The Normal Distribution

Lecture: already delivered

Support Teaching Material

Source: Statistics for Managers by Levine et al

Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value on a continuum (can assume an uncountable number of values)
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches
- These can potentially take on any value depending only on the ability to precisely and accurately measure

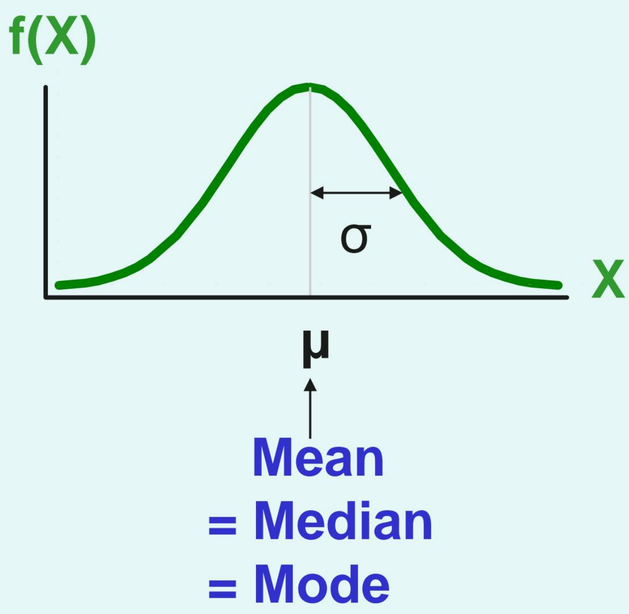
The Normal Distribution

- 'Bell Shaped'
- Symmetrical
- Mean, Median and Mode are Equal

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range: $+\infty$ to $-\infty$



The Normal Distribution Density Function

- The formula for the normal probability density function is

$$f(X) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{(X-\mu)}{\sigma}\right)^2}$$

Where e = the mathematical constant approximated by 2.71828

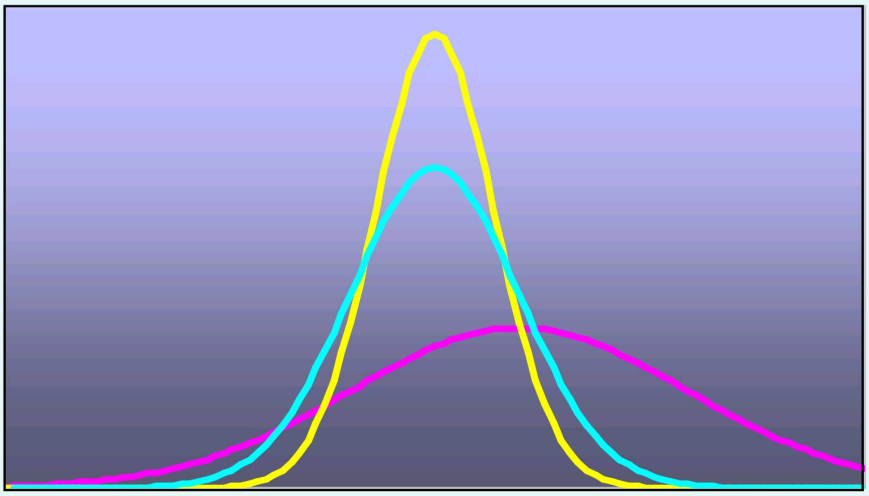
π = the mathematical constant approximated by 3.14159

μ = the population mean

σ = the population standard deviation

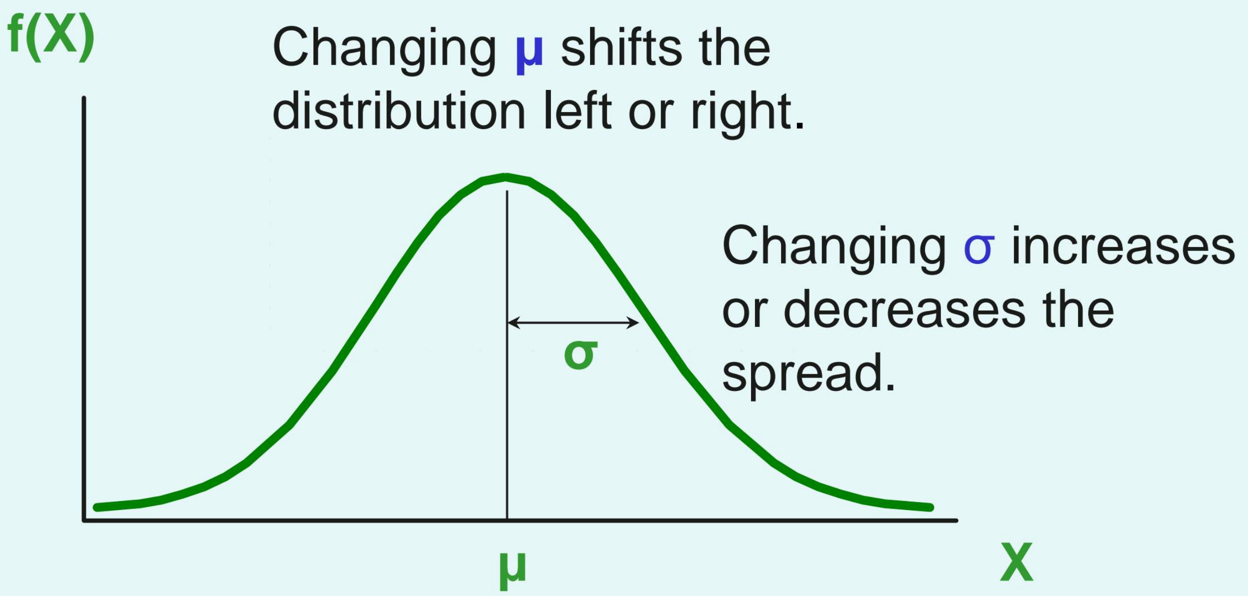
X = any value of the continuous variable

Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

The Normal Distribution Shape



The Standardized Normal

- Any normal distribution (with any mean and standard deviation combination) can be transformed into the standardized normal distribution (Z)

- Need to transform X units into Z units

- The standardized normal distribution (Z) has a mean of 0 and a standard deviation of 1

Translation to the Standardized Normal Distribution

- Translate from X to the standardized normal (the "Z" distribution) by subtracting the mean of X and dividing by its standard deviation:

$$Z = \frac{X - \mu}{\sigma}$$

The Z distribution always has mean = 0 and standard deviation = 1

The Standardized Normal Probability Density Function

- The formula for the standardized normal probability density function is

$$\boxed{f(Z) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)Z^2}}$$

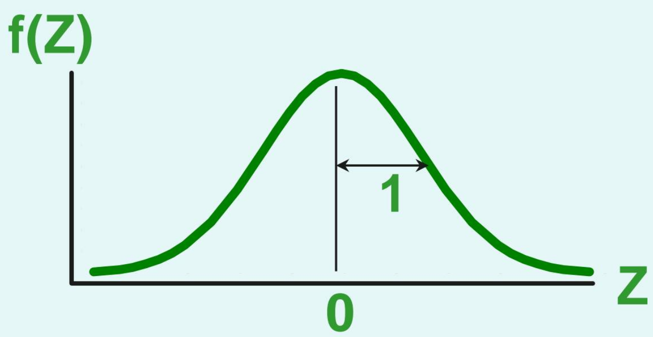
Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

Z = any value of the standardized normal distribution

The Standardized Normal Distribution

- Also known as the “Z” distribution
- Mean is 0
- Standard Deviation is 1



Values above the mean have positive Z-values, values below the mean have negative Z-values

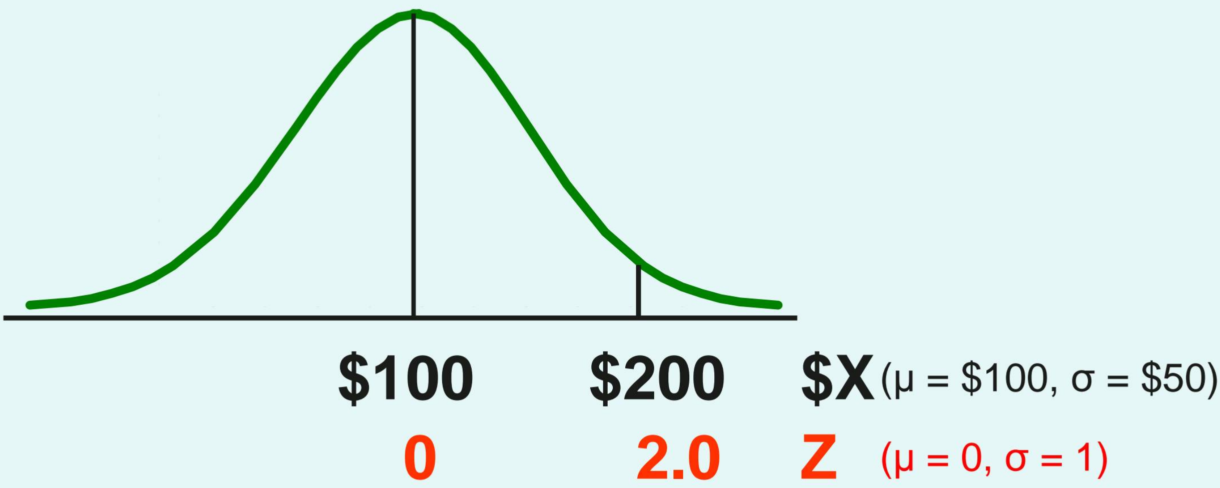
Example

- If X is distributed normally with mean of \$100 and standard deviation of \$50, the Z value for X = \$200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{\$200 - \$100}{\$50} = 2.0$$

- This says that X = \$200 is two standard deviations (2 increments of \$50 units) above the mean of \$100.

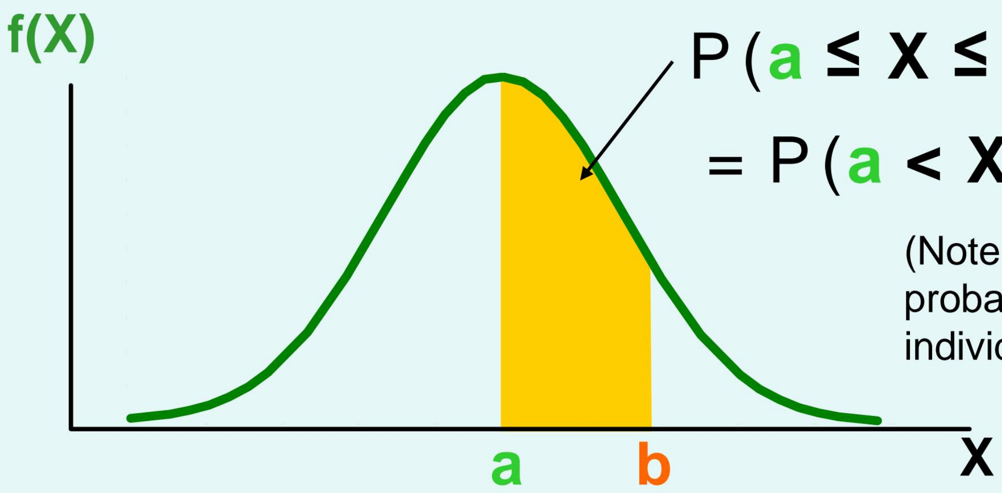
Comparing X and Z units



Note that the shape of the distribution is the same, only the scale has changed. We can express the problem in the original units (X in dollars) or in standardized units (Z)

Finding Normal Probabilities

Probability is measured by the area under the curve

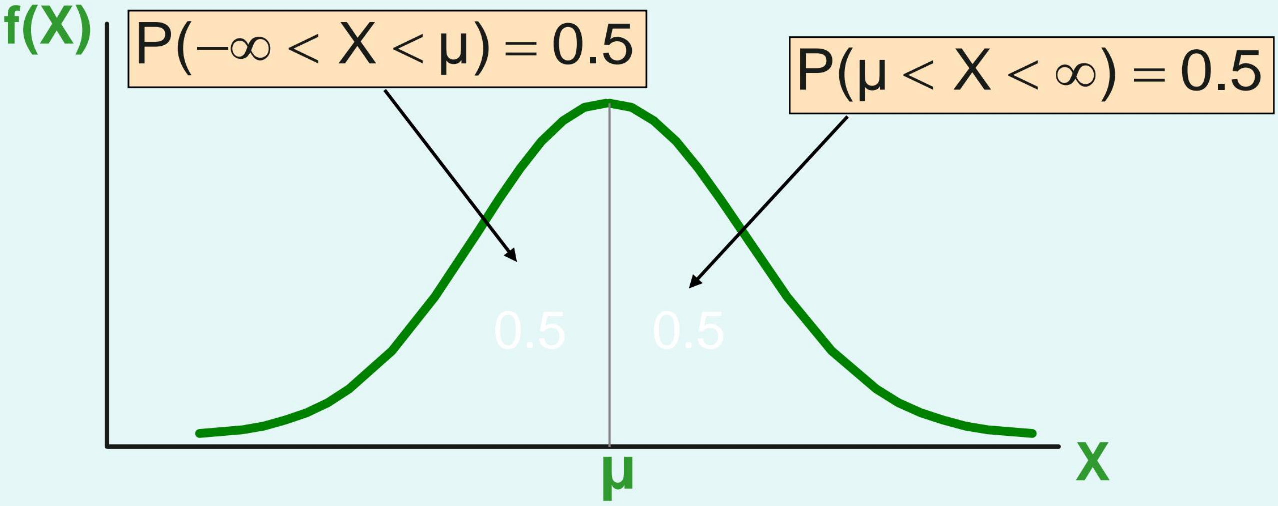


$$P(a \leq X \leq b)$$
$$= P(a < X < b)$$

(Note that the probability of any individual value is zero)

Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



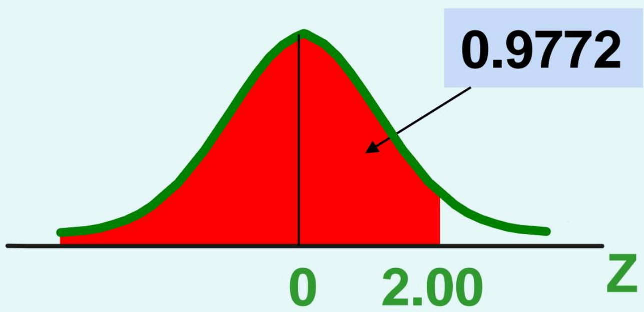
$$P(-\infty < X < \infty) = 1.0$$

The Standardized Normal Table

- The Cumulative Standardized Normal table in the textbook (Appendix table E.2) gives the probability less than a desired value of Z (i.e., from negative infinity to Z)

Example:

P(Z < 2.00) = 0.9772



The Standardized Normal Table

(continued)

The column gives the value of Z to the second decimal point

Z	0.00	0.01	0.02 …
0.0			
0.1			
.			
.			
.			
2.0	.9772		

The row shows the value of Z to the first decimal point

The value within the table gives the probability from Z = – ∞ up to the desired Z value

P(Z < 2.00) = 0.9772

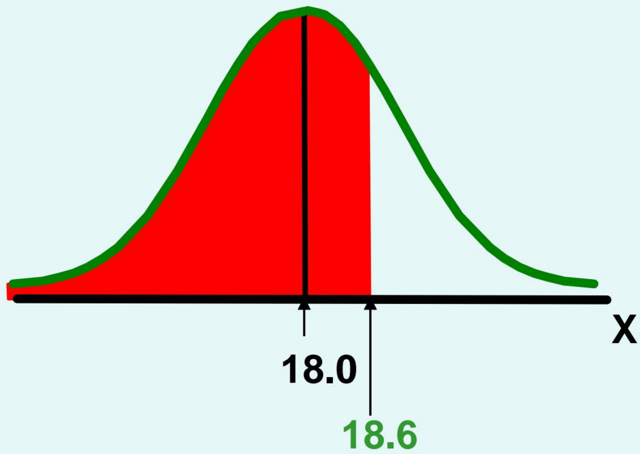
General Procedure for Finding Normal Probabilities

To find $P(a < X < b)$ when X is distributed normally:

- Draw the normal curve for the problem in terms of X
- Translate X-values to Z-values
- Use the Standardized Normal Table

Finding Normal Probabilities

- Let X represent the time it takes (in seconds) to download an image file from the internet.
- Suppose X is normal with a mean of18.0 seconds and a standard deviation of 5.0 seconds. Find $P(X < 18.6)$

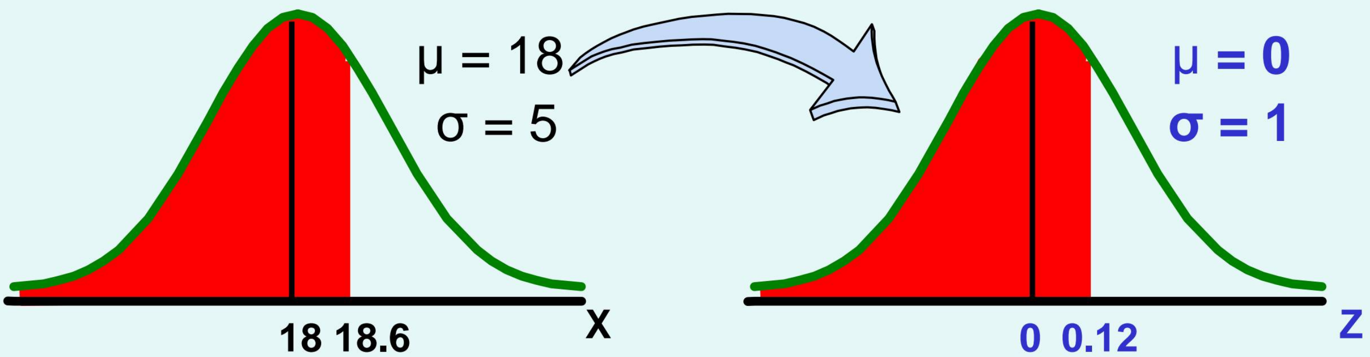


Finding Normal Probabilities

(continued)

- Let X represent the time it takes, in seconds to download an image file from the internet.
- Suppose X is normal with a mean of 18.0 seconds and a standard deviation of 5.0 seconds. Find $P(X < 18.6)$

$$Z = \frac{X - \mu}{\sigma} = \frac{18.6 - 18.0}{5.0} = 0.12$$



$P(X < 18.6)$ $\qquad$ $P(Z < 0.12)$

Solution: Finding P(Z < 0.12)

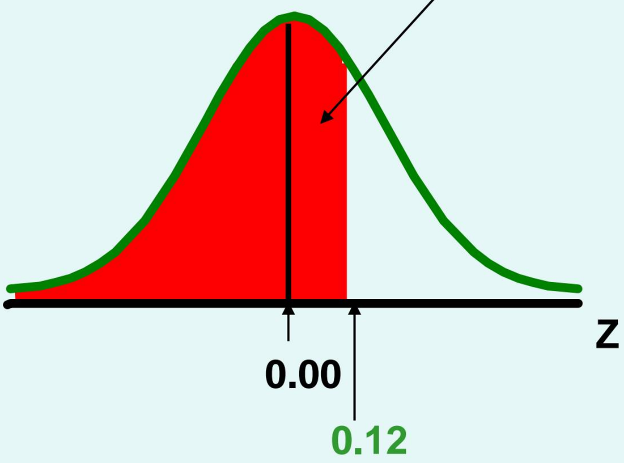
Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.6179	.6217	.6255

$P(X < 18.6)$

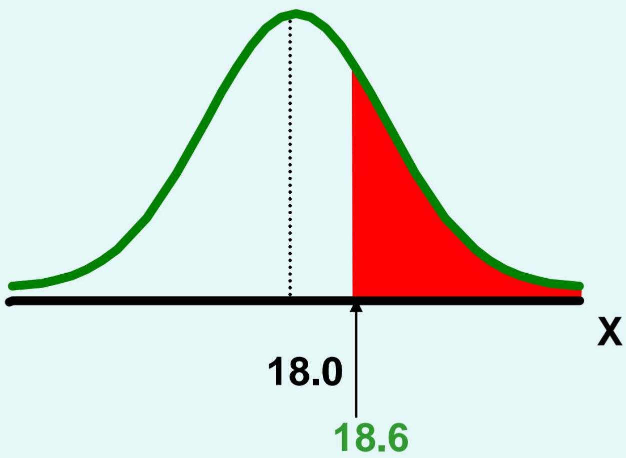
$= P(Z < 0.12)$

0.5478



Finding Normal Upper Tail Probabilities

- Suppose X is normal with mean 18.0 and standard deviation 5.0.
- Now Find P(X > 18.6)



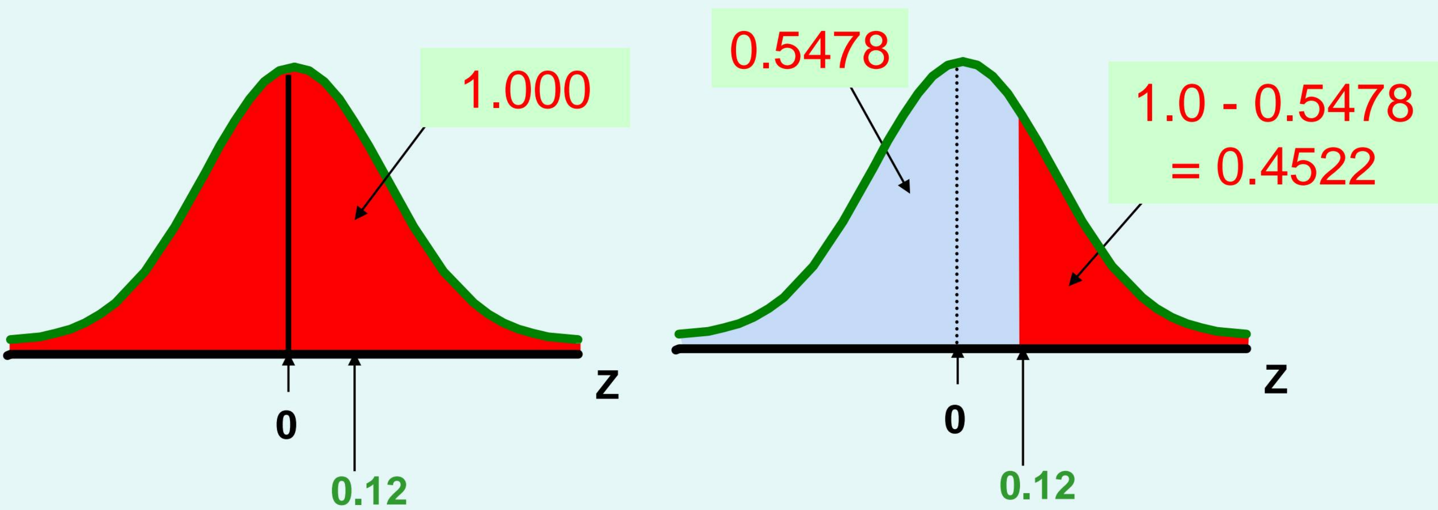
Finding Normal Upper Tail Probabilities

(continued)

- Now Find P(X > 18.6)…

$P(X > 18.6) = P(Z > 0.12) = 1.0 - P(Z \leq 0.12)$

$= 1.0 - 0.5478 = \boxed{0.4522}$



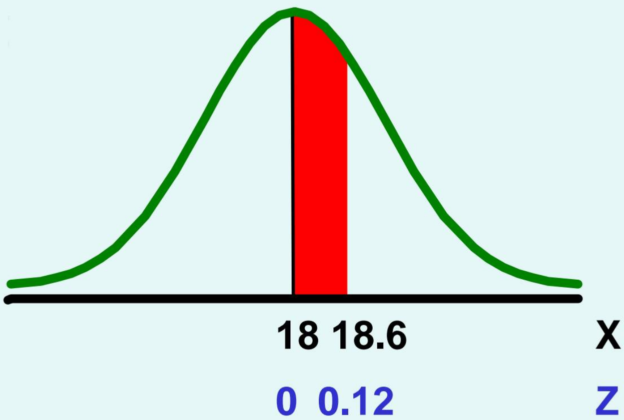
Finding a Normal Probability Between Two Values

- Suppose X is normal with mean 18.0 and standard deviation 5.0. Find P(18 < X < 18.6)

Calculate Z-values:

$$Z = \frac{X - \mu}{\sigma} = \frac{18 - 18}{5} = 0$$

$$Z = \frac{X - \mu}{\sigma} = \frac{18.6 - 18}{5} = 0.12$$



P(18 < X < 18.6)

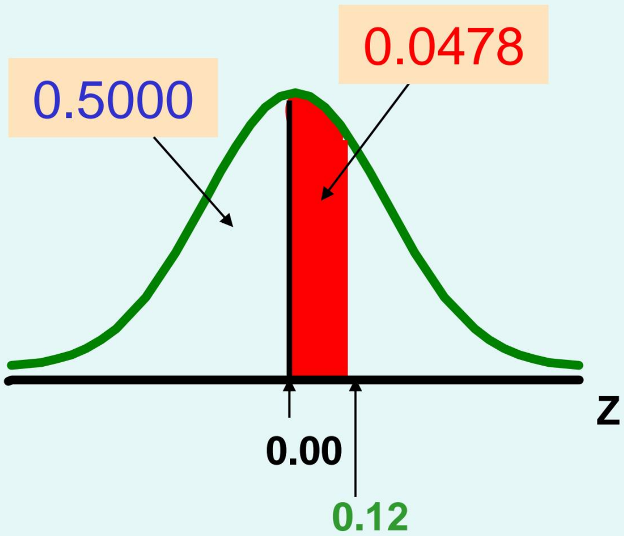
= P(0 < Z < 0.12)

Solution: Finding $P(0 < Z < 0.12)$

Standardized Normal Probability Table (Portion)

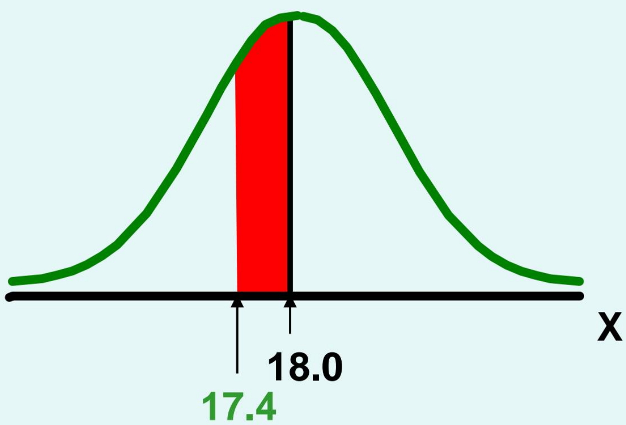
Z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.6179	.6217	.6255

$P(18 < X < 18.6)$
$= P(0 < Z < 0.12)$
$= P(Z < 0.12) - P(Z \leq 0)$
$= 0.5478 - 0.5000 = 0.0478$



Probabilities in the Lower Tail

- Suppose X is normal with mean 18.0 and standard deviation 5.0.
- Now Find P(17.4 < X < 18)



Probabilities in the Lower Tail

(continued)

Now Find P(17.4 < X < 18)…

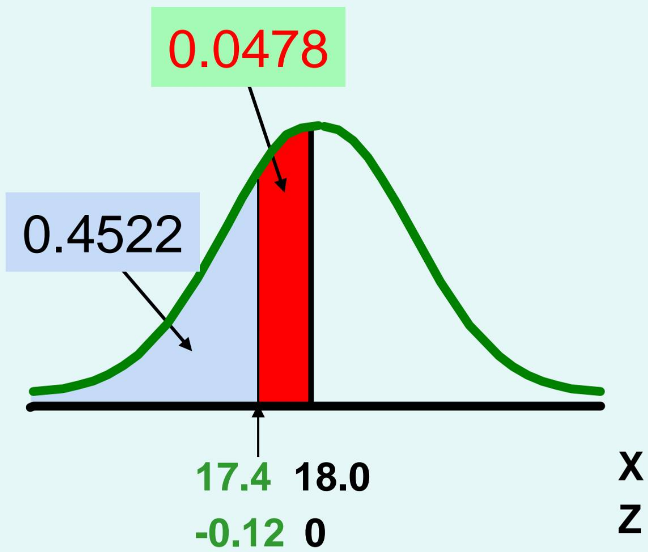
$P(17.4 < X < 18)$

$= P(-0.12 < Z < 0)$

$= P(Z < 0) - P(Z \leq -0.12)$

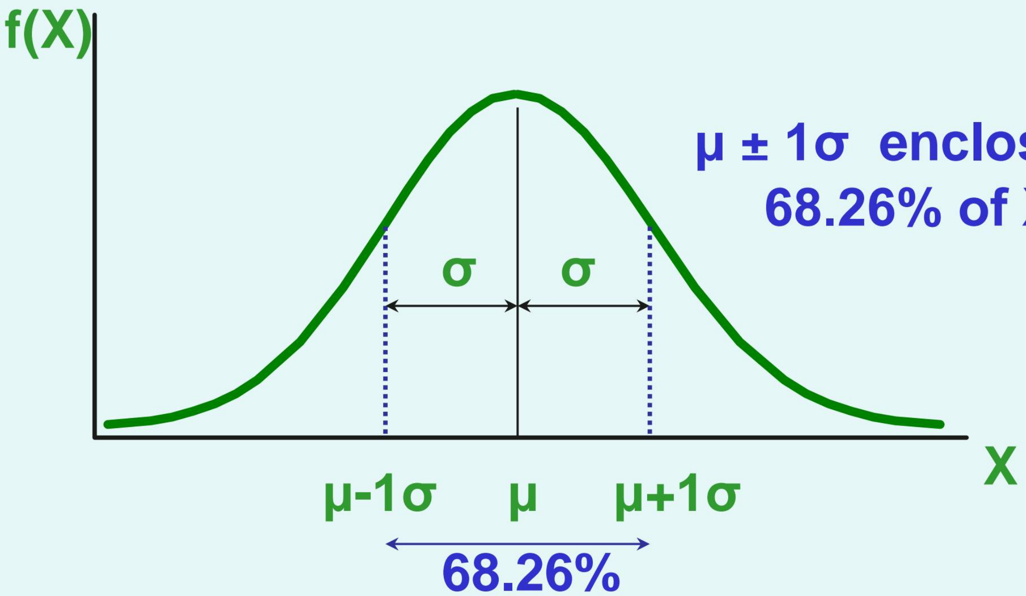
$= 0.5000 - 0.4522 = \boxed{0.0478}$

The Normal distribution is symmetric, so this probability is the same as $P(0 < Z < 0.12)$



Empirical Rules

What can we say about the distribution of values around the mean? For any normal distribution:

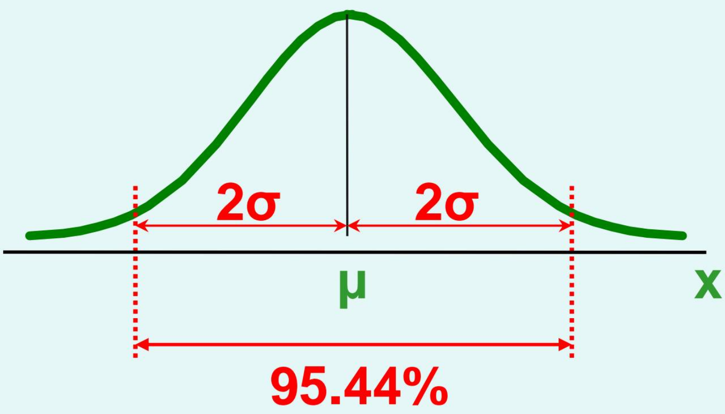


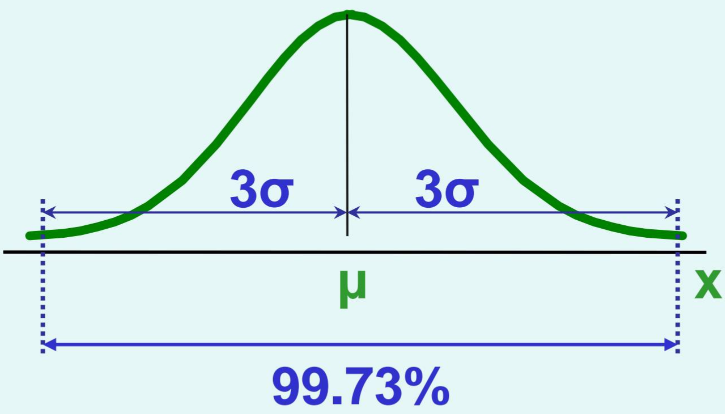
μ ± 1σ encloses about 68.26% of X's

The Empirical Rule

(continued)

- $\mu \pm 2\sigma$ covers about 95% of X's
- $\mu \pm 3\sigma$ covers about 99.7% of X's





Given a Normal Probability Find the X Value

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

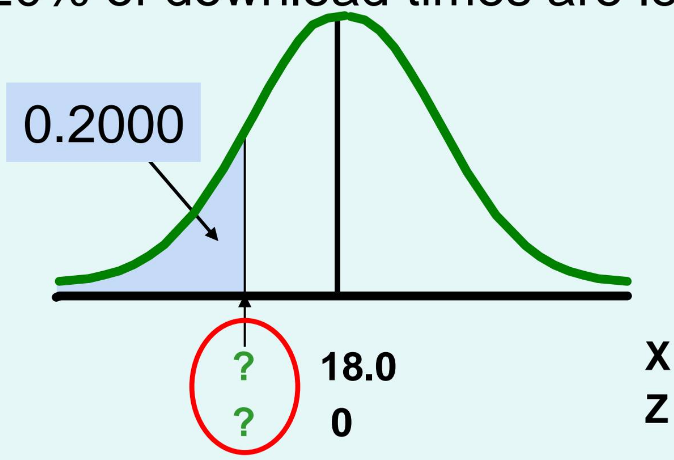
$$\boxed{X = \mu + Z\sigma}$$

Finding the X value for a Known Probability

(continued)

Example:

- Let X represent the time it takes (in seconds) to download an image file from the internet.
- Suppose X is normal with mean 18.0 and standard deviation 5.0
- Find X such that 20% of download times are less than X.



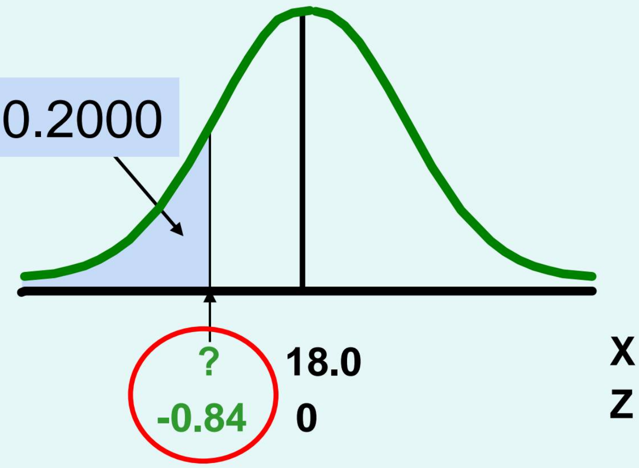
Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

Z	…	.03	.04	.05
-0.9	…	.1762	.1736	.1711
-0.8	…	.2033	.2005	.1977
-0.7	…	.2327	.2296	.2266

- 20% area in the lower tail is consistent with a Z value of -0.84



Finding the X value

2. Convert to X units using the formula:

$$
\begin{aligned}
X &= \mu + Z\sigma \\
&= 18.0 + (-0.84)5.0 \\
&= 13.8
\end{aligned}
$$

So 20% of the values from a distribution with mean 18.0 and standard deviation 5.0 are less than 13.80