Problem Set 4

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Problem 1

With the trapezoidal rule, the approximation of the integral is $I_1 = 4.50656$ with 10 slices and $I_2 = 4.426660000000001$ with 20 slices. Then, using Eq. (5.28), the estimate of the error on the result I_2 is calculated to be $\epsilon_2 = 0.026633333333333137$. The exact error (error between I_2 and the exact value of the integral, 4.4) is $\epsilon_{ex} = 0.026660000000000572$.

Note that there is a small difference ($2.666666666743464 \times 10^{-5}$) between the estimate of the error ϵ_2 and the exact error ϵ_{ex}. The reason for this difference is that the trapezoidal rule should introduce an error of order $O(h^2)$, but in the derivation of Eq. (5.28), we only take the h^2 terms into consideration while neglecting the higher order terms, which are small but non-zero.

Problem 2

(a) The derivation for the expression for T is as follows:

$$\begin{aligned}
E &= \frac{1}{2}m(\frac{dx}{dt})^2 + V(x) \\
V(a) &= \frac{1}{2}m(\frac{dx}{dt})^2 + V(x) \\
(\frac{dx}{dt})^2 &= \frac{2[V(a) - V(x)]}{m} \\
dx &= -\sqrt{\frac{2[V(a) - V(x)]}{m}}dt \\
-\int_a^0 \sqrt{\frac{m}{2[V(a) - V(x)]}}dx &= \int_0^{\frac{T}{4}} dt \\
\sqrt{\frac{m}{2}}\int_0^a \frac{dx}{\sqrt{[V(a) - V(x)]}} &= \frac{T}{4} \\
T &= \sqrt{8m}\int_0^a \frac{dx}{\sqrt{[V(a) - V(x)]}}
\end{aligned}$$

Note that $\frac{dx}{dt}$ is negative, so we add a minus sign when taking the square root of $(\frac{dx}{dt})^2$.

(b) The period versus amplitude graph is shown in Figure 1. The period diverges to infinity as the amplitude approaches 0 but decreases as the amplitude increases.

(c) The anharmonic oscillator gets faster as amplitude increases, but becomes extremely slow when amplitude is small. This result can be explained when the anharmonic oscillator is compared to the harmonic oscillator. In the case of a harmonic oscillator, the period T_0 does not depend on the amplitude, and we know that the restorative force has a linear dependence on position ($F_h \propto x$). Now, for the anharmonic oscillator in this problem which has a potential $V(x) = x^4$, the restorative force is roughly proportional to x^3 ($F_a \propto x^3$). Therefore, for some large enough amplitude $a_1 >> 1$, the restorative force, and therefore the acceleration, is larger than that of the harmonic oscillator case. Therefore, the period for the anharmonic oscillator T_1 should be smaller than T_0. Now, for some even larger amplitude $a_2 > a_1 >> 1$, the acceleration is even larger for the anharmonic oscillator, but since the period of the harmonic oscillator is still T_0, the period of the anharmonic oscillator T_2 should be even smaller than T_1. To sum up, we have $a_2 > a_1 >> 1$, but $T_2 < T_1 < T_0$.

A similar argument can be made about the case where a approaches 0. For some small enough amplitude $a_1' << 1$, the restorative force and therefore the acceleration for the anharmonic oscillator are smaller than those of the harmonic oscillator. Therefore, the period T_1' for the anharmonic oscillator must be larger than T_0. Then for some even smaller amplitude $a_2' < a_1' << 1$, the acceleration is even smaller for the anharmonic oscillator, but the period of the harmonic oscillator is still T_0, so the period of the anharmonic oscillator T_2' should be even larger than T_1'. In other words, we have $a_2' < a_1' << 1$, but $T_2' > T_1' > T_0$.

Problem 3

(a) The wavefunctions for $n = 0, 1, 2, 3$ are shown in Figure 2. The wavefunctions are alternately even and odd starting at $n = 0$, each with $n + 1$ "peaks," as expected.

(b) The wavefunction for $n = 30$ is shown in Figure 3. The wavefunction is even and has $n + 1 = 31$ peaks as expected.

(c) The integral calculation uses the substitution $x = \tan z$ and Eq. (5.75). The root-mean-square position of the particle is calculated to be $2.3452078797796547 \approx 2.3$.

(d) The root-mean-square position of the particle calculated using the Gauss-Hermite quadrature is 2.345207879911714, which is almost equal to the result obtained in part (c). I do not think one can make an exact evaluation of this integral since this integral does not have an analytic solution (with the exponential term e^{-x^2} in the integrand).

My Github link: https://github.com/ziqiu-wang/phys-ua210.

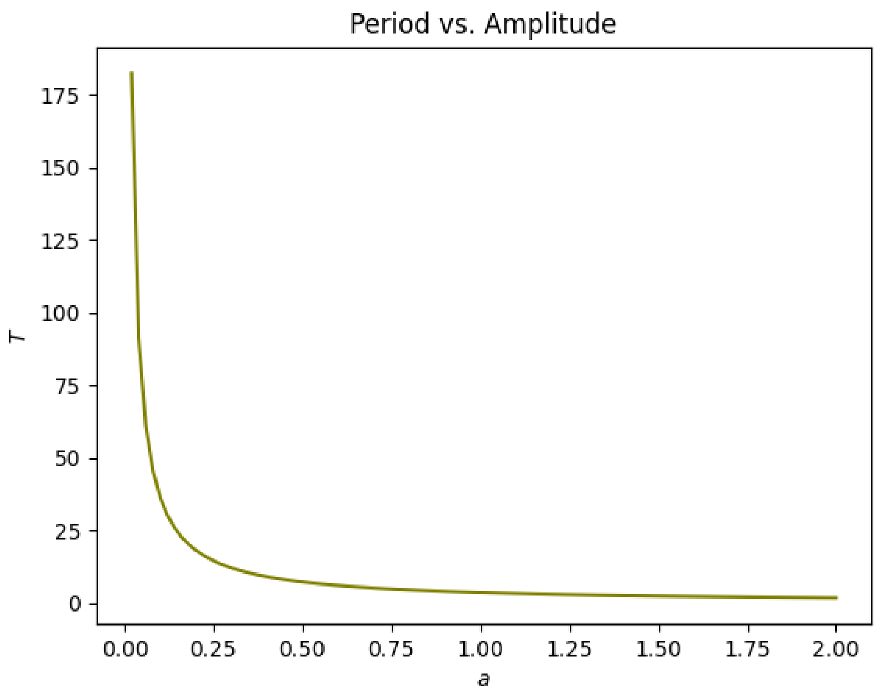

Figure 1: Period vs. Amplitude graph for Problem 2.

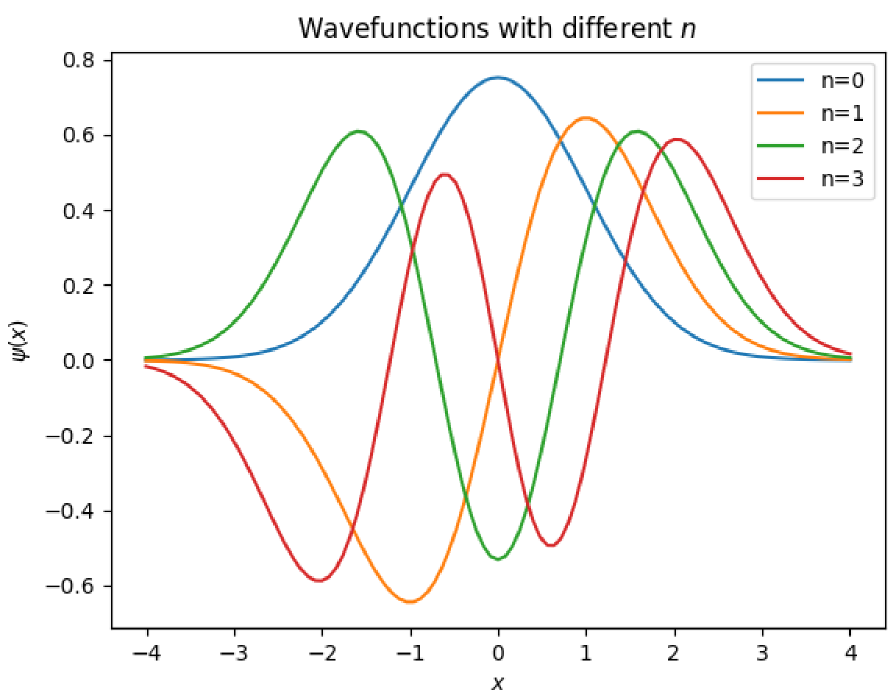

Figure 2: Problem 3 Part (a): wavefunctions with $n = 0, 1, 2, 3$.

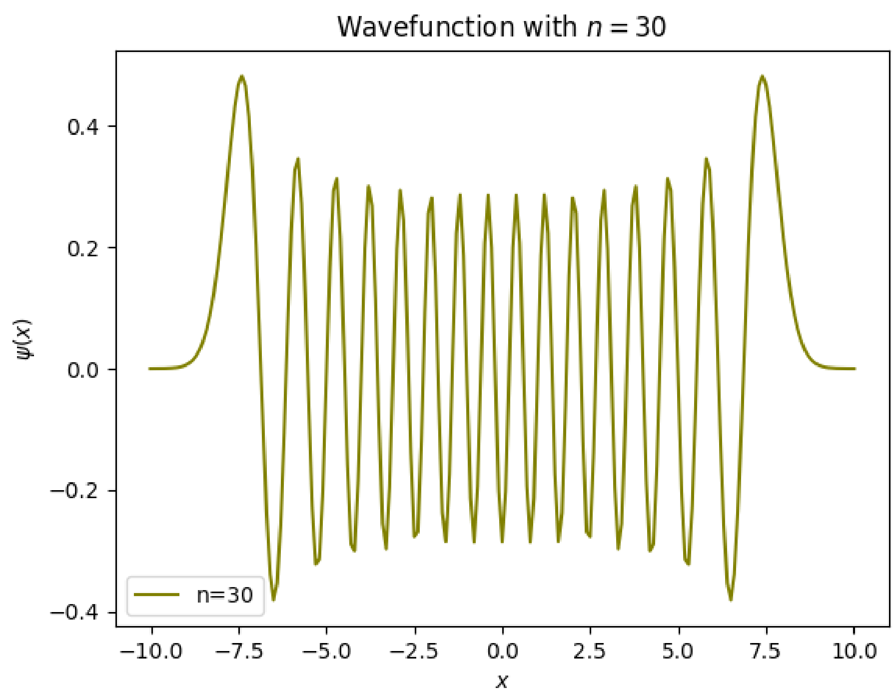

Figure 3: Problem 3 Part (b): wavefunction with $n = 30$.