# Spectral Clustering for Axiom Selection

Zishi Wu

Department of Computer Science
University of Miami

May 17, 2020

# Outline



# Introduction

## Definitions

- What is Automated Theorem Proving (ATP)?
- *Logical formulae* are statements about a domain:
- Show that the *conjecture* is a *logical consequence* of the axioms (a.k.a premises).
- Applications:
  - Formal verification of software - Compilers (e.g. gcc, llvm)
  - Formal verification of hardware - CPU (e.g. 1994 Intel Pentium floating-point division bug)
  - Interactive proof assistants for mathematics (e.g. Isabelle, Mizar)

# Introduction

## Example

- Axiom 1: *All men are mortal*.
- Axiom 2: *Socrates is a man*.
- Conjecture: *Socrates is mortal*.

# Logical Consequence

## Logical Consequence

- Every model of the axioms is a model of the conjecture.
- A set of axioms has a *model* if there is an *interpretation* (assignment of boolean values) to the axioms such that the conjunction of the axioms evaluate to *True*.
- If we list all interpretations of $N$ formulae on a truth table, we get $2^N$ rows. This search space grows exponentially.
- The faster method is to show that the union of the axioms and the negation of the conjecture is *unsatisfiable*. $Ax \cup \neg C = \emptyset$
- In other words, if no model of the axioms is a model of the negated conjecture, then all models of the axioms are models of the conjecture.

# Problem Statement

## Problem Statement

- A *large-theory* problem consists of a conjecture to be proven, and a large number of axioms to be considered.
- However, the solution set(s) usually consist of a few axioms.
- How do we select the necessary axioms? This is known as the problem of *premise selection*.

# Benchmark Data

## MPTP2078 Dataset

- Thousands of Problems for Theorem Provers (TPTP) [Sut17]
  - Standard set of test problems.
- Benchmark dataset of 2078 problems known as the Mizar Problems for Theorem Provers (MPTP2078) [AHK+14].
  - Encodes problems from the Mizar Mathematical Library (MML) of formalized mathematics into first-order logic form.
- There are two versions of each problem:
  - Bushy = smaller version (3 to 40 axioms, 1 to 15 needed)
  - Chainy = larger version (10 to 500 axioms, 2 to 119 needed)
- Premise selection performance compared to state-of-the-art Automated Theorem Provers:
  - E [Sch13]
  - Vampire [KV13]

# Graph Methods

## Minimal Proof Dependency

- Alama et al. [AHK+14]. constructed a knowledge base of minimal proof dependencies that a problem depends on and trained kernel-based machine learning methods on a feature matrix representation of that knowledge base.
- Minimal dependencies encoded as an adjacency matrix, where the $(i,j)^{th}$ cell of the matrix has a value of 1 if the $i^{th}$ formula of a problem is used in the proof of the $j^{th}$ formula of a problem, and 0 otherwise.
- Formulate premise selection as a *ranking problem*:
  - Rank the axioms by how likely they are to prove a conjecture (e.g. shared minimal dependencies).
  - Choose a threshold value.
  - Select all axioms whose score is above that threshold.

# Related Work

## Extended Hutchinson Distance

- To construct an adjacency matrix, we require a measure of similarity or dissimilarity between each pair of nodes in a graph.
- Used a dissimilarity metric between two terms $\Delta_1$ and $\Delta_2$, invented by Qinghua Liu, that extends the Hutchinson distance [Hut97].
- Calculated by finding the Least Generalized Generalization (*lgg*) between two formulae. A term $\Delta$ is the *lgg* of $\Delta_1$ and $\Delta_2$ iff
  - There are substitutions $\theta_1$ and $\theta_2$ such that $\Delta\theta_1 = \Delta_1$ and $\Delta\theta_2 = \Delta_2$.
  - There exists no term $\Delta'$ and substitutions $\sigma$, $\sigma_1$ and $\sigma_2$ such that $\Delta\sigma_1 = \Delta'$ and $\Delta\sigma_2 = \Delta'$.

# Related Work

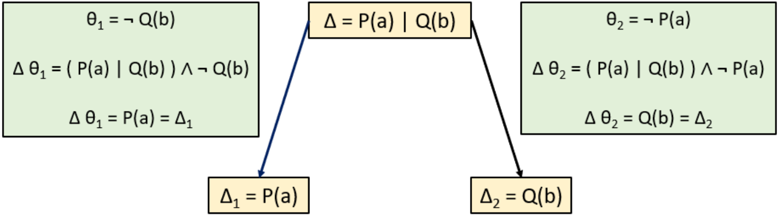
## Extended Hutchinson Distance

- If there is no Least Generalized Generalization $\Delta$ between two terms $\Delta_1$ and $\Delta_2$, then their extended Hutchinson distance is $\infty$. Formally, $\mathit{dissimilarity}(\Delta_1, \Delta_1) = \infty$ iff $\mathit{lgg}(\Delta_1, \Delta_2) = \emptyset$

# Related Work

## Least Generalized Generalization Example

- Note that the substitutions $\theta_1$ and $\theta_2$ are not limited to a single substitution rule. They can also consist of multiple substitution rules occurring one after the other.




- P(a) could represent a predicate: **is_man( Socrates )**
- Q(b) could represent a predicate: **is_mortal ( Man)**

# Methodology

## Data Representation

- NOTE: this should be a comprehensive summary of the entire methodology, not details of one part
- Qinghua designed a dissimilarity metric [LXH17]
- Problem is converted into an undirected fully-connected graph
  - Vertices $V = \{\text{Axioms} \cup \text{Conjecture}\}$
  - Edges $E =$ dissimilarity weights between vertices

# Graph Theory

## Spectral Graph Theory [Chu97]

- Adjacency matrix $A$ consists of similarity values between vertices
- Degree matrix $D$ is a diagonal matrix where the $i^{th}$ element is the sum of the elements of the $i^{th}$ column of $A$
- Un-normalized Graph Laplacian matrix
  - $L = D - A$
- Normalized Graph Laplacian matrix contains *features* of the graph
  - $L_{norm} = I - (D^{-1/2} \; L \; D^{-1/2})$

# Example Graph

## Calculate Normalized Laplacian Matrix

- Adjacency, Degree, and Un-normalized Graph Laplacian

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Normalized Graph Laplacian

$$L_{norm} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Selection Method

## Spectral Clustering [vL07]

- Given graph $G = (V, E)$
- Partition vertices in $V$ into $k$ clusters: $C_1, C_2, ..., C_k$
- Denote the cluster containing the conjecture as $C_C$
- Problem may have more than one set of solutions
- Succesful conjecture cluster only needs to contain one solution

# Spectral Clustering Algorithm

## A Tutorial on Spectral Clustering [vL07]

1. Construct a weight matrix W (i.e. adjacency matrix A)
2. Compute the normalized Laplacian matrix $L_{norm}$ from $W$
3. Compute the first $k$ eigenvectors $v_1, ..., v_k$ of $L_{norm}$ and construct a feature matrix $U$ from those eigenvectors
4. For $i = 1, ..., n$, let $p_i$ be the feature vector for the $i^{th}$ vertex, corresponding to the $i^{th}$ row of $U$
5. Cluster the vertices based on their feature vectors into $k$ clusters: $C_1, C_2, ..., C_k$

# Spectral Clustering Algorithm

## K-Means Initialization Problem

5. Cluster the feature vectors into k clusters: C1, C2, . . . , Ck
   - Each run of k-means chooses a different set of initial centroids for the *k* clusters
   - Results in different clusterings each run
   - We need a deterministic way of clustering that doesn't change over multiple runs of k-means

J. Alama, T. Heskes, D. Külwein, E. Tsivtsivadze, and J. Urban.
Premise Selection for Mathematics by Corpus Analysis and Kernel Methods.
*Journal of Automated Reasoning*, 52(2):191–213, 2014.

F. R. K. Chung.
*Spectral Graph Theory*.
American Mathematical Society, 1997.

A. Hutchinson.
Metrics on Terms and Clauses.
In M. van Someren and G. Widmer, editors, *Proceedings of the 9th European Conference on Machine Learning*, number 1224 in Lecture Notes in Artificial Intelligence, pages 138–145. Springer-Verlag, 1997.

L. Kovacs and A. Voronkov.
First-Order Theorem Proving and Vampire.
In N. Sharygina and H. Veith, editors, *Proceedings of the 25th International Conference on Computer Aided Verification*, number 8044 in Lecture Notes in Artificial Intelligence, pages 1–35. Springer-Verlag, 2013.

Q. Liu, Y. Xu, and X. He.
New terms metric based on substitutions.
In *2017 12th International Conference on Intelligent Systems and Knowledge Engineering (ISKE)*, pages 1–6, 2017.

S. Schulz.
System Description: E 1.8.
In K. McMillan, A. Middeldorp, and A. Voronkov, editors, *Proceedings of the 19th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning*, number 8312 in Lecture Notes in Computer Science, pages 477–483. Springer-Verlag, 2013.

G. Sutcliffe.
The TPTP Problem Library and Associated Infrastructure. From CNF to TH0, TPTP v6.4.0.
*Journal of Automated Reasoning*, 59(4):483–502, 2017.

Ulrike von Luxburg.
A tutorial on spectral clustering.
*Statistics and Computing*, 17(4):395–416, Dec 2007.