Control Engineering Ziraddin Gulumjanli, 2025

PROBLEM 1

1 (a- b- c- d)

System (a)

$$A = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\lambda_1 = -0.3028, \quad \lambda_2 = 3.3028$$

One eigenvalue is positive, the system is **unstable**.

System (b)

$$A = \begin{bmatrix} -2 & -3 & 5 \\ 3 & 2 & -5 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\lambda_1 = -3.0000, \quad \lambda_2 = 0.0000, \quad \lambda_3 = 0.0000$$

The system has double zero eigenvalues, **unstable**.

System (c)

$$A = \begin{bmatrix} -2 & -3 & 5 \\ 6 & 15 & -21 \\ 5 & 14 & -19 \end{bmatrix}$$

$$\lambda_1 = 0.0000, \quad \lambda_2 = -3.0000 + 1.0000i, \quad \lambda_3 = -3.0000 - 1.0000i$$

The system is **Lyapunov Stability** since the eigenvalues of the system matrix A have nonpositive real parts and eigenvalues on the imaginary axis are nonrepeated.

System (d)

$$A = \begin{bmatrix} -2 & -3 & 5 \\ 10 & 20 & -20 \\ 9 & 19 & -18 \end{bmatrix}$$

$$\lambda_1 = -1.0000 + 2.0000i, \quad \lambda_2 = -1.0000 - 2.0000i, \quad \lambda_3 = 2.0000$$

One eigenvalue is positive, the system is **unstable**.

Table Summary

System	**Eigenvalues**	**Stability**
(a)	$-0.3028, 3.3028$	Unstable
(b)	$-3.0000, 0.0000, 0.0000$	Unstable
(c)	$0.0000, -3.0000 \pm 1.0000i$	Stable in LS
(d)	$-1.0000 \pm 2.0000i, 2.0000$	Unstable

To proof our numerical considerations, let's assume the simulation where: the time span is 20 seconds, the input function of $u(t) = e^{-0.3t}\sin(t)$: The state vector x_0 is initialized to zero and dynamically sized to match A for all cases and the control matrix B is structured to influence the last state variable. The state plots visualize the system's dynamic response, as shown in Fig 1. Again we can see that, System c has more stability compared to the others.

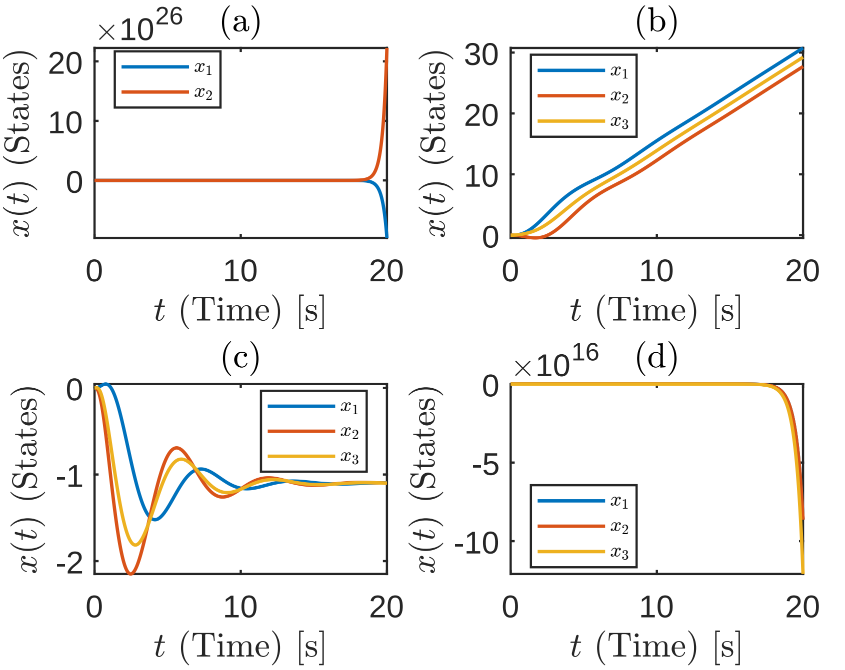

Figure 1: Simulation of $x(t)$ for all scenarios

2 (a)

The observability matrix for an LTI system is given by:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

where:

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Compute CA

$$CA = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \end{bmatrix}$$

Compute CA^2

$$CA^2 = CA \cdot A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & -13.5 \end{bmatrix}$$

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -3 \\ -0.5 & 0 & -13.5 \end{bmatrix}$$

Since the rank of observability matrix is 2, which is less than the number of states (n = 3) **the system is not fully observable** .

2 (b)

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & -1.5 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Observable Subspace Decomposition

$$A_{\text{ob}} = \begin{bmatrix} 2.5 & -1.5 \\ 3 & 2 \end{bmatrix}$$

$$C_{\text{ob}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

To verify that this subsystem is indeed observable.

$$\mathcal{O}_{\text{subspace}} = \begin{bmatrix} C_{\text{ob}} \\ C_{\text{ob}} A_{\text{ob}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

Computing its rank:

$$\text{rank}(\mathcal{O}_{\text{subspace}}) = \text{number of states} = 2$$

To conclude the given system was found to be not fully observable, the system then decomposed into a fully observable subspace. Calculating its rank, it is verified that the observable subspace is indeed fully observable.

3 (a)

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 2 \\ -3 & 3 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -4 \\ 9 & -9 \end{bmatrix} \quad A^3B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 8 \\ -27 & 27 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 2 & 0 & -4 & 0 & 8 \\ 1 & -1 & -3 & 3 & 9 & -9 & -27 & 27 \end{bmatrix}$$

$$\text{Rank}(M_c) = 4$$

$$\text{Eigenvalues:} \quad \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}$$

Since the systems rank is equal tot he number of states and has one positive eigenvalue... The system is **fully controllable** but is **NOT stable**.

3 (b)

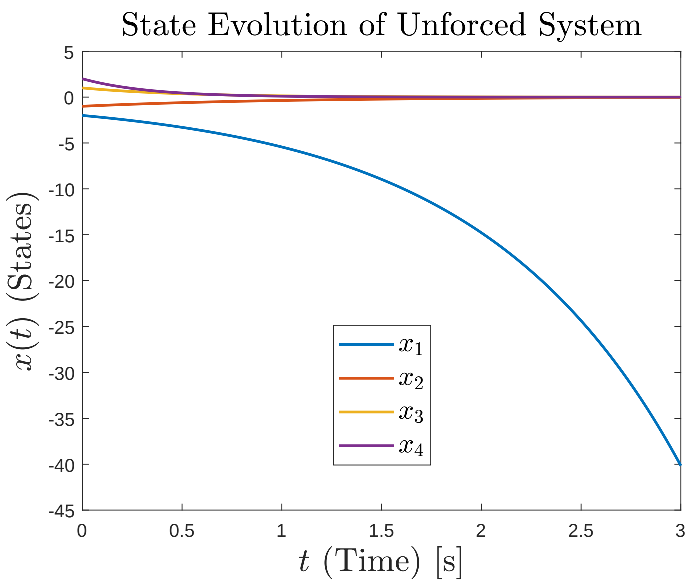

Figure 2: Simulation of the unforced system

The plot shows that one of the states changes exponentially which is an indication of the unstable condition, while, other states decay to zero, aligning with their negative eigenvalues, and that verifies stability. I am confident that this validates our theoretical predictions and confirms the system's mixed stability behavior

3 (c) Comparison analytics

$t_0 = 0$ and $t_f = 3$ and the following initial condition:

$$x_0 = \left[-2; -1; 1; 2\right]$$

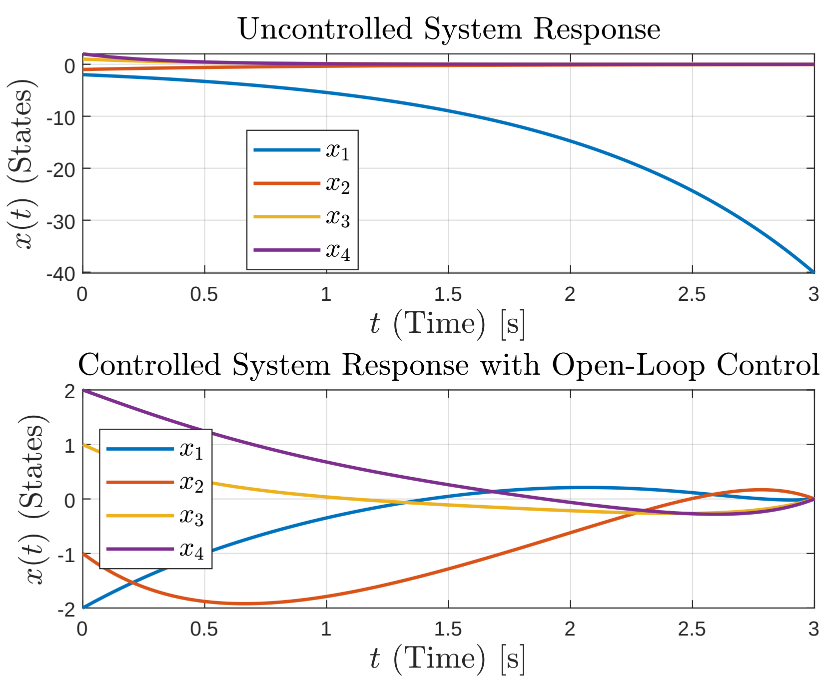

Figure 3: Simulation behavior of the given LTI model using Controllability Theorem 1

The simulation findings validate the open loop's predicted behavior. Without control, the first state expands exponentially due to a positive eigenvalue suggesting instability, but the other states decay. After using open-loop control, the system's behavior changes and all states settle to zero. That is the intended behavior.

However, as noted in the class notes, there are several practical concerns with this controller. I think that because it is based on the accurate computation and inversion of the controllability Gramian, the approach is extremely susceptible to modeling mistakes and numerical inaccuracies. If the Gramian is ill-conditioned, minor numerical errors can multiply, resulting in unreliable control inputs. While Controllability Theorem 1 guarantee that any state may be driven to zero if the Gramian is invertible, numerical tests, such as Kalman's controllability criterion-like approaches, may be more trustworthy because they avoid matrix inversion.

3 (d)

with $t_0 = 0$ and $t_f = 1$ and the following initial condition:

$$x_n = [1; 1; 1; 1]$$

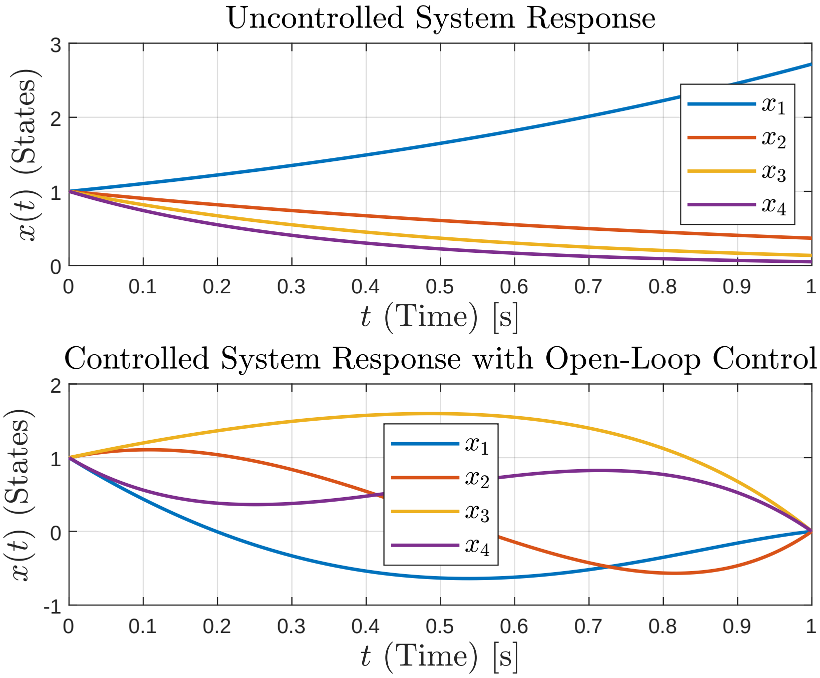

Figure 4: Simulation behavior of the given LTI model using Controllability Theorem 1

4 (a) I believe the number of states in the identified model are 248

4 (b)

The Hankel single value plot gives information on the relevance of each state. The graphic shows that incorporating all states is redundant since the majority of the later states do not contribute significantly to the system's reaction. The shown values show a significant reduction in the first 10-15 states (20 to 30 also would be fair), followed by a more steady decay. This shows that the first 15 dominating states account for the majority of the system's energy, with the other states contributing marginally. Beyond about 20-40 states, the single values become very tiny, implying that those states have little effect on system behavior. To achieve a balance between accuracy and model simplicity, retaining around 20 to 30 states should be recommended and provide a reasonable approximation while significantly reducing system complexity. **NOTE:** Please make sure that `mat` file provided in Canvas is in your directory.

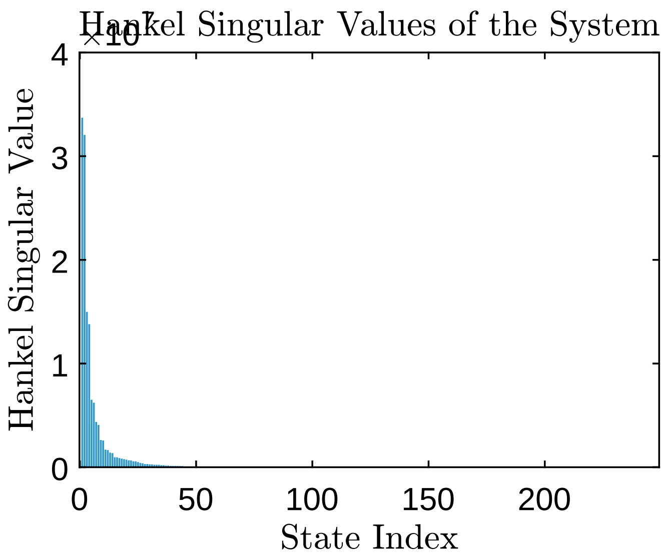

Figure 5: Bar plot of Hankel singular values

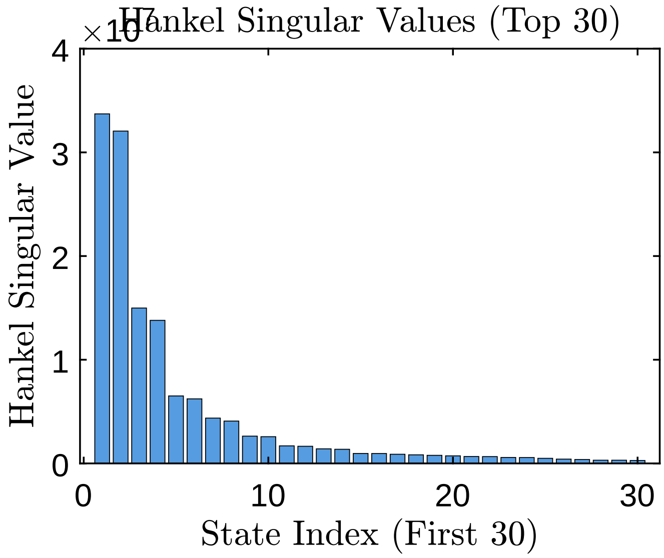

Figure 6: Bar plot of Hankel singular values for first 20 states.

4 (c)

Interpretation of the Results (Figures are on the next page)

In both figures, the 20-state model matches closely with the full model across all frequencies, the 4-state model tries to follow the trend BUT deviates in magnitude and phase, especially near resonance peaks. I believe these plots confirms the previous state number selection process were correct, since 4 state model can provide risky and non-robust results, so choosing 20 would give us similar results with the original states

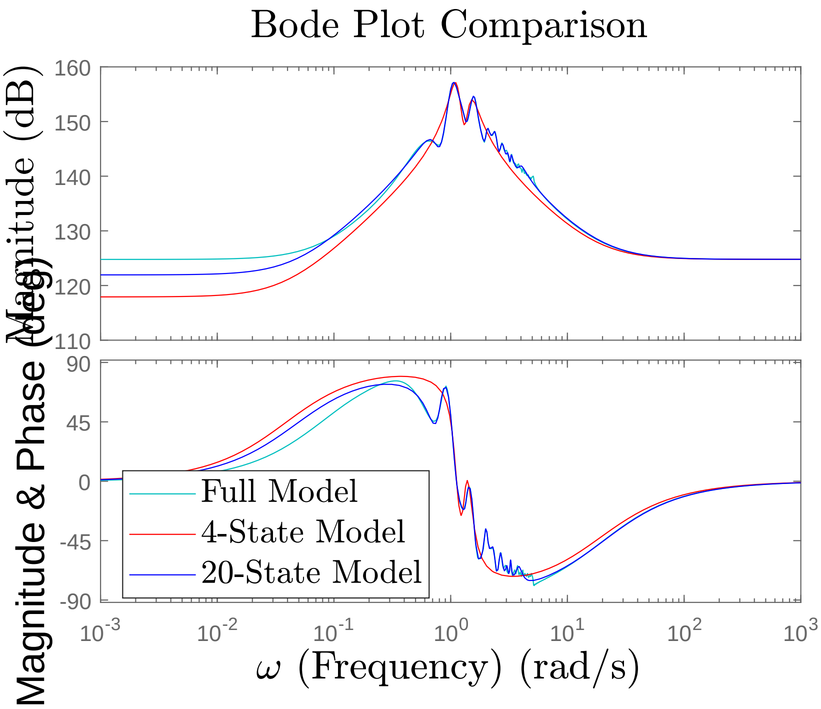

Figure 7: Bode response plot

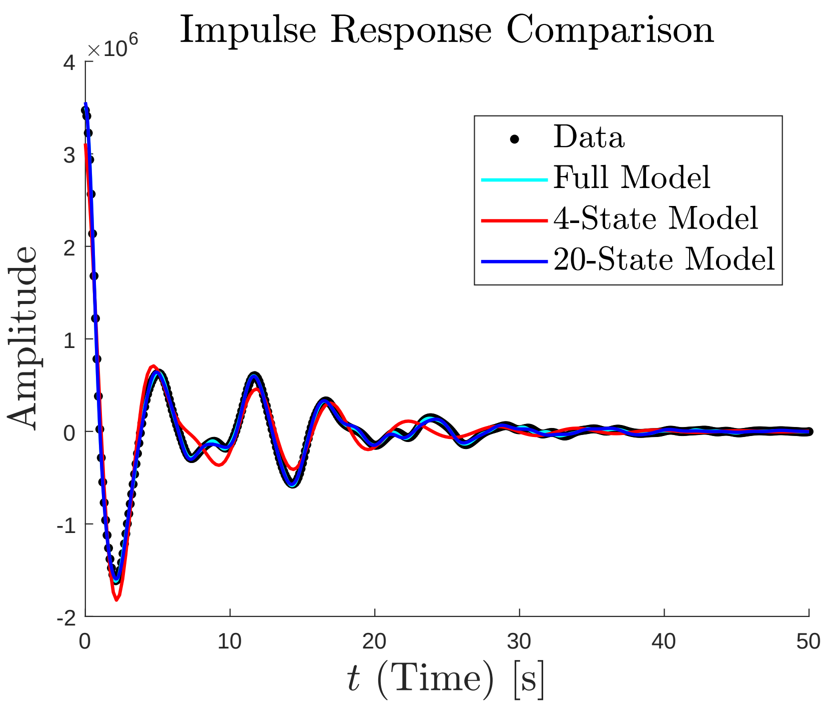

Figure 8: Impulse response plot