Control Engineering / Homework 2 Solutions Ziraddin Gulumjanli, 2025

PROBLEM 1

1 (a): Finding Eigenvalues and Eigenvectors

Given the **state-space system matrix**:

$$A = \begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues λ are found by solving the :

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -4-\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$(-4-\lambda)(-\lambda) - (-3)(1) = 0$$

$$4\lambda + \lambda^2 + 3 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

Factoring:

$$(\lambda + 3)(\lambda + 1) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -1$$

For each eigenvalue λ_i, solve $(A - \lambda I)v = 0$ to find eigenvectors.

Eigenvector for $\lambda_1 = -3$

Solve:

$$\begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Row 1 provides: $-v_1 - 3v_2 = 0 \Rightarrow v_1 = -3v_2$ Choosing $v_2 = 1$, we get:

$$v_1 = -3, \quad v_2 = 1$$

Eigenvector:

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda_2 = -1$

Solve:

$$\begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Row 1 provides: $-3v_1 - 3v_2 = 0 \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$ Choosing $v_2 = 1$, we get:

$$v_1 = -1, \quad v_2 = 1$$

Eigenvector:

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Final Answer

- **Eigenvalues**:

$$\lambda_1 = -3, \quad \lambda_2 = -1$$

- **Eigenvectors (normalized)**:

$$v_1 = \begin{bmatrix} -0.9487 \\ 0.3162 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$$

1 (b): Modal canonical form this system

NOTE: You will see two results here and the reason for that the pdf version assumes raw eigenvectors while matlab code by default uses normalized values. However, that does not affect to next 2 problems (c and d). Compute Modal Canonical Form $J = P^{-1}AP$. The transformation matrix P is formed by placing the eigenvectors as columns from Problem 1a:

$$P = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

The inverse of P is:

$$P^{-1} = \frac{1}{(-3)(1) - (-1)(1)} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-3+1} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

And J becomes:

$$J = P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

The Modal Canonical Form of A is:

$$A_{\text{modal}} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

The modal canonical forms of B and C are:

$$B_{\text{modal}} = P^{-1}B = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$C_{\text{modal}} = CP = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Final Modal Canonical Form

$$\dot{z} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} z$$

Final Modal Canonical Form w/ normalized eigenvectors:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z \end{bmatrix} + \begin{bmatrix} -4.7434 \\ -2.1213 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.3162 & -0.7071 \end{bmatrix} \begin{bmatrix} z \end{bmatrix}$$

1 (c): State transition matrix of the modal system

Considering $A' = A_{\text{modal}}$, to determine the state transition matrix $\Phi(t)$ for the modal system, we use:

$$\Phi(t) = e^{A't}$$

where:

$$A' = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

Compute the Matrix Exponential Since A' is diagonal, the matrix exponential is simply:

$$e^{A't} = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

1 (d): State transition matrix of the original system

Considering $A' = A_{\text{modal}}$

$$\Phi(t) = e^{At} = Pe^{A't}P^{-1}$$
$$= \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Multiplying:

$$\Phi(t) = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{bmatrix}$$

Final Answer

$$\Phi(t) = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{bmatrix}$$

PROBLEM 2

2 (a): linear transformation

The given transformation equations:

$$z = P_x x + Q_x \tag{1}$$
$$w = P_u u \tag{2}$$

Original LTI state-space system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3}$$
$$y(t) = Cx(t) + Du(t) \tag{4}$$

Differentiate the given transformation $z = P_x x + Q_x$:

$$\dot{z} = P_x \dot{x} \tag{5}$$

Substituting $\dot{x}$ from the given state equation:

$$\dot{z} = P_x(Ax + Bu) \tag{6}$$
$$\dot{z} = P_x Ax + P_x Bu \tag{7}$$

Using $x = P_x^{-1}(z - Q_x)$:

$$\dot{z} = P_x A P_x^{-1}(z - Q_x) + P_x Bu \tag{8}$$

Expanding:

$$\dot{z} = P_x A P_x^{-1} z - P_x A P_x^{-1} Q_x + P_x Bu \tag{9}$$

We also have $w = P_u u$, so can solve for $u = P_u^{-1} w$ and substitute:

$$\dot{z} = P_x A P_x^{-1} z - P_x A P_x^{-1} Q_x + P_x B P_u^{-1} w \tag{10}$$

Thus, transformed state-space equation is:

$$\dot{z} = \tilde{A}z + \tilde{B}w + \tilde{Q} \tag{11}$$

where:

$$\tilde{A} = P_x A P_x^{-1}, \quad \tilde{B} = P_x B P_u^{-1}, \quad \tilde{Q} = -P_x A P_x^{-1} Q_x \tag{12}$$

The output equation is:

$$y = Cx + Du \tag{13}$$

Substituting $x = P_x^{-1}(z - Q_x)$ and $u = P_u^{-1} w$:

$$y = CP_x^{-1}(z - Q_x) + DP_u^{-1} w \tag{14}$$

Expanding:

$$y = CP_x^{-1} z - CP_x^{-1} Q_x + DP_u^{-1} w \tag{15}$$

So, the transformed output equation is:

$$y = \tilde{C}z + \tilde{D}w + \tilde{Q}_y \tag{16}$$

where:

$$\tilde{C} = CP_x^{-1}, \quad \tilde{D} = DP_u^{-1}, \quad \tilde{Q}_y = -CP_x^{-1} Q_x \tag{17}$$

Final Transformed State-Space Model

$$\dot{z} = \tilde{A}z + \tilde{B}w + \tilde{Q} \tag{18}$$
$$y = \tilde{C}z + \tilde{D}w + \tilde{Q}_y \tag{19}$$

where:

$$\tilde{A} = P_x A P_x^{-1}, \quad \tilde{B} = P_x B P_u^{-1}, \quad \tilde{C} = CP_x^{-1}, \quad \tilde{D} = DP_u^{-1} \tag{20}$$
$$\tilde{Q} = -P_x A P_x^{-1} Q_x, \quad \tilde{Q}_y = -CP_x^{-1} Q_x \tag{21}$$

2 (b)Simulation Results on original system

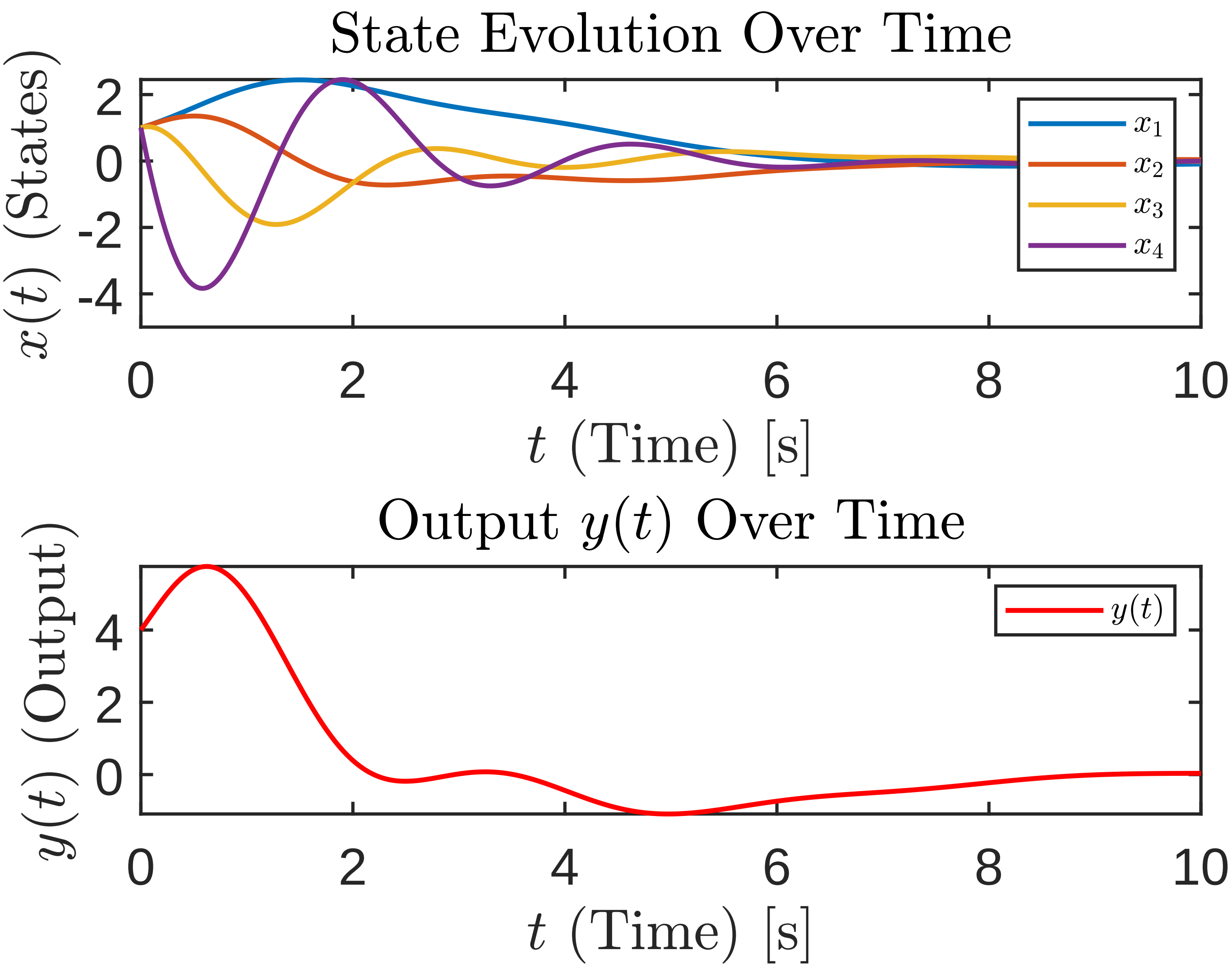

Figure 1: Simulation of $x(t)$ and $y(t)$ for the original state-space system

2 (c)Simulation Results on Transformed system

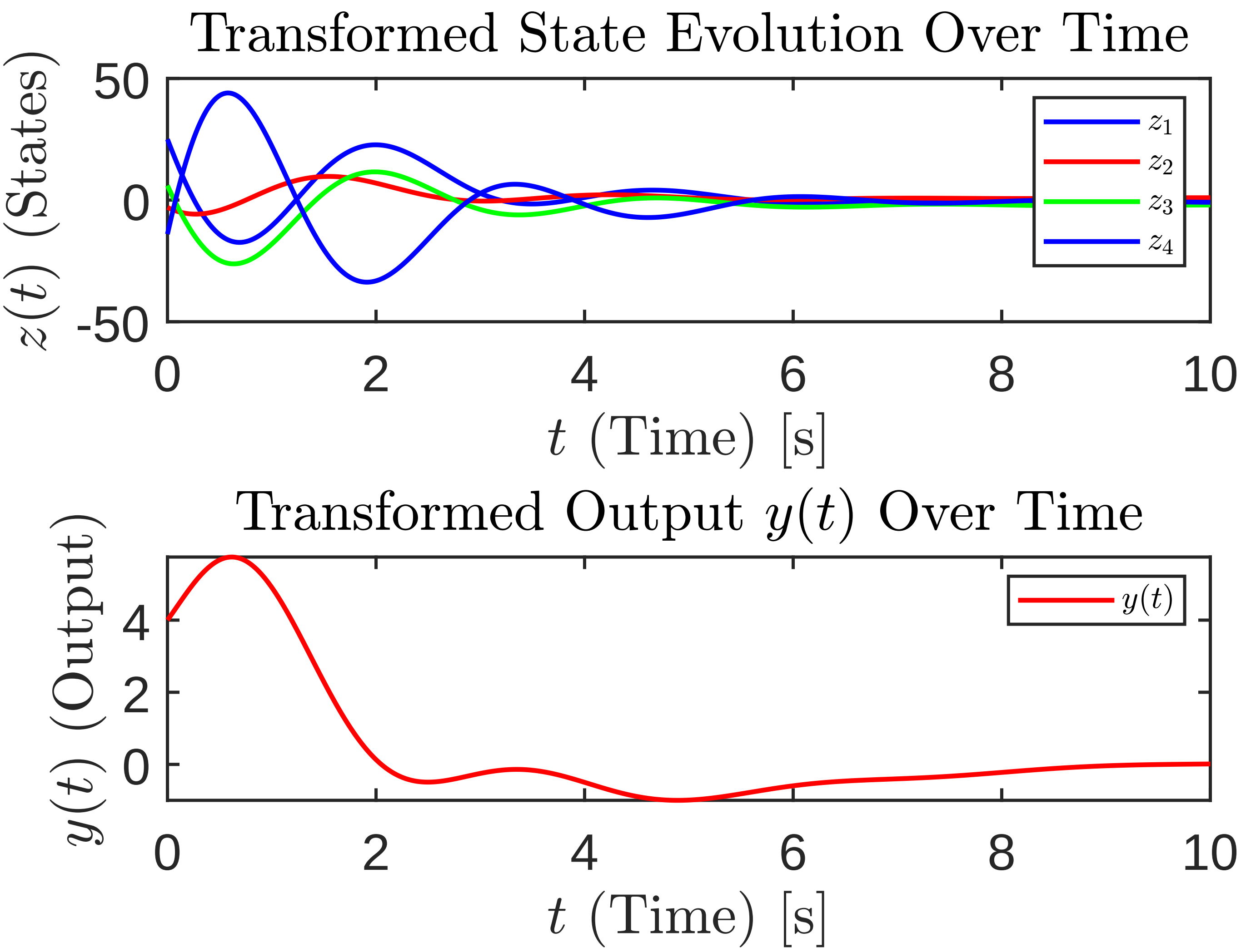

Figure 2: Simulation of $z(t)$ and $y(t)$ for a transformed state-space system

2(d) Comparative Study of Original and Transformed System Simulation Results

Two provided plots, illustrations in Problem 2b and c, respectively illustrate the state evolution and output response of a linear time-invariant (LTI) system before and after applying a state transformation. Fig 1 corresponds to the original state-space representation $x(t)$, whereas Fig 2 visualize the transformed state-space system $z(t)$. In both cases, the state variables exhibit damped oscillatory behavior, indicating a stable system. However, in the transformed system, the individual states z_1, z_2, z_3, z_4 display different -higher magnitudes and initial conditions compared to x_1, x_2, x_3, x_4. The transformation modifies the external representation of state variables, resulting the differences in the graphs in their [0–6] second period. The very interesting part is: The output $y(t)$ in each case remains unchanged, which was expected since state transformations change the representation of state variables but not the output. Overall, the system in both cases appears to be stable after the 6th second.

Thank You!