2
Commodity Mathematics and Products

In this chapter, we shall start with the convenient framework of one-factor models familiar to all students of mathematical finance. After revising the familiar Black–Scholes analysis for traded spot markets, I shall show how this can be extended to forwards and futures markets, such as are encountered in commodities, together with a discussion of some peculiarities of these markets. Having discussed such linear products, we can go on to options – which is what this book is about! So thereafter, I shall describe some of the typical products encountered in commodity options markets together with techniques for their valuation under these models, and the chapter will conclude with a discussion of the more advanced models which appear in the literature.

2.1 SPOT, FORWARDS AND FUTURES

One of the main aspects that differentiates many (but not all) commodities from other asset classes is the absence of a traded spot contract. Typically a spot transaction is for settlement a few good business days in the future (T+2 is customary in FX and, more relevantly, for precious and base metals; T+5 is usual in energy). Let us consider the metals. This means that if a spot trade is entered into on Monday 10 September 2012, then settlement will occur on Wednesday 12 September 2012. By settlement, we mean that the required cashflows and exchanges of commodities of value are scheduled to occur on that date. As the trade date advances forward in time, the spot date rolls forward also, subject to settlement rules and holidays. For example, a spot trade entered into on Tuesday 11 September 2012 will settle on Thursday 13 September 2012.

This is perfectly fine for financial instruments, but one can easily see that trading many physical commodities with such short notice is unrealistic. Oil needs to be shipped from location to location, base metals similarly. For this reason it is absolutely standard in the commodities markets to expect settlement and delivery to occur some time in the future. There are two ways to achieve this. One is effectively an over-the-counter (OTC) **forward contract**, where Counterparty A agrees to trade N units of a particular commodity (of a particular grade) at a known forward price K_{Fwd} with Counterparty B at some prespecified time T in the future. The other is an exchange traded **futures contract**, where a market participant is obliged – and note importantly that the obligation is now with the exchange and not directly with the other counterparty – to deliver or take delivery of (once again) N units of a particular grade of a commodity at some known time T in the future for a known futures price K_{Fut}. The contract sizes and delivery dates are generally standardised to provide sufficient liquidity. The exchange manages the obligation through the use of margin and mark-to-market; we shall discuss how in Section 2.1.3.

Should we therefore attempt to model spot, forwards or futures? Section 2.2 describes two possible ways to get started, both of which are used in practice.

The rest of this section introduces the spot, forward and futures contracts and markets; further details can be found in Carlton (1984), Williams (1986), Chapter 8 of Duffie (1996), Chapter 5 of Pilipović (1998), Pindyck (2001), and Geman (2005) and references therein.

2.1.1 Spot

A spot transaction is generally understood to be the shortest dated transaction that can be entered into in standard fashion. It comprises an agreement to trade a certain amount of a particular commodity at the spot date. For commodities, it only really used for precious metals, where the spot date is two business days after today (allowing for holidays), and for electricity.

We generally use S_t to denote the spot price at time t, a quantity which will make its appearance mathematically speaking in Section 2.2.1. Even where a spot contract does not exist *per se*, it is commonplace to take the shortest dated futures contract (also known as the "prompt future") as a proxy for spot. We shall use the notation $\vec{T}(t)$ to denote the maturity of the prompt future at time t, the arrow indicating that $\vec{T}(t)$ is the first expiry date to the right of t when viewed on a typical time axis, e.g. if we allow the expiry dates of the futures contracts to be denoted T_1, T_2, ... I attempt to show this graphically in Figure 2.1.

$$t \qquad T_1 \equiv \vec{T}(t) \quad \rightarrow \quad T_2 \qquad \longrightarrow \qquad T_3 \quad \rightarrow \quad \dots$$

Figure 2.1 Futures contracts $T_1, T_2, T_3 \dots$ as viewed from time t.

Note that the expiry dates for options on futures are different from the expiry dates of the underlying futures contracts themselves – more on this in Section 2.5.2.

2.1.2 Forwards

Consider a forward contract, entered into at time t with maturity T. In such a trade, Counterparty A has the obligation to deliver 1 unit of the commodity to Counterparty B, and to receive K units of (domestic) cash – both at time T in the future. The contract is an agreement that holds whether it is economically advantageous or disadvantageous for either party. There is no optionality. These contracts can be cash settled or physically settled (I shall assume cash settled, as we are concerned with financial valuation).

The price of such a T-forward contract at time t is denoted $F_{t,T}$. As this contract can have a negative value as well as a forward value, depending on the strike K, it is customary to solve for the value of K such that it makes the forward costless to enter into. This choice of strike K for which this is the case is denoted $F_{0,T}$.

Forward prices are rarely identical and rarely equal to the spot price (where a spot contract exists). In the case where shorter dated forward prices are less than longer dated forward prices, i.e. $F_{0,T_1} < F_{0,T_2}$ with $T_1 < T_2$, we refer to the forward curve as being in *contango*. The other case where shorter dated forward prices are greater than longer dated forward prices, i.e. $F_{0,T_1} > F_{0,T_2}$ with $T_1 < T_2$, is called *backwardation*, reflecting increased demand for contracts with shorter term delivery. More complicated shapes for the term structure of forward curves definitely exist, very possibly including seasonality.

The discrepancy between spot prices and forward prices is often rationalised by appealing to the balance to be obtained between the benefit of holding a commodity until a particular time T in the future and the storage costs required to keep it on hand until that time. Consider a commodity such as heating oil, which has a marked demand peak in the Northern hemisphere winter. Since a higher price can be attained for contracts with delivery in the winter months due to natural seasonal demand, then if storage costs were sufficiently low it would be economically advantageous to buy heating oil in the summer, store it, and deliver it in the winter months.

The fact that the forward curve is not flat indicates that the market has to price in two different effects when comparing contracts with different maturities. The first is the imputed benefit that "accrues to the owner of the physical commodity" (Geman, 2005, p. 24) and the second is the storage and maintenance costs required to keep the commodity to hand.

This is quantified as the **convenience yield** in spot commodity markets, where we refer to the instantaneous benefit (or cost) to the holder of a commodity in terms of an effective rate, which we denote r^f by analogy with the foreign risk-free rate encountered in foreign exchange. Basically this means that in the absence of any spot price movements, we still expect the economic value of a commodity to appreciate or depreciate from S_0 to $S_0 \exp(r^f \delta t)$ over the course of a small time interval δt, as described in Section 2.2 of Geman (2005).

Note that no cashflows take place with a forward contract except at maturity (and at initiation if the strike isn't chosen so that the contract is costless to enter into, which is unusual).

Note that forward contracts are almost always traded over-the-counter (OTC) directly between market participants, and consequently the creditworthiness of the participants should be taken into consideration (this will be outside the scope of this work though).

Having introduced these contracts, we shall obtain the fair price for the forward contract in Section 2.2 under the risk-neutral measure.

2.1.3 Futures

Following the notation of Musiela and Rutkowski (1997), who provide a thorough discussion of forwards and futures in the discrete time setting in Sections 2.5.1, 1.6, 3.2 and 3.3 of their book, we use $F_{t,T}$ for the forward price and $f_{t,T}$ for the futures price.

A futures contract has the same value payout at expiry as a forward contract, but differs from it in two important ways. Firstly, futures contracts are exchange traded and therefore subject to much greater standardisation in terms of which contract sizes and maturity dates are tradeable. Secondly, futures contracts are traded directly with the exchange and "marked to market", which has the benefit of removing credit counterparty risk from the purchaser of the futures contract.[1]

Generally, futures contracts corresponding to the various calendar months are exchange traded, and each month's contract (however it is defined) has a particular ticker symbol associated with it:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
F	G	H	J	K	M	N	Q	U	V	X	Z

These ticker symbols, which seem somewhat arbitrary, are actually chosen by exclusion, as various other commodities had (and often still have) one letter trading codes, e.g. C for corn.

One needs to be careful, though, as the actual maturity date for a particular month's contract sometimes falls into the latter part of the *preceding* month, as seen in an example below. The contracts at the end of each quarter are very often the more liquid, i.e. HMUZ, and indeed out beyond a year or two one often only sees the June (M) and December (Z) contracts quoted.

For poets and quants, a small piece of doggerel may be useful to help remember the codes:

Fragrant garden handshake
Jasmine kettle mandrake
Never quite unseemly
Viewing X-ray zucchini

In addition to the character code for the contract maturity, there is usually a quote code for the particular class of commodity – for example, WTI-NYMEX is denoted CL (for Crude Light). These two, together with the final digit (sometimes final two digits) of the year, are concatenated to produce a quote code.

For example, the FEB12 WTI-NYMEX contract has the code "CL G2" and Table 2.1 shows the following dates of importance for this contract.

On occasion, for the so-called "weekly" options on the CME, it is necessary to specify the particular week on which a contract expires or matures. Taking the example of a Weekly Live Cattle option expiring on the fifth Friday in September 2011 from the CME, the product or telequote code for this is "LC5U11" – where "LC" refers to live cattle options (which expire on Friday, see Section 8.5.2) 5 indicates that the

[1] Except risk relating to the creditworthiness of the exchange itself!

Table 2.1 Contract dates of importance for typical WTI futures contract.

Commodity	Quote code	Futures contract	Maturity date	Settlement date
WTI-NYMEX	CL G2	FEB12	20-Jan-12	27-Jan-12

expiry is on the *fifth* Friday, "U" means September and "11" indicates 2011.

The market conventions for determining these dates vary according to the particular quoted commodity and will be described further in subsequent chapters.

The process of marking to market requires some explanation, details can be found in Exercise 2.17 of Duffie (1996), Chapter 2 of Collins and Fabozzi (1999) and Schneider (2012). Let the price at time t of a commodities future contract with maturity T be denoted by $f_{t,T}$. Corresponding to $t = 0$, the price of the same T-future contract today is denoted by $f_{0,T}$ – meaning that a T-futures contract can be entered into today with a strike of $K = f_{0,T}$ at no cost to enter (we shall discuss the pricing of this in Section 2.2 also).

An investor buys a T-futures contract at the market price $f_{0,T}$. Whereas for the equivalent forward contract no cashflows occur until T, in this case the investor must post an **initial margin** into an (interest earning) margin account. The value of the futures contract is calculated daily by the exchange in a procedure known as "marking to market" – where the change in value of the position over the course of a trading day is called the "variation margin". The margin held on deposit is used to fund changes in the variation margin. If, during the lifetime of the contract, the value is determined to be above the initial margin, then the excess funds can be withdrawn from the margin account. In contrast, if the mark-to-market (MTM) value decreases below the initial margin, then no such withdrawals are possible, and indeed if the value goes down to or below a lower level known as the **maintenance margin** then the margin account must be topped up with additional cash reserves to return the balance to the initial margin.

From this it is immediately apparent that futures contracts are subject to uncertain cashflows over the entire lifetime of the contract, unlike forward contracts. As pointed out by Jarrow and Oldfield (1981), a forward contract only has a cash flow at the expiry date T. In contrast, a futures contract has a number of cash flows throughout the interval $[0, T]$ depending on the mark-to-market of the underlying position. We shall explore this point in a worked example in Section 2.3.

In the presence of stochasticity in the domestic interest rate, the futures contract can have a different value to the equivalent forward contract – though this discrepancy vanishes if the forward contract and domestic bonds have zero correlation (see Cox, Ingersoll and Ross, 1981). Basically, if the price of a traded commodity is positively correlated with interest rates, then profits on a long futures position in that commodity can be reinvested at a higher interest rate than the interest rate used to finance losses on the position (and vice versa for negative correlation). While beyond the scope of this book, discussion of futures convexity adjustments in the context of interest rate options can be found in Flesaker (1993), Vaillant (1995), Pelsser (2000), Piterbarg and Renedo (2006) and Exercise 2.5.2 (2) in Tan (2012).

More directly related to this chapter, Pilz and Schlögl (2013) present a variation of the LIBOR market model for commodity options, wherein the convexity correction relating commodity forwards and commodity futures is discussed, as well as the dependence upon correlation.

Another potential source of discrepancy between forward and futures prices, of course, is counterparty risk, as forward contracts are clearly subject to credit risk. We shall not discuss that here, commodity forward contracts being encountered far less frequently than futures contracts, and merely refer the reader to Section 6.3 of Burger *et al.* (2007) for discussion of credit risk in the context of commodity markets.

For this book, therefore, we shall work under the assumption that futures prices and forward prices coincide, i.e.

$$f_{t,T} = F_{t,T}. \tag{2.1}$$

Note that this does *not* mean that the present value of a T-futures contract with strike K is the same as present value of a T-forward contract with the same maturity T and the same strike K – see Section 2.3 – it does however mean that if the strike K is such that the T-futures contract is costless to enter into, then the T-forward contract is costless to enter into also (and $K = f_{t,T} = F_{t,T}$).

Standard references detailing the similarities and differences between forwards and futures are Jarrow and Oldfield (1981), Cox, Ingersoll and Ross (1981) and, with special reference to commodity markets, Pindyck (2001). Appendix 3B of Sundaram and Das (2010) is also recommended.

2.2 THE BLACK–SCHOLES AND BLACK-76 MODELS

Much of this material will be familiar to the reader who has tackled Baxter and Rennie (1996), Shreve (2004) or indeed my previous book (Clark, 2011). While there are many useful references regarding risk neutrality, two which I have found particularly useful are Sundaram (1997) and Bingham and Kiesel (1998).

Suppose a rational investor has two investment opportunities open for his or her consideration. Such an investor should have no reason to favour one portfolio over another if both are riskless, as this would admit a clear arbitrage opportunity. This leads to the framework of risk-neutral pricing – if we have a risky stock, and a risky derivative on that stock, then by constructing a dynamically rebalanced portfolio which is instantaneously riskless, we can obtain a framework for pricing derivatives. One important note is that when we work through the pricing analysis, the expected drift of the spot process doesn't appear in the partial differential equation, so the drift can be set to the risk-free rate. Further, we shall see that both forward prices and futures prices are martingales.[2]

2.2.1 The Black–Scholes Model

The Black–Scholes analysis for obtaining a partial differential equation governing the price of commodities without any storage costs or imputed benefits from holding the commodity over particular time intervals, is equivalent to the pricing of equity derivatives in the absence of dividends. We therefore follow Black and Scholes (1973) and describe the spot rate by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

with the standard Black–Scholes assumptions, as in Hull (2011):

1. The spot price S_t (in domestic currency) of 1 unit of a tradeable commodity follows a lognormal process (2.2).
2. Short selling is permissible.
3. No transaction costs or taxes.

[2] Technical footnote: forward prices are martingales in the T-forward measure, whereas futures prices are martingales in the domestic risk-neutral measure respectively, though these coincide in the absence of stochastic interest rates.

4. The domestic currency has risk-free rate r^d constant across all maturities; the convenience yield of the commodity is denoted r^f by analogy with the foreign risk-free rate encountered in foreign exchange. Note that r^f can be either positive or negative (and so too these days can r^d).
5. No riskless arbitrage.
6. Trading is continuous between now ($t = 0$) and expiry ($t = T$).

2.2.2 The Black–Scholes Model Without Convenience Yield

Suppose that the price of a contingent claim $V(S_t, t)$, which derives its value from the performance of a tradeable asset with spot price S_t, is known. Let $V_t = V(S_t, t)$ denote the value of the contingent claim at time t, conditional on the asset spot price being S_t at that time. Applying Itô, we have

$$dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS_t^2. \tag{2.3}$$

However we know from (2.2) that $dS_t^2 = \sigma^2 S_t^2 dt$ and consequently at time t, we have

$$dV_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \frac{\partial V}{\partial S}dS_t. \tag{2.4}$$

The term inside the square brackets above is deterministic, whereas the term appearing in front of dS_t is the only stochastic term. We remove the stochastic term by construction of a portfolio Π_t which is long one unit of the contingent claim (with value V_t) and short $\partial V/\partial S$ units of the underlying asset

$$\Pi_t = V_t - \frac{\partial V}{\partial S}S_t. \tag{2.5}$$

This has the SDE

$$d\Pi_t = dV_t - \frac{\partial V}{\partial S}dS_t \tag{2.6}$$

and from (2.4), we have

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt. \tag{2.7}$$

As the growth of Π_t is riskless, we can appeal to risk-neutrality and equate the expected growth of Π_t to that of the domestic risk-free bond B_t^d which is described by

$$dB_t^d = r^d B_t^d dt. \tag{2.8}$$

We therefore put

$$d\Pi_t = r^d \Pi_t dt \tag{2.9}$$

where r^d denotes the domestic interest rate. Equating terms in (2.7) and (2.9) we have

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} &= r^d \Pi_t \\ &= r^d \left[V_t - \frac{\partial V}{\partial S} S \right] \end{aligned} \tag{2.10}$$

leading to the familiar Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r^d S \frac{\partial V}{\partial S} - r^d V_t = 0. \tag{2.11}$$

It is standard, and important, at this point to note that the real world growth rate μ does not appear in the Black–Scholes equation in any form.

2.2.3 The Black–Scholes Model With Convenience Yield

When the commodity is assumed to have a non-zero convenience yield, it is assumed that the value of the tradeable asset is the spot rate multiplied by the returns on that commodity, i.e. $S_t B_t^f$, where

$$dB_t^f = r^f B_t^f dt \tag{2.12}$$

takes into account the convenience yield r^f. By analogy with FX, B_t^f can be thought of as a foreign commodity bond, though it more correctly refers to a continuously rolled over long spot position in a particular commodity. Let us call B_t^f foreign commodity bond for brevity.

As in Section 2.2.2 we suppose that the price of a contingent claim $V(S_t, t)$ is known, which derives its value from the performance of a spot price S_t.

Using V_t to denote the price of the contingent claim at time t, we still have

$$dV_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \frac{\partial V}{\partial S} dS_t \tag{2.13}$$

but the construction of the delta-hedged portfolio is somewhat different. We cannot buy and sell units of the spot commodity without taking on exposure to the convenience yield – so the construction of the hedged portfolio Π_t is obtained by going long one unit of the contingent claim (with value V_t) and short Δ_t units of the underlying foreign commodity bond

$$\Pi_t = V_t - \Delta_t S_t B_t^f. \tag{2.14}$$

The question is what value of Δ_t makes Π_t riskless. We have

$$\begin{aligned}
d\Pi_t &= dV_t - \Delta_t d(S_t B_t^f) \\
&= dV_t - \Delta_t B_t^f dS_t - \Delta_t S_t dB_t^f \\
&= dV_t - \Delta_t B_t^f dS_t - \Delta_t S_t r^f B_t^f dt \\
&= dV_t - \Delta_t B_t^f \left[(r^d - r^f) S_t dt + \sigma S_t dW_t\right] - \Delta_t S_t r^f B_t^f dt \\
&= dV_t - \Delta_t B_t^f \left[r^d S_t dt + \sigma S_t dW_t\right] \\
&= \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + \frac{\partial V}{\partial S} dS_t - \Delta_t B_t^f \left[r^d S_t dt + \sigma S_t dW_t\right] \\
&= \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + \frac{\partial V}{\partial S} dS_t - \Delta_t B_t^f r^d S_t dt - \Delta_t B_t^f \sigma S_t dW_t \\
&= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta_t B_t^f r^d S_t\right] dt + \frac{\partial V}{\partial S} dS_t - \Delta_t B_t^f \sigma S_t dW_t \\
&= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta_t B_t^f r^d S_t\right] dt + \frac{\partial V}{\partial S} \left[(r^d - r^f) S_t dt + \sigma S_t dW_t\right] \\
&\quad - \Delta_t B_t^f \sigma S_t dW_t \\
&= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta_t B_t^f r^d S_t + \frac{\partial V}{\partial S}(r^d - r^f) S_t\right] dt \\
&\quad + \left[\frac{\partial V}{\partial S} - \Delta_t B_t^f\right] \sigma S_t dW_t.
\end{aligned} \tag{2.15}$$

To cancel the dW_t term, we require that Δ_t must satisfy $\Delta_t B_t^f = \partial V/\partial S$, i.e.

$$\Delta_t = \frac{1}{B_t^f}\frac{\partial V}{\partial S}. \tag{2.16}$$

Substituting (2.16) into (2.15) we obtain

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r^d S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}(r^d - r^f)S_t\right] dt. \tag{2.17}$$

Once again, as in Section 2.2.2, we appeal to domestic risk-neutrality and put $d\Pi_t = r^d \Pi_t dt$, where in the context with non-zero convenience yields, using (2.14) and the analysis above, it gives

$$\Pi_t = V_t - \frac{\partial V}{\partial S} S_t. \tag{2.18}$$

We therefore have

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r^d S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}(r^d - r^f)S_t\right] dt = r^d \left[V_t - \frac{\partial V}{\partial S} S_t\right] dt$$

which reduces to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r^d - r^f)S\frac{\partial V}{\partial S} - r^d V = 0. \tag{2.19}$$

We see the convenience yield r^f appear in the convection term (the term containing a multiple of $\frac{\partial V}{\partial S}$) but not in the forcing term (the term containing a multiple of V), and the absence of any μ term.

2.2.4 The Black-76 Model

The standard Black–Scholes analysis presumes that we can price an option that depends on a continuously traded spot rate S_t, as presented above. If, however, we model the T-forward contract $F_{t,T}$ or the T-futures contract $f_{t,T}$ (for fixed T) rather than the spot process S_t then we obtain something different.

2.2.4.1 The Black-76 Model with Respect to Futures

In Black (1976), one assumes that the T-futures price $f_{t,T}$ of a tradeable commodity follows a *driftless* lognormal process

$$df_{t,T} = \sigma f_{t,T} dW_t^d \tag{2.20}$$

where the process should be understood to be driftless with respect to the domestic risk-neutral measure $\mathbf{P}^d$.

For simplicity, let T be fixed *a priori*, we can then use the shorthand $f_t \equiv f_{t,T}$. We then let $V_t = V(f_t, t)$ denote the value of the contingent claim at time t, conditional on the T-futures price being f_t at that time. Applying Itô, we have

$$dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial f}df_t + \frac{1}{2}\frac{\partial^2 V}{\partial f^2}df_t^2. \tag{2.21}$$

However we know from (2.20) that $df_t^2 = \sigma^2 f_t^2 dt$ and consequently at time t, we have

$$dV_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 V}{\partial f^2}\right] dt + \frac{\partial V}{\partial f}df_t. \tag{2.22}$$

In the same manner as Section 2.2.2 we now remove the stochastic term by constructing a portfolio Π_t which is long one unit of the contingent claim (with value V_t) and short $\partial V/\partial f$ units of the underlying T-futures contract $f_t \equiv f_{t,T}$

$$\Pi_t = V_t - \frac{\partial V}{\partial f}u_t, \tag{2.23}$$

where we suppose that such a T-futures contract is entered into at initial cost u_0, and tracks the profit/loss of f_t according to $du_t = df_t + r^d u_t dt$ – i.e. the gain on the futures position plus any interest earned or paid on the margin account.[3] While u_0 will generally be zero at initiation, an observation which can be used to explain the absence of the convection term in (2.29) compared with (2.11), this need not be assumed.

The portfolio Π_t in (2.23) then has the SDE

$$\begin{aligned} d\Pi_t &= dV_t - \frac{\partial V}{\partial f}du_t \\ &= dV_t - \frac{\partial V}{\partial f}[df_t + r^d u_t dt]. \end{aligned} \tag{2.24}$$

From the above and (2.22), we have

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 V}{\partial f^2} - r^d u_t \frac{\partial V}{\partial f}\right] dt. \tag{2.25}$$

[3] While there will be a certain amount of cash in the margin account at all times, we can consider the interest on the gains or losses on the futures position on an incremental basis.

As the growth of Π_t is riskless, we can appeal to risk-neutrality and equate the expected growth of Π_t to that of the domestic risk-free bond B_t^d which is described by

$$dB_t^d = r^d B_t^d dt. \tag{2.26}$$

We therefore put

$$d\Pi_t = r^d \Pi_t dt \tag{2.27}$$

where r^d denotes the domestic interest rate. Equating terms in (2.25) and (2.27), and then using (2.23) we have

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 V}{\partial f^2} - r^d u_t \frac{\partial V}{\partial f} &= r^d \Pi_t \\ &= r^d \left[V_t - \frac{\partial V}{\partial f} u_t \right]. \end{aligned} \tag{2.28}$$

We arrive at the Black-76 equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 V}{\partial f^2} - r^d V = 0. \tag{2.29}$$

2.2.4.2 The Black-76 Model with Respect to Forwards

The analysis above assumes that futures contracts earn interest on the margin, which we know not to be the case for forwards. The analysis of that case is subtly different, and worth going through. We assume that the forwards are driftless under the domestic risk-neutral measure,[4] i.e.

$$dF_{t,T} = \sigma F_{t,T} dW_t^d \tag{2.30}$$

and let $V_t = V(F_t, t)$ denote the value of the contingent claim at time t. By use of Itô we obtain

$$dV_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} \right] dt + \frac{\partial V}{\partial F} dF_t. \tag{2.31}$$

Suppose that a T-forward contract is entered into at initial cost v_0, and tracks the profit/loss of $F_{t,T}$ according to $v_t = e^{-r^d(T-t)} F_{t,T}$ with no

[4] Strictly, we should use the T-forward measure, but unless interest rates are assumed to be stochastic, the domestic risk-neutral measure is equivalent.

interim cashflows – this is why the discounting is required, to obtain the present value at t. Note that, using the shorthand $F_t \equiv F_{t,T}$, we have

$$\begin{aligned} dv_t &= d[e^{-r^d(T-t)}F_t] \\ &= r^d e^{-r^d(T-t)}F_t dt + e^{-r^d(T-t)}dF_t. \end{aligned}$$

We then construct a portfolio $\Pi_t = V_t - \Delta_t v_t$ and attempt to remove the random component by a suitable choice of Δ_t. We can write

$$\begin{aligned} d\Pi_t &= dV_t - \Delta_t dv_t \\ &= dV_t - \Delta_t[r^d e^{-r^d(T-t)}F_t dt + e^{-r^d(T-t)}dF_t] \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - \Delta_t r^d e^{-r^d(T-t)}F_t\right] dt + \left(\frac{\partial V}{\partial F} - \Delta_t e^{-r^d(T-t)}\right) dF_t. \end{aligned}$$

We consequently set $\Delta_t = e^{r^d(T-t)}\frac{\partial V}{\partial F}$ to remove the random term, obtaining

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r^d \frac{\partial V}{\partial F}F_t\right] dt.$$

Being riskless, this can be equated to $r^d \Pi_t dt$, and we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r^d \frac{\partial V}{\partial F}F_t = r^d \Pi_t \tag{2.32}$$

which can be reduced to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r^d \frac{\partial V}{\partial F}F_t = r^d[V_t - \Delta_t v_t].$$

Now, $\Delta_t v_t = e^{r^d(T-t)}\frac{\partial V}{\partial F} \cdot e^{-r^d(T-t)}F_t = \frac{\partial V}{\partial F}F_t$, so we can simplify the above to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r^d \frac{\partial V}{\partial F}F_t = r^d V_t - r^d \frac{\partial V}{\partial F}F_t.$$

The result follows immediately

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r^d V = 0. \tag{2.33}$$

2.2.4.3 Terminal Conditions and Present Value

Black–Scholes type PDEs, such as (2.11), (2.19), (2.29) or (2.33), are able to describe the value of a derivative contract, either with or without path dependency. If path-independent, the value of the contract depends solely on the value at expiry T of the tradeable process S_T, f_{T,T^*} or F_{T,T^*} corresponding to spot, T^*-futures or T^*-forwards respectively.

Considering the case of derivatives on spot for now, at expiry T, i.e. $V_T = V_T(S_T)$, we have the terminal condition

$$V(S_T, T) = V_T(S_T).$$

Given a solution (analytical or numerical) of the PDE, the value V_0 of the derivative contract today (the "present value", or PV) can be directly read off from the $t = 0$ time slice

$$V_0 = V(S_0, 0),$$

and similarly $V_0 = V(f_{0,T^*}, 0)$ and $V_0 = V(F_{0,T^*}, 0)$ for options on futures and forwards respectively.

2.2.4.4 Feynman–Kac and Risk-Neutral Expectation

From Section 2.2, we have a backward parabolic partial differential equation, such as (2.11), (2.19), (2.29) or (2.33), which the price V of a derivative security must obey.

We make the observation that the drift μ of the underlying process for the tradeable asset does not enter into the partial differential equation.

The Feynman–Kac formula makes the connection between the solution of such a partial differential equation, and the expectation of the terminal value of the derivative under an artificial measure – i.e. *not* the real world measure. If we have a backward Kolmogorov equation of the form

$$\frac{\partial g}{\partial t} + a(x,t)\frac{\partial g}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 g}{\partial x^2} = 0 \tag{2.34}$$

with terminal condition $g(x, T) = h(x)$, then the solution of (2.34) can be expressed as an expectation[5]

$$g(x, 0) = \mathbf{E}^d\left[h(X_T)|X_0 = x\right] \tag{2.35}$$

where X_t is described by the diffusion

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t^d. \tag{2.36}$$

[5] We use the notation $\mathbf{E}^d[\cdot]$ as we shall identify this as expectation with respect to the domestic risk-neutral measure in Section 2.2.5.1.

The Feynman–Kac result can be verified by considering the process $g_t = g(X_t, t)$ and constructing the stochastic differential for dg_t

$$dg_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}dX_t^2. \tag{2.37}$$

Substituting (2.36) into (2.37), we have

$$dg_t = \left[a(X_t, t)\frac{\partial g}{\partial x} + \frac{1}{2}b^2(X_t, t)\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t}\right] dt + b(X_t, t)\frac{\partial g}{\partial x}dW_t^d. \tag{2.38}$$

The term in square brackets above vanishes due to (2.34), and so

$$dg_t = b(X_t, t)\frac{\partial g}{\partial x}dW_t^d. \tag{2.39}$$

Integrating from $t = 0$ to T, we have

$$g_T = g_0 + \int_0^T b(X_t, t)\frac{\partial g}{\partial x}dW_t^d.$$

Taking expectations, and recognising that the expectation of the Itô integral above is zero, we have

$$g_0 = \mathbf{E}^d[g_T].$$

We therefore have, as in (2.35), $g(X_0, 0) = \mathbf{E}^d\left[h(X_T)\right]$.

In the presence of a forcing term f, i.e.

$$\frac{\partial V}{\partial t} + a(x, t)\frac{\partial V}{\partial x} + \frac{1}{2}b^2(x, t)\frac{\partial^2 V}{\partial x^2} + f(x, t)V = 0, \tag{2.40}$$

the analysis above gives

$$dV_t + f(X_t, t)Vdt = b(X, t, t)\frac{\partial V}{\partial x}dW_t^d. \tag{2.41}$$

This is amenable to the stochastic integrating factor technique. Multiply the LHS and RHS of (2.41) by $\exp(\int_0^t f(X_s, s)ds)$, and put $\hat{V}_t = \exp(\int_0^t f(X_s, s)ds)V_t$. By the chain rule, we have

$$d\hat{V}_t = \exp\left(\int_0^t f(X_s, s)ds\right) dV_t + \exp\left(\int_0^t f(X_s, s)ds\right) f(X_t, t)Vdt.$$

We therefore have, from (2.41),

$$d\hat{V}_t = \exp\left(\int_0^t f(X_s, s)ds\right) b(X, t, t)\frac{\partial V}{\partial x}dW_t^d.$$

Taking expectations as before and recognising that the expectation of the Itô integral above vanishes, we have $\hat{V}_0 = \mathbf{E}^d[V_T]$, i.e.

$$V_0 = \mathbf{E}^d \left[\exp\left(\int_0^t f(X_s, s) ds \right) V_T \right]. \tag{2.42}$$

In the case where $f(X_s, s)$ is nonstochastic, though potentially a deterministic function of time t, we can take it outside the expectation, obtaining the result $V_0 = \exp(\int_0^t f(X_s, s)ds)\mathbf{E}^d[\hat{V}_T]$.

For Black–Scholes, comparing (2.40) with (2.19), we put $a(S, t) = (r^d - r^f)S$, $b(S, t) = \sigma S$ and $f(S, t) = -r^d$, where we use S in place of x in (2.40). The process (2.36) under which we take the expectation is therefore

$$dS_t = (r^d - r^f)S_t dt + \sigma S_t dW_t^d \tag{2.43}$$

and the result for the present value today of a derivative on a spot contract is the discounted risk-neutral expectation under the domestic risk-neutral measure

$$V_0 = e^{-r^d T}\mathbf{E}^d \left[V_T \right]. \tag{2.44}$$

The Black-76 case is even simpler. Compare (2.40) with (2.29) or (2.33), and put $a(x, t) = 0$, $b(x, t) = \sigma x$ and $f(S, t) = 0$, where x denotes f or F as appropriate (consider f for now). The process (2.36) under which we take the expectation is therefore

$$df_t = \sigma f_t dW_t^d \tag{2.45}$$

(and similarly for F_t) so the present value today of a derivative on either a futures or a forward contract is the risk-neutral expectation under the domestic risk-neutral measure, either with or without discounting:

$$V_0 = e^{-r^d T}\mathbf{E}^d \left[V_T \right], \quad \text{or} \tag{2.46a}$$

$$V_0 = \mathbf{E}^d \left[V_T \right]. \tag{2.46b}$$

Whether to use (2.46a) or (2.46b) depends on whether the derivative instrument pays/receives interest rate linked cashflows between today and maturity T, or whether the derivative *only* pays a cashflow at maturity T.

For a forward contract which only pays out at time T with no other interim cash flows, we use (2.46a) as discounting must be included between the time T cashflow and today. For futures contracts which incorporate margining, since the margin earns interest and incorporates interim cashflows as necessary to fund the position, (2.46b) should be used.

Examples are given below in Sections 2.3.1 and 2.3.2. Note that one should properly use the T-forward measure for forward contracts, but if interest rates are presumed deterministic then the domestic risk-neutral measure is equivalent.

What have we obtained from this exercise? We see that the value today of a derivative written on a spot, forward or futures contract can be expressed in terms of the domestic risk-neutral expectation of the value at expiry.

Fuller discussions of the Feynman–Kac approach can be found in Section 6.2.2.1 of Grigoriu (2002), Chapter VIII of Øksendal (2010), Section 5.8 of Bingham and Kiesel (1998) and Section 4.7 of Lipton (2001).

2.2.5 Risk-Neutral Valuation

Obtaining the Black–Scholes and Black-76 equations for derivatives written on spot, forward and futures contracts, we noticed that the real-world drift term μ does not appear, indicating that all rational market participants can be assumed to price derivatives identically no matter what value of μ is assumed for the expected drift. We saw in Section 2.2.4 above that the present value can be identified as the expectation under a particular choice of measure, which we refer to as the domestic risk-neutral measure.

2.2.5.1 Domestic Risk-Neutral Measure

The domestic investor sees the foreign commodity bond B_t^f as the risky asset, which denominated in domestic currency is valued at $B_t^f S_t$. Construct the ratio of this against the domestic bond

$$\begin{aligned} Z_t &= S_t B_t^f / B_t^d \\ &= S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right) \exp\left((r^f - r^d)t\right) \\ &= S_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) \exp\left((\mu + r^f - r^d)t\right). \end{aligned}$$

To attain the martingale property we require that

$$\mu = \mu^d \equiv r^d - r^f \tag{2.47}$$

so under the domestic risk-neutral measure $\mathbf{P}^d$ we can write

$$S_t = S_0 \exp\left(\sigma W_t^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right) t\right), \tag{2.48}$$
$$dS_t = (r^d - r^f) S_t dt + \sigma S_t dW_t^d.$$

The drift change required is

$$W_t^d = W_t + \frac{\mu - \mu^d}{\sigma} t \tag{2.49}$$

which gives a Radon–Nikodym derivative at time T of

$$\frac{d\mathbf{P}^d}{d\mathbf{P}} = \exp\left(-\gamma^d W_T - \frac{1}{2}[\gamma^d]^2 T\right), \quad \text{where } \gamma^d = \frac{\mu - \mu^d}{\sigma}. \tag{2.50}$$

2.2.5.2 *Foreign Risk-Neutral Measure*

As discussed in Clark (2011), we can also see the domestic risk-free bond as a risky investment from the viewpoint of a foreign denominated investor. Under the foreign risk-neutral measure $\mathbf{P}^f$ we have

$$S_t = S_0 \exp\left(\sigma W_t^f + \left(r^d - r^f + \frac{1}{2}\sigma^2\right) t\right), \tag{2.51}$$
$$dS_t = (r^d - r^f + \sigma^2) S_t dt + \sigma S_t dW_t^f.$$

The applicable drift change required is

$$W_t^f = W_t + \frac{\mu - \mu^f}{\sigma} t, \tag{2.52}$$

which gives a Radon–Nikodym derivative at time T of

$$\frac{d\mathbf{P}^f}{d\mathbf{P}} = \exp\left(-\gamma^f W_T - \frac{1}{2}[\gamma^f]^2 T\right), \quad \text{where } \gamma^f = \frac{\mu - \mu^f}{\sigma} \tag{2.53}$$

and

$$\mu^f \equiv r^d - r^f + \sigma^2. \tag{2.54}$$

2.2.6 Forwards

Forwards, with payout function $V_T = S_T - K$ at time T as illustrated in Figure 2.2, can have a negative value as well as a positive value,

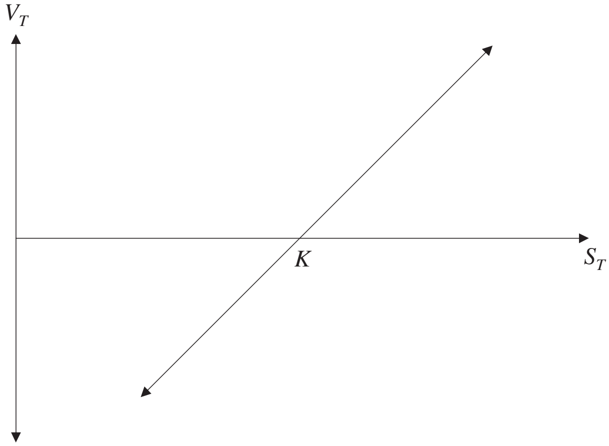

Figure 2.2 Payout function V_T for forward.

depending on the strike K. It is therefore customary to solve for the value of K that makes the forward costless to enter into. This choice of strike K for which $V_0 = e^{-r^d T}\mathbf{E}^d\left[S_T - K\right] = 0$ is denoted $F_{0,T}$, i.e. $F_{0,T} = \mathbf{E}^d\left[S_T\right]$.

By (2.48) we easily obtain

$$\begin{aligned} F_{0,T} &= \mathbf{E}^d\left[S_T\right] \\ &= S_0\mathbf{E}^d\left[\exp\left(\sigma W_T^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)T\right)\right] \\ &= S_0 e^{(r^d - r^f)T}\mathbf{E}^d\left[\exp\left(\sigma W_T^d - \frac{1}{2}\sigma^2 T\right)\right] \\ &= S_0 e^{(r^d - r^f)T} \end{aligned} \tag{2.55}$$

and more generally, taking the conditional expectation at time t we obtain

$$F_{t,T} = S_t e^{(r^d - r^f)(T-t)}. \tag{2.56}$$

From this, a simple application of Itô's Lemma demonstrates that forwards are driftless under the risk-neutral measure. Write (2.56) as $F_{t,T} = h(S_t, t)$ with $h(s,t) = s\exp((r^d - r^f)(T-t)) = s\exp((r^d - r^f)T)\exp((r^f - r^d)t)$. Clearly $h_s = \exp((r^d - r^f)(T-t))$, $h_{ss} = 0$ and

$h_t = (r^f - r^d)s\exp((r^d - r^f)T)$. We then have

$$\begin{aligned}
dF_{t,T} &= d[h(S_t, t)] \\
&= h_s dS_t + h_t dt \\
&= \exp((r^d - r^f)(T-t))dS_t + (r^f - r^d)S_t \exp((r^d - r^f)T)dt \\
&= \exp((r^d - r^f)(T-t))[(r^d - r^f)S_t dt + \sigma S_t dW_t^d] + (r^f - r^d)F_{t,T}dt \\
&= (r^d - r^f)F_{t,T}dt + \sigma F_{t,T}dW_t^d + (r^f - r^d)F_{t,T}dt \\
&= \sigma F_{t,T}dW_t^d \qquad (2.57)
\end{aligned}$$

which verifies the earlier assumed form for the GBM given earlier in (2.20) and (2.30).

2.2.7 The Black–Scholes Term Structure Model

The Black–Scholes model of Section 2.2.1 does not allow any term structure of interest rates or volatility.

We can easily extend the model in (2.2) to

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \qquad (2.58)$$

where both μ_t and σ_t are deterministic processes. By risk-neutrality, we obtain

$$\mu_t = r_t^d - r_t^f. \qquad (2.59)$$

It follows (see Clark, 2011) that the risk-neutral valuation methods of this chapter can be directly applied to valuation of commodity derivatives with expiry T, with r^d, r^f and σ replaced by effective counterparts $\bar{r}_T^d$, $\bar{r}_T^f$ and $\bar{\sigma}_T$, where

$$\bar{r}_T^{d;f} = \frac{1}{T}\int_0^T r_s^{d;f} ds$$

are the effective (continuously compounded) domestic rate and convenience yield respectively over the time interval $[0,T]$, and

$$\bar{\sigma}_T = \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}$$

defines the effective volatility $\bar{\sigma}_T$ applicable over the time interval $[0,T]$.

2.3 FORWARD AND FUTURES CONTRACTS

We are now in position to start introducing actual financial derivative contracts, including the pricing mechanics in the Black case. The simplest are the forward and futures contracts, which we already encountered in Section 2.1. Note, however, that what we introduced there were the forward and futures prices, which are not the same thing as the PVs of a forward or a futures contract. Some explanation is definitely in order.

2.3.1 Forwards

Consider a forward contract, with payout function $V_T = F_{T,T} - K = S_T - K$ at time T shown in Figure 2.2. As introduced in Section 2.2.6, the choice of strike K for which the forward is costless to enter into is denoted $F_{0,T}$, i.e. $F_{0,T} = \mathbf{E}^d\left[S_T\right] = S_t e^{(r^d - r^f)(T-t)}$.

If the forward was not entered into on a costless basis, or if time has passed since the original strike was set and the commodity price has moved, then the strike price K today will not be equal to $F_{0,T}$ and we need to PV it (i.e. to obtain the present value). Quite trivially we obtain

$$V_0 = e^{-r^d T}\mathbf{E}^d\left[S_T - K\right] = e^{-r^d T}[\mathbf{E}^d[S_T] - K] = e^{-r^d T}[F_{0,T} - K]. \quad (2.60)$$

In the event where the final settlement of the forward is at a later time T_{stl} than the fixing T, with $T < T_{\text{stl}}$, we have

$$V_0 = e^{-r^d T_{\text{stl}}}[F_{0,T} - K]. \quad (2.61)$$

2.3.2 Futures

Consider now the T-futures contract, with strike price $K = f_{0,T}$. This is valued similarly to the forward contract treated in Section 2.3.1, but note that the payout function for the futures contract $V_T = f_{T,T} - K = S_T - K$ at time T neglects the interest accrued on the margin account over the lifetime of the futures contract. Consequently we need to adjust for this in obtaining the value of a futures contract today. With $f_{t,T}$ denoting the price of the T-futures contract today, we let $\hat{f}^{(K)}_{t,T}$ denote the value of a portfolio consisting of the T-futures contract with strike K together with any accrued or payable interest on the margin account (we can rebase the maintenance levels without loss of generality). Since this portfolio earns (or is liable for) interest at the instantaneous risk free rate r^d on positive or negative values of $f_{t,T} - K$ respectively, we can write

$$d\hat{f}_{t,T}^{(K)} = df_{t,T} + r^d(f_{t,T} - K)dt. \tag{2.62}$$

Because $f_{t,T}$ is a martingale under the domestic risk-neutral measure, and because r^d only applies a drift correction, we clearly have

$$\mathbf{E}^d\left[\hat{f}_{T,T}^{(K)}\right] = f_{0,T} - K + (f_{0,T} - K)(e^{r^dT} - 1) \tag{2.63a}$$

$$= (f_{0,T} - K)e^{r^dT} \tag{2.63b}$$

where the first two terms on the right hand side of (2.63a) correspond to the expected value of the futures contract at expiry, and the third term corresponds to the expected gain (or loss) due to interest accrued (or paid) on the margin account over $[0, T]$.

Discounting (2.63) appropriately, we obtain

$$V_0 = e^{-r^dT}\mathbf{E}^d\left[\hat{f}_{T,T}^{(K)}\right] = f_{0,T} - K. \tag{2.64}$$

Comparing (2.61) with (2.64), we see that the prices agree up to a discount factor – i.e. PVs for forward contracts are discounted, whereas PVs for equivalent futures contracts are not.

2.3.3 Case Study

We conclude this section with an example. Consider a WTI contract on NYMEX as the example in question. Suppose short positions in two contracts are entered into on 21 October 2010, a forward contract and a DEC10 futures contract, each to be closed out on 19 November 2010 when the futures contract matures.[6] This may seem unusual but it is standard for NYMEX futures, see Chapter 5. Suppose that the US interest rate is 0.25% and that strike $K = 85$.

With initial and variation margins of \$9,788 and \$7,250 respectively and a contract size of 1,000 barrels, i.e. \$9.788 and \$7.25 respectively per barrel, we can tabulate the cashflows on a per-barrel basis for each of these two contracts in Table 2.2, given NYMEX oil prices over that period of interest.

[6] Note that actual final cash settlement is five NYMEX business days later; for this example we presume that positions are closed out on the maturity date.

Table 2.2 Case Study: forward vs futures contract.

t	$f_{t,T}$	P/L	Margin a/c balance c/o	+P/L	+P/L + Margin	Cashflow on Forward	Futures
21-Oct-10	80.36	+4.64	9.788	14.428	14.428		−9.788
22-Oct-10	81.63	−1.27	14.428	13.158	13.158		
25-Oct-10	82.52	−0.89	13.158	12.268	12.268		
26-Oct-10	82.57	−0.05	12.268	12.218	12.218		
27-Oct-10	81.91	+0.66	12.219	12.879	12.879		
28-Oct-10	82.18	−0.27	12.879	12.614	12.614		
29-Oct-10	81.45	+0.72	12.614	13.339	13.339		
1-Nov-10	82.98	−1.53	13.339	11.809	11.809		
2-Nov-10	83.91	−0.93	11.809	10.879	10.879		
3-Nov-10	84.67	−0.76	10.879	10.119	10.119		
4-Nov-10	86.47	−1.80	10.119	8.319	8.319		
5-Nov-10	86.85	−0.38	8.319	7.944	7.944		
8-Nov-10	87.02	−0.17	7.944	7.769	7.769		
9-Nov-10	86.73	+0.29	7.769	8.059	8.059		
10-Nov-10	87.80	−1.07	8.060	6.990	7.250		−0.260
11-Nov-10	87.81	−0.01	7.250	7.240	7.250		−0.010
12-Nov-10	84.91	+2.91	7.250	10.155	10.155		
15-Nov-10	84.84	+0.06	10.155	10.220	10.220		
16-Nov-10	82.34	+2.50	10.220	12.720	12.720		
17-Nov-10	80.46	+1.88	12.720	14.600	14.600		
18-Nov-10	81.87	−1.41	14.601	13.191	13.191		
19-Nov-10	81.51	+0.36	13.191	13.551	13.551	+3.490	+13.551

PV-ing these cashflows and discounting back to 21 October 2010, we obtain prices for the forward and futures contract of \$3.48931 and \$3.48954 per barrel respectively. Note that the futures price is slightly above the forward price.

The forward contract only has a cashflow on 19 November 2010, as one would expect. However the futures contract has the cashflow corresponding to depositing the initial margin on 21 October 2010 and two further injections of cash on 10 and 11 October 2010 into the margin account to bring the balance up to the variation margin, in addition to the final cashflow on 19 November 2010 where the balance (including profits or losses) is retrieved in full.

Given the final oil price of \$81.51/bbl and a strike of 85, we would expect a short futures contract to be valued at \$3.49/bbl and a short forward contract to be valued at \$3.48931/bbl (including discounting at 2.5%). The PV of the forward contract is exactly in line with this, the futures contract is not so because the \$3.49 estimate is purely an *expectation*. The PV for the futures contract of \$3.48954/bbl in this case was obtained from a *particular* asset price trajectory ending at \$81.51/bbl, other trajectories similarly ending at \$81.51/bbl exist which will give PVs for the futures contract higher than \$3.49/bbl and we expect the average to be \$3.49/bbl.

It is not hard to see that the actual value of a futures contract must depend on the trajectory of the asset price process – for example, if the oil price shoots to \$81.51/bbl immediately, that will give a different P/L for the margin account than if the oil price only goes up to \$81.51/bbl in late October.

2.4 COMMODITY SWAPS

A futures contract is exposed to only one price fixing, on the expiry date of the futures itself. This may be too much fixing risk for a market participant, given the volatilities in commodity markets. For this reason it is quite common for products to be fixed against the average of a futures contract, and since a particular contract needs to be chosen, the one usually referenced is the prompt futures contract.

As discussed in Das (2005), Section 8.3.1 of Flavell (2009) and Section 2.1.3 of Burger *et al.* (2007), a commodity swap involves exchanging floating commodity prices against a fixed known commodity price K, with cash settlement either at the end of the swap or on a monthly basis. A long position in a commodity swap involves collecting the cash amount equivalent to the arithmetic difference between the (variable) fixing and the (fixed) strike K at each of the fixing dates t_i, i.e.

$$V_i = f_{t_i,\vec{T}(t_i)} - K$$

where t_i refers to the time of the i-th fixing and $\vec{T}(t_i)$ denotes[7] the expiry of the prompt future at time t_i.

Note that these values V_i are computed at each of the times t_i and the value today of the swap is equivalent to

$$V_0 = e^{-r^d T_{\mathrm{stl}}} \mathbf{E}^d \left[\frac{1}{n} \sum_{i=1}^{n} V_i \right].$$

[7] As introduced in Section 2.1.1, the arrow is meant to denote moving along the time axis, in the direction of increasing time, from time t_i to the first futures expiry $\vec{T}(t_i)$ after t_i.

It is customary to solve for the strike that makes the swap costless to enter into, i.e. solving for $V_0 = 0$, thereby obtaining

$$\begin{aligned} K_{\text{Swap}} &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} f_{0,\vec{T}(t_i)} \end{aligned} \tag{2.65}$$

since futures prices are martingales under the domestic risk-neutral measure. We sometimes call this the "crossing" level.

Note that this "swap price" is not the same as the PV of a swap, in the same way as the forward/futures prices are not the PVs of the equivalent forward or futures contract. In fact it is nothing more than the arithmetic average of the futures contracts computed using today's futures prices

$$A^{A;f}_{0;\{t_1,t_n\}} = \frac{1}{n}\sum_{i=1}^{n} f_{0,\vec{T}(t_i)}. \tag{2.66}$$

Note that $A^{A;f}_{0;\{t_1,t_n\}}$ is a deterministic quantity that can be computed today, being a linear combination of today's futures prices. It should not be confused with the arithmetic averages of either spot or futures prices that we can construct as stochastic processes, i.e.

$$A^{A,s}_{t_1,t_n} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{2.67a}$$

$$A^{A,f}_{t_1,t_n} = \frac{1}{n}\sum_{i=1}^{n} f_{t_i,\vec{T}(t_i)} \tag{2.67b}$$

which can only be known deterministically once all n observations have taken place. We will see more of these in Section 2.7 on Asian options.

In the event of a swap longer dated than a calendar month, with settlement each month, we can use $T_{\text{stl};i}$ to denote the settlement dates and then have

$$V_0 = \frac{1}{n}\mathbf{E}^d\left[\sum_{i=1}^{n} e^{-r^d T_{\text{stl};i}} V_i\right]. \tag{2.68}$$

Swaps for durations in excess of one calendar month can easily be decomposed into the various constituent months.

Now, since the prompt futures contract generally rolls during the course of a calendar month rather than neatly at the end, this means that a commodity swap is effectively a weighted average of the two futures contracts which comprise the prompt future over that particular month. One sometimes sees the terminology "calendar month average" (CMA) used for this.

The only troublesome aspect of pricing commodity swaps is working out the specifics of handling the roll date. It is commonplace to use Swap(1,0) and Swap(1,1) to denote two different variants, the first being the case where the prompt future rolls at the end of the expiry date, the second being where the prompt future rolls at the beginning of the expiry date.

Cash settlement for NYMEX swaps (and most other energy swaps) is usually on the 5th business day of the subsequent month, i.e. five business days after the final fixing.

A case study of commodity swap pricing with respect to oil can be found in Section 5.1.2.

2.5 EUROPEAN OPTIONS

Forward, futures and swap contracts are all directional trades, being the obligation to buy (or sell) a particular commodity at a particular time in the future, where the price is either fixed at the maturity date or by averaging the price over a number of fixing dates. As such, any of these trades can have positive or negative value to the buyer (or indeed the seller).

Potentially, however, a market participant may be happy to accept a higher price for an asset he or she is selling, but wants to avoid the possibility of lower prices. Alternatively, a purchaser of oil may be quite happy with lower prices, but wants to avoid higher prices. Both of these possibilities can be catered for with option contracts, which typically convey the right but not the obligation to buy (or sell) an asset at a prespecified strike price K at some known time in the future. Note that the option can *only* be exercised at this specific time, i.e. on the exercise date.

In spite of the fact that most commodity options are options on futures, we can equivalently consider the case of options on the forward $F_{t,T}$ given the assumption for all the models considered in this work: that futures and forwards prices coincide, or options on the spot process S_t. In order to be able to describe all of these, we shall discuss options

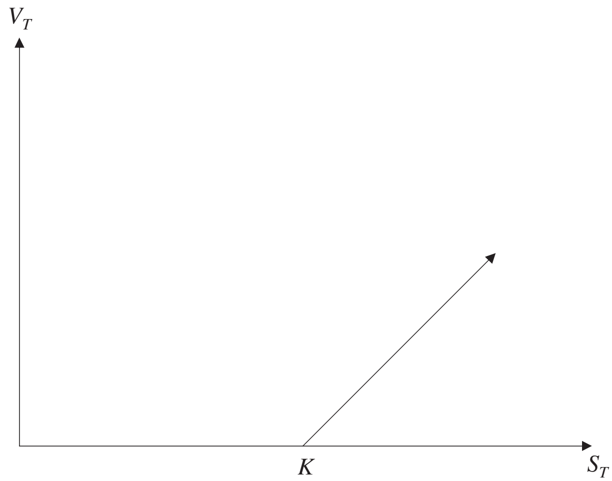

Figure 2.3 Payout function V_T for European call.

on the spot process S_t, since driftless processes for $F_{t,T}$ and $f_{t,T}$ can be handled as special cases of S_t with the convenience yield r^f set equal to the interest rate r^d, to kill the drift.

2.5.1 European Options on Spot

Consider a European call option, with payout function $V_T = \max(S_T - K, 0) = (S_T - K)^+$ at time T shown in Figure 2.3.

One can compute the price of such an option today by constructing the following risk-neutral expectation in the domestic risk-neutral measure:

$$
\begin{aligned}
V_0 &= e^{-r^d T}\mathbf{E}^d\left[(S_T - K)^+\right] \\
&= e^{-r^d T}\mathbf{E}^d\left[(S_T - K)\mathbf{1}_{\{S_T \geq K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d\left[S_T\mathbf{1}_{\{S_T \geq K\}} - K\mathbf{1}_{\{S_T \geq K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d\left[S_T\mathbf{1}_{\{S_T \geq K\}}\right] - Ke^{-r^d T}\mathbf{E}^d\left[\mathbf{1}_{\{S_T \geq K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d\left[S_T\mathbf{1}_{\{S_T \geq K\}}\right] - Ke^{-r^d T}\mathbf{P}^d\left[S_T \geq K\right]. \qquad (2.69)
\end{aligned}
$$

Evaluating $\mathbf{P}^d\left[S_T \geq K\right]$, which is nothing more than the domestic risk-neutral probability that $S_T \geq K$, is straightforward as we know the distribution of S_T. The second term requires a change of measure. In Section 2.2.5 we had expressions for the Radon–Nikodym derivatives relating the domestic and foreign risk-neutral measures to the real-world measure, so the Radon–Nikodym derivative relating one risk-neutral measure to the other needs to be obtained. From (2.50) and (2.53) we have

$$\frac{d\mathbf{P}^f}{d\mathbf{P}^d} = \exp\left(-\gamma^{f,d} W_T^d - \frac{1}{2}[\gamma^{f,d}]^2 T\right), \quad \text{where } \gamma^{f,d} = \frac{\mu^d - \mu^f}{\sigma}. \quad (2.70)$$

In short,

$$W_t^f = W_t^d + \frac{\mu^d - \mu^f}{\sigma} t = W_t^d - \sigma t. \quad (2.71)$$

From (2.47) and (2.54) we have $\mu^f - \mu^d = \sigma^2$ and consequently $\gamma^{f,d} = (\mu^d - \mu^f)/\sigma = -\sigma$, which gives

$$\frac{d\mathbf{P}^f}{d\mathbf{P}^d} = \exp\left(\sigma W_T^d - \frac{1}{2}\sigma^2 T\right). \quad (2.72)$$

Let us now use (2.72) to complete (2.69). The term requiring attention is $\mathbf{E}^d\left[S_T \mathbf{1}_{\{S_T \geq K\}}\right]$, which admits the following reduction

$$\begin{aligned}
\mathbf{E}^d\left[S_T \mathbf{1}_{\{S_T \geq K\}}\right] &= \mathbf{E}^d\left[S_0 \exp\left(\left(r^d - r^f - \frac{1}{2}\sigma^2\right) T + \sigma W_T^d\right) \mathbf{1}_{\{S_T \geq K\}}\right] \\
&= S_0 e^{(r^d - r^f)T} \mathbf{E}^d\left[\exp\left(\sigma W_T^d - \frac{1}{2}\sigma^2 T\right) \mathbf{1}_{\{S_T \geq K\}}\right] \\
&= S_0 e^{(r^d - r^f)T} \mathbf{E}^d\left[\frac{d\mathbf{P}^f}{d\mathbf{P}^d} \mathbf{1}_{\{S_T \geq K\}}\right] \\
&= S_0 e^{(r^d - r^f)T} \mathbf{E}^f\left[\mathbf{1}_{\{S_T \geq K\}}\right] \\
&= S_0 e^{(r^d - r^f)T} \mathbf{P}^f\left[S_T \geq K\right]. \quad (2.73)
\end{aligned}$$

This yields

$$V_0 = S_0 e^{-r^f T} \mathbf{P}^f\left[S_T \geq K\right] - K e^{-r^d T} \mathbf{P}^d\left[S_T \geq K\right]. \quad (2.74)$$

Next, we calculate the two risk-neutral probabilities (in $\mathbf{P}^d$ and $\mathbf{P}^f$) that $S_T \geq K$. Recall from (2.48) and (2.51) that

$$S_T = S_0 \exp\left(\sigma W_T^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right) T\right), \quad (2.75a)$$

$$S_T = S_0 \exp\left(\sigma W_T^f + \left(r^d - r^f + \frac{1}{2}\sigma^2\right) T\right). \quad (2.75b)$$

To enable us to perform the computations in both measures, introduce the index i which takes values in $\{1, 2\}$ and $X(\cdot)$ defined such that $X(1) \equiv f$ and $X(2) \equiv d$. We then have

$$S_T = S_0 \exp\left(\sigma W_T^{X(i)} + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right). \quad (2.76)$$

The probabilities in (2.74) for domestic and foreign risk-neutral measures are obtained by computing $\mathbf{P}^{X(i)}\left[S_T \geq K\right]$ for $i = 1, 2$. We then use (2.76) to write

$$\begin{aligned}
&\mathbf{P}^{X(i)}\left[S_0 \exp\left(\sigma W_T^{X(i)} + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right) \geq K\right] \\
&= \mathbf{P}^{X(i)}\left[\exp\left(\sigma W_T^{X(i)} + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right) \geq \frac{K}{S_0}\right] \\
&= \mathbf{P}^{X(i)}\left[\sigma W_T^{X(i)} + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T \geq \ln\left(\frac{K}{S_0}\right)\right] \\
&= \mathbf{P}^{X(i)}\left[\sigma W_T^{X(i)} \geq \ln\left(\frac{K}{S_0}\right) - \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right] \\
&= \mathbf{P}^{X(i)}\left[\sigma W_T^{X(i)} \leq \ln\left(\frac{S_0}{K}\right) + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right] \\
&= \mathbf{P}\left[\sigma\sqrt{T}\xi \leq \ln\left(\frac{S_0}{K}\right) + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T\right] \\
&= \mathbf{P}\left[\xi \leq \frac{\ln\left(\frac{S_0}{K}\right) + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T}{\sigma\sqrt{T}}\right]
\end{aligned}$$

where ξ is a standard $N(0, 1)$ normal distribution. Let us now define d_1 and d_2 by

$$d_i = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r^d - r^f + \left[\frac{1}{2} - (i-1)\right]\sigma^2\right)T}{\sigma\sqrt{T}}, \quad (2.77)$$

which lets us write

$$\mathbf{P}^{X(i)}\left[S_T \geq K\right] = N(d_i), \quad (2.78)$$

where $N(x) = \int_{-\infty}^{x} n(u)du$ is the cumulative distribution function and

$$n(u) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}u^2\right) \tag{2.79}$$

is the probability density function for the standard normal distribution $N(0, 1)$.

Finally, substituting (2.78) into (2.74) yields the standard Garman–Kohlhagen (Garman and Kohlhagen, 1983) formula for a European call

$$V_0^C = S_0 e^{-r^f T} N(d_1) - K e^{-r^d T} N(d_2) \tag{2.80}$$

with

$$d_{1,2} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r^d - r^f \pm \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}. \tag{2.81}$$

The result is familiar.

The same argument applies for the European put option, with payout function $V_T = \max(K - S_T, 0) = (K - S_T)^+$ at time T shown in Figure 2.4, for which we obtain

$$V_0^P = K e^{-r^d T} N(-d_2) - S_0 e^{-r^f T} N(-d_1). \tag{2.82}$$

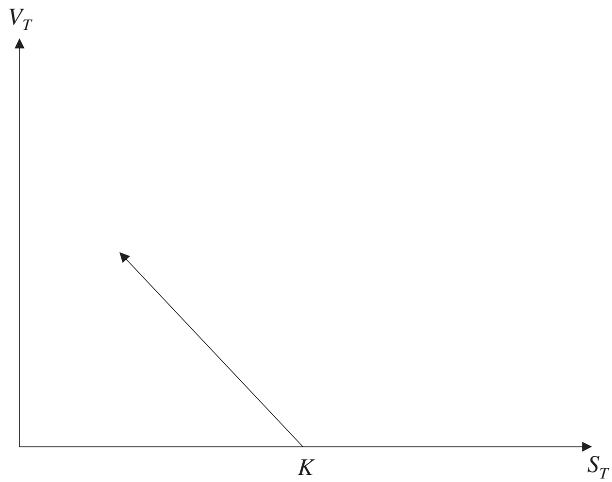

Figure 2.4 Payout function V_T for European put.

Both can be expressed more concisely by introducing a variable ω which is +1 for a call and −1 for a put, we then have

$$V_0^{C/P} = \omega S_0 e^{-r^f T} N(\omega d_1) - \omega K e^{-r^d T} N(\omega d_2). \tag{2.83}$$

2.5.2 European Options on Futures

If, instead of options on spot S_T, we have options on the T^*-forward or a T^*-futures contract, then instead of dS_t as in (2.43) we have df_{t,T^*} or dF_{t,T^*} as in (2.20) or (2.30). Let's suppose without loss of generality that we are dealing with options on the T^*-forward. Supposing that the volatility σ of the T^*-forward is known, we have

$$V_0^{C/P} = \omega e^{-r^d T} [F_{0,T^*} N(\omega d_1) - K N(\omega d_2)] \tag{2.84a}$$

with

$$d_{1,2} = \frac{\ln\left(F_{0,T^*}/K\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}. \tag{2.84b}$$

2.5.3 Settlement Adjustments

Finally, let us suppose that an option has expiry date T but actual settlement possibly later on. We shall use T_{stl} to denote the time to settlement, where $T_{\mathrm{stl}} \geq T$. If the option is cash settled, then this just means we have some additional discounting, corresponding to the extra period we have to wait to receive the cash. If, however, the option is physically settled and it is an option on a spot process (i.e. not on a futures contract or on a forward), then we also need to adjust the strike for options on spot to take into account the expected drift of the process between T and T_{stl}. Define K' by

$$K' = \begin{cases} K \cdot D^d_{T,T_{\mathrm{stl}}} / D^f_{T,T_{\mathrm{stl}}}, & \text{physically settled option on } S_T \\ K, & \text{otherwise.} \end{cases}$$

As a result, in the case of European options with late settlement we merely use

$$V_0^{C/P;\mathrm{dd}}(K,T) = D^d_{T,T_{\mathrm{stl}}} \cdot V_0^{C/P}(K',T), \tag{2.85}$$

where dd denotes "delayed delivery".

2.6 AMERICAN OPTIONS

While European options are easily priced in closed form, at least under a simple model such as the Black–Scholes or the Black-76 model, it is commonplace in the markets for options on futures to be exercisable continuously up to and on the exercise date. Some approximations do exist for this, such as Barone-Adesi and Whaley (1987), Bjerksund and Stensland (1993) and Bjerksund and Stensland (2002), but it is equally commonplace to use a binomial tree or PDE lattice method.

2.6.1 Barone-Adesi and Whaley (1987)

Firstly, I shall introduce the Barone-Adesi and Whaley (1987) approximation, in the special case where the drift of the stochastic process identically vanishes (such as it does in the Black-76 model). Let $V_A(F, T)$ denote the price of an American option with maturity T and $V_E(F, T)$ denote the price of an equivalent European option without the early exercise feature. We shall suppose that these are options on the forward F.

As both American and European options satisfy the same linear partial differential equation (2.33), we know by the principle of superposition that $\epsilon = V_A(F, T) - V_E(F, T)$ must also satisfy (2.33).

We therefore require

$$\frac{\partial \epsilon}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 \epsilon}{\partial F^2} - r^d \epsilon = 0 \tag{2.86}$$

or, after multiplying by $2/\sigma^2$ and denoting $M = 2r^d/\sigma^2$, we have

$$F^2 \frac{\partial^2 \epsilon}{\partial F^2} - \frac{M}{r^d}\frac{\partial \epsilon}{\partial \tau} - M\epsilon = 0 \tag{2.87}$$

where we switch to time reversed coordinates $\tau = T - t$.

We then use a separation of variables approach and posit the following form for the early exercise premium

$$\epsilon(F, k) = k(\tau) g(F, k). \tag{2.88}$$

Note that k above bears no relationship to strike – basically we're saying that k depends on τ and then $\epsilon(F, \tau) = \epsilon(F, k(\tau)) = k(\tau)g(F, k(\tau))$.

We have

$$\frac{\partial^2 \epsilon}{\partial F^2} = k\frac{\partial^2 g}{\partial F^2} \quad \text{and} \quad \frac{\partial \epsilon}{\partial \tau} = \frac{\partial k}{\partial \tau} g + k \frac{\partial k}{\partial \tau}\frac{\partial g}{\partial k}. \tag{2.89}$$

Substituting (2.89) into (2.87) we get

$$F^2 \frac{\partial^2 g}{\partial F^2} - Mg \left[1 + \frac{1}{r^d k} \left(1 + \frac{k}{g} \frac{\partial g}{\partial k} \right) \frac{\partial k}{\partial \tau} \right] = 0. \tag{2.90}$$

The Barone-Adesi and Whaley approach now puts $k(\tau) = 1 - e^{-r^d \tau}$, and so $k' = r^d e^{-r^d \tau}$. With this, we can simplify (2.90) somewhat. Let us take a fragment of the term on the right hand side and apply some simple algebra to it. We have

$$\begin{aligned} g \left[1 + \frac{1}{r^d k} \left(1 + \frac{k}{g} \frac{\partial g}{\partial k} \right) \frac{\partial k}{\partial \tau} \right] &= \left(1 + \frac{1}{r^d k} \frac{\partial k}{\partial \tau} \right) g + \frac{1}{r^d} \frac{\partial k}{\partial \tau} \frac{\partial g}{\partial k} \\ &= \left(1 + \frac{1}{r^d} \frac{k'}{k} \right) g + e^{-r^d \tau} \frac{\partial g}{\partial k} \\ &= \left(1 + \frac{e^{-r^d \tau}}{1 - e^{-r^d \tau}} \right) g + (1 - k) \frac{\partial g}{\partial k} \\ &= \left(\frac{1}{k} \right) g + (1 - k) \frac{\partial g}{\partial k}, \end{aligned}$$

where the final step follows from $1 + \frac{e^{-r^d \tau}}{1 - e^{-r^d \tau}} = \frac{1 - e^{-r^d \tau} + e^{-r^d \tau}}{1 - e^{-r^d \tau}} = \frac{1}{1 - e^{-r^d \tau}} = \frac{1}{k}$. Consequently we arrive at

$$F^2 \frac{\partial^2 g}{\partial F^2} - \frac{M}{k} g - M(1 - k) \frac{\partial g}{\partial k} = 0. \tag{2.91}$$

Barone-Adesi and Whaley (1987) then make the approximation that the final $M(1 - k)\frac{\partial g}{\partial k}$ term in (2.91) can be supposed to be negligible (particularly so when τ is very small or very large), thereby arriving at

$$F^2 \frac{\partial^2 g}{\partial F^2} - \frac{M}{k} g = 0. \tag{2.92}$$

Note that (2.91) is a partial differential equation but (2.92) is now an ordinary differential equation since $g(\cdot)$ is now a univariate function of q.

This can be solved by substituting $g(F) = F^q$ into (2.92) which reduces (as the second derivative $g''(F) = q(q - 1)F^{q-2}$ trivially) to

$$q(q - 1)F^q - \frac{M}{k} F^q = 0. \tag{2.93}$$

This requires solution of a quadratic in q

$$q^2 - q - \frac{M}{k} = 0 \tag{2.94}$$

which has solutions $q_1 = \frac{1}{2}(1 - \sqrt{1 + 4M/k})$, $q_2 = \frac{1}{2}(1 + \sqrt{1 + 4M/k})$. Our solution of (2.93) is

$$g(F) = a_1 F^{q_1} + a_2 F^{q_2}$$

where a_1 and a_2 are real constants and $q_1 < 0$ and $q_2 > 0$. For an American call we require $a_1 = 0$ in order that the price not explode as $F \to 0$, and for an American put we require $a_2 = 0$ so that the price is similarly well behaved for large values of F.

We know that for sufficiently in-the-money options, it is optimal to exercise immediately (so as to obtain the intrinsic value today rather than having to wait until T). What this means is that for American calls, there must exist an F^* such that the value of the American call is equal to the (positive) intrinsic value in $[F^*, \infty)$, i.e. $V_A(F, t) = F - K \;\forall\; F > F^*$. For American puts, the exercise region is $[0, F^*]$, i.e. the value of the American put $V_A(F, t) = K - F \;\forall\; F < F^*$.

The task is to find F^*. Since we are dealing with a diffusion equation, we know that the function and its first derivative must be continuous across the domain transition at F^*. For the American call we have

$$F^* - K = V_E^C(F^*, T) + ka_2(F^*)^{q_2} \tag{2.95a}$$

$$\begin{aligned} 1 &= \frac{\partial}{\partial F^*}\left(V_E^C(F^*, T)\right) + ka_2 q_2 (F^*)^{q_2 - 1} \\ &= e^{-r^f T} N(d_1(F^*)) + ka_2 q_2 (F^*)^{q_2 - 1} \end{aligned} \tag{2.95b}$$

and for the American put

$$K - F^* = V_E^P(F^*, T) + ka_1(F^*)^{q_1} \tag{2.96a}$$

$$\begin{aligned} -1 &= \frac{\partial}{\partial F}\left(V_E^P(F^*, T)\right) + ka_1 q_1 (F^*)^{q_1 - 1} \\ &= -e^{-r^f T} N(-d_1(F^*)) + ka_1 q_1 (F^*)^{q_1 - 1}. \end{aligned} \tag{2.96b}$$

As Barone-Adesi and Whaley (1987) point out, either (2.95) or (2.96) constitute a system of two equations in two unknowns. Consider the American call. We can use (2.95b) to solve for a_2, obtaining

$$ka_2(F^*)^{q_2} = \frac{F^*}{q_2}\left[1 - e^{-r^f T} N(d_1(F^*))\right] \tag{2.97}$$

which plugs directly into (2.95a), giving

$$F^* - K = V_E^C(F^*, T) + \frac{F^*}{q_2}\left[1 - e^{-r^f T}N(d_1(F^*))\right]. \qquad (2.98)$$

As (2.98) is an implicit equation in F^*, solving for the critical value F^* (above which the value of the American call is equal to its intrinsic value) requires a numerical scheme such as Newton–Raphson to be used. We refer the reader to the original paper by Barone-Adesi and Whaley (1987), Section 7.2 of Briys, Bellalah, Mai and de Varenne (1998) or Chapter 3 of Haug (2007) for further discussion.

Once F^* has been determined, the value of an American call option can then be computed directly by addition of the early exercise premium ϵ to the European call price $V_E^C(F^*, T)$ by use of (2.97). We have

$$\begin{aligned}
\epsilon &= kg(F) = ka_2 F^{q_2} \\
&= \left(\frac{1}{F^*}\right)^{q_2} \frac{F^*}{q_2}\left[1 - e^{-r^f T}N(d_1(F^*))\right] F^{q_2} \\
&= A_2\left(\frac{F}{F^*}\right)^{q_2},
\end{aligned}$$

where $A_2 = \frac{F^*}{q_2}\left[1 - e^{-r^f T}N(d_1(F^*))\right]$. The price of the American option can then be written as

$$V_A(F, T) = V_E(F, T) + A_2\left(\frac{F}{F^*}\right)^{q_2}. \qquad (2.99)$$

We make the final remark that other approximations, such as Bjerksund and Stensland (1993) and Bjerksund and Stensland (2002), are also worth investigating.

2.6.2 Lattice Methods

While fast, approximations like the above are inexact. For this reason, lattice methods, such as binomial trees or a finite difference PDEs, are often used. The formulation of these is well beyond the scope of this book and will very likely be familiar to some extent; we refer the reader to standard PDE references such as (but by no means limited to) Wilmott, Dewynne and Howison (1993) and/or Wilmott, Howison and Dewynne (1995), Tavella and Randall (2000) and Duffy (2006) as well as my previous book for discussion of the finite difference approach to option pricing.

For these methods, at each step of the backward induction, a comparison is made between the PV of the option (conditional on spot and time) and the intrinsic value of the product. For each node on the finite difference grid, if the intrinsic value is greater, then one replaces the lattice PV with the intrinsic value. The use of PDEs in option pricing is discussed in Chapter 8 of Clark (2011) with the specific condition relating to value monitoring for American options in Section 8.7.2.

For each timestep where the American option is early exercisable, we apply the following condition, where ω is 1 for a call and -1 for a put.

$$V(S, t^-) = \max(V(S, t^+) - K_C, (\omega(S_t - K)^+)). \tag{2.100}$$

2.7 ASIAN OPTIONS

One of the features that differentiates commodities from other asset classes is the extent to which Asian options (in this context meaning average price options) are used. Part of this is because commodities tend to experience greater volatility than many other asset classes, so there is a high degree of fixing risk attached to having an option which expires on one particular day and has a payout which is only a function of the marginal distribution at that one time. Additionally, the demand for commodities is relatively continuous – market participants tend to produce or consume commodities on an ongoing basis.

There are, however, two other features attached to the Asian options encountered in commodity markets (and in industry in general): firstly, the averaging is on the arithmetic average[8] and secondly the averaging is invariably over a set of discrete fixings rather than a continuous average (integral). These two features complicate the analysis, so let's start with something simple.

2.7.1 Geometric Asian Options – Continuous Averaging

Angus (1999) provides a very clear and easy to follow introduction to mathematical option pricing theory for Asian options, purely in the case where the average is a geometric average.

[8] Harmonic averages are additionally encountered in FX as the spot rate can be interpreted in one of two ways depending on which currency the notional is in.

As usual, we suppose that X_t denotes the logspot, i.e. $X_t = \ln S_t$. From (2.75a) we have

$$X_t = X_0 + \sigma W_t^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)t. \tag{2.101}$$

First we construct the logspot integral I_t^X, which is given by

$$I_t^X = \int_0^t X_u du$$

and then we use this to express the continuous geometric average $A_t^{G;s} = \exp\left(I_t^X/t\right)$. A geometric average price call option, for example, will have payout $V_T = (A_T^{G;s} - K)^+$, whereas a geometric average strike option will have payout $V_T = (S_T - A_T^{G;s})^+$. To be able to price both styles of Asian options, we need to consider the joint distribution of (S_T, I_t^X), but for average price options we need only the statistical properties of I_t^X. Since average strike features are quite rare in the commodity markets, we shall restrict ourselves to considering only average price options, which make up the vast majority of the Asian options traded.

By stochastic integration of dX_t we have

$$X_t = X_0 + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)t + \sigma W_t^d$$

so we can write

$$\begin{aligned} I_t^X &= \int_0^t X_u du \\ &= \int_0^t \left[X_0 + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)u + \sigma W_u^d\right] du \\ &= tX_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t^2 + \sigma \int_0^t W_u^d du. \end{aligned}$$

From this, we can compute first and second moments. The first is given by

$$\mathbf{E}^d\left[I_t^X\right] = tX_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t^2$$

since $\mathbf{E}^d\left[\int_0^t W_u^d du\right] = 0$. Let us now define $\hat{I}_t^X = I_t^X - \mathbf{E}^d\left[I_t^X\right] = \sigma \int_0^t W_u^d du$, which we will use to compute $\mathbf{Var}^d\left[I_t^X\right] = \mathbf{E}^d\left[(\hat{I}_t^X)^2\right]$.

The analysis is simple:

$$\begin{aligned}
\mathbf{E}^d\left[(\hat{I}_t^X)^2\right] &= \sigma^2\mathbf{E}^d\left[\left(\int_0^t W_s^d ds\right)\left(\int_0^t W_u^d du\right)\right] \\
&= \sigma^2\mathbf{E}^d\left[\int_0^t\int_0^t W_s^d W_u^d dsdu\right] \\
&= \sigma^2\int_0^t\int_0^t \mathbf{E}^d\left[W_s^d W_u^d\right] dsdu \\
&= \sigma^2\int_0^t\int_0^t \min(s,u)dsdu.
\end{aligned}$$

Note that the final line can be obtained by assuming (without loss of generality) $s \leq u$ and writing $W_u = W_s + (W_u - W_s)$, and writing $\Delta W = W_u - W_s$, where W_s and ΔW are independent. We then have $\mathbf{E}\left[W_s W_u\right] = \mathbf{E}\left[W_s(W_s + \Delta W)\right] = \mathbf{E}\left[W_s^2\right] + \mathbf{E}\left[W_s \Delta W\right] = \mathbf{E}\left[W_s^2\right] = s$. If, however, $u \leq s$, then by the same argument $\mathbf{E}\left[W_s W_u\right] = u$, so the general result $\mathbf{E}\left[W_s W_u\right] = \min(s,u)$ follows.

Now let us evaluate $\int_0^t\int_0^t \min(s,u)dsdu$, using (x, y) instead of (s, u).

$$\begin{aligned}
\int_0^t\int_0^t \min(x,y)dxdy &= 2\int_0^t\int_0^x \min(x,y)dxdy \\
&= 2\int_0^t\int_0^x ydydx \\
&= 2\int_0^t \left[\frac{1}{2}y^2\right]_0^x dx \\
&= \int_0^t x^2 dx \\
&= \left[\frac{1}{3}x^3\right]_0^t = \frac{1}{3}t^3.
\end{aligned}$$

The first and second moments follow:

$$\mathbf{E}^d\left[I_t^X\right] = tX_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t^2, \tag{2.102a}$$

$$\mathbf{Var}^d\left[I_t^X\right] = \frac{1}{3}\sigma^2 t^3, \tag{2.102b}$$

i.e. with $\ln A_T^{G;s} = I_T^X / T$ we have

$$\mathbf{E}^d \left[\ln A_T^{G;s}\right] = X_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right) T, \tag{2.103a}$$

$$\mathbf{Var}^d \left[\ln A_T^{G;s}\right] = \frac{1}{3}\sigma^2 T. \tag{2.103b}$$

We can therefore write

$$I_t^X = tX_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right) t^2 + \sigma\sqrt{\frac{1}{3}t^3} \cdot \xi \tag{2.104}$$

where $\xi \sim N(0, 1)$ under the domestic risk-neutral measure.

We now have the machinery to price the continuously monitored average price call option, with payoff at expiry

$$V_T = \left(A_T^{G;s} - K\right)^+$$

where $A_T^{G;s} = \exp\left(I_T^X / T\right)$. Following the risk-neutral approach, we construct the fair price today for the option by

$$\begin{aligned}
V_0 &= e^{-r^d T}\mathbf{E}^d \left[(A_T^{G;s} - K)^+\right] \\
&= e^{-r^d T}\mathbf{E}^d \left[(A_T^{G;s} - K)\mathbf{1}_{\{A_T^{G;s} \geq K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d \left[\left(\exp\left(I_T^X/T\right) - K\right)\mathbf{1}_{\{\exp(I_T^X/T) \geq K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d \left[\left(\exp\left(I_T^X/T\right) - K\right)\mathbf{1}_{\{I_T^X \geq T\ln K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d \left[\exp\left(I_T^X/T\right)\mathbf{1}_{\{I_T^X \geq T\ln K\}}\right] - Ke^{-r^d T}\mathbf{E}^d \left[\mathbf{1}_{\{I_T^X \geq T\ln K\}}\right] \\
&= e^{-r^d T}\mathbf{E}^d \left[\exp\left(I_T^X/T\right)\mathbf{1}_{\{I_T^X \geq T\ln K\}}\right] - Ke^{-r^d T}\mathbf{P}^d \left[I_T^X \geq T\ln K\right].
\end{aligned} \tag{2.105}$$

We commence by calculating the domestic risk-neutral probability in the second of the terms in (2.105). Write

$$\begin{aligned}
\mathbf{P}^d \left[I_T^X \geq T\ln K\right] &= \mathbf{P}^d \left[TX_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right) T^2 + \sigma\sqrt{\frac{1}{3}T^3} \cdot \xi \geq T\ln K\right] \\
&= \mathbf{P}^d \left[\sigma\sqrt{\frac{1}{3}T^3} \cdot \xi \geq T\ln K - TX_0 - \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right) T^2\right] \\
&= \mathbf{P}^d \left[\sigma\sqrt{\frac{1}{3}T} \cdot \xi \geq \ln K - X_0 - \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right) T\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{P}^d \left[\sigma \sqrt{\frac{1}{3} T} \cdot \xi \leq \ln S_0 - \ln K + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T \right] \\
&= \mathbf{P}^d \left[\xi \leq \frac{\ln S_0 - \ln K + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{\frac{1}{3} T}} \right] \\
&= N(d_2^G)
\end{aligned}
$$

with

$$
d_2^G = \frac{\ln(S_0/K) + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{\frac{1}{3} T}}.
$$

It remains to calculate the first term in (2.105) which requires a change of measure argument. We need to compute

$$
\mathbf{E}^d \left[\exp\left(I_T^X / T \right) \mathbf{1}_{\{I_T^X \geq T \ln K\}} \right]
$$

with

$$
\begin{aligned}
\frac{I_T^X}{T} &= X_0 + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T + \sqrt{\frac{1}{3} T} \cdot \sigma \xi \\
&= X_0 + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T + \sqrt{\frac{1}{3}} \cdot \sigma \hat{W}_T^d,
\end{aligned}
$$

where $\hat{W}_t^d$ is a Brownian motion under the domestic risk-neutral measure – not to be confused with the Brownian motion W_t^d driving the logspot process. We then have

$$
\begin{aligned}
\exp\left(\frac{I_T^X}{T} \right) &= S_0 \exp\left(\frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T + \sqrt{\frac{1}{3}} \cdot \sigma \hat{W}_T^d \right) \\
&= S_0 \exp\left(\sqrt{\frac{1}{3}} \cdot \sigma \hat{W}_T^d - \frac{1}{6}\sigma^2 T + \frac{1}{6}\sigma^2 T + \frac{1}{2} \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) T \right) \\
&= S_0 \exp\left(\sqrt{\frac{1}{3}} \cdot \sigma \hat{W}_T^d - \frac{1}{6}\sigma^2 T \right) \exp\left(\frac{1}{2} \left(r^d - r^f \right) T - \frac{1}{12}\sigma^2 T \right).
\end{aligned}
$$

The reason for this somewhat unusual grouping of terms is so that we can put $\alpha = \frac{1}{\sqrt{3}}\sigma$, and thereby identify the first of the two exponential terms above as a Radon–Nikodym derivative

$$\frac{d\mathbf{P}^*}{d\mathbf{P}^d} = \exp\left(\alpha \hat{W}_T^d - \frac{1}{2}\alpha^2 T\right).$$

Note that $\hat{W}_T^d$ is no longer a martingale under $\mathbf{P}^*$, but $\hat{W}_T^* = \hat{W}_T^d - \alpha t$ is most definitely a $\mathbf{P}^*$-martingale. We then have, for an event ε

$$\begin{aligned}\mathbf{E}^d\left[\exp\left(I_T^X/T\right)\mathbf{1}_\varepsilon\right] &= S_0 \exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right)\mathbf{E}^d\left[\frac{d\mathbf{P}^*}{d\mathbf{P}^d}\mathbf{1}_\varepsilon\right] \\ &= S_0 \exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right)\mathbf{E}^*\left[\mathbf{1}_\varepsilon\right]\end{aligned}$$

and of course $\mathbf{E}^*\left[\mathbf{1}_\varepsilon\right] = \mathbf{P}^*\left[\varepsilon\right]$.

For an average price call option, $\varepsilon = \{I_T^X \geq T\ln K\}$, which means that we need to compute $\mathbf{P}^*\left[I_T^X/T \geq \ln K\right]$. Expressing I_T^X/T in terms of the new Brownian $\hat{W}_T^*$, we have

$$\begin{aligned}\frac{I_T^X}{T} &= X_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)T + \sqrt{\frac{1}{3}}\cdot\sigma[\hat{W}_T^* + \alpha T] \\ &= X_0 + \frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{4}\sigma^2 T + \frac{1}{\sqrt{3}}\sigma\hat{W}_T^* + \frac{1}{\sqrt{3}}\sigma\alpha T \\ &= X_0 + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{\sqrt{3}}\sigma\hat{W}_T^* + \frac{1}{12}\sigma^2 T.\end{aligned}$$

We therefore have

$$\begin{aligned}\mathbf{P}^*\left[I_T^X/T \geq \ln K\right] &= \mathbf{P}^*\left[X_0 + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{\sqrt{3}}\sigma\hat{W}_T^* + \frac{1}{12}\sigma^2 T \geq \ln K\right] \\ &= \mathbf{P}^*\left[\frac{1}{\sqrt{3}}\sigma\hat{W}_T^* \geq \ln\left(\frac{K}{S_0}\right) - \frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right] \\ &= \mathbf{P}^*\left[\hat{W}_T^* \geq \frac{\ln\left(\frac{K}{S_0}\right) - \frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T}{\frac{1}{\sqrt{3}}\sigma}\right] \\ &= \mathbf{P}^*\left[\hat{W}_T^* \leq \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{12}\sigma^2 T}{\frac{1}{\sqrt{3}}\sigma}\right]\end{aligned}$$

$$= \mathbf{P}\left[\xi \leq \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{12}\sigma^2 T}{\frac{1}{\sqrt{3}}\sigma\sqrt{T}}\right] \text{with } \xi \sim N(0,1)$$

$$= N(d_1^G)$$

where

$$d_1^G = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{12}\sigma^2 T}{\frac{1}{\sqrt{3}}\sigma\sqrt{T}}.$$

Putting this together, we have

$$\mathbf{E}^d\left[\exp\left(I_T^X/T\right)\mathbf{1}_{\{I_T^X \geq T\ln K\}}\right] = S_0\exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right)N\left(d_1^G\right)$$

which we can substitute into (2.105) to obtain the closed form price for a continuously monitored geometric average price call option

$$V_0^{GAC} = e^{-r^d T}\left[S_0\exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right)N\left(d_1^G\right) - KN\left(d_2^G\right)\right]. \tag{2.106}$$

Comparing this with equation (2.8) in Angus (1999) in the special case where $t = 0$, we see the result is proven.

The continuous geometric average price put is priced along the same lines, an exercise which we encourage readers to work through for themselves. The result is

$$V_0^{GAP} = e^{-r^d T}\left[KN\left(-d_2^G\right) - S_0\exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right)N\left(-d_1^G\right)\right]. \tag{2.107}$$

Expressions for the price of partially seasoned geometric average price options and geometric average strike options can be found in Angus (1999), along with average price and average strike binary options. This is discussed also in Section 6.2 of Geman (2005).

2.7.2 Arithmetic Asian Options – Continuous Averaging

Zhang (1998) provides a solid discussion of the various types of mean that can be used for computing the payouts and prices of Asian options, using the general mean

$$M(\gamma|S) = \left(\frac{1}{n}\sum_{i=1}^{n} S_i^{\gamma}\right)^{1/\gamma}.$$

Note that $\lim_{\gamma\to 0} M(\gamma|S) = \left(\prod_{i=1}^{n} S_i\right)^{1/n}$, which is nothing more than the discrete geometric average, whereas $M(1|S) = \frac{1}{n}\sum_{i=1}^{n} S_i$ is the discrete arithmetic average and the case $\gamma = -1$, i.e. $M(-1|S) = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{S_i}\right)^{-1}$, corresponds to the harmonic mean. However, we're still working in the continuous time framework, so we begin by introducing the continuous arithmetic average via the spot integral I_t^S

$$I_t^S = \int_0^t S_u du.$$

With this, the continuous arithmetic average is given by $A_t^{A;s} = I_t^S/t$.

Zhang (1998) demonstrates that the arithmetic average can be approximated by the geometric average together with a correction, so in the continuous case

$$A_T^{A;s} = \kappa A_T^{G;s}$$

where

$$\kappa_c = 1 + \frac{1}{24}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)^2 T^2 + \frac{1}{576}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)^4 T^4,$$

the averaging period being understood to be $[0, T]$.

We arrive at

$$V_0^{AAC} = e^{-r^d T}\left[\kappa_c S_0 \exp\left(\frac{1}{2}\left(r^d - r^f\right)T - \frac{1}{12}\sigma^2 T\right) N\left(d_1^A\right) - KN\left(d_2^A\right)\right] \tag{2.108}$$

where

$$d_1^A = \frac{\ln(\kappa_c S_0/K) + \frac{1}{2}\left(r^d - r^f\right)T + \frac{1}{12}\sigma^2 T}{\sigma\sqrt{\frac{1}{3}T}}$$

and

$$d_2^A = \frac{\ln(\kappa_c S_0/K) + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{\frac{1}{3}T}}.$$

For further details, we refer the reader to Section 6.7 of Zhang (1998).

2.7.3 Geometric Average Options – Discrete Fixings – Kemna and Vorst (1990)

Section 2.7.1 demonstrates how closed form solutions can be found in the simplest case, where the averaging is continuous and performed using the geometric average. This rests on the property that the continuous geometric average of a logspot process is itself lognormally distributed. In fact, the same is true for a geometric average computed over a discrete set of time points. This was first described in the original paper by Kemna and Vorst (1990), who recognised that the continuous geometric average of a lognormal process is itself lognormal and used this feature to obtain a closed form expression for the price. A useful compendium of results for Asian options can be found in Chapters 5 to 7 of Zhang (1998) – our aim here is to cover a subset of that work.

Let us suppose a sequence of n observation times $\{t_1, \ldots, t_n\}$ is given over the interval $[0, T]$ with $0 < t_1 < t_2 < \cdots < t_n \leq T$.

For the sake of exposition, though, let us suppose that the timepoints are equally spaced – purely to simplify the algebra below (the computations can easily be performed for arbitrary fixing schedules)

$$t_i = T - (n - i)h. \tag{2.109}$$

Note that depending on the magnitude of h, the period over which the averaging is performed can be a subset of $[0, T]$ or can even be a superset, i.e. extending into the past ($t < 0$). We shall not concern ourselves with this possibility here, though.

From (2.75a) after taking logs we have

$$\ln S_{t_i} = \ln S_0 + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)t_i + \sigma W_{t_i}^d. \tag{2.110}$$

We can now define the geometric average over these observations

$$
\begin{aligned}
A^{G;s}_{t_1,t_n} &= \left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} \\
&= \exp\left(\frac{1}{n}\ln\left(\prod_{i=1}^{n} S_{t_i}\right)\right) \\
&= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\ln S_{t_i}\right) \\
&= \exp\left(\ln S_0 + \frac{1}{n}\sum_{i=1}^{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t_i + \frac{1}{n}\sum_{i=1}^{n}\sigma W^d_{t_i}\right)
\end{aligned}
$$

and by inspection we can see that $A^{G,s}_{t_1,t_n}$ is lognormally distributed.

$$
\ln A^{G,s}_{t_1,t_n} = \ln S_0 + \frac{1}{n}\sum_{i=1}^{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t_i + \frac{1}{n}\sum_{i=1}^{n}\sigma W^d_{t_i}. \quad (2.111)
$$

Let us therefore try to extract the first and second moments of the distribution of $\ln A^{G,s}_{t_1,t_n}$. We use the notation of Section 5.3 of Zhang (1998) and restrict ourselves to the case where no historical fixings have taken place yet (i.e. with $j = 0$ in his notation). The first moment is trivial:

$$
\begin{aligned}
\mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n} - \ln S_0\right] &= \frac{1}{n}\sum_{i=1}^{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)t_i \\
&= \frac{1}{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)\sum_{i=1}^{n}\left[T - (n-i)h\right] \\
&= \frac{1}{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)\left[nT - n^2h + h\sum_{i=1}^{n} i\right] \\
&= \frac{1}{n}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)\left[nT - n^2h + h\frac{n(n+1)}{2}\right] \\
&= \left(r^d - r^f - \frac{1}{2}\sigma^2\right)\left[T - \frac{1}{2}h(n-1)\right]
\end{aligned}
$$

i.e.

$$\mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \ln S_0 + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)\left[T - \frac{1}{2}h(n-1)\right]. \quad (2.112)$$

The second moment is not. From (2.111) we have

$$\begin{aligned}
\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right] &= \mathbf{Var}^d\left[\frac{1}{n}\sum_{i=1}^{n}\sigma W^d_{t_i}\right] \\
&= \frac{\sigma^2}{n^2}\mathbf{Var}^d\left[\sum_{i=1}^{n} W^d_{t_i}\right] \\
&= \frac{\sigma^2}{n^2}\mathbf{E}^d\left[\sum_{i,j=1}^{n} W^d_{t_i}W^d_{t_j}\right] \\
&= \frac{\sigma^2}{n^2}\sum_{i,j=1}^{n}\min(t_i, t_j) \\
&= \frac{\sigma^2}{n^2}\sum_{i=1}^{n} t_i[2(n-i)+1].
\end{aligned}$$

We now use the timepoint specification (2.109) to evaluate the term above.

$$\begin{aligned}
\sum_{i=1}^{n} t_i[2(n-i)+1] &= \sum_{i,j=1}^{n}[T-(n-i)h][2(n-i)+1] \\
&= n(T-nh)(2n+1) + (4nh - 2T + h)\sum_{i=1}^{n} i - 2h\sum_{i=1}^{n} i^2 \\
&= n^2T - \frac{1}{6}nh(4n+1)(n-1)
\end{aligned}$$

via $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ and $\sum_{i=1}^{n} i = \frac{1}{6}n(n+1)(2n+1)$. From this we obtain

$$\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \sigma^2 T - \frac{1}{6}\frac{\sigma^2 h}{n}(4n+1)(n-1). \quad (2.113)$$

Note that this is in agreement with the terms in Theorem 5.1 of Zhang (1998) in the case where $j = 0$.

In the case where the discrete time points are equally distributed over the interval $[0, T]$, i.e. $t_1 = h$ and $t_n = T$, we have $h = T/n$ which when substituted into (2.112) and (2.113) gives

$$\mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \ln S_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)\left[1 + \frac{1}{n}\right]T, \tag{2.114a}$$

$$\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \sigma^2 T\left[1 - \frac{1}{6}\frac{(4n+1)(n-1)}{n^2}\right]. \tag{2.114b}$$

Note that in the limit as $n \to \infty$, (2.7.3) reduces to

$$\mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \ln S_0 + \frac{1}{2}\left(r^d - r^f - \frac{1}{2}\sigma^2\right)T, \tag{2.115a}$$

$$\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right] = \frac{1}{3}\sigma^2 T, \tag{2.115b}$$

which is entirely consistent with (2.7.1) in the limit.

Similarly to (2.104), but with different first and second moments, we can therefore write

$$\frac{I^X_T}{T} = \mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n}\right] + \sqrt{\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right]}\cdot\xi \tag{2.116}$$

where the first and second moments are obtained from (2.112) and (2.113) and $\xi \sim N(0, 1)$ under the domestic risk-neutral measure. From that point on, the pricing of discretely monitored geometric average price options proceeds the same way as in Section 2.7.1.

2.7.3.1 Discretely Monitored Geometric Average Options on Forwards

In reality, commodity Asian options are always specified with a schedule of fixings, and very often with each fixing being referenced against the prompt futures contract on the fixing date (just as in commodity swaps). What this means is that the drift terms corresponding to r^d and r^f are suppressed and the terms $\mathbf{E}^d\left[\ln A^{G,s}_{t_1,t_n}\right]$ and $\mathbf{Var}^d\left[\ln A^{G,s}_{t_1,t_n}\right]$ are adjusted to take into account the relevant prompt future at each of the fixing times, i.e.

$$\mathbf{E}^d\left[\ln A^{G,f}_{t_1,t_n}\right] = \frac{1}{n}\sum_{i=1}^{n}\left[\ln f_{t_i,\vec{T}(t_i)} - \frac{1}{2}[\sigma_{\mathrm{imp}}(\vec{T}(t_i))]^2 t_i\right]. \tag{2.117}$$

Since the time points are arbitrary, it is straightforward to handle the case where the fixings occur over a particular period some way away in the future.

Note that, similarly to commodity swaps, the roll date can be handled one of two ways – so we use Asian(1,0) and Asian(1,1) to denote two different variants, the first being the case where the prompt future (and the associated volatility) rolls at the end of the expiry date, the second being where the roll occurs at the start of the expiry date. Clearly these two possibilities for handling the roll date associated with discrete fixings can be applied to either geometric average options or arithmetic average options. We've met the former and now we introduce the latter.

2.7.4 Arithmetic Average Options – Discrete Fixings – Turnbull and Wakeman (1991)

The overwhelming majority of Asian options in actual traded markets are with arithmetic averaging not geometric averaging. Recall from (2.67) in Section 2.4 that discretely monitored arithmetic averages over the observations $\{t_1, \ldots, t_n\}$ have already been used to value commodity swaps. We shall be using the same arithmetic averages here.

We introduce the presentation in Section 10.2 of West (2009) and the similar presentation of Schneider (2012), the first being with regard to averages on spot, the second being with averages on the futures. The first will assume perfect correlation across the entire futures curve derived from the spot, the second permits some decorrelation across the futures contracts, which is more representative of actual traded commodity markets.

2.7.4.1 Turnbull–Wakeman for Spot Averages

Since $\mathbf{E}^d[S_t] = f_{0,t}$, we clearly have

$$\mathbf{E}^d\left[A^{A,s}_{t_1,t_n}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}^d[S_{t_i}] = \frac{1}{n}\sum_{i=1}^{n} f_{0,t_i} \tag{2.118}$$

which we already know to be the fair value of the commodity swap (i.e. the strike that makes it costless to enter into).

Also from (2.67a) we have

$$\left[A^{A,s}_{t_1,t_n}\right]^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} S_{t_i}S_{t_j} \tag{2.119}$$

which can be expanded to give

$$\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]^2\right] = \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbf{E}^d[S_{t_i}S_{t_j}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}^d\left[S^2_{t_i}\right] + \frac{2}{n^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathbf{E}^d[S_{t_i}S_{t_j}] \tag{2.120}$$

where $i < j$ by the choice of summation. The term $\mathbf{E}^d[S_{t_i}S_{t_j}]$ can be resolved in the following manner

$$\begin{aligned}
\mathbf{E}^d[S_{t_i}S_{t_j}] &= \mathbf{E}^d\left[\mathbf{E}^d[S_{t_i}S_{t_j}|\mathcal{F}_{t_i}]\right]\\
&= \mathbf{E}^d\left[S_{t_i}\mathbf{E}^d[S_{t_j}|\mathcal{F}_{t_i}]\right]\\
&= \mathbf{E}^d\left[S^2_{t_i}\right]\exp(r^d - r^f)(t_j - t_i))]\\
&= f_{0,t_i}f_{0,t_j}\exp\left(\sigma_i^2 t_i\right)
\end{aligned} \tag{2.121}$$

where σ_i denotes the implied volatility for time t_i.

We thereby obtain

$$\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]^2\right] = \frac{1}{n^2}\sum_{i=1}^{n}f^2_{0,t_i}\exp\left(\sigma_i^2 t_i\right) + \frac{2}{n^2}\sum_{j=2}^{n}f_{0,t_j}\sum_{i=1}^{j-1}f_{0,t_i}\exp\left(\sigma_i^2 t_i\right). \tag{2.122}$$

We now apply the moment matching technique. Assume that $A^{A,s}_{t_1,t_n}$ is lognormally distributed, i.e. that $\ln A^{A,s}_{t_1,t_n} \sim N(\mu_A, \sigma_A^2 t_n)$, where by the properties of lognormal distributions we have

$$\sigma_A^2 t_n = \ln\frac{\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]^2\right]}{\left(\mathbf{E}^d\left[[A^{A,s}_{t_1,t_n}]\right]\right)^2} \tag{2.123}$$

$$\mu_A = \ln\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]^2\right] - \frac{1}{2}\sigma_A^2 \tag{2.124}$$

which is equivalent to seeing that an arithmetic average price option can be priced using (2.84) from the Black-76 model on a synthetic asset with a volatility of σ_A given by

$$\sigma_A = \sqrt{\frac{1}{t_n}\left[\ln \frac{\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]^2\right]}{\left(\mathbf{E}^d\left[\left[A^{A,s}_{t_1,t_n}\right]\right]\right)^2}\right]}, \tag{2.125}$$

an initial futures price of $\mathbf{E}^d\left[[A^{A,s}_{t_1,t_n}]\right]$, a strike of K and a settlement time T_{stl} using (2.118) and (2.122) together with the settlement adjustment described in Section 2.5.3.

Note that σ_A is referred to as the Asian volatility.

2.7.4.2 Turnbull–Wakeman for Futures Averages

Suppose instead of an average of spot price fixings we have an average of futures prices over an averaging period. This average is as defined in (2.67b), i.e.

$$A^{A,f}_{t_1,t_n} = \frac{1}{n}\sum_{i=1}^{n} f_{t_i,\vec{T}(t_i)} \tag{2.126}$$

where $t_1 \ldots t_n$ comprise the set of fixing times and $\vec{T}(t)$ denotes the maturity time of the prompt future at each time t.

As we know from Section 2.4, the averaging of a strip of futures can be constructed in one of two ways depending on the roll date. Let us continue to use the same notation, supposing that we have a strip of futures with maturities T_k (we use capitals here to avoid confusing the futures maturity times T_i with the fixing times t_i) – each of which is described by a stochastic differential equation

$$df_{t,T_k} = \sigma_k f_{t,T_k} dW^k_t,$$

where dW^k_t is the driving Brownian motion for f_{t,T_k} and $\langle dW^k_t, dW^l_t\rangle = \rho_{kl}dt$ describes the correlation structure of the futures. Along with $\vec{T}(t)$, we introduce a maturity index $m(i)$ defined such that $T_{m(i)} = \vec{T}(t_i)$, i.e. $f_{t_i,T_{m(i)}} = f_{t_i,\vec{T}(t_i)}$. Note that the maturity index $m(\cdot)$ is a mapping from the fixing time index to the futures time index, and these are *not* the same.

For arbitrary t_i and t_j in the set of time fixings, i.e. $1 \leq i \leq j \leq n$ (note $i \leq j$) we have

$$\begin{aligned}
\mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)}\right] &= f_{0,\vec{T}(t_i)} \\
\mathbf{E}^d\left[f^2_{t_i,\vec{T}(t_i)}\right] &= f^2_{0,\vec{T}(t_i)} \exp\left(\int_0^{t_i} \sigma^2_{m(i)} ds\right) = f^2_{0,\vec{T}(t_i)} \exp\left(\sigma^2_{m(i)} t_i\right) \\
\mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)} f_{t_j,\vec{T}(t_j)}\right] &= f_{0,\vec{T}(t_i)} f_{0,\vec{T}(t_j)} \exp\left(\int_0^{t_i} \rho_{m(i),m(j)} \sigma_{m(i)} \sigma_{m(j)} ds\right) \\
&= f_{0,\vec{T}(t_i)} f_{0,\vec{T}(t_j)} \exp\left(\rho_{m(i),m(j)} \sigma_{m(i)} \sigma_{m(j)} t_i\right).
\end{aligned}$$

Note that in the case where $\rho_{m(i),m(j)} = 1$ and $\sigma_{m(i)} = \sigma$ for all i, j the equations above reduce to $\mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)}\right] = f_{0,\vec{T}(t_i)}$, $\mathbf{E}^d\left[f^2_{t_i,\vec{T}(t_i)}\right] = f^2_{0,\vec{T}(t_i)} \exp\left(\sigma^2 t_i\right)$ and $\mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)} f_{t_j,\vec{T}(t_j)}\right] = f_{0,\vec{T}(t_i)} f_{0,\vec{T}(t_j)} \exp\left(\sigma^2 t_i\right)$ which are very reminiscent of $\mathbf{E}^d[S_t] = f_{0,t}$ and (2.121) in the preceding section.

The first moment is immediate by taking the expectation of (2.126), i.e.

$$\mathbf{E}^d\left[A^{A,f}_{t_1,t_n}\right] = \frac{1}{n} \sum_{i=1}^{n} f_{0,\vec{T}(t_i)} \tag{2.127}$$

while the second moment is given by

$$\begin{aligned}
\mathbf{E}^d\left[[A^{A,f}_{t_1,t_n}]^2\right] &= \mathbf{E}^d\left[\left(\frac{1}{n}\sum_{i=1}^{n} f_{t_i,\vec{T}(t_i)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} f_{t_j,\vec{T}(t_j)}\right)\right] \qquad (2.128) \\
&= \frac{1}{n^2}\mathbf{E}^d\left[\sum_{i,j=1}^{n} f_{t_i,\vec{T}(t_i)} f_{t_j,\vec{T}(t_j)}\right] \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n} \mathbf{E}^d\left[f_{t_i,\vec{T}(t_i)} f_{t_j,\vec{T}(t_j)}\right] \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n} f_{0,\vec{T}(t_i)} f_{0,\vec{T}(t_j)} \exp\left(\rho_{m(i),m(j)} \sigma_{m(i)} \sigma_{m(j)} t_{\min(\{i,j\})}\right).
\end{aligned}$$

From this we can use Black-76 with a volatility of

$$\sigma_A = \sqrt{\frac{1}{t_n}\left[\ln\frac{\mathbf{E}^d\left[\left[A^{A,f}_{t_1,t_n}\right]^2\right]}{\left(\mathbf{E}^d\left[\left[A^{A,f}_{t_1,t_n}\right]\right]\right)^2}\right]} \tag{2.129}$$

and an initial futures price $f_{0,T} = \mathbf{E}^d\left[\left[A^{A,f}_{t_1,t_n}\right]\right] = A^{A;f}_{0;\{t_1,t_n\}}$, using (2.127) and (2.128) together with the settlement adjustment from Section 2.5.3.

Other approximations are well known, which involve matching higher order moments. The practical benefit of these is somewhat limited and the two-moment Turnbull–Wakeman (TW2) approximation is very standard in the world of commodity derivatives.

Further reading on the topic of Asian options can be found in Kwok (1998) and Zhang (1998), together with original references including – but by no means limited – to Turnbull and Wakeman (1991), Levy (1992), Carverhill and Clewlow (1990), Levy and Turnbull (1992) and Curran (1994).

2.8 COMMODITY SWAPTIONS

Järvinen and Toivonen (2004) and Larsson (2011) introduce these products, which are not covered extensively in the literature by any means, but which are relevant particularly for the coal market.

An Asian option is an option on the arithmetic average which pays the positive component of the difference at expiry between the discretely monitored arithmetic average over the fixing times $\{t_1, \ldots, t_n\}$ and strike (subject to different sign convention for calls and puts). Note that the exercise decision is made at the point in time when the last fixing is determined, and generally settlement occurs five business days later.

In contrast, a swaption is an option on a forward swap which requires the holder to elect whether to exercise the swaption *before* the swap commences (unlike an Asian option, which is more like a European option referenced against an underlying swap). An Asian option is exercised (or not) once all the fixings have been determined, whereas the decision rule for a commodity swaption necessarily has to be with respect to an underlying forward swap.

Let $t_0 = T$ denote the expiry date of the swaption, and $t_1, \ldots, t_n$ the fixing dates for the underlying commodity swap, with $t_0 < t_1 < \ldots < t_n$.

Typically, the swap will be marked against the prompt futures, so each cashflow will be of the form $V_i = f_{t_i,\vec{T}(t_i)} - K$. We shall not concern ourselves here with swaptions referenced against spot, as these are hardly ever encountered in practice.

We can thereby consider (2.68) but at time $t_0 > 0$, i.e.

$$V_{t_0} = \frac{1}{n}\mathbf{E}^d\left[\sum_{i=1}^{n} e^{-r^d(T_{\mathrm{stl};i}-t_0)} \cdot V_i \middle| \mathcal{F}_{t_0}\right] \tag{2.130}$$

and split this into floating and fixed components, i.e.

$$V_{\mathrm{float}}(t_0) = \frac{1}{n}\sum_{i=1}^{n} e^{-r^d(T_{\mathrm{stl};i}-t_0)} f_{t_i,\vec{T}(t_i)}, \tag{2.131a}$$

$$V_{\mathrm{fixed}}(t_0) = \frac{1}{n}\sum_{i=1}^{n} e^{-r^d(T_{\mathrm{stl};i}-t_0)} K. \tag{2.131b}$$

A commodity payer swaption (or call on the underlying swap) then has value at t_0 equal to

$$V_{t_0}^{\mathrm{C}} = \max(V_{\mathrm{float}}(t_0) - V_{\mathrm{fixed}}(t_0), 0)$$

whereas a commodity receiver swaption (or put on the underlying swap) has value at t_0 equal to

$$V_{t_0}^{\mathrm{P}} = \max(V_{\mathrm{fixed}}(t_0) - V_{\mathrm{float}}(t_0), 0).$$

The terminology comes from the holder of the swaption having the right to pay a fixed price, or to receive a fixed price, which offsets against the floating price of the commodity in question.

This should be quite reminiscent of the Turnbull–Wakeman result in the previous section. Compare (2.131a) against (2.67b) and we see that $V_{\mathrm{float}}(t_0)$ is the same as $A_{t_1,t_n}^{A,f}$ except with n discount factors included.

In fact, it is not unusual for settlement to be rolled up into one cashflow at one particular time in the future (generally aligned with the settlement rules for swaps and Asians with the same fixing schedule), in which case we have

$$V_{\mathrm{float}}(t_0) = \frac{1}{n} e^{-r^d(T_{\mathrm{stl}}-t_0)} \sum_{i=1}^{n} f_{t_0,\vec{T}(t_i)} \tag{2.132a}$$

$$V_{\mathrm{fixed}}(t_0) = e^{-r^d(T_{\mathrm{stl}}-t_0)} K. \tag{2.132b}$$

The value of the commodity payer swaption at time t_0 is therefore

$$\begin{aligned}
V_{t_0}^{\mathrm{C}} &= \max(V_{\mathrm{float}}(t_0) - V_{\mathrm{fixed}}(t_0), 0) \\
&= e^{-r^d(T_{\mathrm{stl}}-t_0)} \max\left(\frac{1}{n}\sum_{i=1}^{n} f_{t_i,\vec{T}(t_i)} - K, 0\right) \\
&= e^{-r^d(T_{\mathrm{stl}}-t_0)} \max\left(A_{t_1,t_n}^{A,f} - K, 0\right)
\end{aligned}$$

with $A_{t_1,t_n}^{A,f}$ as defined in (2.67b).

We know from Section 2.7.4 that $A_{t_1,t_n}^{A,f}$ is approximately lognormally distributed, with mean $\mathbf{E}^d\left[[A_{t_1,t_n}^{A,f}]\right] = A_{0;\{t_1,t_n\}}^{A;f}$ and effective volatility σ_A as given in (2.129).

We then aim to price the swaption as

$$\begin{aligned}
V_0^C &= e^{-r^d t_0}\mathbf{E}^d\left[V_{t_0}^C\right] \\
&= e^{-r^d t_0}\mathbf{E}^d\left[e^{-r^d(T_{\mathrm{stl}}-t_0)} \max\left(A_{t_1,t_n}^{A,f} - K, 0\right)\right] \\
&= e^{-r^d t_0} e^{-r^d(T_{\mathrm{stl}}-t_0)}\mathbf{E}^d\left[\mathbf{E}^d\left[\max\left(A_{t_1,t_n}^{A;f} - K, 0\right)\middle|\mathcal{F}_{t_0}\right]\right] \\
&= e^{-r^d T_{\mathrm{stl}}}\mathbf{E}^d\left[\max\left(A_{t_0;\{t_1,t_n\}}^{A;f} - K, 0\right)\right].
\end{aligned}$$

From the above, we see that we need to construct the distribution of $A_{t_0;\{t_1,t_n\}}^{A;f}$ at time t_0. A useful practitioner's approximation is to use the Asian volatility σ_A as defined in (2.125) and to presume that this is applicable over the time interval $[0, t_0]$. Basically, this is saying that we use the Turnbull–Wakeman moment matching technique to estimate the mean and variance of the swap (upon which an Asian option depends), and then to presume that the instantaneous forward variance of the swap is evenly distributed over the time interval $[0, t_0]$.

An energy swaption can therefore be approximately valued under the presumption of a one-factor geometric Brownian motion as a European option, with the initial asset level equal to $A_{t_0;\{t_1,t_n\}}^{A;f}$, volatility set to σ_A, time to expiry of t_0, and delayed settlement to T_{stl}.

More complex and realistic techniques certainly exist, but are beyond the scope of this introductory discussion. We refer the reader to Järvinen and Toivonen (2004) and Larsson (2011), together with Riedhauser (2005a, 2005b) and Huang (2007).

2.9 SPREAD OPTIONS

Spread options are options which pay a certain amount according to the difference of two (or sometimes more) financial quantities. These quantities can either be futures contracts with different maturities on the same commodity, or futures contracts with the same maturity, but on different commodities. The first is known as an intra-commodity calendar spread option (often called calendar spread option, or CSO), the second is known as an inter-commodity futures spread (there are many frequently encountered flavours of these such as crack spreads, which will be introduced later in the text).

As such, the payout of a two-asset spread option at expiry is given by

$$V_T = \max\left(S_T^{(1)} - S_T^{(2)} - K, 0\right) \tag{2.133a}$$

where $S_T^{(1)}$ and $S_T^{(2)}$ are two financial quantities observed at time T.

This can, of course, be extended to a multi-asset spread option

$$V_T = \max\left(\sum_{i=1}^{n} \omega_i S_T^{(i)} - K, 0\right) \tag{2.133b}$$

where $\min_i\{\omega_i\} < 0$ and $\max_i\{\omega_i\} > 0$. Note that the presence of negative weights in (2.133b) is what differentiates spread options from basket options with strictly positive weights.

Note that the financial observables need not be stochastic all the way out to time T; for example, one can construct a forward starting option with the strike set at $t_1 < T$ as a special case of a calendar spread option, i.e. with $V_T = \max\left(S_T - S_{t_1} - K, 0\right)$, where clearly S_{t_1} is stochastic up to t_1 and then constant thereafter.

Further it is quite commonplace in commodities for the quantities to be arithmetic averages, for example, an Asian calendar spread option which pays the difference between two averages

$$V_T = \max\left(A_{t_1,t_{n1}}^{A,f} - A_{t_{n1+1},t_{n1+n2}}^{A,f} - K, 0\right) \tag{2.134}$$

or an inter-commodity Asian futures spread

$$V_T = \max\left(A_{t_1,t_n}^{A,f_1} - A_{t_1,t_n}^{A,f_2} - K, 0\right). \tag{2.135}$$

We shall restrict ourselves here to the case of two risky assets, described by the SDEs

$$dS_t^{(1)} = [r^d - r^{f;1}]S_t^{(1)}dt + \sigma^{(1)}S_t^{(1)}dW_t^{(1;d)} \tag{2.136a}$$

$$dS_t^{(2)} = [r^d - r^{f;2}]S_t^{(2)}dt + \sigma^{(2)}S_t^{(2)}dW_t^{(2;d)} \tag{2.136b}$$

where $W_t^{(1;d)}$ and $W_t^{(2;d)}$ are Brownian motions with respect to the domestic risk-neutral measure $\mathbf{P}^d$, subject to correlation $\langle dW_t^{(1;d)}, dW_t^{(2;d)}\rangle = \rho_{12;d}dt$.

2.9.1 Margrabe Exchange Options

The analysis is simplest in the case of two-asset spread options where $K = 0$. In this case we have the option to exchange the second asset and receive the first, with no extra payment being required. The payout at expiry, for an option to surrender the second risky asset and receive the first risky asset, is

$$V_T = \max\left(S_T^{(1)} - S_T^{(2)}, 0\right). \tag{2.137}$$

as discussed in Margrabe (1978).

By adopting the second asset as numeraire, this can be viewed as a call on the first asset – there is a nice parallel interpretation in the context of foreign exchange, which I discuss in Chapter 10 of Clark (2011). Pricing of this instrument proceeds exactly as in Section 2.2. Rubinstein (1991) quotes the price

$$V_0^{\text{exch}} = S_0^{(1)}e^{-r^{f;1}T}N(d_1) - S_0^{(2)}e^{-r^{f;2}T}N(d_2) \tag{2.138a}$$

where

$$d_{1;2} = \frac{\ln\left(\frac{S_0^{(1)}e^{-r^{f;1}T}}{S_0^{(2)}e^{-r^{f;2}T}}\right) \pm \frac{1}{2}[\sigma^{(1;2)}]^2T}{\sigma^{(1;2)}\sqrt{T}} \tag{2.138b}$$

with

$$[\sigma^{(1;2)}]^2 = [\sigma^{(1)}]^2 + [\sigma^{(2)}]^2 - 2\rho_{12;d}\sigma^{(1)}\sigma^{(2)}$$

through construction of the cross $S_t^{(1;2)} = S_t^{(1)}/S_t^{(2)}$.

The same result can be found in Equation (2) of Venkatramanan and Alexander (2011).

However, for commodity options, the presumption that $K = 0$ is too restrictive. We therefore continue by introducing a common approximation used in the energy markets for two-asset spread options: the Kirk approximation.

2.9.2 The Kirk Approximation

The Margrabe approach above works nicely because we are considering the difference between two lognormal quantities, namely $S_T^{(1)} - S_T^{(2)}$. If the strike is nonzero, we have

$$\begin{aligned} V_T &= \max\left(S_T^{(1)} - S_T^{(2)} - K, 0\right) \\ &= \max\left(S_T^{(1)} - (S_T^{(2)} + K), 0\right) \end{aligned}$$

The approximation of Kirk (1995) is that when $K \ll S_T^{(2)}$ we can regard $S_T^{(2)} + K$ as being approximately lognormal. The ratio of two lognormal processes X_t and Y_t is itself lognormal; suppose $\ln X_t \sim N(\mu_X, \sigma_X^2)$ and $\ln Y_t \sim N(\mu_Y, \sigma_Y^2)$ and $\mathbf{Cov}(\ln X_t, \ln Y_t) = \varrho\sigma_X\sigma_Y$. Note that ϱ is the terminal correlation between $\ln X_t$ and $\ln Y_t$, *not* the correlation between $S_t^{(1)}$ and $S_t^{(2)}$. We then construct the ratio $\ln(X_t/Y_t) = \ln X_t - \ln Y_t$ which gives $\ln(X_t/Y_t) \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2 - 2\varrho\sigma_X\sigma_Y)$.

Let us follow Venkatramanam and Alexander (2011) and define two new stochastic processes Y_t and Z_t by

$$Y_t = S_t^{(2)} + Ke^{-r^d(T-t)}, \tag{2.139a}$$

$$Z_t = S_t^{(1)}/Y_t, \tag{2.139b}$$

noting that at expiry, $Y_T = S_T^{(2)} + K$. With this, we can straightforwardly put

$$\begin{aligned} \max\left(S_T^{(1)} - (S_T^{(2)} + K), 0\right) &= \max\left(S_T^{(1)} - Y_T, 0\right) \\ &= \max\left(Y_T\left(\frac{S^{(1)}}{Y_T} - 1\right), 0\right) \\ &= Y_T \max\left(\left(Z_T - 1\right), 0\right). \end{aligned}$$

In fact it is shown in Venkatramanam and Alexander (2011) that

$$\frac{dZ_t}{Z_t} = (r^d - \tilde{r}^d - (r^{f;1} - \tilde{r}^{f;2}))dt + \sigma_Z dW_t^*$$

where $\tilde{r}^d = r^d \frac{S_t^{(2)}}{Y_t}$, $\tilde{r}^{f;2} = r^{f;2} \frac{S_t^{(2)}}{Y_t}$, $\tilde{\sigma}^{(2)} = \sigma^{(2)} \frac{S_t^{(2)}}{Y_t}$ and

$$\sigma_Z = \sqrt{[\sigma^{(1)}]^2 + [\tilde{\sigma}^{(2)}]^2 - 2\varrho_{12;d}\sigma^{(1)}\tilde{\sigma}^{(2)}}.$$

Note that W_t^* is a Brownian motion under a different probability measure $\mathbf{P}^*$ related to $\mathbf{P}^d$ by

$$\frac{d\mathbf{P}^*}{d\mathbf{P}^d} = \exp\left(-\frac{1}{2}\left[\tilde{\sigma}^{(2)}\right]^2 T + \tilde{\sigma}^{(2)} dW_t^{(2;d)}\right).$$

We can now construct the price for a spread option with payoff $V_T = (S_T^{(1)} - S_T^{(2)} - K)^+$

$$\begin{aligned} V_0 &= e^{-r^d T}\mathbf{E}^d\left[\max\left(S_T^{(1)} - S_T^{(2)} - K, 0\right)\right] \\ &= e^{-r^d T}\mathbf{E}^d\left[Y_T\left(Z_T - 1\right)^+\right] \\ &= e^{-r^d T}\mathbf{E}^d\left[(S_T^{(1)} - Y_T)^+\right]. \end{aligned}$$

Upon shifting to the measure corresponding to using Y_t as numeraire, which is just a portfolio of the second asset $S_T^{(2)}$ together with an amount of cash that FVs to amount K at time T, we obtain

$$V_0 = S_0^{(1)} e^{-r^{f;1}T} N\left(d_1^Z\right) - Y_0 e^{-(r^d - (\tilde{r}^d - \tilde{r}^{f;2}))T} N\left(d_2^Z\right)$$

where

$$d_2^Z = \frac{\ln Z_0 + (r^d - r^{f;1} - (\tilde{r}^d - \tilde{r}^{f;2}) - \frac{1}{2}\sigma_Z^2)T}{\sigma_Z\sqrt{T}} \tag{2.140a}$$

$$d_1^Z = d_2^Z + \sigma_Z\sqrt{T}. \tag{2.140b}$$

Note that if we kill the drift terms, as is standard in Black-76, we get

$$V_0 = e^{-r^d T}\left[F_{0,T}^{(1)} N\left(d_1^Z\right) - Y_0 N\left(d_2^Z\right)\right]$$

with $d_1^Z = (\ln Z_0 + \frac{1}{2}\sigma_Z^2 T)/\sigma_Z\sqrt{T}$ and $d_2^Z = (\ln Z_0 - \frac{1}{2}\sigma_Z^2 T)/\sigma_Z\sqrt{T}$. In this case $Z_0 = S_0^{(1)}/(S_0^{(2)} + K)$.

2.9.3 Calendar Spread Options

2.9.3.1 Calendar Spread Options on Spot

The analysis above presumes that we have two separate but correlated assets which are presumed to evolve stochastically over $[0, T]$, and we have payoff $V_T = (S_T^{(1)} - S_T^{(2)} - K)^+$. It is quite common in commodities to have options which depend on the asset values for a single commodity, but evaluated at two particular times, e.g.

$$V_T = (S_T - S_{t_1} - K)^+ \tag{2.141}$$

where $t_1 < T$. In the case where $K = 0$, this reduces to a forward starting option, for which a closed form expression is quoted in Equation (8.5) of Zhang (1998)

$$V_0 = S_0 \left[e^{-r^f T} N(d_1^{\text{fwd}}) - e^{-r^d(T-t_1)-r^f t_1} N\left(d_2^{\text{fwd}}\right) \right], \tag{2.142}$$

with

$$d_1^{\text{fwd}} = d_2^{\text{fwd}} + \sigma\sqrt{T - t_1} \tag{2.143a}$$

$$d_2^{\text{fwd}} = \frac{r^d - r^f - \frac{1}{2}\sigma^2}{\sigma}\sqrt{T - t_1}. \tag{2.143b}$$

Now we can identify S_T with $S_T^{(1)}$ and S_{t_1} with $S_T^{(2)}$ for a stochastic process $S_t^{(2)}$ that has volatility $\sigma^{(2)}$ for $0 \le t \le t_1$, and zero volatility and drift thereafter. Note that $S_T^{(2)}$ has total variance $[\sigma^{(2)}]^2 t_1$, which is the same as $\left[\bar{\sigma}^{(2)}\right]^2 T$ for $\bar{\sigma}^{(2)} = \sigma^{(2)}\sqrt{t_1/T}$. This means we can assume (these products not being path dependent) that (2.141) can be expressed equivalently as

$$V_T = (S_T^{(1)} - S_T^{(2)} - K)^+. \tag{2.144}$$

Now we can apply the Kirk approximation but with $\bar{\sigma}^{(2)} = \sigma^{(2)}\sqrt{t_1/T}$ used in (2.136b).

2.9.3.2 Calendar Spread Options on Futures

Of course, if we are modelling driftless futures, then in reality the futures contracts being observed at the two dates t_1 and T will be the two corresponding to whichever will be the prompt future at t_1 and T. We already have the notation $f_{t,\vec{T}(t_1)}$ and $f_{t,\vec{T}(T)}$ for these. Let us suppose that the volatilities for each of these two can be denoted by $\sigma^{(1)}$ and $\sigma^{(2)}$ respectively, in which case the Kirk approximation is still applicable, but with initial asset levels $f_{0,\vec{T}(t_1)}$ and $f_{0,\vec{T}(T)}$ instead of $S_0^{(1)}$ and $S_0^{(2)}$.

2.9.4 Asian Spread Options

In reality, in the commodities markets, calendar spread options are rarely determined according to the difference of prompt futures at only two dates. More often, the spread is computed by taking the arithmetic average over the far month, minus the arithmetic average over the near month. This, as introduced in (2.134), can be expressed as

$$V_T = \max\left(A^{A,f}_{t_1,t_{n_1}} - A^{A,f}_{t_{n_1+1},t_{n_1+n_2}} - K, 0\right) \tag{2.145}$$

with

$$A^{A,f}_{t_1,t_{n_1}} = \frac{1}{n_1}\sum_{i=1}^{n_1} f_{t_i,\vec{T}(t_i)}, \text{ and} \tag{2.146a}$$

$$A^{A,f}_{t_{n_1+1},t_{n_1+n_2}} = \frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2} f_{t_i,\vec{T}(t_i)}. \tag{2.146b}$$

We can quite straightforwardly apply the Turnbull–Wakeman adjustment of Section 2.7.4 to (2.146a) and (2.146b), obtaining effective Asian volatilities $\sigma^{(1)}$ and $\sigma^{(2)}$. These can then be subjected to the same approach as detailed in Section 2.9.3 – noting that it is the short end volatility, i.e. $\sigma^{(1)}$ that needs to be transformed into $\bar{\sigma}^{(1)} = \sigma^{(1)}\sqrt{(t_n/T)}$.

We shall provide a case study example on WTI oil calendar spreads in Chapter 5.

For intra-commodity spread options with payoff as described in (2.135), we just use the moment matching approach without needing to adjust the near end volatility, but where the near and far end terms will be driven off different futures curves and volatilities.

See Sections 6.6 and 7.5 for how these spreads arise in practice in commodity markets.

2.10 MORE ADVANCED MODELS

The products we have presented in this chapter have, so far, all been discussed solely with reference to the standard Black-76 and Black–Scholes analysis. Our intent in this chapter has been to provide the reader with a technical introduction which he or she can use to guide the development of more complex models, under which these products (and more) can be priced. Let us sketch some directions in which work can proceed.

There are many shortcomings to the use of geometric Brownian motion for commodity option pricing. The first is the absence of volatility smile. This can be remedied for simple products by looking up the volatility σ from an implied volatility surface $\sigma_{\text{imp}}(K, T)$ – depending on the strike K and the maturity (or fixing time) T. For more complicated path-dependent products, we should properly construct a local volatility surface or engage a stochastic volatility (or local stochastic volatility) model, though this is complicated when one needs to model several futures contracts.

2.10.1 Mean Reverting Models

The second problem with pure geometric Brownian motion is the absence of mean reversion. It is well known that many commodities experience some degree of seasonality – and while it is clearly possible to avoid having to deal with this by just pricing using today's future curve (incorporating seasonal expectations) as a reference, and working with a family of futures curves $f_{t,T_1}, \ldots, f_{t,T_n}$, this requires estimation of the correlation matrix connecting the stochastic evolution of all of these tradeables.

2.10.1.1 The Schwartz (1997) One-Factor Model

Schwartz (1997) introduces a one-factor model ("Model 1" in the original paper) that incorporates mean reversion into spot price, expressed in mean reversion in logspot.[9] The process followed is

$$dS_t = \kappa(\alpha_S - \ln S_t)S_t dt + \sigma S_t dW_t \tag{2.147a}$$

or equivalently, for $X_t = \ln S_t$

$$dX_t = \kappa(\alpha_X - X_t)dt + \sigma dW_t \tag{2.147b}$$

with $\alpha_X = \alpha_S - \sigma^2/2\kappa$. The proof of this is a simple exercise in Itô calculus. Put $X_t = f(S_t)$ with $f(x) = \ln x$, where $f'(x) = x^{-1}$ and $f''(x) = -x^{-2}$.

[9] Pilipović (1998) discusses in Section 4.3.2.2 a model where the mean reversion is in terms of spot rather than logspot.

From Itô, we obtain $dX_t = f'(S_t)dS_t + \frac{1}{2}f''(S_t)dS_t^2$. But from (2.147a) we have $dS_t/S_t = \kappa(\alpha_S - X_t)dt + \sigma dW_t$ and therefore $dS_t^2/S_t^2 = \sigma^2 dt$. We thereby have $dX_t = \kappa(\alpha_S - X_t)dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$ which can be written as (2.147b) with the substitution $\alpha_X = \alpha_S - \sigma^2/2\kappa$.

Note that (2.147b) is the same SDE as encountered in the Vasicek model, in the context of interest rate modelling

$$dr_t = (\theta - \kappa r_t)dt + \sigma dW_t. \tag{2.148}$$

The intent here is to capture the general trend that when commodity spot prices rise, the futures curves tend to experience backwardation, and low spot prices more often occur together with contango markets – this basically being the observation that the short end of the futures curve fluctuates a lot more than the long end. This model pulls the spot price back to the long term level $S_\infty = \exp(\alpha_S)$.

We introduce a stochastic integrating factor $\exp(\kappa t)$, and write

$$\hat{X}_t = \exp(\kappa t)X_t$$

under which we have, by the Itô product rule,

$$\begin{aligned}
d\hat{X}_t &= e^{\kappa t}dX_t + \kappa e^{\kappa t}X_t dt \\
&= e^{\kappa t}dX_t + \kappa\hat{X}_t dt \\
&= e^{\kappa t}(\kappa(\alpha_X - X_t)dt + \sigma dW_t) + \kappa\hat{X}_t dt \\
&= \kappa e^{\kappa t}[\alpha_X dt - X_t dt] + \sigma e^{\kappa t}dW_t + \kappa\hat{X}_t dt \\
&= \kappa e^{\kappa t}\alpha_X dt - \kappa e^{\kappa t}X_t dt + \sigma e^{\kappa t}dW_t + \kappa\hat{X}_t dt \\
&= \kappa e^{\kappa t}\alpha_X dt - \kappa\hat{X}_t dt + \sigma e^{\kappa t}dW_t + \kappa\hat{X}_t dt \\
&= \kappa e^{\kappa t}\alpha_X dt + \sigma e^{\kappa t}dW_t.
\end{aligned}$$

This can of course be integrated. We have

$$\begin{aligned}
\hat{X}_T &= \hat{X}_0 + \int_0^T d\hat{X}_t \\
&= \hat{X}_0 + \kappa\alpha_X \int_0^T e^{\kappa t}dt + \sigma\int_0^T e^{\kappa t}dW_t \\
&= \hat{X}_0 + \alpha_X[e^{\kappa T} - 1] + \sigma\int_0^T e^{\kappa t}dW_t.
\end{aligned}$$

Consider now the integral $\sigma\int_0^T e^{\kappa t}dW_t$. Since Brownian motion is driftless, this integral is a martingale with expectation equal to zero. We can compute the variance of $\int_0^T e^{\kappa t} dW_t$, however, by use of the Itô isometry, namely

$$\mathbf{E}\left[\left(\int_0^T \Phi_t dW_t\right)^2\right] = \mathbf{E}\left[\int_0^T \Phi_t^2 dt\right]$$

where Φ_t is an adapted process. Putting $\Phi_t = e^{\kappa t}$, which is deterministic and therefore trivially adapted, we have no need of expectations and can simply compute

$$\int_0^T e^{2\kappa t} dt = \left[\frac{e^{2\kappa t}}{2\kappa}\right]_{t=0}^T = \frac{e^{2\kappa T} - 1}{2\kappa}.$$

We therefore have (note that $\hat{X}_0 = X_0$)

$$\mathbf{E}\left[\hat{X}_T\right] = X_0 + \alpha_X[e^{\kappa T} - 1] \tag{2.149a}$$

$$\mathbf{Var}\left[\hat{X}_T\right] = \frac{\sigma^2}{2\kappa}\left[e^{2\kappa T} - 1\right]. \tag{2.149b}$$

Finally, since $X_T = e^{-\kappa T}\hat{X}_T$, we have

$$\mathbf{E}\left[X_T\right] = e^{-\kappa T}\mathbf{E}\left[\hat{X}_T\right] = X_0 e^{-\kappa T} + \alpha_X[1 - e^{-\kappa T}] \tag{2.150a}$$

and

$$\mathbf{Var}\left[X_T\right] = e^{-2\kappa T}\mathbf{Var}\left[\hat{X}_T\right] = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa T}\right]. \tag{2.150b}$$

Note that this gives rise to a lognormally distributed S_T, i.e. X_T has a lognormal distribution characterised by first and second moments which can be used in exactly the same manner as the moments arising from a standard geometric Brownian motion for option pricing. We have

$$V_0^{C/P} = \omega e^{-r^d T}[F_{0,T}N(\omega d_1) - KN(\omega d_2)] \tag{2.151a}$$

with

$$d_{1,2} = \frac{\ln\left(F_{0,T}/K\right) \pm \frac{1}{2}\sigma_{0,T}^2 T}{\sigma_{0,T}\sqrt{T}}. \tag{2.151b}$$

where, from (2.150b),

$$\sigma_{0,T} = \sqrt{\frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa T}\right]}. \tag{2.151c}$$

To match T-forward prices, i.e. to choose α_X such that $\mathbf{E}^d\left[S_T\right] = F_{0,T}$, we use the result quoted in (3.11) of Burger et al. (2007)

$$\mathbf{E}\left[S_T\right] = \exp\left(\mathbf{E}\left[X_T\right] + \frac{1}{2}\mathbf{Var}\left[X_T\right]\right). \tag{2.152}$$

Substituting (2.150) into (2.152), we obtain

$$\mathbf{E}\left[S_T\right] = \exp\left(X_0 e^{-\kappa T} + \alpha_X[1 - e^{-\kappa T}] + \frac{\sigma^2}{4\kappa}\left[1 - e^{-2\kappa T}\right]\right). \tag{2.153}$$

One final adjustment is required, to convert the expectation to the expectation under the risk-neutral measure. If we let λ denote the market price of risk, as in Burger *et al.* (2007), then we write $\hat{\alpha}_X = \alpha_X - \lambda/\kappa$ and then we have

$$\mathbf{E}^d\left[S_T\right] = \exp\left(X_0 e^{-\kappa T} + \hat{\alpha}_X[1 - e^{-\kappa T}] + \frac{\sigma^2}{4\kappa}\left[1 - e^{-2\kappa T}\right]\right). \tag{2.154}$$

Taking the limit as $T \to \infty$, we have

$$\begin{aligned}\lim_{T\to\infty} F_{0,T} &= \exp\left(\hat{\alpha}_X + \frac{\sigma^2}{4\kappa}\right) \\ &= \exp\left(\alpha_S - \frac{\sigma^2}{4\kappa} - \frac{\lambda}{\kappa}\right)\end{aligned} \tag{2.155}$$

while for finite T we have

$$F_{0,T} = [S_0]^{e^{-\kappa T}} \cdot \exp\left(\hat{\alpha}_X[1 - e^{-\kappa T}] + \frac{\sigma^2}{4\kappa}\left[1 - e^{-2\kappa T}\right]\right) \tag{2.156}$$

and more generally

$$F_{t,T} = [S_0]^{e^{-\kappa(T-t)}} \cdot \exp\left(\hat{\alpha}_X[1 - e^{-\kappa(T-t)}] + \frac{\sigma^2}{4\kappa}\left[1 - e^{-2\kappa(T-t)}\right]\right). \tag{2.157}$$

If the domestic rate r^d and convenience yield r^f to time T are known, then we can write $F_{0,T} = S_0 e^{(r^d - r^f)T}$ and solve (2.156), obtaining

$$\hat{\alpha}_X = \ln S_0 + \frac{\left[(r^d - r^f)T + \frac{\sigma^2}{4\kappa}\left[e^{-2\kappa T} - 1\right]\right]}{1 - e^{-\kappa T}}. \tag{2.158}$$

Note that $\sigma^2_{0,T}$ is the integrated variance from 0 to T. One can follow Clewlow and Strickland (2000) and Geman (2005) in differentiating (2.157) to obtain the instantaneous proportional volatility of $F_{t,T}$, i.e. that quantity $\sigma_F(t, T)$ such that $\frac{dF_{t,T}}{F_{t,T}} = \sigma_F(t, T) dW_t^F$ for a suitably chosen Brownian motion W_t^F.

Alternatively, one can recognise that $\sigma_{0,T}^2 = \int_0^T \sigma_F^2(t,T)dt$, or more generally, $\sigma_{t,T}^2 = \int_t^T \sigma_F^2(u,T)du$ with

$$\sigma_{t,T}^2 = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(T-t)}\right]. \tag{2.159}$$

Differentiating under the integral sign, we have

$$\frac{\partial}{\partial t}\int_t^T \sigma_{t,T}^2 du = -\sigma_F^2(t,T).$$

However, using (2.189) directly we have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_t^T \sigma_{t,T}^2 du &= \frac{\partial}{\partial t}\left[\frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(T-t)}\right]\right] \\
&= \frac{\sigma^2}{2\kappa}\frac{\partial}{\partial t}\left[1 - e^{-2\kappa(T-t)}\right] \\
&= -\frac{\sigma^2}{2\kappa}e^{-2\kappa T}\frac{\partial}{\partial t}\left[e^{2\kappa t}\right] \\
&= -\frac{\sigma^2}{2\kappa}e^{-2\kappa T}2\kappa e^{2\kappa t} \\
&= -\sigma^2 e^{-2\kappa(T-t)}.
\end{aligned}$$

Equating, we have $\sigma_F^2(t,T) = \sigma^2 e^{-2\kappa(T-t)}$, from which the result follows

$$\sigma_F(t,T) = \sigma e^{-\kappa(T-t)}. \tag{2.160}$$

Note that this is consistent with the Samuelson effect, where forward contracts of greater maturity are often observed to have smaller volatilities than shorter dated forward contracts.

2.10.1.2 Mean Reverting One-Factor Models with Term Structure

Mean reverting models are also discussed in Section 33.4 of Hull (2011), where the mean reversion level is allowed to have a term structure, i.e.

$$dX_t = (\theta_t - \kappa X_t)dt + \sigma dW_t \tag{2.161}$$

This is analogous to the Hull–White model in interest rate modelling, and similar techniques can be used to fit θ_t to the term structure of observed futures prices. This can be done on trees, as discussed in Hull and White (1996) in the context of interest rate derivatives, and is also discussed in the commodities context in Section 33.4 of Hull (2011), where the logspot process is modelled as the sum of a process mean reverting around zero, plus a time-dependent drift.

This approach is often known as the "deterministic shift decomposition," and is described in Ludkovski and Carmona (2004) in the context of adding term structure to the mean reversion level of the Gibson–Schwartz model. For the Schwartz (1997) one-factor model, a term structure of mean reversion level is handled by integrating (2.161). This proceeds using the same stochastic integrating factor technique. Write $\hat{X}_t = \exp(\kappa t)X_t$ and employ $d\hat{X}_t = e^{\kappa t}dX_t + \kappa e^{\kappa t}X_t dt$. Using (2.161) we obtain

$$d\hat{X}_t = e^{\kappa t}\theta_t dt + \sigma e^{\kappa t} dW_t.$$

Integrating from 0 to T we have

$$\hat{X}_T = \hat{X}_0 + \int_0^T e^{\kappa t}\theta_t dt + \int_0^T \sigma e^{\kappa t} dW_t.$$

Multiplying by $e^{-\kappa T}$ we obtain

$$X_T = X_0 e^{-\kappa T} + \int_0^T e^{-\kappa(T-t)}\theta_t dt + \sigma \int_0^T e^{-\kappa(T-t)} dW_t. \tag{2.162}$$

The Itô integral $\int_0^T e^{-\kappa(T-t)} dW_t$ is once again a martingale, from which we have

$$\mathbf{E}\left[X_T\right] = X_0 e^{-\kappa T} + \int_0^T e^{-\kappa(T-t)}\theta_t dt. \tag{2.163a}$$

Since no term structure for κ nor σ is supposed at this stage, we still have, from (2.150b),

$$\mathbf{Var}\left[X_T\right] = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa T}\right]. \tag{2.163b}$$

Alternatively, let us now define A_t by

$$dA_t = -\kappa A_t dt + \sigma dW_t, \tag{2.164}$$

with $A_0 = 0$, and let us attempt to decompose X_t into the sum $X_t = A_t + \alpha_t$. We have $\alpha_t = X_t - A_t$, so

$$\begin{aligned} d\alpha_t &= dX_t - dA_t \\ &= (\theta_t - \kappa X_t)dt + \sigma dW_t + \kappa A_t dt - \sigma dW_t \\ &= (\theta_t - \kappa(X_t - A_t))dt \\ &= (\theta_t - \kappa\alpha_t)dt \end{aligned} \tag{2.165}$$

which is clearly nonstochastic. Introduce an integrating factor $e^{\kappa t}$ and write $\hat{\alpha}_t = \alpha_t e^{\kappa t}$, as a result of which we have $d\hat{\alpha}_t = e^{\kappa t} d\alpha_t + \kappa e^{\kappa t} \alpha_t dt = e^{\kappa t}[d\alpha_t + \kappa \alpha_t dt]$. From (2.165) we know $d\alpha_t + \kappa \alpha_t dt = \theta_t dt$, so we need only solve $d\hat{\alpha}_t = e^{\kappa t} \theta_t dt$. Integration of this is simple, we have

$$\hat{\alpha}_t = \hat{\alpha}_0 + \int_0^t e^{\kappa s} \theta_s ds$$

or, since $\alpha_t = \hat{\alpha}_t e^{-\kappa t}$ and $\alpha_0 = X_0$,

$$\alpha_t = X_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-s)} \theta_s ds. \tag{2.166}$$

Since $\mathbf{E}\left[A_t\right] = 0$, we have $\mathbf{E}\left[X_t\right] = \alpha_t$, which is consistent with (2.163a).

We can now use (2.152) together with (2.163) to compute $\mathbf{E}^d\left[S_T\right]$ and infer suitable values for θ_t in order to solve $\mathbf{E}^d\left[S_T\right] = F_{0,T}$ and thereby recover correct forward prices $F_{0,T}$.

2.10.1.3 The Clewlow–Strickland (1999) One-Factor Model for Forwards

There are two approaches, basically, to modelling commodities, as we already know. One is to model an underlying spot price process[10] and then to infer the dynamics of the forwards. The second approach is to model the forward curve directly. This is the approach introduced in Clewlow and Strickland (1999a), who proposed a one-factor model to describe the stochastic evolution of the forward curve. A general form of the one-factor model can be written as

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma_F(t, T) dW_t. \tag{2.167}$$

What the authors recognised was that, if the instantaneous proportional volatility of $F_{t,T}$ under the one-factor Schwartz (1997) model is known to be (2.160), then the same volatility specification can be used to build a forward curve based model, but where the initial forward curve can be specified exogenously rather than needing to be determined using today's spot and model parameters.

[10] Whether it exists or not is another story.

Under this specific model, we have

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma_F(t, T)dW_t = \sigma e^{-\kappa(T-t)}dW_t. \tag{2.168}$$

Note that, unlike the one-factor Schwartz (1997) model, the only required parameters are the short term volatility σ and the mean reversion rate κ. One important observation: the mean reversion parameter κ is *embedded* in the volatility function, there is no actual stochastic mean reversion term in (2.168).

We follow Section 8.5 of Clewlow and Strickland (2000) and integrate (2.167). Squaring this gives

$$\frac{dF_{t,T}^2}{F_{t,T}^2} = \sigma_F^2(t, T)dt. \tag{2.169}$$

Simple application of Itô's lemma to $\ln F_{t,T} = f(F_{t,T})$ with $f(x) = \ln x$ gives

$$\begin{aligned} d\ln F_{t,T} &= \frac{dF_{t,T}}{F_{t,T}} - \frac{1}{2}\frac{dF_{t,T}^2}{F_{t,T}^2} \\ &= \sigma_F(t, T)dW_t - \frac{1}{2}\sigma_F^2(t, T)dt. \end{aligned}$$

We integrate this expression, obtaining

$$\ln F_{t,T} = \ln F_{0,T} - \frac{1}{2}\int_0^t \sigma_F^2(s, T)ds + \int_0^t \sigma_F(s, T)dW_s. \tag{2.170}$$

If one sets $T = t$ then (2.170) reduces to

$$\ln S_t = \ln F_{t,t} = \ln F_{0,t} - \frac{1}{2}\int_0^t \sigma_F^2(s, t)ds + \int_0^t \sigma_F(s, t)dW_s. \tag{2.171}$$

i.e.

$$\begin{aligned} S_t &= F_{0,t} \cdot \exp\left(-\frac{1}{2}\int_0^t \sigma_F^2(s, t)ds + \int_0^t \sigma_F(s, t)dW_s\right) \\ &= F_{0,t} \cdot \exp\left(I_t\right) \end{aligned} \tag{2.172}$$

with $I_t = -\frac{1}{2}\int_0^t \sigma_F^2(s, t)ds + \int_0^t \sigma_F(s, t)dW_s$. We can now write

$$\begin{aligned} dS_t &= dF_{0,t} \cdot \exp\left(I_t\right) + F_{0,t} \cdot d\exp\left(I_t\right) \\ &= dF_{0,t}\frac{S_t}{F_{0,t}} + F_{0,t}\exp\left(I_t\right) \cdot d[I_t] \end{aligned}$$

from which we obtain

$$\begin{aligned}\frac{dS_t}{S_t} &= \frac{dF_{0,t}}{F_{0,t}} + d[I_t] \\ &= \frac{1}{F_{0,t}}\frac{\partial F_{0,t}}{\partial t}dt + d[I_t] \\ &= \frac{\partial \ln F_{0,t}}{\partial t}dt + d[I_t].\end{aligned}$$

It remains to calculate

$$\begin{aligned}d[I_t] &= d\left(-\frac{1}{2}\int_0^t \sigma_F^2(s,t)ds + \int_0^t \sigma_F(s,t)dW_s\right) \\ &= \left[-\int_0^t \sigma_F(s,t)\frac{\partial \sigma_F(s,t)}{\partial t}ds + \int_0^t \frac{\partial \sigma_F(s,t)}{\partial t}dW_s\right]dt + \sigma_F(t,t)dW_t\end{aligned}$$

where the second line follows via differentiating under the integral sign. We therefore obtain (A.4) from Clewlow and Strickland (1999a), i.e.

$$\begin{aligned}\frac{dS_t}{S_t} = &\left[\frac{\partial \ln F_{0,t}}{\partial t} - \int_0^t \sigma_F(s,t)\frac{\partial \sigma_F(s,t)}{\partial t}ds\right. \\ &\left. + \int_0^t \frac{\partial \sigma_F(s,t)}{\partial t}dW_s\right]dt + \sigma_F(t,t)dW_t. \end{aligned} \tag{2.173}$$

It is shown in the original paper that (2.173), with $\sigma_F(t,T) = \sigma e^{-\kappa(T-t)}$, is equivalent to

$$\frac{dS_t}{S_t} = [\mu_t - \kappa \ln S_t]dt + \sigma dW_t \tag{2.174a}$$

with

$$\mu_t = \frac{\partial \ln F_{0,t}}{\partial t} + \kappa \ln F_{0,t} + \frac{\sigma^2}{4}\left(1 - e^{-2\kappa t}\right). \tag{2.174b}$$

Note that, as a result, the drift term in (2.174a) is obtained implicitly in terms of the initial forward curve.

Option pricing under this model is straightforward; the same equations (2.197) as for the Schwartz model are applicable, but where $F_{0,T}$ are obtained directly from the market and there is no need to calibrate a mean reversion level to the forwards.

The mean reversion rate κ and volatility σ can now easily be imbued with a term structure (i.e. κ_t and σ_t), the only computation required being the integral

$$\sigma_{0,T} = \sqrt{\int_0^T \sigma_t e^{-2\kappa_T(T-t)} dt}. \tag{2.175}$$

2.10.2 Multi-Factor Models

Single factor models, while simple, are inadequate for several reasons. Firstly, they can only describe perfect correlation between non-coinciding forward contracts. As discussed in Chapter 5 of Eydeland and Wolyniec (2003), even a generic one-factor forward model of the form $dF_{t,T} = \sigma(t,T)F_{t,T}dW_t$ has $\rho(t, T_1, T_2) = 1$ describing the correlation between $\ln F_{t,T_1}$ and $\ln F_{t,T_2}$.

Secondarily, only a limited variety of volatility term structures can be captured, all of which decay to zero for increasingly long-dated forwards.

2.10.2.1 The Schwartz–Smith (2000) Two-Factor Model

In order to tackle some of these concerns, Schwartz and Smith (2000) introduced a two-factor commodities spot model, under which logspot is described as the sum of two factors

$$\ln S_t = \chi_t + \xi_t$$

where

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dW_t^\chi \tag{2.176a}$$

$$d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^\xi \tag{2.176b}$$

and $\langle dW_t^\chi, dW_t^\xi \rangle = \rho dt$.

The first factor χ_t is a short-term mean reverting factor analogous to the Schwartz (1997) one-factor model, as is particularly evident from comparing (2.176a) to (2.164), while the second factor ξ_t captures the long-term dynamics. While previously we had $X_t = A_t + \alpha_t$ with α_t being deterministic, now we have $X_t = \chi_t + \xi_t$ with ξ_t also being stochastic. This is a two-factor model, driven by two correlated Brownian motions W_t^χ and W_t^ξ.

From $X_t = \ln S_t = \chi_t + \xi_t$, by summation of the two stochastic differential components in (2.176) we have

$$\begin{aligned} dX_t &= (\mu_\xi - \kappa\chi_t)dt + \sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi \\ &= (\mu_\xi - \kappa(X_t - \xi_t))dt + \sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi \\ &= (\mu_\xi + \kappa\xi_t - \kappa X_t)dt + \sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi. \end{aligned} \tag{2.177}$$

If we add a constant offset to ξ_t, making $\theta_t = \xi_t + \mu_\xi/\kappa$, then clearly $d\xi_t$ obeys the same SDE as (2.176b), i.e.

$$d\theta_t = \mu_\xi dt + \sigma_\xi dW_t^\xi \tag{2.178}$$

and we can write (2.177) as

$$\begin{aligned} dX_t &= (\mu_\xi + \kappa\xi_t - \kappa X_t)dt + \sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi \\ &= \kappa(\theta_t - X_t)dt + \sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi \\ &= \kappa(\theta_t - X_t)dt + \sigma_X dW_t^X \end{aligned} \tag{2.179}$$

where $W_t^X = \frac{1}{\sigma_X}[\sigma_\chi dW_t^\chi + \sigma_\xi dW_t^\xi]$. Consequently, the Schwartz–Smith two-factor model can be seen to be equivalent to a mean-reverting model, but where the mean reversion level is itself stochastic.

The original paper derives the following expressions for the first and second moments of χ_t and ξ_t:

$$\mathbf{E}\left[\chi_t\right] = e^{-\kappa t}\chi_0 \tag{2.180a}$$

$$\mathbf{E}\left[\xi_t\right] = \xi_0 + \mu_\xi t \tag{2.180b}$$

and

$$\mathbf{Cov}\left[\chi_t, \xi_t\right] = \begin{bmatrix} \left(1 - e^{-2\kappa t}\right)\frac{\sigma_\chi^2}{2\kappa} & \left(1 - e^{-\kappa t}\right)\frac{\rho\sigma_\chi\sigma_\xi}{\kappa} \\ \left(1 - e^{-\kappa t}\right)\frac{\rho\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2 t \end{bmatrix} \tag{2.181}$$

from which one can derive the result that X_T is normally distributed with

$$\mathbf{E}\left[X_T\right] = \xi_0 + \mu_\xi T + e^{-\kappa T}\chi_0 \tag{2.182a}$$

$$\mathbf{Var}\left[X_T\right] = \left(1 - e^{-2\kappa T}\right)\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2\left(1 - e^{-\kappa T}\right)\frac{\rho\sigma_\chi\sigma_\xi}{\kappa}. \tag{2.182b}$$

Equation (2.152) can then be used to express the first and second moments of the lognormal distribution describing S_T. Note, however, that for pricing one needs to correct for the market price of risk. This can be done by introducing λ_χ and λ_ξ (market risk factors) and putting $\hat{\mu}_\chi = -\lambda_\chi$ and $\hat{\mu}_\xi = \mu_\xi - \lambda_\xi$, then writing

$$d\chi_t = \left(\hat{\mu}_\chi - \kappa\chi_t\right) dt + \sigma_\chi dW_t^{(\chi;d)} \tag{2.183a}$$

$$d\xi_t = \hat{\mu}_\xi dt + \sigma_\xi dW_t^{(\xi;d)} \tag{2.183b}$$

where we use $W_t^{(\chi;d)}$ and $W_t^{(\xi;d)}$ to denote Brownian motions under the domestic risk-neutral measure. Note that the drift adjustment only affects the first moment, i.e.

$$\mathbf{E}^d\left[X_T\right] = \xi_0 + \hat{\mu}_\xi T + e^{-\kappa T}\chi_0 + \frac{\hat{\mu}_\chi}{\kappa}(1 - e^{-\kappa T}) \tag{2.184a}$$

$$\mathbf{Var}^d\left[X_T\right] = \left(1 - e^{-2\kappa T}\right)\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2\left(1 - e^{-\kappa T}\right)\frac{\rho\sigma_\chi\sigma_\xi}{\kappa}. \tag{2.184b}$$

We can obtain the result in (2.184a) using the familiar stochastic integrating factor technique. From (2.183a), write

$$d\chi_t + \kappa\chi_t dt = \hat{\mu}_\chi dt + \sigma_\chi dW_t^{(\chi;d)}.$$

By putting $\hat{\chi}_t = e^{\kappa t}\chi_t$ we have

$$d[\hat{\chi}_t] = d[e^{\kappa t}\chi_t] = e^{\kappa t}d\chi_t + \kappa e^{\kappa t}\chi_t dt = \hat{\mu}_\chi e^{\kappa t}dt + \sigma_\chi e^{\kappa t}dW_t^{(\chi;d)} \tag{2.185}$$

Integrating (2.185) we have

$$\begin{aligned}\hat{\chi}_T &= \hat{\chi}_0 + \hat{\mu}_\chi\int_0^T e^{\kappa t}dt + \sigma_\chi\int_0^T e^{\kappa t}dW_t^{(\chi;d)} \\ &= \hat{\chi}_0 + \frac{\hat{\mu}_\chi}{\kappa}\left[e^{\kappa T} - 1\right] + \sigma_\chi\int_0^T e^{\kappa t}dW_t^{(\chi;d)}.\end{aligned} \tag{2.186}$$

Since the Itô integral has zero expectation, we have

$$\mathbf{E}^d\left[\hat{\chi}_T\right] = \hat{\chi}_0 + \frac{\hat{\mu}_\chi}{\kappa}\left[e^{\kappa T} - 1\right]$$

and, after multiplying by $e^{-\kappa T}$ (since $\chi_T = e^{-\kappa T}\hat{\chi}_T$) we obtain

$$\mathbf{E}^d\left[\chi_T\right] = \chi_0 e^{-\kappa T} + \frac{\hat{\mu}_\chi}{\kappa}\left[1 - e^{-\kappa T}\right]. \tag{2.187}$$

Since $\mathbf{E}^d\left[\xi_T\right] = \xi_0 + \hat{\mu}_\xi T$ by straightforward integration of (2.183b), the result (2.184a) follows. We can therefore obtain the prices for forwards in the Schwartz–Smith (2000) two-factor model

$$\begin{aligned}F_{0,T} &= \mathbf{E}^d\left[S_T\right] = \exp\left(\mathbf{E}^d\left[X_T\right] + \frac{1}{2}\mathbf{Var}^d\left[X_T\right]\right)\\ &= \exp\Bigg(\xi_0 + \chi_0 e^{-\kappa T} + \left(\mu_\xi - \lambda_\xi + \frac{1}{2}\sigma_\xi^2\right)T\\ &\quad + \frac{\rho\sigma_\chi\sigma_\xi - \lambda_\chi}{\kappa}\left(1 - e^{-\kappa T}\right) + \frac{\sigma_\chi^2}{4\kappa}\left(1 - e^{-2\kappa T}\right)\Bigg). \qquad (2.188)\end{aligned}$$

Finally, these values for $F_{0,T}$ can be used in the Black equation together with a volatility $\sigma_{0,T}$ obtained from (2.182b)

$$\sigma_{0,T}^2 = \left(1 - e^{-2\kappa T}\right)\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2\left(1 - e^{-\kappa T}\right)\frac{\rho\sigma_\chi\sigma_\xi}{\kappa}. \qquad (2.189)$$

Further discussion of the Schwartz–Smith model can be found in the original paper, Example 5.1 in Eydeland and Wolyniec (2003), and in Section 3.2.3 of Burger, Graeber and Schindlmayr (2007).

2.10.2.2 The Burger–Graeber–Schindlmayr (2007) Two-Factor Model

As presented earlier in our discussion of the Clewlow–Strickland (1999) model, we can model spot processes or we can model forwards. An interesting two-factor model is presented in Burger, Graeber and Schindlmayr (2007), which takes the latter approach. In this model, we have

$$\frac{dF_{t,T}}{F_{t,T}} = e^{-\kappa(T-t)}\sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)} \qquad (2.190)$$

where $\langle dW_t^{(1)}, dW_t^{(2)}\rangle = \rho dt$. Squaring (2.190), we have

$$\frac{dF_{t,T}^2}{F_{t,T}^2} = \left(\sigma_1^2 e^{-2\kappa(T-t)} + 2\rho\sigma_1\sigma_2 e^{-\kappa(T-t)} + \sigma_2^2\right)dt. \qquad (2.191)$$

Simple application of Itô's lemma to $\ln F_{t,T} = f(F_{t,T})$ with $f(x) = \ln x$ gives

$$\begin{aligned}d\ln F_{t,T} &= \frac{dF_{t,T}}{F_{t,T}} - \frac{1}{2}\frac{dF_{t,T}^2}{F_{t,T}^2}\\ &= e^{-\kappa\tau}\sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)} - \frac{1}{2}\left(\sigma_1^2 e^{-2\kappa\tau} + 2\rho\sigma_1\sigma_2 e^{-\kappa\tau} + \sigma_2^2\right)dt\end{aligned}$$

where $\tau = T - t$. We integrate this expression, obtaining

$$\ln F_{t,T} = \ln F_{0,T} - \frac{1}{2}\int_0^t \left(\sigma_1^2 e^{-2\kappa(T-s)} + 2\rho\sigma_1\sigma_2 e^{-\kappa T-s} + \sigma_2^2\right) dt + \chi_t^T + \xi_t^T$$

where $\chi_t^T = \sigma_1 \int_0^t e^{-\kappa(T-s)} dW_s^{(1)} ds$ and $\xi_t^T = \sigma_2 \int_0^t dW_s^{(2)} ds$. With a little algebra, we obtain

$$\begin{aligned}\ln F_{t,T} = \ln F_{0,T} - \sigma_1^2 e^{-2\kappa T}\frac{e^{2\kappa t}-1}{4\kappa} - \rho\sigma_1\sigma_2 e^{-\kappa T}\frac{e^{\kappa t}-1}{\kappa} \\ -\frac{1}{2}\sigma_2^2 t + \chi_t^T + \xi_t^T.\end{aligned}$$

In their work, it is shown that the spot price process imputed by this, i.e. with $S_t = F_{t,t}$, is consistent with a Schwartz–Smith type model, but with time dependent parameters, of the form

$$\frac{dS_t}{S_t} = \kappa(\alpha_t + \xi_t - \ln S_t)dt + \sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)}. \quad (2.192a)$$

Under this model, we have

$$\ln S_t = A_t + \chi_t + \xi_t \quad (2.193a)$$

where

$$A_t = \ln F_{0,T} - \left(\frac{\sigma_1^2}{4\kappa}(1 - e^{-2\kappa t}) + \frac{\rho\sigma_1\sigma_2}{\kappa} + \frac{1}{2}\sigma_2^2 t\right) \quad (2.193b)$$

$$d\chi_t = -\kappa\chi_t dt + \sigma_1 dW_t^{(1)} \quad (2.193c)$$

$$d\xi_t = \sigma_2 dW_t^{(2)}. \quad (2.193d)$$

Note the similarity of (2.193c) and (2.193d) to (2.176). Since forwards are directly presumed to be martingales under the risk-neutral measure, no adjustment to remove the market price of spot risk is required.

2.10.2.3 The Clewlow–Strickland (1999) Multi-Factor Model for Forwards

Clewlow and Strickland (1999b) extended their earlier work to cover a sequence of models to describe the stochastic evolution of the forward curve. While for the one-factor model we have $\frac{dF_{t,T}}{F_{t,T}} = \sigma_F(t, T)dW_t$, the multi-factor model is given by

$$\frac{dF_{t,T}}{F_{t,T}} = \sum_{i=1}^{n} \sigma_i(t, T)dW_t^{(i)} \tag{2.194}$$

where $\{W_t^{(1)}, \ldots, W_t^{(n)}\}$ are n independent Brownian motions.

This can be integrated in the same manner as the Clewlow and Strickland one-factor model

$$\ln F_{t,T} = \ln F_{0,T} + \sum_{i=1}^{n} \left[-\frac{1}{2} \int_0^t \sigma_i^2(s, T)ds + \int_0^t \sigma_i(s, T)dW_s \right]. \tag{2.195}$$

Note that the presence of the $\int_0^t \frac{\partial \sigma_F(s,t)}{\partial t} dW_s$ term in the drift means that this model is non-Markovian, i.e. it has a memory and the stochastic evolution of the spot process S_t is not purely determined by the value of the stochastic variables at time t. Taking exponentials of (2.195), we have

$$F_{t,T} = F_{0,T} \cdot \exp\left(\sum_{i=1}^{n} \left[-\frac{1}{2} \int_0^t \sigma_i^2(s, T)ds + \int_0^t \sigma_i(s, T)dW_s \right] \right) \tag{2.196}$$

from which it is apparent that $F_{t,T}$ is lognormally distributed. As a result, as stated in Clewlow and Strickland (1999b), European style options on forwards can easily be priced needing "only univariate integrations involving the volatility functions of the forward prices"

$$V_0^{C/P} = \omega e^{-r^d T}[F_{0,T}N(\omega d_1) - KN(\omega d_2)] \tag{2.197a}$$

with

$$d_{1,2} = \frac{\ln\left(F_{0,T}/K\right) \pm \frac{1}{2}\sigma_{0,T}^2 T}{\sigma_{0,T}\sqrt{T}} \tag{2.197b}$$

where

$$\sigma_{0,T}^2 = \frac{1}{T} \sum_{i=1}^{n} \left[\int_0^T \sigma_i^2(u, s)du \right]. \tag{2.197c}$$

2.10.3 Convenience Yield Models

As well as introducing extra stochastic factors to capture extra degrees of freedom in the volatility, it is certainly quite possible to introduce extra stochastic factors for other terms in the model, such as convenience yield.

2.10.3.1 The Gibson–Schwartz (1990) Two-Factor Model

The first work in this area was Gibson and Schwartz (1990), who introduced a two-factor model – the first factor being commodity spot, and the second factor being convenience yield δ_t

$$\frac{dS_t}{S_t} = (\mu_t - \delta_t)dt + \sigma^{(1)} dW_t^{(1)} \tag{2.198a}$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma^{(2)} dW_t^{(2)} \tag{2.198b}$$

with $\langle dW_t^{(1)}, dW_t^{(2)} \rangle = \rho dt$. The same model is presented as "Model 2" in Schwartz (1997). In this model, α is the long term convenience yield, $\sigma^{(1)}$ is the spot volatility and $\sigma^{(2)}$ is the volatility of the convenience yield. The κ term gives the speed of mean reversion and ρ allows chances in convenience yield to be correlated with movements in the spot process.

Upon transforming to the domestic risk-neutral measure, we have

$$\frac{dS_t}{S_t} = (r_t^d - \delta_t)dt + \sigma^{(1)} dW_t^{(1;d)} \tag{2.199a}$$

$$\begin{aligned} d\delta_t &= (\kappa(\alpha - \delta_t) - \lambda_\delta)dt + \sigma^{(2)} dW_t^{(2;d)} \\ &= (\kappa(\hat{\alpha} - \delta_t))dt + \sigma^{(2)} dW_t^{(2;d)} \end{aligned} \tag{2.199b}$$

with $\hat{\alpha} = \alpha - \lambda_\delta/\kappa$, where λ_δ denotes the market price of convenience yield risk. Note that the market price per unit of convenience yield risk λ in Bjerksund (1991) is equal to $\lambda_\delta/\sigma^{(2)}$ in our notation. The Brownians $W_t^{(1;d)}$ and $W_t^{(2;d)}$ are now Brownian motions with respect to the (domestic) risk-neutral measure, such that $\langle dW_t^{(1;d)}, dW_t^{(2;d)} \rangle = \rho dt$.

As discussed in Schwartz (1997), as originally derived in Jamshidian and Fein (1990) and Bjerksund (1991), this can be solved to obtain the forward price

$$F_{0,T} = S_0 \cdot \exp\left(-\delta_0 \frac{1 - e^{-\kappa T}}{\kappa} + A(T) \right) \tag{2.200a}$$

where

$$A_T = \left(r^d - \hat{\alpha} + \frac{[\sigma^{(2)}]^2}{2\kappa^2} - \frac{\rho\sigma^{(1)}\sigma^{(2)}}{\kappa} \right) T + \frac{1}{4}[\sigma^{(2)}]^2 \frac{1 - e^{-2\kappa T}}{\kappa^3}$$
$$+ \left(\hat{\alpha}\kappa + \rho\sigma^{(1)}\sigma^{(2)} - \frac{[\sigma^{(2)}]^2}{\kappa} \right) \frac{1 - e^{-\kappa T}}{\kappa^2}. \qquad (2.200b)$$

Upon changing to the risk-neutral measure and removing the market price of risk, we have

$$V_0^{S_T} = e^{-r^d T}\mathbf{E}^d \left[S_T\right]$$
$$= S_0 \exp\left(\hat{\mu} + \frac{1}{2}\hat{\sigma}^2 \right) \qquad (2.201)$$

where

$$\hat{\mu} = \left(-\frac{1}{2}[\sigma^{(1)}]^2 + \hat{\alpha} \right) T + \left(\hat{\alpha} - \delta_0 \right) \frac{1 - \theta}{\kappa} \qquad (2.202a)$$

$$\hat{\sigma} = \left[[\sigma^{(2)}]^2 - \frac{2\rho\sigma^{(1)}\sigma^{(2)}}{\kappa} + \frac{[\sigma^{(2)}]^2}{\kappa^2} \right] T \qquad (2.202b)$$
$$+2\left(\frac{\rho\sigma^{(1)}\sigma^{(2)}}{\kappa^2} - \frac{[\sigma^{(2)}]^2}{\kappa^3} \right)(1-\theta) + \frac{[\sigma^{(2)}]^2}{2\kappa^3}(1-\theta^2)$$

with $\theta = \exp(-\kappa T)$. This can be used directly for European option pricing, following Bjerksund (1991) we have

$$V_0^C = V_0^{S_T} N(d_1) - Ke^{-r^d T} N(d_2) \qquad (2.203)$$

where

$$d_{1;2} = \frac{\ln(V_0^{S_T}/K) + r^d T \pm \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}. \qquad (2.204)$$

The convenience yield does not appear directly in (2.203), as it is already accounted for in computation of $V_0^{S_T}$ via $\hat{\mu}$.

Finally, and as a prelude to the next model, Cortazar and Schwartz (2003) noted that (2.198) can be simplified somewhat by defining the so-called "demeaned" convenience yield $y_t = \delta_t - \alpha$ (clearly $dy_t = d\delta_t$). With this, we transform (2.198b) as follows

$$dy_t = d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma^{(2)}dW_t^{(2)}$$
$$= -\kappa y_t dt + \sigma^{(2)}dW_t^{(2)}. \qquad (2.205)$$

As for (2.198a), we introduce a "long-term" price return $\nu_t = \mu_t - \alpha$. Since both terms y_t and ν_t are adjusted by the same constant factor, we have $\nu_t - y_t = \mu_t - \delta_t$, and therefore

$$\begin{aligned}\frac{dS_t}{S_t} &= (\mu_t - \delta_t)dt + \sigma^{(1)}dW_t^{(1)} \\ &= (\nu_t - y_t)dt + \sigma^{(1)}dW_t^{(1)}.\end{aligned} \tag{2.206}$$

Note that ν_t, like μ_t, is deterministic.

2.10.3.2 The Schwartz (1997) Three-Factor Model

Schwartz (1997) also presents a three-factor model, basically an extension of the Gibson–Schwartz model with an additional Ornstein–Uhlenbeck process for the instantaneous (domestic) short rate, i.e. $dr_t^d = a(m - r_t^d)dt + \sigma^{(3)}dW_t^{(3)}$, meaning we have a system of SDEs:

$$\frac{dS_t}{S_t} = (r_t^d - \delta_t)dt + \sigma^{(1)}dW_t^{(1;d)} \tag{2.207a}$$

$$d\delta_t = \kappa(\hat{\alpha} - \delta_t)dt + \sigma^{(2)}dW_t^{(2;d)} \tag{2.207b}$$

$$dr_t^d = a(m - r_t^d)dt + \sigma^{(3)}dW_t^{(3;d)} \tag{2.207c}$$

where $\langle dW_t^{(i;d)}, dW_t^{(j;d)} \rangle = \rho_{ij}dt$. Equation (2.207c) is nothing other than the Vasicek interest rate model. Typically a rates model will be calibrated separately to caplet and swaption volatilities. These parameters will be used in calibrating the three-factor model to the commodities market. This model and the next are discussed in the original papers – additionally, a good technical overview of both can be found in Hosseini (2007).

2.10.3.3 The Cortazar–Schwartz (2003) Three-Factor Model

The three-factor model of Cortazar and Schwartz (2003) is a very simple extension, in principle, to the Gibson–Schwartz model. We have the same dynamics (2.206) and (2.205) for spot and convenience yield respectively, but we allow ν_t, the long-term price return, to be stochastic, and model it by (2.208c) below – the three SDEs being

$$\frac{dS_t}{S_t} = (\nu_t - y_t)dt + \sigma^{(1)}dW_t^{(1)} \tag{2.208a}$$

$$dy_t = -\kappa y_t dt + \sigma^{(2)}dW_t^{(2)} \tag{2.208b}$$

$$d\nu_t = a(\bar{\nu} - \nu_t)dt + \sigma^{(3)}dW_t^{(3)} \tag{2.208c}$$

where $\langle dW_t^{(i)}, dW_t^{(j)} \rangle = \rho_{ij}dt$. Note that, as before, the market price of long-term price return is adjusted to correct for the market price of risk. Futures prices can be computed under this model, we refer the reader to equation (32) in the original paper.

We now progress to a discussion of how the theory presented in this chapter can be applied to the various traded commodities markets. Note that while we have been careful to discriminate between forward and futures contracts in this chapter, we shall be working under the assumption of (2.1) henceforth, in line with typical market terminology where futures curves and forward curves are often regarded as synonymous.