Tracer gas methods

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June 17, 2023

Contents

1 Tracer gas methods

There are three different injection methods:
(1) **The pulse method**. A small amount m of tracer gas or contaminant is introduced to the supply duct or at a point within the room. The recorded concentration versus time curve gives the density function (f or ϕ_p) of the corresponding age distribution.
(2) **The step-up method**. A continuous and constant flow of tracer gas or contaminant is introduced to the supply air duct or at a point within the room. The recorded growth of concentration versus time gives the cumulative age distribution (F or Φ_p).
(3) **The step-down (decay,washout) method**. After the concentrations have reached their equilibrium values in a step-up procedure, the steady state gas or contaminant addition is stopped. The recorded decay of concentration versus time gives us the complementary cumulative age distribution function(1-F or 1-Φ_p).
For simplicity, we will only focus on the situation that injection point is at supply duct and sampled at extract duct.

Injection point	Supply duct		In the room	
	Location of measuring point			
Injection procedure	Extract duct	In the room	Extract duct	In the room
Pulse m (m^3)	$f(\tau)=\dfrac{C_e(\tau)}{\int_0^\infty C_e(\tau)\,d\tau}=\dfrac{C_e(\tau)}{(m/Q)}$	$\phi_P(\tau)=\dfrac{C_P(\tau)}{\int_0^\infty C_P(\tau)\,d\tau}=\dfrac{C_P(\tau)}{(m/Q)}$	$f(\tau)=\dfrac{C_e(\tau)}{\int_0^\infty C_e(\tau)\,d\tau}=\dfrac{C_e(\tau)}{(m/Q)}$	$\phi_P(\tau)=\dfrac{C_P(\tau)}{\int_0^\infty C_P(\tau)\,d\tau}$
Step-up $\dot{m}$ (m^3/s)	$F(\tau)=\dfrac{C_e(\tau)}{C_e(\infty)}=\dfrac{C_e(\tau)}{(m/Q)}$	$\Phi_P(\tau)=\dfrac{C_P(\tau)}{C_P(\infty)}=\dfrac{C_P(\tau)}{(m/Q)}$	$F(\tau)=\dfrac{C_e(\tau)}{C_e(\infty)}=\dfrac{C_e(\tau)}{(\dot{m}/Q)}$	$\Phi_P(\tau)=\dfrac{C_P(\tau)}{C_P(\infty)}$
Step-down	$1-F(\tau)=\dfrac{C_e(\tau)}{C_e(0)}$	$1-\Phi_P(\tau)=\dfrac{C_P(\tau)}{C_P(0)}$	$1-F(\tau)=\dfrac{C_e(\tau)}{C_e(0)}$	$1-\Phi_P(\tau)=\dfrac{C_P(\tau)}{C_P(0)}$
Population	Air leaving the room	Local population of air at point P	Contaminant leaving the room	Local population of contaminant at point P

Figure 1: Age distribution obtained from different injection procedures

Table A1.1 Summary of mean age equations

	Tracer gas mixed with supply air or fully mixed in the room		Tracer gas source positioned in the room
	Local mean age of air $\overline{\tau}_p$	Room mean age of air $\langle\overline{\tau}\rangle$	Turnover time for the contaminant, τ_t^c
Tracer step-down method	$\overline{\tau}_p = \int_0^\infty \frac{c_p(t)}{c_e(0)} \cdot dt$	$\langle\overline{\tau}\rangle = \frac{\int_0^\infty t \cdot c_e(t) \cdot dt}{\int_0^\infty c_e(t) \cdot dt} = \frac{1}{\tau_n} \int_0^\infty t \cdot \frac{c_e(t)}{c_e(0)} \cdot dt$	$\tau_t^c = \int_0^\infty \frac{c_e(t)}{c_e(0)} \cdot dt$
Tracer step-up method	$\overline{\tau}_p = \int_0^\infty \left(1 - \frac{c_p(t)}{c_e(\infty)}\right) \cdot dt$	$\langle\overline{\tau}\rangle = \frac{1}{\tau_n} \int_0^\infty t \cdot \left(1 - \frac{c_e(t)}{c_e(\infty)}\right) \cdot dt$	$\tau_t^c = \int_0^\infty \left(1 - \frac{c_e(t)}{c_e(\infty)}\right) \cdot dt$
Pulse method	$\overline{\tau}_p = \frac{\int_0^\infty t \cdot c_p(t) \cdot dt}{\int_0^\infty c_p(t) \cdot dt}$	$\langle\overline{\tau}\rangle = \frac{1}{2 \cdot \tau_n} \frac{\int_0^\infty t^2 \cdot c_e(t) \cdot dt}{\int_0^\infty c_e(t) \cdot dt}$	$\tau_t^c = \frac{\int_0^\infty t \cdot c_e(t) \cdot dt}{\int_0^\infty c_e(t) \cdot dt}$

Figure 2: Local mean age of air of different tracer gas methods

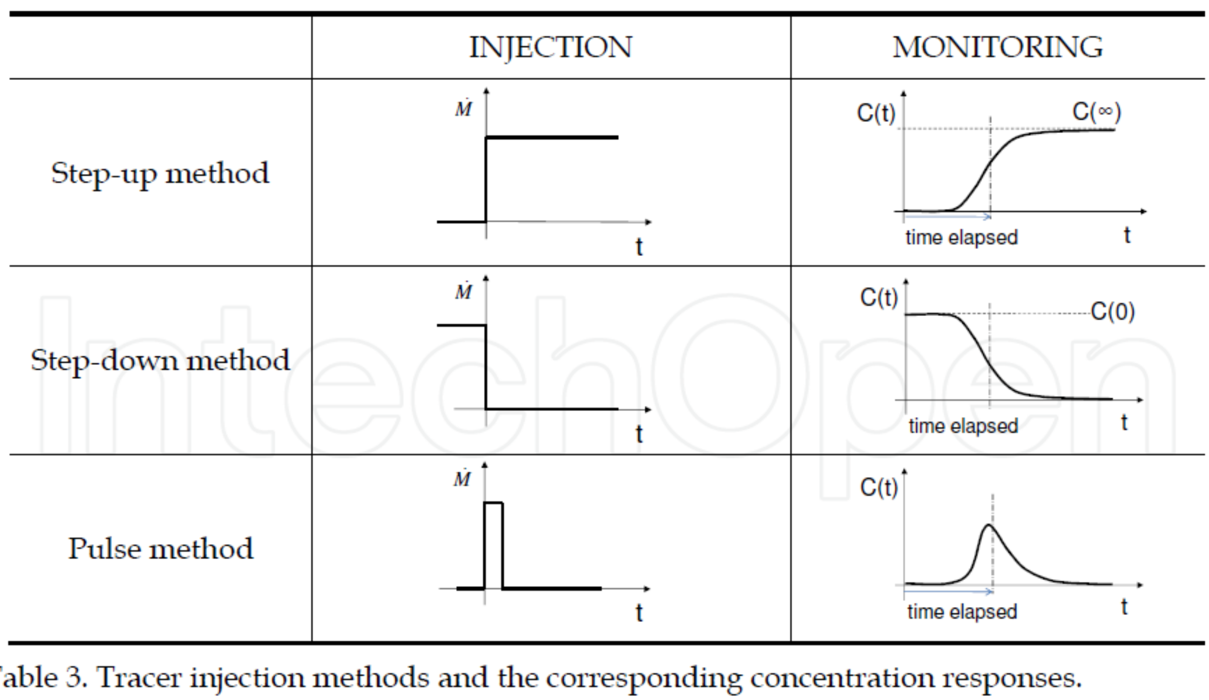

Table 3. Tracer injection methods and the corresponding concentration responses.

Figure 3: Concentration responses of different tracer gas methods

2 Pulse method

2.1 What does the concentration graph mean?

The concentration $C(t)$ at time t means the age of the amount of air ($C(t)$) is equal to t.

2.2 How to express it by frequency function?

$$\o_P(\tau) = \frac{C(\tau)}{\int_0^\infty C(\tau)} \tag{1}$$

2.3 How to calculate mean average of air?

Now, we have the density function, the mean local age of air can be calculated:

$$\overline{\tau_P} = \int_0^\infty \tau f(\tau) d\tau \tag{2}$$

If we substitute $\o_P(\tau)$ by Eq1, we can get:

$$\overline{\tau} = \frac{\int_0^\infty \tau C(\tau) d\tau}{\int_0^\infty C(\tau) d\tau} \tag{3}$$

2.4 How to calculate ventilation rates?

According to mass balance, the mass of emission tracer gas is equal to collected mass of tracer gas, therefore:

$$\begin{gathered} D(\tau) = \int_0^\tau C(\tau) d\tau = \frac{M(\tau)}{Q} \\ Q = \frac{M(\infty)}{\int_0^\infty C(\tau) d\tau} \end{gathered} \tag{4}$$

where $D(\tau)$ is the time integral of the concentration up to time τ, $M(\tau)$ is the mass of concentration collected before τ and Q is the sampling flow rate.

3 Step-up

3.1 What does the concentration graph mean?

The recorded growth of concentration versus time gives the cumulative age distribution (F or Φ_p).

3.2 How to express it by CDF?

It can be converted to cumulative distribution function(CDF) by following equation:

$$F(\tau) = \frac{C(\tau)}{C(\infty)} \tag{5}$$

3.3 How to calculate mean value from cumulative distribution function?

The relations between a cumulative distribution function(CDF) and corresponding frequency distribution:

$$\frac{\partial F}{\partial \tau} = f(\tau) \ or \ \int_0^{\tau} f(\tau) d\tau = f(\tau) \tag{6}$$

As aforementioned, the mean can be calculated from frequency function similar to Eq.2:

$$\overline{\tau} = \int_0^{\infty} \tau f(\tau) d\tau \tag{7}$$

In order to get mean local of age air from cumulative distribution, we simply replace the frequency function with cumulative distribution as Eq.6:

$$\begin{aligned}
\overline{\tau} &= \int_0^{\infty} \tau f(\tau) d\tau \\
&= \int_0^{\infty} \tau \frac{dF(\tau)}{d\tau} d\tau \\
&= \int_0^{\infty} \tau dF(\tau) \\
&= -\int_0^{\infty} \tau d(1 - F(\tau)) \\
&= -\tau(1 - F(\tau))|_0^{\infty} + \int_0^{\infty} (1 - F(\tau)) d\tau \\
&= \int_0^{\infty} (1 - F(\tau)) d\tau \\
&= \int_0^{\infty} (1 - \frac{C(\tau)}{C(\infty)}) d\tau
\end{aligned} \tag{8}$$

Find expected value using CDF 1
Find expected value using CDF 2
Integration By Parts

3.4 How to calculate ventilation rates?

At time τ=0, a contaminant generation at rate $\dot{m}$ (m^3/s or kg/s) starts within the room. The concentration in the extract duct is recorded. At an arbitrary time, τ, the mass balance for the whole system gives the relation:

$$\dot{m}\tau - Q \int_0^{\tau} C(\tau) d\tau = M(\tau) \tag{9}$$

By differentiating Eq.9, we obtain

$$\dot{m} - Q \cdot C(\tau) = \frac{\partial M(\tau)}{\partial \tau} \tag{10}$$

At equilibrium a quantity of contaminant equal to that which is generated leaves the room, $M(\tau)/\partial\tau = 0$, and Eq.10 gives the equilibrium concentration, $C(\infty)$ in the concentration at steady state:

$$C(\infty) = \frac{\dot{m}}{Q} \tag{11}$$

4 Step-down(tracer gas decay method)

4.1 What does the concentration graph mean?

The recorded decay of concentration versus time gives us the complementary cumulative age distribution function(1-F or 1-Φ_p). At time t, a certain amount of tracer gas is replaced by fresh air delivered from supply duct, therefore, the difference between the initial concentration and concentration at time t ($C(t)$) is the CDF of local age of air

4.2 How to express it by CDF?

$$F(\tau) = 1 - \frac{C(\tau)}{C(0)} \tag{12}$$

4.3 How to calculate mean value from cumulative distribution function?

Similar to Eq.8, the mean local age of air can be calculated:

$$\overline{\tau} = \int_0^\infty \frac{C(\tau)}{C(0)} d\tau \tag{13}$$

4.4 How to calculate ventilation rates?

There are two ways calculate ventilation rates, two-point decay method and multi-points decay method. According to mass balance, we can get following equation,

$$\begin{aligned} QC(\tau) &= \frac{dC(\tau)}{dt} \cdot V \\ \frac{Q}{V}C(\tau) &= \frac{dC(\tau)}{dt} \end{aligned} \tag{14}$$

By integrating the above equation, we can get:

$$\frac{Q}{V}t = In(C(\tau)) + C \tag{15}$$

C is a constant. By two-point method, we choose two points (usually initial concentration and one more point), for simplicity, $N = \frac{Q}{V}$ (N is air change rate).

$$\begin{aligned} NT_2 - NT_1 &= In(C(\tau_2)) - In(C(\tau_1)) \\ N &= \frac{In(C(\tau_2)) - In(C(\tau_1))}{\Delta T} \end{aligned} \tag{16}$$

The multi-point method simply uses least-square regression.

$$N = \frac{\sum \tau_i \sum In(C_i) - n \sum \tau_i In(C_i)}{n \sum \tau_i^2 - (\sum \tau_i)^2} \tag{17}$$

4.5 Least-square regression

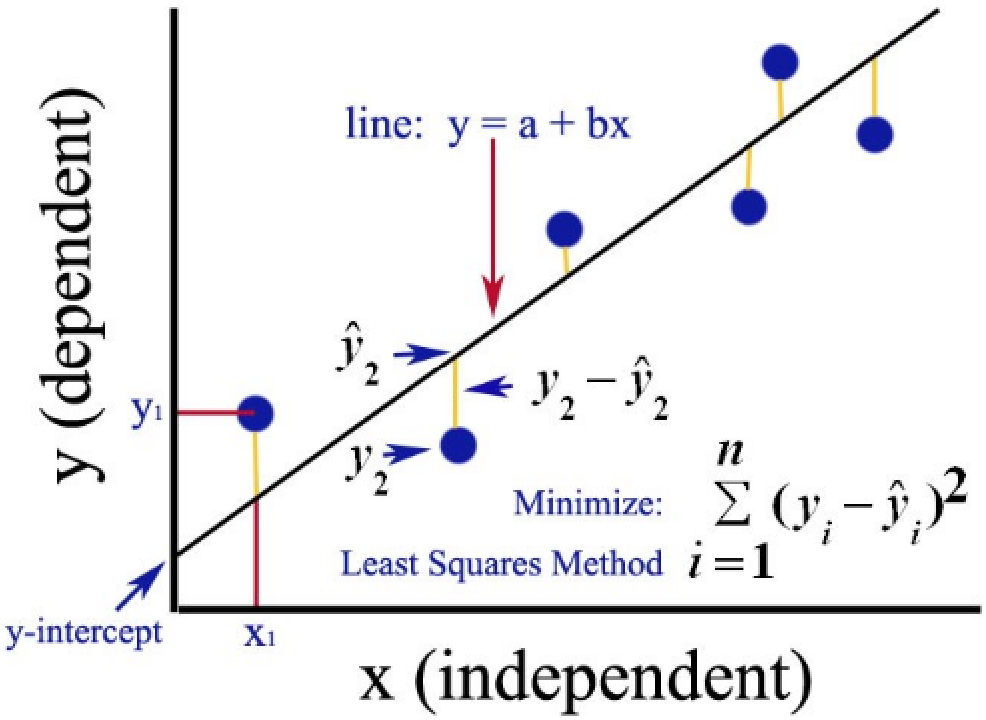

Figure 4: Least Regression

For example, y=kx+b.The convergence criteria is:

$$S(k, b) = \sum_{i=1}^{n} (y_i - kx_i - b)^2 \tag{18}$$

It should satisfy

$$\frac{\partial S}{\partial b} = -2 \sum_{i=1}^{n} (y_i - kx_i - b) = 0 \tag{19}$$

and

$$\frac{\partial S}{\partial k} = -2 \sum_{i=1}^{n} (y_i - kx_i - b)x_i = 0 \tag{20}$$

4.5.1 The solution of b

$$
\begin{aligned}
-2\sum_{i=1}^{n}(y_i - kx_i - b) = 0 \\
\sum_{i=1}^{n} y_i = nb + k\sum_{i=1}^{n} x_i \\
b = \frac{\sum_{i=1}^{n} y_i - k\sum_{i=1}^{n} x_i}{n} \\
b = \overline{y} - k\overline{x}
\end{aligned}
\tag{21}
$$

4.5.2 The solution of k

$$
\begin{aligned}
-2\sum_{i=1}^{n}(y_i - kx_i - b)x_i = 0 \\
\sum_{i=1}^{n}(x_i y_i - kx_i^2 - bx_i) = 0 \\
\sum_{i=1}^{n}(x_i y_i - kx_i^2 - (\overline{y} - k\overline{x})x_i) = 0 \\
\sum_{i=1}^{n}((y_i - \overline{y}) + k(\overline{x} - x_i)) = 0 \\
k = \frac{\sum_{i=1}^{n}(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})}
\end{aligned}
\tag{22}
$$

Least Square Method
Ordinary Least Square (OLS) Method for Linear Regression