84)90022-7.

[Tei87] L. Teirlinck. "Non-trivial t-designs without repeated blocks exist for all t". In: *Discrete Mathematics* 65.3 (1987), pp. 301–311.

[Wag91] G. Wagner. "On averaging sets". In: *Monatshefte für Mathematik* 111.1 (1991), pp. 69–78. DOI: 10.1007/BF01299278.

[Wal92] P. G. Walsh. "A quantitative version of Runge's theorem on Diophantine equations". In: *Acta Arithmetica* 62.2 (1992), pp. 157–172. URL: http://eudml.org/doc/206487.

[Wil72a] R. M. Wilson. "An existence theory for pairwise balanced designs I. Composition theorems and morphisms". In: *Journal of Combinatorial Theory, Series A* 13.2 (1972), pp. 220–245. DOI: 10.1016/0097-3165(72)90028-3.

[Wil72b] R. M. Wilson. "An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures". In: *Journal of Combinatorial Theory, Series A* 13.2 (1972), pp. 246–273. DOI: 10.1016/0097-3165(72)90029-5.

[Wil75] R. M. Wilson. "An existence theory for pairwise balanced designs, III: Proof of the existence conjectures". In: *Journal of Combinatorial Theory, Series A* 18.1 (1975), pp. 71–79. DOI: 10.1016/0097-3165(75)90067-9.

[Xia12] Z. Xiang. "A Fisher type inequality for weighted regular t-wise balanced designs". In: *Journal of Combinatorial Theory, Series A* 119.7 (2012), pp. 1523–1527. DOI: 10.1016/j.jcta.2012.04.008.

[Xia18a] Z. Xiang. "Explicit spherical designs". In: (2018). URL: http://ziqing.org/data/Explicit%20spherical%20designs.pdf.

[Xia18b] Z. Xiang. "New lower bound on the sizes of designs". In: (2018).

[Xia18c] Z. Xiang. "Nonexistence of nontrivial tight 8-designs". In: *Journal of Algebraic Combinatorics* 47.2 (2018), pp. 301–318. DOI: 10.1007/s10801-017-0776-0.