1. VAJE

① a) Pretvori $101110_{(2)}$ in $11001.001_{(2)}$ v desetiški sistem

b) Pretvori $1362_{(10)}$ in $2.71875_{(10)}$ v dvojiški sistem.

a) $101110_{(2)} = 0\cdot2^0 + 1\cdot2^1 + 1\cdot2^2 + 1\cdot2^3 + 0\cdot2^4 + 1\cdot2^5 = 0+2+4+8+0+32 = 46_{(10)}$

$\times2^5 \quad 2^4 \quad \times2^3 \quad \times2^2 \quad \times2^1 \quad \times2^0$

$11001.001 = 1\cdot2^{-3} + 1\cdot2^0 + 1\cdot2^3 + 1\cdot2^4 = \frac{201}{8} = 25{,}125_{(10)}$

$\times2^{-1} \quad \times2^{-2} \quad \times2^{-3}$; $\quad 1/8 \quad 2 \quad 4 \quad 16$

b) Iz desetiškega v dvojiški si pomagamo z ostanki pri deljenju

$1362 : 2 = 681 + 0^{ost}$

16
02

$681 : 2 = 340 + 1^{ost}$

$340 : 2 = 170 + 0$

$170 : 2 = 85 + 0$

$85 : 2 = 42 + 1$ $\Longrightarrow$ Preberemo od spodaj navzgor

$42 : 2 = 21 + 0$ $\quad 10101010010_{(2)}$

$10 + 1$

$5 + 0$

$2 + 1$

$1 + 0$

$1 : 2 = 0 + 1$

• $2{,}71875_{(10)} =$

$2 : 2 = 1 + 0$ $\quad \Rightarrow 2_{(10)} = 10_{(2)}$

$1 : 2 = \underline{0} + 1$

za decimalke

$0{,}71875 \cdot 2 = 1{,}4375 = 1 + 0{,}4375$ $\quad \Rightarrow 0{,}71875_{(10)} = 0{,}10111_{(2)}$

$0{,}4375 \cdot 2 = 0 + 0{,}875$

$0{,}875 \cdot 2 = 1{,}75 = 1 + 0{,}75$

$0{,}75 \cdot 2 = 1{,}5 = 1 + 0{,}5$

$0{,}5 \cdot 2 = 1 + 0$

$$\boxed{2{,}71875_{(10)} = 10{,}10111_{(2)}}$$

Predstavljiva števila iz množice $P(b,t,L,U)$ so oblike $\pm m \cdot b^e$, $L \leq e \leq U$, $m = 0.c_1c_2\dots c_t$, $c_1 \neq 0$

IEEE754 STANDARD

- enojna natančnost: 32 bitov
  - → 1: predznak
  - → 8: eksponent → ker je $2^7 = 128$
  - → 23: mantisa

$(-1)^p \cdot (1.m) \cdot 2^{e-127}$

zato da bodo vsi možni eksponenti vedno pozitivni

↓

to bi mi zapisali kot $(2, 23, -126, 128)$

- dvojna natančnost: 64 bitov
  - → 1: predznak
  - → 11: eksponent → $2^{10} = 1024$
  - → 52: mantisa

ker je to največje možno število, ki ga zapišeš z 11-imi biti

$(-1)^p (1.m) \cdot 2^{e-1023}$

② a) Zapiši število 2.71875 v IEEE 754 standardu z dvojno natančnostjo.

b) Zapiši število 11 v IEEE754 z dvojno natančnostjo

a) ① najprej damo v dvojiški

$2{,}71875_{(10)} = 10{,}10111_{(2)} = 1{,}010111_{(2)} \cdot 2^1$

premaknemo vejico (podobno kot v desetiškem: $10{,}1_{(10)} = 1{,}01 \cdot 10^1$)

$\underbrace{010111}_{m}$

② predznak: $p = 0$

③ mantisa: $m = 010111$

④ eksponent: $(1 = e - 1023)$ $e = 1024_{(10)} = 1 \cdot 2^{10} + 0 \cdot 2^9 + 0 \cdot 2^8 + \dots + 0 \cdot 2^0 = 10000000000_{(2)}$

↑ ker more računalnik shraniti v dvojiškem zapisu

b) $11_{(10)} = 1011_{(2)} = 1{,}011_{(2)} \cdot 2^3$

$11 : 2 = 5 + 1$
$5 : 2 = 2 + 1$
$2 : 2 = 1 + 0$
$1 : 2 = 0 + 1$

$\Rightarrow p = 0$

$\Rightarrow m = 011$

$\Rightarrow e = 3 + 127 = 130 \xrightarrow{\text{v dvojiškega}}$

$130 : 2 = 65 + 0$ $= 10000010$
$65 : 2 = 32 + 1$
$32 : 2 = 16 + 0$
$16 : 2 = 8 + 0$
$8 : 2 = 4 + 0$
$4 : 2 = 2 + 0$
$2 : 2 = 1 + 0$
$1 : 2 = 0 + 1$

Poglejmo kako bi sedaj mi nazaj prišli do 11

računalnik: p: $\boxed{0}$ e: $\boxed{10000010}$ (8) m: $\boxed{01100\dots}$ (23)

p: ⇓ $(-1)^0$ ⇓ 1

e: ⇓ v desetiško 130 ⇓ $130 - 127 = 3$ ⇓ 2^3

m: ⇓ $1.011_{(2)} = 1 \cdot 2^0 + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = \frac{11}{8}$

$\Rightarrow 1 \cdot \frac{11}{8} \cdot 2^3 = 11$

osnovna zaokrožitvena napaka (zato se lahko največ zmotiš)
$u = \frac{1}{2} \cdot b^{1-t}$

③ V formatu $P(2,7,-10,10)$ zapiši število $x=13,7$. Izračunaj relativno napako, tj. $\frac{|x-fl(x)|}{|x|}$ in jo primerjaj z osnovno zaokrožitveno napako u, v danem formatu

• pretvorimo v dvojiški sistem

$13 : 2 = 6 + 1$
$6 : 2 = 3 + 0$
$3 : 2 = 1 + 1$
$1 : 2 = 0 + 1$

$13_{(10)} = 1101_{(2)}$

$13,7_{(10)} = 1101,1\overline{0110}$

$0,7 \cdot 2 = 1,4 = 1 + 0,4$
$0,4 \cdot 2 = 0,8 = 0 + 0,8$
$0,8 \cdot 2 = 1,6 = 1 + 0,6$
$0,6 \cdot 2 = 1,2 = 1 + 0,2$
$0,2 \cdot 2 = 0,4 = 0 + 0,4$
$0,4 \cdot 2 = 0,8 = 0 + 0,8$
$0,8 \cdot 2 = 1,6 = 1 + 0,6$
$1 + 0,2$
$0 + 0,4$
$\vdots$

$0,7_{(10)} = 0,10110\overline{0110}_{(2)}$

• premaknimo vejico $1101,1\overline{0110} = 0,1101101\overline{10} \cdot 2^4$

– zaokroževanje

$\geq \frac{b}{2} = 1$

To enko sedaj premikamo naprej do prve ničle, ki se spremeni v 1, vse ostale desno pa so 0

→ OP: če bi bilo namesto te 1 0, se ne bi pri zaokroževanju nič spremenilo, le zadnji dve številki bi zbrisali

zaokroženo: $\boxed{0,1101110 \cdot 2^4} = fl(x)$

Osnovno zaokroževanje je na 7 mest

sedaj poskusimo dati fl nazaj v desetiški zapis

$fl(x) = 1101,110_{(2)} = 1 \cdot 2^0 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^{-1} + 1 \cdot 2^{-2} = 13,75_{(10)}$

$1 \cdot 2^0 = 1$, $2^2 = 4$, $2^3 = 8$, $2^{-1} = 1/2$, $2^{-2} = 1/4$

relativna napaka: $\frac{|13,75-13,7|}{|13,7|} = \frac{0,05}{13,75} \doteq 0,00365$

poglejmo še osnovno zaokrožitveno napako zgornjega formata: $u = \frac{1}{2} \cdot 2^{1-7} = \frac{1}{2} \cdot 2^{-6} = 2^{-7} = 0,00781$

④ Naj bo $x = 0{,}1$

a) Pokaži, da velja $x = \sum_{i=1}^{\infty}(2^{-4i} + 2^{-4i-1}) = \sum_{i=1}^{\infty}(2^{-4i-1}(2+1)) = 3\cdot\sum_{i=1}^{\infty} 2^{-4i-1} = \frac{3}{2}\cdot\sum_{i=1}^{\infty} 2^{-4i} = \frac{3}{2}\cdot\sum_{i=1}^{\infty}(2^{-4})^{i} =$

$= \frac{3}{2}\cdot\sum_{i=1}^{\infty}\left(\frac{1}{16}\right)^{i+1} = \frac{3}{2}\cdot\frac{1}{16}\cdot\sum_{i=1}^{\infty}\left(\frac{1}{16}\right)^{i} =$

$= \frac{3}{32}\cdot\frac{1}{1-\frac{1}{16}} = \frac{3}{32}\cdot\frac{16}{15} = \frac{3}{30} = \frac{1}{10} = 0{,}1$

$\boxed{\text{geometrijska vrsta: } \sum_{i=0}^{\infty} q^i = \frac{1}{1-q} \quad ; |q| < 1}$

b) Zapiši binarni zapis za x s pomočjo točke (a)

15.10

$0{,}1_{(10)} = \underbrace{1\cdot 2^{-4} + 1\cdot 2^{-5}}_{i=1} + \underbrace{2^{-8} + 2^{-9}}_{i=2} + \ldots$

$0\cdot 2^{0} + 0\cdot 2^{-1} + 0\cdot 2^{-2} + 0\cdot 2^{-3}$

$= 0.\overline{0001}1_{(2)}$

c) Zapiši x v IEEE standardu z enojno in dvojno natančnostjo

$0.0\overline{0011}_{(2)} = 1.1001\overline{100} \cdot 2^{-4}$

$= \underbrace{1.100\overline{1100}\,\overline{1100}\,\overline{1100}\,\overline{1100}\,\overline{1100}\,1100}_{23 \text{ za enojno natančnost}} \,|\, 1100$

zaokrožimo

$= 1.100\,\underbrace{1100}_{\times 4}\,1101$

$p = 0$

$m = 100\,\underbrace{1100}_{\times 4}\,1101$

$e: \quad e - 127 = -4 \implies e = 123_{(10)} = 1111011_{(2)}$

$123 : 2 = 61 + 1$	
$61 : 2 = 30 + 1$	
$30 : 2 = 15 + 0$	
$15 : 2 = 7 + 1$	
$7 : 2 = 3 + 1$	
$3 : 2 = 1 + 1$	
$1 : 2 = 0 + 1$	

dvojiški sistem

DVOJNO: $m = 100\,\overbrace{1100}^{\times 11}1100\,1|100$ zaokroženo

$m = 100\,\underbrace{1100}_{11\times}\,11010$

$p = 0$

$e: \quad e - 1023 = -4 \implies e = 1019_{(10)} = 1111111011_{(2)}$

$1019 : 2 = 509 + 1$	
$509 : 2 = 254 + 1$	
$254 : 2 = 127 + 0$	
$127 : 2 = 63 + 1$	
$63 : 2 = 31 + 1$	
$31 : 2 = 15 + 1$	
$15 : 2 = 7 + 1$	
$7 : 2 = 3 + 1$	
$3 : 2 = 1 + 1$	
$1 : 2 = 0 + 1$	

VAJE 2

ZAOKROŽITVENE NAPAKE

Se lahko pojavijo pri:
- odštevanju približno enako velikih števil
- deljenju z majhnim številom

① Povej, kje nastopi težava in pretvori v stabilnejšo obliko

a) $x(\sqrt{x+1}-\sqrt{x})$

težava pri velikem x: saj bi se $\sqrt{x+1}-\sqrt{x}$ se odštelo v 0

$$\frac{x(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \frac{x(x+1-x)}{\sqrt{x+1}+\sqrt{x}} = \frac{x}{\sqrt{x+1}+\sqrt{x}}$$

b) $\sqrt{x+1}-1$

težava pri majhnem x-u

$$\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} = \frac{x+1-1}{\sqrt{x+1}+1} = \frac{x}{\sqrt{x+1}+1}$$

c) $\frac{1-\cos x}{x^2}$

Problem pri majhnem x-u

$$\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)} \xrightarrow{x\to 0} \frac{1}{2}$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

REŠEVANJE NELINEARNIH ENAČB

Vsako enačbo lahko pretvorimo na metodo za iskanje ničel : $f(x)=0$

- BISEKCIJA :

→ pogoji : f, $[a,b]$, $\text{sign}(f(a)) \neq \text{sign}(f(b))$ → v krajiščih more biti funkcija drugačno predznačena

→ vhod : f, $[a,b]$

```
for i=1, 2, ...
    c = (a+b)/2
    if sign(f(a)) = sign(f(c))
        a = c
    else :
        b = c
    end
end
```

→ izhod : približek za ničlo je sredina zadnjega intervala

② S tremi koraki bisekcije na intervalu $[1,2]$ določite približek za število $\sqrt{2}$.

si izmislimo funkcijo : $f(x) = x^2 - \sqrt{2}$

1) $f(1) = -1$

$f(2) = 2$

$c = 3/2 \quad f(3/2) = \frac{9}{4} - \frac{8}{4} = \frac{1}{4}$

2) $[1, 3/2]$

$c_2 = \frac{3/2 + 1}{2} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$

$f(5/4) = \frac{25}{16} - \frac{32}{16} = -\frac{7}{16}$

3) $[5/4, 3/2]$

$c_3 = \frac{1}{2}\left(\frac{5}{4} + \frac{6}{4}\right) = \frac{\frac{11}{4}}{2} = \frac{11}{8}$

$f(11/8) = \frac{121}{64} - \frac{128}{64} = -\frac{7}{64}$

$\Downarrow$

$[11/8, 3/2]$

→ polovična točka tega intervala bo naša aprox. $\frac{1}{2}\left(\frac{11}{8} + \frac{3}{2}\right) = \frac{\frac{11+12}{8}}{2} = \frac{23}{16} \doteq 1{,}438$

$\Downarrow$

$\boxed{\sqrt{2} \doteq 1{,}41}$

NAVADNA ITERACIJA

→ $f(x)=0 \iff g(x)=x$ iteracijska funkcija

→ uhod: g, x_0 → približek ?

for $r=0,1,\dots$

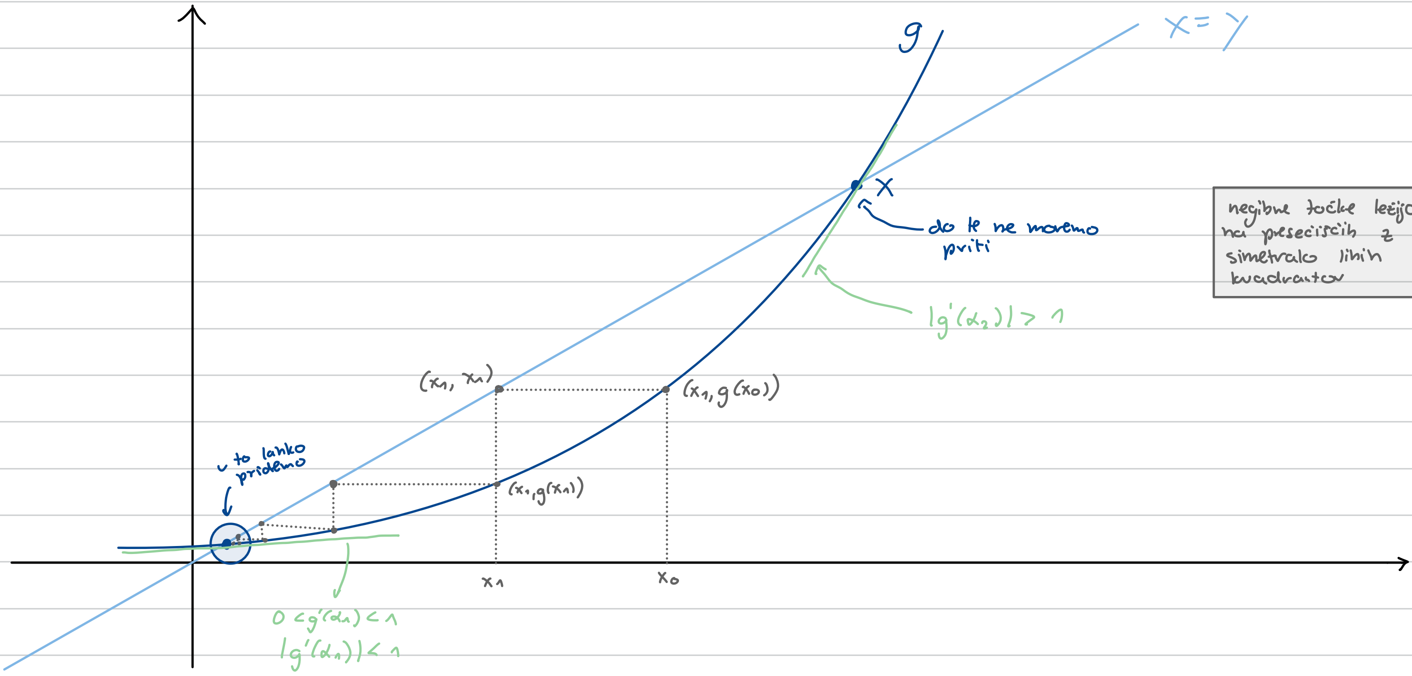
$x_{r+1}=g(x_r)$

end

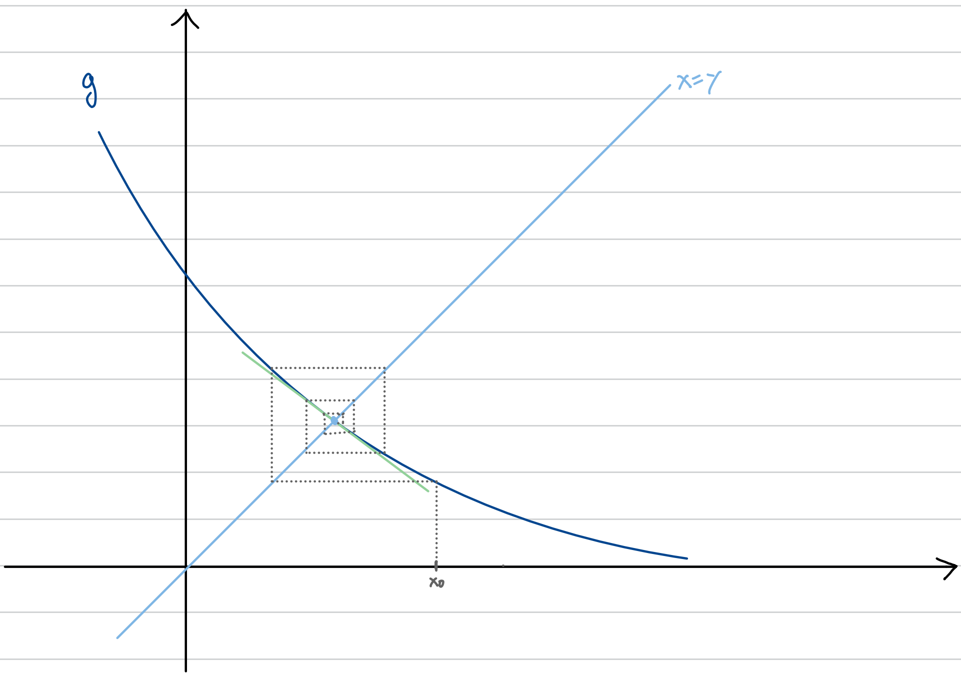
→ preprosta izbira bi bila $g(x)=x-f(x)$

Naj bo α negibna točka funkcije g $(g(\alpha)=\alpha)$:

- $|g'(\alpha)|<1$: α privlačna negibna točka (z ustreznim začetnim približkom x_0 bomo prišli v α)
- $|g'(\alpha)|>1$: α odbojna negibna točka



negibne točke ležijo na presečiščih z simetralo lihih kvadrantov



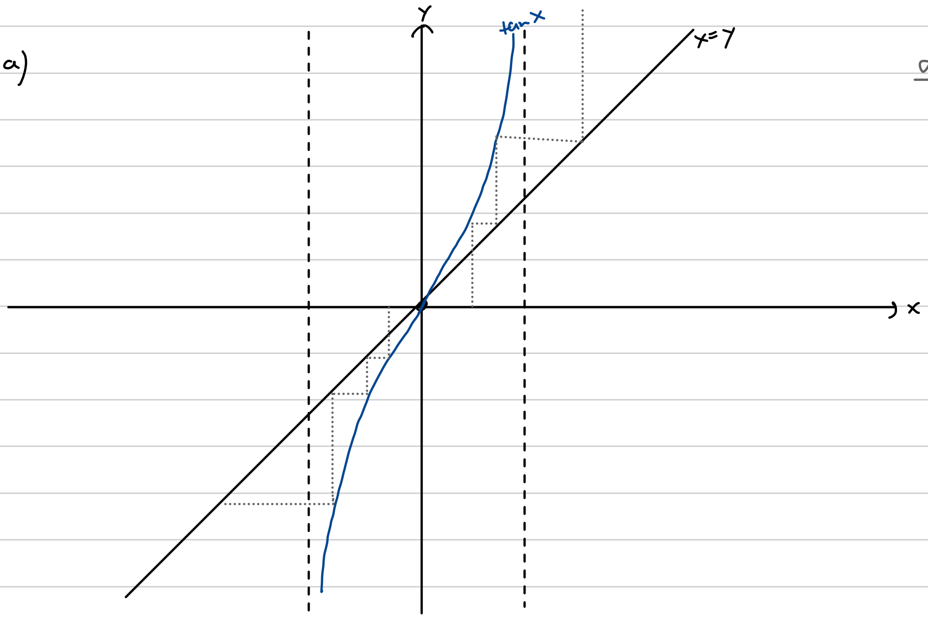
③ Za reševanje enačbe $x - \tan x = 0$ v okolici 0 sta dani dve iteracijski funkciji

a) $g_1(x) = \tan x$

b) $g_2(x) = \arctan x$

katera je primernejša

a)



α je odbojna negibna točka

↓

preverimo to še računsko

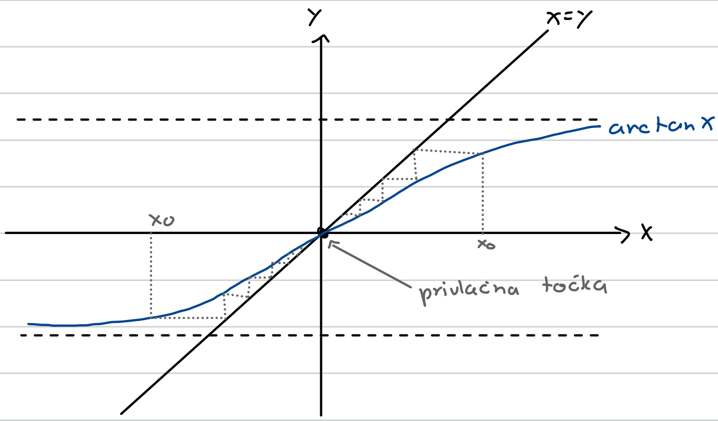
$\tan' x = \frac{1}{\cos^2 x} > 1$ za $\forall x$ razen $x = 0$

$\Downarrow$

$|g_1'(x)| > 1$ za $\forall x$ razen $x = 0$

$|g_1'(0)| = 1$

b)



$|g_2'(x)| < 1$

$\arctan' x = \frac{1}{1+x^2}$ za $\forall x \in \mathbb{R}$ razen $x = 0$

$|g_2'(0)| = 1$

22.10

KONVERGENCA:

- Naj bo g zvezno odvedljiva na $I = [\alpha - \delta, \alpha + \delta]$ kjer je α negibna točka, in naj velja $|g'(x)| < 1$ za $\forall x \in I$.
  Potem za $\forall x_0 \in I$ zaporedje $x_{r+1} = g(x_r)$ konvergira k α.
- $g(\alpha) = g'(\alpha) = \ldots = g^{(p-1)}(\alpha) = 0$; $g^{(p)}(\alpha) \neq 0$
  $\Rightarrow$ red konvergence je p

① Iteracijska funkcija je podana kot $g(x) = -x^2+8x-12$

a) Določite negibne točke in privlačnost / odbojnost (to smo naredili že prejšnji teden)

$g(x) = x$

$-x^2+8x-12 = x$

$x^2-7x+12=0$

$-(x-3)(x-4)=0 \rightarrow x_1=4$, $x_2=3$

$g'(x) = -2x+8$

$g'(4) = 0 < 1$ privlačna

$g'(3) = 2 > 1$ odbojna

b) Za katere x_0 nam konvergenčni izrek zagotavlja konvergenco?

$g'(x) = -2x+8$

$|-2x+8| < 1 \longrightarrow$ ① $-2x+8 \geq 0$

$-2x+8 < 1$

$-2x < -7 \quad :(-2)$

$x > 7/2$

② $-2x+8 < 0$

$2x-8 < 1$

$2x < 9$

$x < 9/2$

$\Rightarrow x \in (7/2, 9/2)$

če zberemo katerokoli točko iz tega intervala, bo konvergirala k 4.

c) Poišči še ostale x, za katere konvergira k negibni točki

narišimo lepše graf

$g(x) = -x^2+8x-12$

ničle: $-x^2+8x-12=0$

$-(x^2-8x+12)=0$

$-(x-6)(x-2)=0$

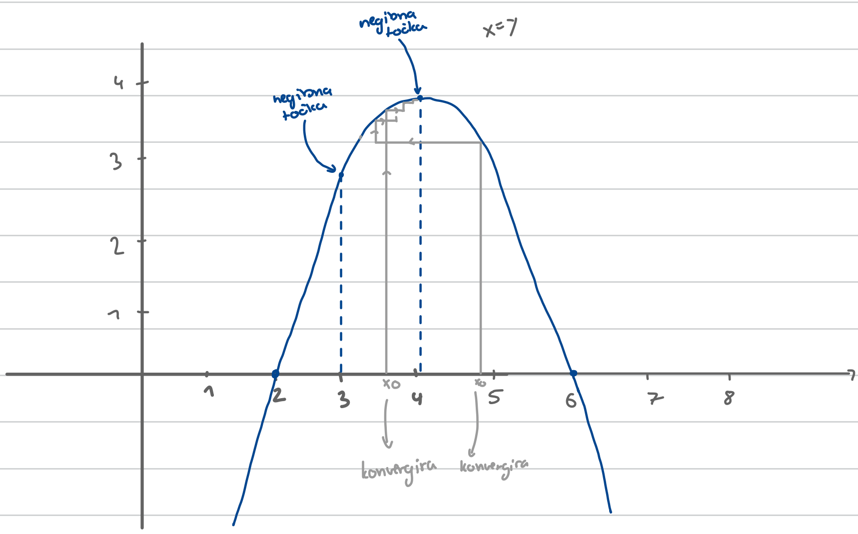
$x_1=6 \quad x_2=2$

teme: $-2x+8=0$

$-2x=-8$

$x=4$

$f(4) = -16+32-12 = 4$



$x_0 \in (3,5)$ konvergira

$x_0 \in (3,5) \Rightarrow \lim_{r\to\infty} x_{r+1} = 4$

$\lim_{r\to\infty} (x_{r+1}-4) = 0$

$x_{r+1} = g(x_r)$

$x_{r+1} = -x_r^2+8x_r-12$

$x_{r+1}-4 = -x_r^2+8x_r-16 = -(x_r^2-8x_r+16) = -(x_r-4)^2 = -(-(x_{r-1}-4)^2)^2 = -(x_{r-1}-4)^{2^2}$

$\Rightarrow x_r-4 = -(x_{r-1}-4)^2$

če bi ta postopek še r-1 krat ponovili, da dobimo x_0, bi dobili:

$= -(x_0-4)^{2^{r+1}}$

$\lim_{r\to\infty} (x_{r+1}-4) = \lim_{r\to\infty} \underbrace{(x_0-4)}_{\text{je po } |\cdot|<1}{}^{2^{r+1}} = 0$

$x_0 \in (3,5) \Rightarrow x_0-4 \in (-1,1)$

$\boxed{\text{analiza: } \lim_{r\to\infty} g^r = 0 \ ; \ |g|<1}$

ko smo izračunali, da je lim=0, smo dokazali, da na intervalu (3,5) sigurno konv proti 4

Tako so rešili c) nalogo lani

Kaj pa ostale točke, ki niso v tem intervalu?

Desna: $x_0 \in (-\infty, 3)$

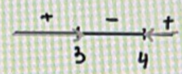
$x_1 < x_0$

$x_1 = 8x_0 - 12x_0 - x_0^2$

$8x_0 - 12x_0 - x_0^2 < x_0$

$x_0^2 - 7x_0 + 12x_0 > 0$

$(x_0 - 3)(x_0 - 4) > 0$



$\Rightarrow$ Divergira

Leva: $x_0 \in (4, \infty)$

$x_1 < 3$

$8x_0 - 12x_0 - x_0^2 < 3$

$x_0^2 - 8x_0 + 15 > 0$

$(x_0 - 3)(x_0 - 5) > 0$ $\Rightarrow$ Divergira, ker je $x_0 > 5$ (mogel bi pa bit samo $x_1 < 3$)

$x_0 \in (3, 5)$

Pokažimo, da je x_1 bližje 4 kot x_0

$|4 - x_1| < |4 - x_0|$

$|(x_0 - 4)^2| < |4 - x_0|$

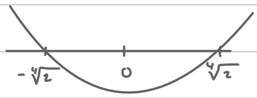
$|x_0 - 1| < 1$ $\Rightarrow$ Konvergira

② Ničlo funkcije $f(x) = x^5 - 10x + 1$ iščemo z iteracijo $g(x) = \frac{x^5+1}{10}$

a) Pokažite, da ima f na intervalu $[0, 0.2]$ natanko eno ničlo (α).

b) Ali začetni približek $x_0 = 0$ zagotavlja konvergenco k α?

c) Pri $x_0 = 0$ določite x_1 in x_2



$f(0) = 1 \quad > 0$

$f(1/5) < 0$

} ker se spremeni predznak, je tam ničla

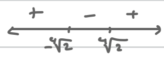
$f'(x) = 5x^4 - 10 \longrightarrow$ to bo negativno za $\forall x \in [0, 0.2]$, torej ves čas pada in je ena ničla

$f'(0) < 0$

$f'(1/5) < 0$

če izračunaš ničle k funkcije

$0 = 5(x^4 - 2) = 5(x - \sqrt[4]{2})(x + \sqrt[4]{2})(x^2 + \sqrt{2}) \rightarrow$



Na območju med $-\sqrt[4]{2}$ in $\sqrt[4]{2}$ je $f'(x) < 0$ ✓

b) Najdimo okolico, ki jo uporabimo v konvergenčnem izreku, in poglejmo, če je 0 na tem intervalu.

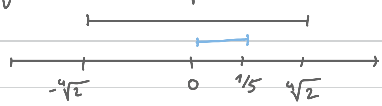
$|g'(x)| < 1$

$\frac{5x^4}{10} < 1$

$x^4 < 2$

$-\sqrt[4]{2} < x < \sqrt[4]{2} \longrightarrow$ Vsi ti x zagotavljajo da bomo iz točke konvergirali proti ničli

Torej začetni približek $x_0 = 0$ zagotovo konv k α.



c) $x_1 = g(0) = \frac{1}{10}$

$x_2 = g(x_1) = g\left(\frac{1}{10}\right) = \frac{\left(\frac{1}{10}\right)^5 + 1}{10} = \frac{1}{10^6} + \frac{1}{10} = \frac{100001}{1000000}$

③ a) Pokažite, da lahko $\sqrt{a}$ izračunamo z iteracijo:

$$x_{r+1} = x_r \cdot \frac{x_r^2+3a}{3x_r^2+a}$$

b) Določi red konvergence

c) Pokažite, da metoda konvergira za $\forall x_0 > 0$

a) $g(x) = x\,\frac{x^2+3a}{3x^2+a}$

poglejmo če je sploh iteracija vredu → torej more biti $\sqrt{a}$ negibna točka

1) $\sqrt{a}$ je negibna točka : $\sqrt{a} = g(\sqrt{a})$

2) $\sqrt{a}$ je privlačna

poiščimo vse negibne točke

$$x = x\,\frac{x^2+3a}{3x^2+a}$$

$$3x^3+xa = x^3+3xa$$

$$2x^3 = 2xa$$

$$x^3 = xa$$

$$x^2 = a$$

$$x = \pm\sqrt{a}$$

Imamo 3 negibne točke: $\sqrt{a}$, $-\sqrt{a}$, 0

ali je $\sqrt{a}$ privlačna

$$g'(x) = \frac{x^2+3a}{3x^2+a} + x\,\frac{2x(3x^2+a)-(x^2+3a)6x}{(3x^2+a^2)} =$$

$$= \frac{x^2+3a}{3x^2+a} + x\,\frac{6x^3+2xa-6x^3-18xa}{(3x^2+a)^2} =$$

$$= \frac{(x^2+3a)(3x^2+a)-16x^2a}{(3x^2+a)^2} =$$

$$= \frac{3x^4+10x^2a+3a^2-16x^2a}{(3x^2+a)^2} =$$

$$= \frac{3x^4-6x^2a+3a^2}{(3x^2+a)^2} =$$

$$= \frac{3(x^2-a)^2}{(3x^2+a)^2}$$

$|g'(\sqrt{a})| = 0 < 1$ ✓ → je privlačna

b) $$g''(x) = \frac{(3(x^2-a)2x\cdot 2)(3x^2+a)^2 - 3(x^2-a)^2(3x^2+a)12x}{(3x^2+a)^4} =$$

$$= \frac{12x(x^2-a)(3x^2+a)((3x^2+a)-3(x^2-a))}{(3x^2+a)^4}$$

$$= \frac{12x(x^2-a)4a}{(3x^2+a)^3}$$

$|g''(\sqrt{a})| = 0$

to bomo odvajali po pravilu za produkt

$$g'''(x) = 2x\,\frac{12x\,4a}{(3x^2+a)^3} + (x^2-a)\cdot\ldots$$

$$g'''(\sqrt{a}) = \frac{96a^2}{(4a)^3} = \frac{96a^2}{64a^3} = \frac{3}{2}a \neq 0$$ ⇒ konvergenca je kubična

c) $|g'(x)| < 1$

$$\left|\frac{3(x^2-a)^2}{(3x^2+a)^2}\right| < 1$$

$\forall x_0$ iteracija konvergira

$$g'(x) = \frac{3(x^2-a)^2}{(3x^2+a)^2}$$

$x_0 \in (0,\sqrt{a}) \Rightarrow x_{r+1} > x_r$ in $x_r < \sqrt{a}$ → naraščajoče navzgor omejeno zap $\Rightarrow$ ima limito

$x_{r+1} = g(x_r)$

- $x_{r+1} = x_r \frac{x_r^2+3a}{3x_r^2+a} - \sqrt{a} < 0$

$$\frac{x_r^3+3x_ra-3x_r^2\sqrt{a}-a^{3/2}}{3x_r^2+a \; >0} < 0$$

$$(x_r-\sqrt{a})^3 < 0$$

$$x_r < \sqrt{a} \checkmark$$

- $x_{r+1} > x_r$

$$x_r \frac{x_r^2+3a}{3x_r^2+a} > x_r$$

$$x_r\left(\frac{x_r^2+3a}{3x_r^2+a} - 1\right) > 0$$

$$x_r\left(\frac{-2x_r^2+2a}{3x_r^2+a}\right) > 0$$

$$x_r\left(\frac{a-x_r^2}{3x_r^2+a \; >0}\right) > 0$$

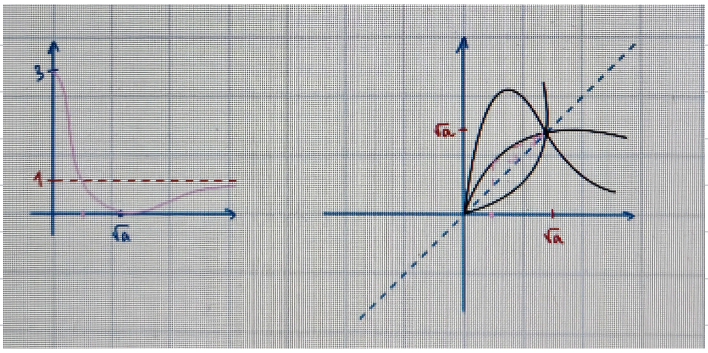
$x_r(a-x_r^2) > 0$, $x_r > 0$

$a - x_r^2 > 0 \longrightarrow x_r < \sqrt{a}$

$\Downarrow$

$x_{r+1} > x_r \checkmark$

$$\begin{bmatrix} x_0 < \sqrt{a} \Rightarrow x_1 > x_0 \\ x_1 < \sqrt{a} \Rightarrow x_2 > x_1 \end{bmatrix}$$



① Enačbo $x^3-A=0$ $(A\neq 0)$ rešujemo z nastavkom $x_{r+1}=\beta_1 x_r+\frac{\beta_2}{x_r^2}$ za $r=0,1,\dots$. Določite β_1 in β_2 tako, da bo konvergenca vsaj kvadratična $(g'(\alpha)=0)$. Ali je konvergenca kubična?

α negibna točka : $g'(\alpha)=\dots=g^{(p-1)}(\alpha)=0$
$g^{(p)}(\alpha)\neq 0$: konvergenca je p

$g(x)=\beta_1 x+\frac{\beta_2}{x^2}$ $\qquad$ $x^3-A=0 \longrightarrow x=\sqrt[3]{A}$

$|g'(x)|<1$, negibna

$g'(x)=\beta_1+\frac{2\beta_2}{x^3}$

$g'(\sqrt[3]{A})=\beta_1-\frac{2\beta_2}{A}$

$\beta_1-\frac{2\beta_2}{A}=0 \implies \boxed{\beta_1=\frac{2\beta_2}{A}}$

$\sqrt[3]{A}$ negibna $\quad (g(\sqrt[3]{A})=\sqrt[3]{A})$

$g(x)=\beta_1 x+\frac{\beta_2}{x^2}$

$g(\sqrt[3]{A})=\beta_1\sqrt[3]{A}+\frac{\beta_2}{(\sqrt[3]{A})^2}$

$\beta_1\sqrt[3]{A}+\frac{\beta_2}{(\sqrt[3]{A})^2}=\sqrt[3]{A}$

$\beta_1+\frac{\beta_2}{A}=1 \longrightarrow \beta_1=1-\frac{\beta_2}{A}$

sistem dveh enačb

$1-\frac{\beta_2}{A}=\frac{2\beta_2}{A}$

$\frac{3\beta_2}{A}=1$

$\beta_2=\frac{A}{3} \longrightarrow \beta_1=\frac{2(\frac{A}{3})}{A}=\frac{2}{3}$

$x_{r+1}=\frac{2}{3}x_r+\frac{A}{3x_r^2}$

$g(x)=\frac{2}{3}x+\frac{A}{3x^2}$

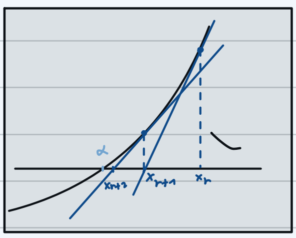
$g'(x)=\frac{2}{3}-\frac{2A}{3x^3}$

$g'(\sqrt[3]{A})=\frac{2}{3}-\frac{2}{3}=0$

$g''(x)=\frac{9x^2\cdot 2A}{9x^6}=\frac{2A}{x^4}$

$g''(\sqrt[3]{A})=\frac{2\sqrt[3]{A}}{A^{4/3}}\neq 0 \implies$ kvadratična konvergenca

TANGENTNA METODA



enačba tangente v $(x_r, f(x_r))$:

$y-f(x_r)=f'(x_r)(x-x_r)$

ničla : $-f(x_r)=f'(x_r)(x-x_r) \implies \boxed{x_{r+1}=x_r-\frac{f(x_r)}{f'(x_r)}}$

↳ iteracijska funkcija g za tangentno metodo

② a) Izpeljite metodo za računanje števila $\sqrt{a}$ s pomočjo tangentne metode

b) Kakšen je red konvergence?

c) Dokažite, da metoda konvergira za $\forall x_0 > 0$

a) $f(x) = x^2 - a$

$$g(x) = x - \frac{x^2-a}{2x} = \frac{2x^2-x^2+a}{2x} = \frac{x^2+a}{2x}$$

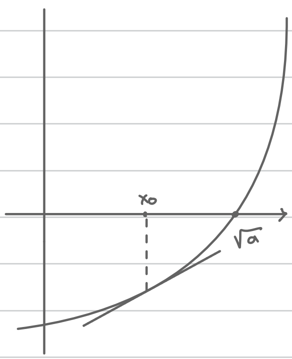
b) $$g'(x) = \frac{2x\cdot 2x - 2x^2 - 2a}{4x^2} = \frac{2(x^2-a)}{4x^2} = \frac{(x^2-a)}{2x^2}$$

$$g'(\sqrt{a}) = \frac{a-a}{a} = 0$$

$$g''(x) = \frac{2x\cdot 2x^2 - 2x(x^2-a)}{4x^4} = \frac{4xa}{4x^4} = \frac{a}{x^3}$$

$g''(\sqrt{a}) \neq 0 \longrightarrow$ red je 2

c)



$x_0 \in (0, \sqrt{a}) \Rightarrow x_1 > \sqrt{a}$

$\rightarrow x_0 \in (\sqrt{a}, \infty) \Rightarrow \sqrt{a} < x_{r+1} < x_r$

$\rightarrow x_1 = x_0 - \frac{x_0^2-a}{2x_0} = \frac{x^2+a}{2x_0} > \sqrt{a}$

$$\frac{x_0^2+a-2x_0\sqrt{a}}{2x_0} > 0$$

$x_0^2 + a - 2x_0\sqrt{a} = 0$

$\longrightarrow D = 4a - 4a = 0 \quad \Rightarrow x_{1,2} = \frac{2\sqrt{a}}{2} = \sqrt{a}$

$\rightarrow x_{r+1} = \frac{x_r^2+a}{2x_r} < x_r$

$$\frac{x_r^2+a-2x_r^2}{2x_r} < 0$$

$$\frac{-x_r^2+a}{2x_r} < 0$$

$-x_r^2 + a < 0$

$\sqrt{a} < x_r$

$\rightarrow x_{r+1} > \sqrt{a}$

$$\frac{x_r^2+a}{2x_r} > \sqrt{a}$$

$$\frac{x_r^2+a-2\sqrt{a}\cdot x_r}{2x_r} > 0$$

$$\frac{(x_r-\sqrt{a})^2}{2x_r} > 0 \quad ✓$$

③ Naj bo $a>0$. Podobna je funkcija $f(x)=x^3-a$

a) Ničlo funkcije f iščemo s tangentno metodo. Zapišite iteracijsko funkcijo g in se prepričajte, da velja $g'(\sqrt[3]{a})=0$

$$g(x)=x-\frac{x^3-a}{3x^2}=x-\frac{x}{3}+\frac{a}{3x^2}$$

$$g'(x)=1-\frac{1}{3}+\frac{a}{3}(-2)x^{-3}=\frac{2}{3}-\frac{2a}{3x^3}$$

$$g'(\sqrt[3]{a})=\frac{2}{3}-\frac{2a}{3a}=0 \checkmark$$

b) Naj bo $g_2=g\circ g$. Pokažite, da je g_2 tudi ustrezna za iskanje ničle funkcije f. Nato pokažite, da je red konvergence pri uporabi g_2 vsaj 4.

Potrebno je pokazati, da velja $g_2(\sqrt[3]{a})=\sqrt[3]{a}$ in $|g_2'(\sqrt[3]{a})|<1$

$$g\circ g=g(g(x))$$

$$(g\circ g)(x)=g_2(x)$$

$$g_2(\sqrt[3]{a})=g(g(\sqrt[3]{a}))=g(\sqrt[3]{a})=\sqrt[3]{a} \checkmark$$

$\rightarrow (g(g(x)))'=g'(g(x))\cdot g'(x)$

- $g'(g(x))=g'(\sqrt[3]{a})$
- $g_2'(\sqrt[3]{a})=g'(\sqrt[3]{a})\cdot g'(\sqrt[3]{a})=0$

$\rightarrow g_2''(x)=g''(g(x))\,g'(x)\cdot g'(x)+g'(g(x))\cdot g''(x)$

- $g_2''(\sqrt[3]{a})=0$

$$\rightarrow g_2'''(x)=g'''(g(x))\,(g'(x))^3+2g''(g(x))\,g''(x)\,g'(x)+g''(g(x))\cdot g'(x)\cdot g''(x)+g'(g(x))\cdot g'''(x)=$$
$$=g'''(g(x))\,(g'(x))^3+3g''(g(x))\,g'(x)\,g''(x)+g'(g(x))\,g'''(x)$$

- $g_2'''(\sqrt[3]{a})=0$

$\Longrightarrow$ red konvergence je res vsaj 4

KONVERGENCA TANGENTNE METODE

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)\cdot f'(x) - f(x)\cdot f''(x)}{(f'(x))^2} = \cancel{1} - \cancel{1} + \frac{f(x)\ f''(x)}{(f'(x))^2}$$

$$g'(\alpha) = \frac{\overset{=0}{f(\alpha)}\cdot f''(\alpha)}{(f'(\alpha))^2}$$

$= 0$: v primeru, da je α enostavna ničla, saj je $f'(\alpha) \neq 0$ } vsaj kvadratična

$= ?$: v primeru, da je α večkratna ničla

① Naj bo α m-kratna ničla funkcije f. Dokažite, da velja:

$$\lim_{x\to\alpha} g'(x) = 1 - \frac{1}{m}$$

razpišimo s Taylorjevo vrsto:

$$TV(f)(x) = \underbrace{f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \dots + \frac{f^{(m-1)}(\alpha)}{(m-1)!}(x-\alpha)^{m-1}}_{=0} + \frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m \dots$$

vemo, da je $f(\alpha) = 0$ in ker imamo, da je m-kratna ničla vemo:

$f'(\alpha) = 0$

$\vdots$

$f^{(m-1)}(\alpha) = 0$

Torej bo prvih $(m-1)$ členov v TV enako 0.

• Poglejmo sedaj limito $f(x)$

→ $\lim_{x\to\alpha} \frac{f(x)}{(x-\alpha)^m}$

→ razpišimo s Taylorjevo vrsto : $\lim_{x\to\alpha} \frac{\frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m + \dots}{(x-\alpha)^m} = \underbrace{\lim_{x\to\alpha} \frac{f^{(m)}(\alpha)}{m!}}_{= \frac{f^{(m)}(\alpha)}{m!}} + \underbrace{\lim \frac{f^{(m+1)}(\alpha)}{(m+1)!}(x-\alpha) + \dots}_{\to 0}$ $\neq 0$

→ Torej smo dobili $\lim_{x\to\alpha} \frac{f(x)}{(x-\alpha)^m} = \frac{f^{(m)}(\alpha)}{m!}$

→ Poglejmo še limito $\lim_{x\to\alpha} \frac{f'(x)}{m(x-\alpha)^{m-1}} = \frac{f^{(m)}(\alpha)}{m!}$

Sedaj lahko na tej limiti uporabimo L'hopitala : $\lim_{x\to\alpha} \frac{f(x)}{(x-\alpha)^m} \overset{L'H}{\underset{\text{"}\frac{0}{0}\text{"}}{=}} \lim_{x\to\alpha} \frac{f'(x)}{m(x-\alpha)^{m-1}} = \frac{f^{(m)}(\alpha)}{m!}$

→ Torej smo ugotovili $\lim_{x\to\alpha} \frac{f(x)}{(x-\alpha)^m} = \frac{f^{(m)}(\alpha)}{m!} = \lim_{x\to\alpha} \frac{f'(x)}{m(x-\alpha)^{m-1}}$

• Sedaj želimo pokazati, da je $\lim_{x\to\alpha} g(x) = \alpha$

$$\lim_{x\to\alpha} \left(\underset{\to\alpha}{x} - \frac{f(x)}{f'(x)}\right) = \alpha - \lim_{x\to\alpha} \frac{f(x)}{f'(x)} \underbrace{\frac{(x-\alpha)^{m-1}\,(x-\alpha)\,m}{(x-\alpha)^{m-1}\,(x-\alpha)\,m}} = \alpha - \underline{0}$$

$\left(\frac{f^{(m)}(\alpha)}{m!}\right)^{-1}$ $\qquad$ $\frac{f^{(m)}(\alpha)}{m!}$ $\qquad$ $(x-\alpha) \to 0$

to želimo "na silo" vriniti, da upeljemo zgornje limite

Torej smo pokazali, da je $\lim_{x\to\alpha} g(x) = \alpha$

· Sedaj moramo dokazati samo še da velja $\lim_{x\to\alpha} g'(x) = 1 - \frac{1}{m}$

$$\lim_{x\to\alpha} g'(x) = \lim_{x\to\alpha}\left(\lim_{h\to 0} \frac{1}{h}\left(g(h+x)-g(x)\right)\right) =$$

po def odvoda

$$= \lim_{x\to\alpha}\left(\lim_{x\to\alpha} \frac{g(\alpha)-g(x)}{\alpha-x}\right)$$

preoblikujemo, da bo šel v limiti $x\to\alpha$

$$= \lim_{x\to\alpha} \frac{\alpha-g(x)}{\alpha-x}$$

$$= \lim_{x\to\alpha} \frac{\alpha-x+\frac{f(x)}{f'(x)}}{\alpha-x}$$

ker uporabljamo tangentno metodo, vemo kaj je $g(x)$

$$= \lim_{x\to\alpha} 1 + \underbrace{\frac{f(x)}{f'(x)(\alpha-x)}}$$

Sedaj želimo dokazati da gre to proti $\frac{1}{m}$, spet uporabimo tisti dve limiti

$$= 1 + \lim_{x\to\alpha} \frac{f(x)}{f'(x)(\alpha-x)}$$

$$= 1 + \lim_{x\to\alpha} \frac{1}{(\alpha-x)} \underbrace{\frac{f(x)}{(x-\alpha)^m}}_{\frac{f^{(m)}(\alpha)}{m!}} \underbrace{\frac{m(x-\alpha)^{m-1}}{f'(x)}}_{\frac{m!}{f^{(m)}(\alpha)}} \frac{(x-\alpha)}{m}$$

$$= 1 + \lim_{x\to\infty}\left(\frac{f^{m}(\alpha)}{m!}\right)^0 \cdot \frac{1}{m}$$

$$= 1 + 1\cdot\frac{1}{m}$$

$$= 1 + \frac{1}{m}$$

Popravek tangentne metode, da bo reda vsaj 2:

$$\boxed{g_1(x) = x - m\cdot\frac{f(x)}{f'(x)}}$$

želimo, da velja $g'(\alpha)=0$

$$g_1'(x) = 1 - m\cdot\frac{(f'(x))^2 - f(x)\cdot f''(x)}{(f'(x))^2} = 1 - m + m\,\frac{f(x)\cdot f''(x)}{(f'(x))^2} = 1 - m + m g'(x)$$

$$g_1'(\alpha) = 1 - m + m\cdot\underbrace{\lim_{x\to\alpha} g'(x)}_{\text{to smo dokazali v prejšnji nalogi}} = 1 - m + m\left(1-\frac{1}{m}\right) = 0$$ ✓

② Naj bo $f \in \mathcal{C}^2$ in α njena enostavna ničla

a) Pokažite, da metoda z iteracijsko funkcijo

$$g(x) = x - \frac{2 \cdot f(x) \cdot f'(x)}{2 \cdot (f'(x))^2 - f(x) \cdot f''(x)}$$

ustreza tangentni metodi za funkcijo $F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$

$$g(x) = x - \frac{F(x)}{F'(x)} =$$

$$F'(x) = \left(\frac{f(x)}{\sqrt{|f'(x)|}}\right)' = \frac{f'(x)\sqrt{|f'(x)|} - f(x) \cdot \frac{1}{2}\frac{1}{\sqrt{|f'(x)|}} \cdot \frac{f'(x)}{|f'(x)|} \cdot f''(x)}{|f'(x)|} = \frac{\frac{f(x)\, 2\, {}^3\sqrt{|f'(x)|}\sqrt{|f'(x)|} \cdot |f'(x)| - f(x) \cdot f'(x) \cdot f''(x)}{2\sqrt{|f'(x)|}\ |f'(x)|}}{|f'(x)|} = \frac{\cancel{f'(x)}\left(2(f'(x))^2 - f(x) \cdot f''(x)\right)}{2\sqrt{|f'(x)|} \cdot f'(x)^{\cancel{2}}}$$

$$\left(\sqrt{|f'(x)|}\right)' = \frac{1}{2}\frac{1}{\sqrt{|f'(x)|}} \cdot \frac{f'(x)}{|f'(x)|} \cdot f''(x)$$

#ko daš $|abs|^2$ lahko abs spustimo

$$\boxed{|x|' = \begin{cases} 1 & ; \text{ če } x > 0 \\ -1 & ; \text{ če } x < 0 \end{cases} = \frac{x}{|x|}}$$

to je trik za odvajanje abs. vr.

$$g(x) = x - \frac{f(x)}{\sqrt{|f'(x)|}} \frac{2\sqrt{|f'(x)|} \cdot f'(x)}{(2(f'(x))^2 - f(x)f''(x))} = x - \frac{2f(x) \cdot f'(x)}{2(f'(x))^2 - f(x) \cdot f''(x)}$$

b) Poenostavite metodo za $f(x) = x^2 - a$

$f(x) = x^2 - a$

$f'(x) = 2x$

$f''(x) = 2$

$$g(x) = x - \frac{2(x^2-a)(2x)}{2(4x^2) - (x^2-a) \cdot 2} =$$

$$= \frac{4x^3 - x^3 + ax - 2x^3 + 2ax}{4x^2 - x^2 + a}$$

$$= \frac{x^3 + 3ax}{3x^2 + a}$$

TO JE HALEYEVA METODA

③ V nekem naselju je 1250 stanovanj, v katerih živi 2000 oseb. Naslednjih 20 let bo število ljudi v tem naselju opisovala funkcija $f(x) = 1000 + 1000 \cdot 2^{\frac{x}{20}}$

Oblasti načrtujejo gradnjo novih stanovanj s konstantno hitrostjo 10 stanovanj na leto.
Zanima nas čez koliko časa bo število oseb enako 2x števila stanovanj.
Opiši kako bi rešitev tega problema poiskali s sekantno metodo

x ... pomeni čas
število ljudi: $f_1(x) = 1000 + 1000 \cdot 2^{\frac{x}{20}}$
število stanovanj: $f_2(x) = 1250 + 10 \cdot x$

Želimo, da velja $\underline{f_1(x) = 2 \cdot f_2(x)}$

$$1000 + 1000 \cdot 2^{\frac{x}{20}} = 2 \cdot (1250 + 10 \cdot x)$$
$$1000 + 1000 \cdot 2^{\frac{x}{20}} = 2500 + 20x$$
$$1000 \cdot 2^{\frac{x}{20}} - 20x = 1500$$
$$-1500 + 1000 \cdot 2^{\frac{x}{20}} - 20x = 0$$

Torej mi bomo numerično iskali ničlo funkcije $F(x) = -1500 + 1000 \cdot 2^{\frac{x}{20}} - 20x$ na intervalu $x \in [0, 20]$

- Najprej preverimo, da $F(x)$ sploh ima ničlo

$$\left.\begin{aligned} F(0) &= -1500 + 1000 \cdot 2^0 - 0 = -500 \\ F(20) &= -1500 + 1000 \cdot 2^1 - 20 \cdot 20 = -1500 + 1000 \cdot 2 - 400 = 100 \end{aligned}\right\} \Rightarrow$$ na tem intervalu ničla je, saj je F zvezna

- <u>Kako pa bi poiskali ničlo</u>

Pri sekantni metodi odvod nadomestimo s kvocientom, torej namesto $f'(x_r)$ uporabimo $\frac{f(x_r) - f(x_{r-1})}{x_r - x_{r-1}}$, ostalo pa je enako kot v formuli za tangentno metodo

tangentna: $g(x) = x - \frac{f(x)}{f'(x)}$ $\longrightarrow$ $\boxed{\text{sekantna: } g(x) = x - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})} = x_{r+1}}$

V tej nalogi bi naprimer vzeli $x_0 = 20$ in $x_1 = 19$ in potem bi po nekaj korakih prišli v 17,86

NORME

- **Vektorske norme**:
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$
$$\|x\|_\infty = \max |x_i|$$
$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

- **Matrične norme**: $A \in \mathbb{C}^{n \times n}$
$$\|A\|_1 = \max_{j=1,\dots,n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$
$$\|A\|_\infty = \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$
$$\|A\|_2 = \sqrt{\lambda_{max}(A^H A)} = \sigma_{max}(A)$$
$$\|A\|_F = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}|^2} = \sqrt{tr(A^H A)}$$

Za $\|A\|_2$ veljajo ocene:

1) $\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F$
2) $\frac{1}{\sqrt{n}} \|A\|_\infty \le \|A\|_2 \le \sqrt{n}\, \|A\|_\infty$
3) $\frac{1}{\sqrt{n}} \|A\|_1 \le \|A\|_2 \le \sqrt{n}\, \|A\|_1$
4) $\|A\|_2 \le \sqrt{\|A\|_1 \cdot \|A\|_\infty}$

Očitno:
- $\|A^H\|_F = \|A\|_F$
- $\|A^H\|_1 = \|A\|_\infty$
- $\|A^H\|_2 = \|A\|_2$

① Naj bo $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & -1 & 2 \end{bmatrix}$. Izračunajte $\|A\|_1$, $\|A\|_\infty$ in $\|A\|_F$ ter čim bolj natančno ocenite $\|A\|_2$

- $\|A\|_1 = \max\{2+5+|-2|,\ |-1|+4+|-1|,\ 3+1+2\} =$
$= \max\{9, 6, 6\} =$
$= 9$

- $\|A\|_\infty = \max\{6, 10, 5\} = 10$

- $\|A\|_F = tr\left(\begin{bmatrix} 2 & 5 & -2 \\ -1 & 4 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & -1 & 2 \end{bmatrix} \right) = tr \begin{bmatrix} 4+25+4 & & \\ & 1+16+1 & \\ & & 9+1+4 \end{bmatrix} = 33+18+14 = 65$

$\|A\|_F^2 = \sum_{j=1}^{3}\sum_{i=1}^{3} |a_{ij}|^2 = 4+25+4+1+16+1+9+1+4 = 65$
$\|A\|_F = \sqrt{65}$

ZA MATLAB:

```
norm(A) = norm(A,2)
norm(A,'fro')
norm(A,1)
norm(A,'inf')

norm(x)
norm(x,1)
norm(x,'inf')
```

- Ocenimo

1) $\frac{1}{\sqrt{3}} \sqrt{65} \le \|A\|_2 \le \sqrt{65}$
2) $\frac{1}{\sqrt{3}} 10 \le \|A\|_2 \le \sqrt{3} \cdot 10$
3) $\frac{1}{\sqrt{3}} 9 \le \|A\|_2 \le \sqrt{3} \cdot 9$
4) $\|A\|_2 \le \sqrt{90}$

najmanjša zgornja meja: $\min\{\sqrt{65}, \sqrt{3}\cdot 10, \sqrt{3}\cdot 9, \sqrt{90}\} = \sqrt{65} \doteq 8{,}06$
največja spodnja meja: $\max\left\{\frac{\sqrt{65}}{\sqrt{3}}, \frac{10}{\sqrt{3}}, \frac{9}{\sqrt{3}}\right\} = \frac{10}{\sqrt{3}} \doteq 5{,}77$

$\Longrightarrow 5{,}77 \le \|A\|_2 \le 8{,}06$

② Izračunaj $\|A\|_2$ za $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^H A)}$$

$$A^H A = A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$$

$$\det(A^H A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 6 \\ 0 & 6 & 10-\lambda \end{vmatrix} = 1-\lambda \begin{vmatrix} 4-\lambda & 6 \\ 6 & 10-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left((4-\lambda)(10-\lambda) - 36\right) = (1-\lambda)(40 - 14\lambda + \lambda^2 - 36)$$

$$= (1-\lambda)(\lambda^2 - 14\lambda + 4)$$

$$\downarrow$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{14 \pm \sqrt{196 - 16}}{2} = \frac{14 \pm \sqrt{180}}{2} = 7 \pm 3\sqrt{5}$$

$\implies$ $\lambda_1 = 1$

$\boxed{\lambda_2 = 7 + 3\sqrt{5}}$ ta je max

$\lambda_3 = 7 - 3\sqrt{5}$

$$\implies \|A\|_2 = \sqrt{7 + 3\sqrt{5}} = 3{,}70$$

<u>Operatorska norma</u> $(\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty)$

$$\boxed{\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}}$$

<u>Lema</u>: Za vsako matrično normo in poljubno lastno vrednost matrike A velja

$$|\lambda| \leq \|A\|.$$

③ Dokaži neenakosti 1) – 4)

$\boxed{1}$ $\frac{1}{\sqrt{n}}\|A_n\|_F \leq \|A\|_2 \leq \|A\|_F$

- $\frac{1}{\sqrt{n}}\|A_n\|_F \leq \|A\|_2$

help: Matrika B, $tr(B) = \sum_i \lambda_i$ za λ_i je lastna vrednost

$\frac{1}{n}\|A_n\|_F^2 \leq \|A\|_2^2$

$\frac{1}{n} tr(A^H A) \leq \lambda_{max}(A^H A)$

$\underbrace{\frac{1}{n}\sum_i \lambda_i}$ $\quad \max \lambda_i$

$\Longrightarrow \frac{1}{n} tr(A^H A) \leq \lambda_{max}(A^H A)$ ✓

za $\forall \lambda_i \leq \max \lambda_i$

$\Rightarrow \sum_i \lambda_i \leq n \cdot \max \lambda_i$

$\frac{1}{n}\sum \lambda_i \leq \max \lambda_i$

- $\|A\|_2^2 \leq \|A\|_F^2$

$\|A\|_2^2 = \lambda_{max}(A^H A)$, $\quad \|A\|_F^2 = tr(A^H A) = \sum \lambda_i$

ker je $\forall \lambda_i \geq 0$ potem velja: $\lambda_{max}(A^H A) \leq \sum \lambda_i$

ker je $A^H A$ simetrična pozitivno def matrika

preveri še dodatno

saj ta vsota vsebuje vsaj največjo lastno vr. in lahko še kakšno drugo poz. lastno vr.

$\boxed{2}$ $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{n}\|A\|_\infty$

za vektorsko normo vemo $\underbrace{\|x\|_\infty \leq \overbrace{\|x\|_2}^{*}}_{*} \overbrace{\leq \sqrt{n}\cdot\|x\|_\infty}$

za dokaz 2. neenakosti bomo uporabili to neenakost in operatorsko normo

- $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2$

vektor

$$\frac{1}{\sqrt{n}}\|A\|_\infty = \frac{1}{\sqrt{n}}\max_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{1}{\sqrt{n}}\max_{x\neq 0}\|Ax\|_\infty \frac{1}{\|x\|_\infty} \leq \frac{1}{\sqrt{n}}\max_{x\neq 0}\frac{\|Ax\|_2\sqrt{n}}{\|x\|_2} = \max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2$$

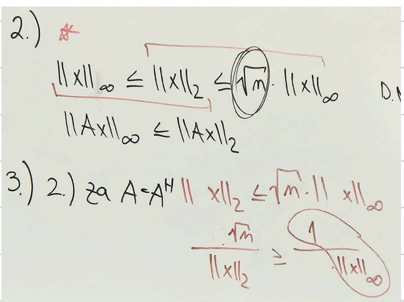
Iz * vemo $\|Ax\|_\infty \leq \|Ax\|_2$

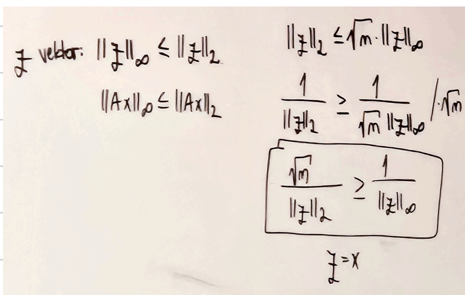
Iz * vemo $\frac{1}{\|x\|_\infty} \leq \frac{\sqrt{n}}{\|x\|_2}$

- $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$

$$\|A\|_2 = \max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x\neq 0}\frac{\|Ax\|_\infty\sqrt{n}}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty$$

iz * je $\|x\|_2 \leq \sqrt{n}\|x\|_\infty \Longrightarrow \|Ax\|_2 \leq \sqrt{n}\|Ax\|_\infty$


2.) *
$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{m}\cdot\|x\|_\infty$ D.
$\|Ax\|_\infty \leq \|Ax\|_2$
3.) 2.) za $A = A^H$ $\|x\|_2 \leq \sqrt{m}\cdot\|x\|_\infty$
$\frac{\sqrt{m}}{\|x\|_2} \geq \frac{1}{\|x\|_\infty}$


z vektor: $\|z\|_\infty \leq \|z\|_2$
$\|Ax\|_\infty \leq \|Ax\|_2$
$\|z\|_2 \leq \sqrt{m}\cdot\|z\|_\infty$
$\frac{1}{\|z\|_2} \geq \frac{1}{\sqrt{m}\|z\|_\infty} \quad /\cdot\sqrt{m}$
$\boxed{\frac{\sqrt{m}}{\|z\|_2} \geq \frac{1}{\|z\|_\infty}}$
$z = x$

3 $\frac{1}{\sqrt{n}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\,\|A_1\|$

- uporabimo * za $A = A^H \implies \|x\|_2 \le \sqrt{n}\,\|x\|_\infty \implies \frac{\sqrt{n}}{\|x\|_2} \geq \frac{1}{\|x\|_\infty}$
- ter lastnosti: $\|A^H\|_1 = \|A\|_\infty$, $\|A^H\|_\infty = \|A\|_1$, $\|A^H\|_2 = \|A\|_2$

$$\frac{1}{\sqrt{n}}\|A^H\|_\infty \le \|A^H\|_2 \le \sqrt{n}\,\|A^H\|_\infty$$
$$\frac{1}{\sqrt{n}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\,\|A\|_1$$

4 $\|A\|_2 \le \sqrt{\|A\|_1 \cdot \|A\|_\infty}$

uporabili bomo Lemo

submultiplikativnost

$$\|A_2\|^2 = \lambda_{max}(A^H A) \le \|A^H A\|_1 \le \|A^H\|_1\,\|A\|_1 = \|A\|_\infty \cdot \|A\|_1$$

$\|A^H\|_1 = \|A\|_\infty$

④ Naj bosta x in y neničelna vektorja, $x, y \in \mathbb{R}^n$. Naj bo $A = x \cdot y^T$. Izračunajte $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ in $\|A\|_2$

- $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad y^T = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}$

- $x \cdot y^T = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix}$

- $\|A\|_1 = \max\limits_{j=1,\dots,n} \left(\sum\limits_{i=1}^n |a_{ij}|\right) = \max\limits_{j=1,\dots,n}\left(\sum\limits_{i=1}^n |x_i y_j|\right) = \max\limits_{j=1,\dots,n}\left(|y_j| \cdot \underbrace{\sum\limits_{i=1}^n |x_i|}_{\|x\|_1}\right) = \max\limits_{j=1,\dots,n}\left(|y_j| \cdot \underbrace{\|x\|_1}_{\text{število}}\right) = \|x\|_1 \cdot \underbrace{\max\limits_{j=1,\dots,n} |y_j|}_{= \|y\|_\infty} = \|x\|_1 \cdot \|y\|_\infty$

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- $\|A\|_\infty = \max\limits_i \left(\sum\limits_{j=1}^n |a_{ij}|\right) = \max\limits_i \left(\sum\limits_{j=1}^n |x_i y_j|\right) = \max\limits_i \left(|x_i| \sum\limits_{j=1}^n |y_j|\right) = \max\limits_i \left(|x_i| \cdot \|y\|_1\right) = \|y\|_1 \cdot \max\limits_i |x_i| = \|y\|_1 \cdot \|x\|_\infty$

$\|$

opomba: $\|A^T\|_1$

> opomba: $(AB)^T = B^T A^T$
>
> $\hookrightarrow$ če $A = x y^T \implies A^T = (x \cdot y^T)^T = y \cdot x^T$

- $\|A\|_F^2 = \sum\limits_{i=1}^n \sum\limits_{j=1}^n (|a_{ij}|^2) = \sum\limits_{i=1}^n \sum\limits_{j=1}^n (|x_i y_j|^2) = \sum\limits_{i=1}^n x_i^2 \cdot \sum\limits_{j=1}^n y_j^2 = (\|x\|_2\ \|y\|_2)^2$

$\implies \|A\|_F = \|x\|_2 \cdot \|y\|_2$

- $\|A\|_2^2 = \lambda_{max}(A^TA)$
  $\overset{\|}{\max_\lambda}$

$\boxed{\text{namig}: \|A\|_F = \|A\|_2 \ !}$

$A^T = y \cdot x^T$

$A^TA = y \cdot x^T \cdot x \cdot y^T = y \cdot \|x\|_2^2 \cdot y^T = \|x\|_2^2 \cdot y \cdot y^T$

$x^Tx = [\quad][\ ] = \text{skalar} = x_1^2 + x_2^2 + \ldots + x_n^2 = \sum x_i^2 = \|x\|_2^2$

$\Longrightarrow \|A\|_2^2 = \lambda_{max}(A^TA) = \lambda_{max}(\|x_2\|^2 \cdot y y^T)$
$= \|x_2\|^2 \cdot \lambda_{max}(y y^T)$

- Vemo, da je $\|A\|_F^2 = \|A\|_2^2$ [iz namiga]
  $\|A\|_F^2 = \|x\|_2^2 \, \|y\|_2^2$, $\quad \|A\|_2^2 = \|x\|_2^2 \cdot \lambda \max(yy^T)$

Torej želimo sedaj dokazati $\underline{\|y\|_2^2 = \lambda \max(yy^T)}$

$$yy^T = \begin{bmatrix} y_1 y_1 & \cdots & y_1 y_n \\ y_2 y_1 & & \vdots \\ \vdots & & \\ y_n y_1 & \cdots & y_n y_n \end{bmatrix}$$

> opomba: $C = B^TB \rightarrow$ simetrična: $C = C^T$, $\lambda_i \in \mathbb{R}$
> $\rightarrow$ semi-pozitivno definitna: $\lambda_i \geq 0$

$\underline{\lambda_{max}(yy^T) = \|y\|_2^2}$

1) $\|y\|_2^2$ je lastna vrednost za lastni vektor y.

$y(y^T \cdot y) = \|y\|_2^2 \cdot y$

$y \cdot \|y\|_2^2 = \|y\|_2^2 y$ ✓

2) $\|y\|_2^2$ je največja lastna vrednost od yy^T

ker so vrstice samo večkratniki prve, zato je rang $(yy^T) = 1$,

torej je samo 1 lastna vrednost $\neq 0$,

torej je ta res maksimalna, saj je matrika pozitivno semi definitna

> opomba: za simetrično matriko velja, da je rang enak številu od 0 različnih lastnih vrednosti

$\Longrightarrow \|A\|_2 = \|x\|_2 \|y\|_2$

LU RAZCEP

- $A = LU$,

  L je spodnje trikotna matrika z enicami na diagonali, U pa zgornje trikotna matrika matrika (nesingularna: $u_{ii} \neq 0$)
- število operacij: $\frac{2}{3}n^3 + \mathcal{O}(n^2)$
- reševanje sistema $Ax = b$:

  $Ax = b \rightsquigarrow L\underbrace{Ux}_{y} = b$

  1) $Ly = b$: prema substitucija (n^2 operacij)

  2) $Ux = y$: obratna substitucija (n^2+n operacij)

① Izračunaj LU razcep za:

$$A = \begin{bmatrix} 2^* & 1 & 3 & -4 \\ -4 & -1 & -4 & 7 \\ 2 & 3 & 5 & -3 \\ -2 & -2 & -7 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & -4 \\ -2 & \boxed{1} & 2 & -1 \\ 1 & 2 & 2 & 1 \\ -1 & -1 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ -2 & 1 & 2 & -1 \\ 1 & 2 & \boxed{-2} & 3 \\ -1 & -1 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & -4 \\ -2 & 1 & 2 & -1 \\ 1 & 2 & -2 & 3 \\ -1 & -1 & 1 & \boxed{1} \end{bmatrix}$$

$:2^*$ $\quad -1-1\cdot(-2) \quad -7-(3\cdot(-1)) \quad 2-(2\cdot 2)=-2 \quad 4-(3\cdot 1)=1$

postopek:
- prvo vrstico prepišemo
- ostale elemente prvega stolpca delimo z diagonalnim
- el. v novem A: (el. v starem A na istem mestu) − (el. v novem A (skrajno zgornji) · (skrajno levi))
- ponovimo to za manjšo matriko

L: pod diagonalo enako in 1ke na diag.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

U: vse nad diag. iz diag. vrt

$$U = \begin{bmatrix} 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② Preko LU razcepa rešite sistem enačb

$$2x+2y-4z+w=-2$$
$$2z+x=9$$
$$2y+x-w=1$$
$$y-z+1=0$$

- $b=\begin{bmatrix}-2\\9\\1\\-1\end{bmatrix}$ $\quad A=\begin{bmatrix}2&2&-4&1\\1&0&2&0\\1&2&0&-1\\0&1&-1&0\end{bmatrix}\sim\begin{bmatrix}2&2&-4&1\\1/2&-1&4&-1/2\\1/2&1&2&-3/2\\0&1&-1&0\end{bmatrix}\sim\begin{bmatrix}2&2&-4&1\\1/2&-1&4&-1/2\\1/2&-1&6&-2\\0&-1&3&-1/2\end{bmatrix}$

$$\sim\begin{bmatrix}2&2&4&1\\1/2&-1&4&-1/2\\1/2&-1&6&-2\\0&-1&1/2&1/2\end{bmatrix}$$

$$L=\begin{bmatrix}1&&&\\1/2&1&&\\1/2&-1&1&\\0&-1&1/2&1\end{bmatrix}\qquad U=\begin{bmatrix}2&2&-4&1\\&-1&4&-1/2\\&&6&-2\\&&&1/2\end{bmatrix}$$

- "$Ly=b$"

$$\begin{bmatrix}1&&&\\1/2&1&&\\1/2&-1&1&\\0&-1&1/2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}-2\\9\\1\\-1\end{bmatrix}$$

$y_1=-2$

$-1+y_2=9\rightarrow y_2=10$

$-1-10+y_3=1\rightarrow y_3=12$

$-10+6+y_4=-1\rightarrow y_4=3$

$$y=\begin{bmatrix}-2\\10\\12\\3\end{bmatrix}$$

- "$Ux=y$"

$$\begin{bmatrix}2&2&-4&1\\&-1&4&-1/2\\&&6&-2\\&&&1/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2\\10\\12\\3\end{bmatrix}$$

$x_4=6$

$6x_3-12=12\rightarrow x_3=4$

$-x_2+16-3=10\rightarrow x_2=3$

$2x_1+6-16+6=-2\rightarrow x_1=1$

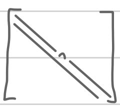
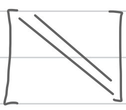
$$x=\begin{bmatrix}1\\3\\4\\6\end{bmatrix}\begin{matrix}=x\\=y\\=z\\=w\end{matrix}$$

③ Poiščite LU razcep za matriko. Kaj opazite

$$A=\begin{bmatrix}1&3&0&0\\4&5&2&0\\0&2&1&6\\0&0&1&2\end{bmatrix}$$ tridiagonalna matrika

$$\sim\begin{bmatrix}1&3&0&0\\4&-7&2&0\\0&2&1&6\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&3&0&0\\4&-7&0&0\\0&-\frac{2}{7}&\frac{11}{7}&6\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&3&0&0\\4&-7&0&0\\0&-\frac{2}{7}&\frac{11}{7}&6\\0&0&\frac{7}{11}&-\frac{20}{11}\end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0&0\\4&1&0&0\\0&-\frac{2}{7}&1&0\\0&0&\frac{7}{11}&1\end{bmatrix}\qquad U=\begin{bmatrix}1&3&0&0\\0&-7&2&0\\0&0&\frac{11}{7}&6\\0&0&0&-\frac{20}{11}\end{bmatrix}$$

opazimo: $L=$  $U=$ 

lahko izračunaš samo prvo diagonalo: $n-1$ elementov

lahko izračunamo samo diagonalo, brez 1. el: $n-1$ elementov

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b) Zapišite algoritem (Thomasov algoritem) za LU razcep tridiagonalne matrike A in preštejte število operacij. Zapišite postopek za reševanje sistema $Ax=z$ in preštejte št. operacij (v splošnem: $2/3\,n^3+\mathcal{O}(n^2)$)

$$A=\begin{bmatrix}a_1&b_1&&&\\c_1&a_2&b_2&&\\&c_2&a_3&b_3&\\&&c_3&\ddots&\ddots\\&&&\ddots&b_{n-1}\\&&&c_{n-1}&a_n\end{bmatrix}$$

Vhod so a,b,c

Izhod pa l (dolžine $n-1$) in u (dolžine n)

želimo si zapisati vektorja l in u

$$A\sim\begin{bmatrix}a_1&b_1&&&\\\frac{c_1}{a_1}&a_2-\frac{c_1}{a_1}b_1&b_2&&\\&c_2&\ddots&\ddots&\\&&\ddots&\ddots&b_{n-1}\\&&&c_{n-1}&a_n\end{bmatrix}\sim\begin{bmatrix}u_1&b_1&&&\\l_1&u_2&b_2&&\\&\frac{c_2}{u_2}&a_3-\frac{c_2}{u_2}b_2&\ddots&\\&&\ddots&\ddots&b_{n-1}\\&&&c_{n-1}&a_n\end{bmatrix}$$

↳ nova elementa: $u_1=a_1$, $l_1=\frac{c_1}{a_1}$, $u_2=a_2-l_1b_1$

nova sprem el: $l_2=\frac{c_2}{u_2}$, $u_3=a_3-l_2b_2$

Opazimo, da izmenično računamo elemente u in l

$$l=\begin{bmatrix}l_1\\l_2\\\vdots\\l_{n-1}\end{bmatrix}=\begin{bmatrix}\frac{c_1}{a_1}\\\frac{c_2}{a_2}\\\vdots\\\ \end{bmatrix}\begin{matrix}2\\4\\\\\ \end{matrix}\qquad u=\begin{bmatrix}u_1\\u_2\\u_3\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}a_1\\a_2-l_1b_1\\a_3-l_2b_2\\\vdots\\\ \end{bmatrix}\begin{matrix}1\\3\\5\\\\\ \end{matrix}$$

$l_i=\frac{c_i}{a_i}$ za $i=1,\dots,n-1$ $\qquad u_1=a_1,\ u_i=a_i-l_{i-1}b_{i-1}$ $i=1,2,\dots n$

ALGORITEM (za v Matlab)

$u(1) = a(1)$

for $i = 1, 2, \dots, n$

$\quad l(i) = \frac{a(i)}{u(i)}$ → $(n-1)$ operacij za dobit $(n-1)$ elementov

$\quad u(i+1) = a(i+1) - l(i)\, b(i)$

za en element dve operaciji, torej za izračun $(n-1)$ el. je tu $2(n-1)$ operacij

end

$\Longrightarrow$ skupno število operacij je $(n-1) + 2(n-1) = 3(n-1)$

Splošna matrika: $A = LU$

$Ax = z$

$L\underbrace{Ux}_{y} = z$

1) $Ly = z$ prema substitucija (n^2 operacij)

2) $Ux = y$ obratna substitucija ($n^2 + n$ operacij)

Tridiagonalna matrika:

1) $Ly = z$ $\quad 2(n-1)$ operacij

2) $Ux = y$ $\quad 3n-2$ operacij

$Ly = z$

$$\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & l_{n-1} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ l_1 y_1 + y_2 \\ l_2 y_2 + y_3 \\ \vdots \\ l_{n-1} y_{n-1} + y_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

→ $y_1 = z_1$

→ $l_1 y_1 + y_2 = z_2 \Longrightarrow y_2 = z_2 - l_1 y_1$

→ $l_2 y_2 + y_3 = z_3 \Longrightarrow y_3 = z_3 - l_2 y_2$

$\vdots$

→ $y_n = z_n - l_{n-1} y_{n-1}$

$y_i = z_i - l_{i-1} y_{i-1} \qquad i = 1, 2, \dots, n$

število operacij : $2(n-1)$

$Ux = y$

$$\begin{bmatrix} u_1 & b_1 & & \\ & u_2 & b_2 & \\ & & \ddots & \ddots \\ & & & b_{n-1} \\ & & & u_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \begin{bmatrix} u_1 x_1 + b_1 x_2 \\ u_2 x_2 + b_2 x_3 \\ \vdots \\ u_{n-1} x_{n-1} + b_{n-1} x_n \\ u_n x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

→ $u_n x_n = y_n \Longrightarrow x_n = \frac{y_n}{u_n}$

→ $u_{n-1} x_{n-1} + b_{n-1} x_n = y_{n-1} \Longrightarrow x_{n-1} = \frac{y_{n-1} - b_{n-1} x_n}{u_{n-1}}$

$x_i = \frac{y_i - b_i x_{i+1}}{u_i} \qquad i = n-1, \dots, 1$

število operacij : $3(n-1) - 1$

LU RAZCEP Z DELNIM PIVOTIRANJEM

- Dopuščamo zamenjavo vrstic
- $PA = LU$
  - P → permutacijska matrika (I z ustrezno zamenjanimi vrsticami)
- $Ax = b$

$PAx = Pb$

$LUx = Pb$ 1) $Ly = Pb$

2) $Ux = y$

(4) Podana je matrika A. Določite LU razcep z delnim pivotiranjem. Nato izračunajte $\det(A)$ in rešite sistem $Ax = \begin{bmatrix} 12 \\ 20 \\ 9 \\ 4 \end{bmatrix}$.

$$A = \begin{bmatrix} 0 & 4 & 12 & 12 \\ 12 & 4 & 18 & 0 \\ 0 & 1 & 9 & 18 \\ 6 & 6 & 8 & 8 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 12 & 4 & 18 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1 & 9 & 18 \\ 6 & 6 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & \underline{4} & 12 & 12 \\ 0 & 1 & 9 & 18 \\ \underline{1/2} & \underline{4} & \underline{4} & 8 \end{bmatrix} \sim \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1/4 & 6 & 15 \\ 1/2 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1/4 & -8 & -4 \\ 1/2 & 1 & 6 & 15 \end{bmatrix} \sim$$

gledamo absolutne vrednosti elementov

$$\sim \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1/4 & -8 & -4 \\ 1/2 & 1 & -3/4 & 12 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 1/2 & 1 & 1 & \\ 0 & 1/4 & -3/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 12 & 4 & 8 & 0 \\ & 4 & 12 & 12 \\ & & -8 & -4 \\ & & -3/4 & 12 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\det(L) = 1$ $\qquad \det(U) = 12 \cdot 4 \,(-8)\, 12 = -4608$ $\qquad \det(P) = (-1)^n$, n = št. menjav vrstic

$\det(P) = 1$

$$\det(A) = \frac{\det(L)\cdot\det(U)}{\det(P)} = \frac{1\cdot(-4608)}{1} = -4608$$

$$Ly = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 1/2 & 1 & 1 & \\ 0 & 1/4 & -3/4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 12 \\ 20 \\ 9 \\ 4 \end{bmatrix} \qquad Ux = \begin{bmatrix} 12 & 4 & 8 & 0 \\ & 4 & 12 & 12 \\ & & -8 & -4 \\ & & -3/4 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 9 \\ 4 \end{bmatrix}$$

$y_1 = 20$

$y_2 = 12$

$\frac{1}{2} y_1 + y_2 + y_3 = 14 \rightarrow y_3 = -8$

$\frac{1}{4} y_2 - \frac{3}{4} y_3 + y_4 = 9 \rightarrow y_4 = 0$

$$y = \begin{bmatrix} 20 \\ 12 \\ -8 \\ 0 \end{bmatrix}$$

$12 x_4 = 0 \longrightarrow x_4 = 0$

$-8x_4 - 4x_4 = -8 \longrightarrow x_3 = 1$

$4x_2 + 12x_3 + 12x_4 = 12 \longrightarrow x_2 = 0$

$12x_1 + 8x_3 = 20 \longrightarrow x_1 = 1$

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

LU RAZCEP Z KOMPLETNIM PIVOTIRANJEM

- dopuščanje zamenjave vrstic in stolpcev
- $PAQ = LU$

zamenjave vrstic (P), zamenjave stolpcev (Q)

① Izračunajte LU razcep s kompletnim pivotiranjem za matriko $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix}$ in rešite sistem $Ax = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} -6 & 2 & 4 \\ 1/2 & 1 & -1 \\ -1/2 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} -6 & 2 & 4 \\ -1/2 & 4 & -1 \\ 1/2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -6 & 2 & 4 \\ -1/2 & 4 & -1 \\ 1/2 & 1/4 & -3/4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -6 & 2 & 4 \\ 0 & 4 & -1 \\ 0 & 0 & -3/4 \end{bmatrix}$$

1) $Ax = b \quad / \cdot P$

$PAQ = LU \rightarrow PA = LUQ^{-1}$

Iz 1) $PAx = Pb$

$PAx = LUQ^{-1}x = Pb$

REŠEVANJE SISTEMA:

- delno: $Ax = b \quad / \cdot P$
  - $PAx = Pb$
  - $LUx = Pb$
  - 1) $Ly = Pb$
  - 2) $Uz = y$
  - 3) $x = Qz$
- kompletno: $PAQ = LU$
  - ⋮
  - $LU\underbrace{Q^{-1}x}_{z} = Pb$
  - $L\underbrace{Uz}_{y} = Pb$
  - 1) $Ly = Pb$
  - 2) $Uz = y$
  - 3) $(Q^{-1}x = z) \quad x = Qz$

1) $Ly = Pb$

$$Pb = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$$Ly = \begin{bmatrix} 1 & & \\ -1/2 & 1 & \\ 1/2 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ -\frac{1}{2}y_1 + y_2 \\ \frac{1}{2}y_1 + \frac{1}{4}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$y_1 = 14$

$-7 + y_2 = 0 \rightarrow y_2 = -7$

$7 - 7/4 + y_3 = 8 \rightarrow y_3 = -3/4$

2) $$Uz = \begin{bmatrix} -6 & 2 & 4 \\ 0 & 4 & -1 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -6z_1 + 2z_2 + 4z_3 \\ 4z_2 - z_3 \\ -3/4\, z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \\ -3/4 \end{bmatrix}$$

$z_3 = 1$

$4z_2 - 1 = 7 \rightarrow z_2 = 2$

$4 + 4 - 6z_1 = 14 \rightarrow z_1 = -1$

$$z = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

② Opišite postopek za reševanje sistema linearnih enačb oblike $\begin{bmatrix} U & -I \\ B & L \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, kjer je B nesingularna matrika z danim LU razcepom. Preštejte število operacij.

$$\left.\begin{array}{l} B \in \mathbb{R}^{n\times n} \\ L, U \in \mathbb{R}^{n\times m} \end{array}\right\} B = LU$$

$x, y, a, b \in \mathbb{R}^n$

<u>v splošnem</u>: $LU = \frac{2}{3}n^3 + \mathcal{O}(n^2)$

$Ly = b \;: n$

$Ux = y \;: n^2+n$

$$\left[\begin{array}{cc} \overset{U}{\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ 0 & u_{22} & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots\, 0 & u_{nn} \end{bmatrix}} & \overset{-I}{\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \vdots \\ \vdots & & \ddots & -1\ \ 0 \\ 0 & \cdots & & 0\ -1 \end{bmatrix}} \\ \underset{B=LU}{\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}} & \underset{L}{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ l_{n1} & \cdots & l_{n(n-1)} & 1 \end{bmatrix}} \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} Ux - Iy \\ Bx + Ly \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$Ux - Iy = a \longrightarrow \overline{LUx - Ly = La}$

$Bx + Ly = b \longrightarrow \underline{LUx + Ly = b}$

$2LUx = La + b \longrightarrow$ ta način nam ni všeč, probamo se izognit množenju s matrikami

Pomagajmo si s preoblikovanjem zapisov enačb: $Ux - Iy = a$

$LUx + Ly = b \longrightarrow L\underbrace{(Ux+y)}_{z\ \to \text{ smo si zmislili}} = b$

$$\left.\begin{array}{l} Ux + y = z \\ Ux - y = a \end{array}\right\} \Longrightarrow$$

1) $Lz = v$ $\qquad : n^2$

2) $2Ux = a + z$ $\qquad : \frac{a+z}{2} = n + n = 2n$ (deljenje z 2, seštevanje)

$Ux = \frac{1}{2}(a+z)$

$Ux = n^2 + n$

3) $Ux - y = a$

$y = Ux - a = \frac{a+z}{2} - a = \frac{z-a}{2}$ $\qquad : 2n$

$\Longrightarrow$ vse operacije skupaj: $n^2 + 2n + n^2 + 2n + n = 2n^2 + 5n$

③ Dana je matrika $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & 1+a \end{bmatrix}$, $0 < a < 1$

a) Izračunajte LU razcep z delnim pivotiranjem za matriko A. Kako moramo izbrati a, da bo $\|U\|_F = \frac{9}{4}$?

$$A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & 1+a \end{bmatrix} \sim \begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 1 & 1+a \end{bmatrix} \sim \begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 1 & 1+a \end{bmatrix} \sim \begin{bmatrix} a & 0 & 1 \\ 0 & 1 & 1+a \\ 0 & a & 0 \end{bmatrix} \sim \begin{bmatrix} a & 0 & 1 \\ 0 & 1 & 1+a \\ 0 & a & -a(1+a) \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \qquad U = \begin{bmatrix} a & 0 & 1 \\ 0 & 1 & 1+a \\ 0 & 0 & -a(1+a) \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\|U\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} (u_{ij})^2} = \sqrt{a^2+1+1+(1+a)^2+(-a(1+a))^2} = \sqrt{2a^2+3+2a+a^4+a^2+2a^3}$$

$$\sqrt{a^4+2a^3+3a^2+2a+3} = 9/4$$

$$a^4+2a^3+3a^2+2a+3 = 81/16$$

$$16a^4+32a^3+48a^2+32a+48 = 81$$

vemo, da je $a \in (0,1)$ $\quad \left.\begin{matrix} p(0) = -33 < 0 \\ p(1) > 0 \end{matrix}\right\}$ vemo, da bo tudi ničla $\in (0,1)$

kandidati: $\left.\begin{matrix} 16 ; 1,2,4,8,16 \\ 33 ; 1,3,11,33 \end{matrix}\right\}$ $1/2, 1/4, 1/8, 1/16, 3/4, 3/8, 3/16, 11/16$ $\longrightarrow$ upoštevali smo, da bo kandidat za 1. ničlo na intervalu $(0,1)$

	16	32	48	32	-33
1/2	↓	8	20	34	33
	16	40	68	66	0

$\longrightarrow$ 1. ničla je $1/2$

$$16a^3+40a^2+68a+66 = 0$$

$8a^3+20a^2+34a+33 = 0 \longrightarrow p(0) = 33$

$p_1(-1) = -8+20-34+33 > 0$

$p_2(2) < 0$ ni več pozitivnih ničel $\Longrightarrow a = 1/2$ je edina pozitivna ničla

b) Izračunajte LU razcep s kompletnim pivotiranjem za matriko A. Glede na vrednost parametra a določite determinanto matrik P in Q.

$$\begin{bmatrix} 0 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & 1+a \end{bmatrix} \sim \begin{bmatrix} 1+a & 1 & 0 \\ 1 & 0 & a \\ 0 & a & 0 \end{bmatrix} \sim \begin{bmatrix} 1+a & 1 & 0 \\ \frac{1}{1+a} & \boxed{-\frac{1}{1+a} \quad a} & \\ 0 & \boxed{a \quad 0} & \end{bmatrix} \sim \begin{bmatrix} 1+a & 1 & 0 \\ 0 & & \\ \frac{1}{1+a} & & \end{bmatrix}$$

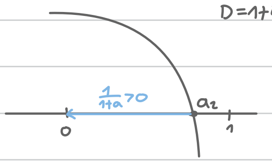
$\max \left\{ \left|-\frac{1}{1+a}\right|, |a|, |a|, |0| \right\}$

$$\frac{1}{1+a} - a = \frac{1-a(1+a)}{1+a} = \frac{1-a-a^2}{1+a}$$

Kdaj je to pozitivno? $1-a-a^2>0$

$D = 1+4 = 5 \qquad a_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \longrightarrow a_1 = \frac{1+\sqrt{5}}{-2}$

$\longrightarrow a_2 = \frac{1-\sqrt{5}}{-2} > 0$



1) $\frac{1}{1+a} > a$: $\begin{bmatrix} 1+a & 1 & 0 \\ 0 & -\frac{1}{1+a} & a \\ \frac{1}{1+a} & -a(1+a) & a^2(1+a) \end{bmatrix}$ ni menjav

$0 < a < \frac{\sqrt{5}-1}{2} \implies \det(P) = 1 \qquad \det(Q) = -1$

2) $\frac{1}{1+a} < a$ $\quad 2 \leftrightarrow 3$

$$\begin{bmatrix} 1+a & 1 & 0 \\ 0 & a & 0 \\ \frac{1}{1+a} & -\frac{1}{1+a} & a \end{bmatrix} \sim \begin{bmatrix} 1+a & 1 & 0 \\ 0 & a & 0 \\ \frac{1}{1+a} & -\frac{1}{a(1+a)} & a \end{bmatrix}$$

$\frac{\sqrt{5}-1}{2} < a < 1 \implies \det(P) = 1 \qquad \det(Q) = -1$

10.12

REŠEVANJE SISTEMOV NELINEARNIH ENAČB

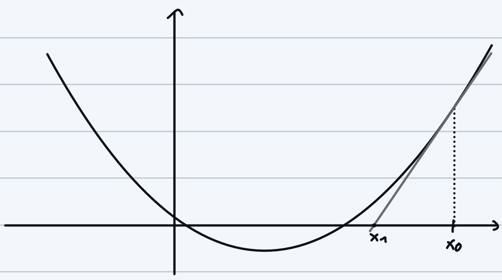
$f_1(x_1, x_2, \dots, x_n) = 0$

$f_2(x_1, x_2, \dots, x_n) = 0$

$\vdots$

$f_n(x_1, x_2, \dots, x_n) = 0$

• Reševanje linearnih enačb : $f(x) = 0$



$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

$$x^{r+1} = x^r - JF^{-1}(x^{(r)}) F(x^{(r)}) \quad / \cdot JF(x^{(r)})$$

$$JF(x^{(r)}) x^{r+1} = JF(x^{(r)}) x^{(r)} - F(x^{(r)})$$

$$JF(x^{(r)}) \underbrace{(x^{(r+1)} - x^{(r)})}_{\Delta x^{(r)}} = -F(x^{(r)})$$

• Newtonova metoda

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad F(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

$$\underset{JF(x)}{\overset{\parallel}{JF(x_1, x_2, \dots, x_n)}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \dots, x_n) \\ \frac{\partial f_2}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial f_2}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1, \dots, x_n) \end{bmatrix}$$

1) $JF(x^{(r)}) \Delta x^{(r)} = -F(x^{(r)})$

2) $x^{(r+1)} = \Delta x^{(r)} + x^{(r)}$

1 a) Naredite dva koraka Newtonove metode za sistem enačb $x^2+y^2=2$, $x^2-2xy+y=2$ z začetnim približkom $x^{(0)}=\begin{bmatrix}1\\1\end{bmatrix}$

$f_1(x,y)=x^2+y^2-2$

$f_2(x,y)=x^2-2xy+y-2$

$JF=\begin{bmatrix}2x & 2y\\ 2x-2y & -2x+1\end{bmatrix}$ $F=\begin{bmatrix}x^2+y^2-2\\ x^2-2xy+y-2\end{bmatrix}$ $JF(x^{(0)})=\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}$ $F(x^{(0)})=\begin{bmatrix}0\\-2\end{bmatrix}$

$$\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}\underbrace{\begin{bmatrix}\Delta x_1\\ \Delta y_1\end{bmatrix}}_{\Delta x^{(1)}}=\begin{bmatrix}0\\2\end{bmatrix}$$

$$\begin{bmatrix}2\Delta x_1+2\Delta y_1\\ -\Delta y_1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix} \longrightarrow \Delta y_1=-2$$

$2\Delta x_1+2\Delta y_1=0 \longrightarrow 2\Delta x_1=4 \longrightarrow \Delta x_1=2$

$$\Rightarrow \Delta x^{(0)}=\begin{bmatrix}2\\-2\end{bmatrix}$$

$$\Rightarrow x^{(1)}=\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{x^{(0)}}+\underbrace{\begin{bmatrix}2\\-2\end{bmatrix}}_{\Delta x^{(0)}}=\begin{bmatrix}3\\-1\end{bmatrix}$$

$JF(x^{(1)})=\begin{bmatrix}6 & -2\\ 8 & -5\end{bmatrix}$ $F(x^{(1)})=\begin{bmatrix}8\\12\end{bmatrix}$

$$\begin{bmatrix}6 & -2\\ 8 & -5\end{bmatrix}\begin{bmatrix}\Delta x_2\\ \Delta y_2\end{bmatrix}=\begin{bmatrix}-8\\-12\end{bmatrix}$$

$6\cdot\Delta x_2-2\Delta y_2=-8$

$8\cdot\Delta x_2-5\cdot\Delta y_2=-12$

$30\cdot\Delta x_2-16\Delta x_2=-40+24$

$14\cdot\Delta x_2=-16$

$\Delta x_2=-16/14=-8/7 \longrightarrow -8\Delta y_2+15\Delta y_2=-32+36$

$7\cdot\Delta y_2=4$

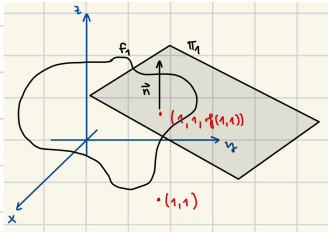
$\Delta y_2=4/7$

$$\Rightarrow \Delta x^{(1)}=\begin{bmatrix}-8/7\\4/7\end{bmatrix}$$

$$\Rightarrow x^{(2)}=\begin{bmatrix}13/7\\-3/7\end{bmatrix}$$

b) Približek za $x^{(1)}$ iz a) določite še na geometrijski način

• Določi tangentni ravnini na ploskvi $f_1(x,y)$ in $f_2(x,y)$ v točki $x^{(0)}=(1,1)$



$\vec{n} = \left(\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1), -1\right)$

$f_1(x,y) = x^2+y^2-2$

$f_2(x,y) = x^2-2xy+y-2$

$\Rightarrow \vec{n}_1 = (2,2,-1) \qquad \vec{n}_2 = (0,-1,-1)$

$\left.\begin{array}{l} T_1(1,1,0) \\ T_2(1,1,-2) \end{array}\right\}$ točke na ploskvi

$2x+2y-z=d$

$\vec{n}$ a T

$\pi_1: 2+2=d \qquad \pi_2: -1+2=d$

$2x+2y-z=4 \qquad -y-z=1 \longrightarrow y+z=-1$

$\pi: ax+by+cz=d \longrightarrow \vec{n}=(a,b,c)$

• presek π_1 in π_2 z $z=0$

presek π_1 in $z=0$: $2x+2y=4$ $\qquad$ presek π_2 in $z=0$: $y=-1$

$x+y=2$

• presek premic je $x^{(1)}$

presek : $x-1=2 \longrightarrow x=3$ $\qquad \Longrightarrow x^{(1)} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$y=1$

② Naredite en korak Newtonove metode za sistem enačb $x^2+y^2+z^2=1$, $2x^2+y^2-4z=0$, $3x^2-4y+z^2=0$

z začetnim približkom $x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$f_1(x,y,z)=x^2+y^2+z^2-1$

$f_2(x,y,z)=2x^2+y^2-4z$

$f_3(x,y,z)=3x^2-4y+z^2$

$$JF=\begin{bmatrix}2x & 2y & 2z\\ 4x & 2y & -4\\ 6x & -4 & 2z\end{bmatrix} \qquad F=\begin{bmatrix}x^2+y^2+z^2-1\\ 2x^2+y^2-4z\\ 3x^2-4y+z^2\end{bmatrix}$$

$$JF(x^{(0)})=\begin{bmatrix}2 & 2 & 2\\ 4 & 2 & -4\\ 6 & -4 & 2\end{bmatrix} \qquad F(x^{(0)})=\begin{bmatrix}2\\-1\\0\end{bmatrix}$$

1. korak

$$\begin{bmatrix}2 & 2 & 2\\ 4 & 2 & -4\\ 6 & -4 & 2\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix} \xrightarrow{\text{sistem}}$$

$2\Delta x+2\Delta y+2\Delta z=-2$

$4\Delta x+2\Delta y-4\Delta z=1$

$6\Delta x-4\Delta y+2\Delta z=0$

rešitev sistema: $\Delta x=-1/12$ $\Delta y=-7/18$ $\Delta z=-19/36$

$$\Longrightarrow \Delta x^{(0)}=\begin{bmatrix}-1/12\\-7/18\\-19/36\end{bmatrix}$$

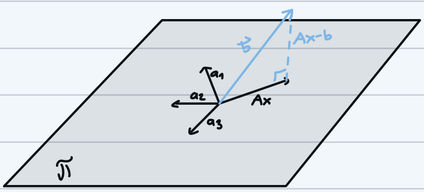
$$\Longrightarrow x^{(1)}=\begin{bmatrix}11/12\\11/18\\17/36\end{bmatrix}$$

LINEARNI PROBLEM NAJMANJŠIH KVADRATOV

Imamo matriko $A\in\mathbb{R}^{m\times n}$, $m>n$ in vektor $b\in\mathbb{R}^n$. Rešujemo $Ax=b$. Minimiziramo $\|Ax-b\|_2$. Torej iščemo $x\in\mathbb{R}^n$, ki minimizira $\|Ax-b\|_2$ (x imenujemo rešitev po metodi najmanjših kvadratov)

NORMALNI SISTEM

Vektorji Ax ležijo v $\underbrace{\text{ravnini}}_{\pi}$, ki jo razpenjajo stolpci matrike A



Najbližji element v π na dani b je pravokotna projekcija b na π. Iščemo tak $x\in\mathbb{R}^n$, da bo $(Ax-b)\perp\pi$, torej

$A^T(Ax-b)=0$

$$A=\begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix} \qquad \underbrace{\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}}_{A^T}\underbrace{\begin{bmatrix}Ax-b\end{bmatrix}}_{(Ax-b)}=0$$

Dobimo: $\boxed{A^TAx=A^Tb}$ NORMALNI SISTEM

③ Naj bosta $A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}$ in $b = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

a) Določite x, ki minimizira $\|Ax-b\|_2$ po normalnem sistemu

b) Pokažite, da velja $Ax = \text{proj}_{\mathcal{U}} b$

a) $A^TAx = A^Tb$

$$\begin{bmatrix} 1 & 4 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$3x_1 + 7x_2 = 5$

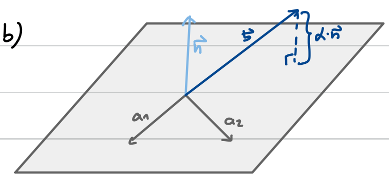
$7x_1 + 21x_2 = 21$

$-2x_1 = 6 \longrightarrow x_1 = -3$

$-9 + 7x_2 = 5 \longrightarrow x_2 = 2$

$\Longrightarrow x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

b)



$a_1, a_2 \in \mathcal{U}$

enačba ravnine:

$$\vec{n} = \vec{a}_1 \times \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Longrightarrow \mathcal{U}: 2x + y + 3z = 0$

projekcija: $\text{proj}_{\mathcal{U}} \vec{b} = \vec{b} - \alpha \vec{n} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} - \alpha \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3-2\alpha \\ 7-\alpha \\ 5-3\alpha \end{bmatrix}$

vstavimo v $\mathcal{U}$: $2(3-2\alpha) + 7 - \alpha + 3(5-3\alpha) = 0$

$28 - 4\alpha - \alpha - 9\alpha = 0$

$14\alpha = 28$

$\alpha = 2$ $\Longrightarrow \text{proj}_{\mathcal{U}} \vec{b} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$

preverimo enakost

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

✓

RAZCEP CHOLESKEGA

- $A \in \mathbb{R}^{n \times n}$ simetrična ($A^T = A$) & pozitivno definitna ($x^T A x > 0 \quad \forall x \neq 0$)

- $A = VV^T$, kjer je V faktor Choleskega

  V je spodnje-trikotna matrika s pozitivnimi diagonalnimi elementi

- <u>ALGORITEM</u>

  for $j = 1 : n$

  $\quad v_{ii} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} v_{jk}^2}$

  $\quad$ for $i = j+1 : n$

  $\quad\quad v_{ij} = \frac{1}{v_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} v_{ik} \cdot v_{jk} \right)$

  $\quad$ end

  end

① Izračunaj faktor Choleskega za

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 8 & -2 & 8 \\ -2 & -2 & 14 & -11 \\ 3 & 8 & -11 & 15 \end{bmatrix}$$

$$V = \begin{bmatrix} \sqrt{1-0}=1 & & & \\ 2 & 2 & & \\ -2 & 1 & 3 & \\ 3 & 1 & -2 & 1 \end{bmatrix}$$

$\frac{a_{21}}{v_{11}}$

$v_{22} = \sqrt{8 - 2^2} = 2$

$v_{32} = \frac{1}{2} \left(-2 - \boxed{(-2 \cdot 2)} \right) = 1$

$v_{4,2} = \frac{1}{2} \left(8 - \boxed{(3 \cdot 2)} \right) = 1$

$v_{3,3} = \sqrt{14 - ((-2)^2 + 1^2)} =$

$v_{4,3} = \frac{1}{\boxed{3}} \left(-11 - (-2 \cdot 3 + 1 \cdot 1) \right) = -2$

$v_{44} = \sqrt{15 - (9 + 1 + 4)} = 1$

② Izračunaj faktor Choleskega za

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & 1 \\ 2 & 13 & 23 & 8 & 8 \\ 4 & 23 & 77 & 32 & 32 \\ 1 & 8 & 32 & 30 & 30 \\ 1 & 8 & 32 & 30 & 55 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & & & & \\ 2 & 3 & & & \\ 4 & 5 & 6 & & \\ 1 & 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$v_{22} = \sqrt{13 - 4} = 3$

$v_{32} = \frac{1}{3} (23 - (4 \cdot 2)) = \frac{1}{3} \cdot 15 = 5$

$v_{42} = \frac{1}{3} (8 - (1 \cdot 2)) = 2$

$v_{52} = v_{42} = 2$

$v_{33} = \sqrt{77 - 5^2 - 4^2} = \sqrt{36} = 6$

$v_{43} = \frac{1}{6} \left(32 - (1 \cdot 4) - (2 \cdot 5) \right) = \frac{1}{6} \cdot 18 = 3$

$v_{53} = 1/6 \left(32 - (1 \cdot 4) - (2 \cdot 5) \right) = 3$

$v_{44} = \sqrt{30 - 1^2 - 2^2 - 3^2} = \sqrt{16} = 4$

$v_{45} = \frac{1}{4} \left(30 - (1 \cdot 1) - (2 \cdot 2) - (3 \cdot 3) \right) = \frac{16}{4} = 4$

$v_{55} = \sqrt{55 - (1 + 4 + 9 + 16)} = \sqrt{25} = 5$

A je simetrična in pozitivno definitna $\Longrightarrow$ razcep Choleskega uspe

③ Podana sta matrika A in vektor b

$$A = \begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 15 \\ 2 \\ 1 \end{bmatrix}$$

a) Za katere α je A pozitivno definitna?

b) Naj bo $\alpha = 23$. Rešite sistem $Ax = b$

a) $$V = \begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ -1 & -1 & 1 & \\ -2 & 3 & 1 & \sqrt{\alpha - 14} \end{bmatrix}$$ $\alpha > 14$

$v_{22} = \sqrt{18-9} = 3$

$v_{32} = \frac{1}{3}(0 - 1 \cdot 3) = -1$

$v_{42} = 1/3\,(3 + 3 \cdot 2) = 3$

$v_{33} = \sqrt{3-1-1} = 1$

$v_{43} = (-4 - (-2 \cdot 1 - 1 \cdot 3)) = 1$

$v_{44} = \sqrt{\alpha - 4 - 9 - 1} = \sqrt{\alpha - 14}$

b) $\alpha = 23$, $Ax = b$

$A = VV^T$

$$V^T = \begin{bmatrix} 2 & 3 & 1 & -2 \\ & 3 & -1 & 3 \\ & & 1 & 1 \\ & & & 3 \end{bmatrix}$$

$Ax = b$

$V\underbrace{V^T x}_{y} = b \longrightarrow$ 1) $Vy = b$

2) $V^T x = y$

1) $Vy = b$

$$\begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ 1 & -1 & 1 & \\ -2 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 2 \\ 1 \end{bmatrix}$$

$y_1 = 3$

$3 \cdot 3 + 3y_2 = 15 \longrightarrow y_2 = 2$

$3 - 2 + y_3 = 2 \longrightarrow y_3 = 1$

$-6 + 6 + 1 + 3y_4 = 1 \longrightarrow y_4 = 0$

$$\Longrightarrow y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

2) $V^T x = y$

$$\begin{bmatrix} 2 & 3 & 1 & -2 \\ & 3 & -1 & 3 \\ & & 1 & 1 \\ & & & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$x_4 = 0$

$x_3 = 1$

$-1 + 3x_2 = 2 \longrightarrow x_2 = 1$

$2x_1 + 3 + 1 = 3 \longrightarrow x_1 = -1/2$

$$\Longrightarrow x = \begin{bmatrix} -1/2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Metoda najmanjših kvadratov

$Ax = b$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Iščemo rešitev po MNK, torej minimiziramo $\|Ax-b\|_2$

Iskanje x, za katerega je ta minimum dosežen je ekvivalentno reševanja sistema $\boxed{A^TAx = A^Tb}$ NORMALNI SISTEM

Vemo, da je A^TA je simetrična in poz. definitna, zato lahko uporabimo Choleskega

↳ DOKAZ: $A^TA = B$

· simetričnost: $B^T = B$, ker $(A^TA)^T = A^T(A^T)^T = A^TA$

· poz. def:

④ Poišči parabolo $p(x) = c + bx + ax^2$, ki se po MNK najbolje prilagaja funkciji f, za katero poznamo vrednosti:

$f(-1) = 11/4$

$f(0) = 7/4$

$f(1) = 1/4$

$f(2) = 13/4$

želimo, da se te točke čim bližje paraboli po drugi normi

$p(x_i) = f(x_i)$

$p(-1) = c - b + a$

$p(0) = c$

$p(1) = c + b + a$

$p(2) = c + 2b + 4a$

$$\begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \end{array} \underbrace{\begin{bmatrix} c \\ b \\ a \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 11/4 \\ 7/4 \\ 1/4 \\ 13/4 \end{bmatrix}}_{b}$$

$$A^TA: \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

↳ mora biti simetrična in poz. definitna

$$A^Tb: \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 11/4 \\ 7/4 \\ 1/4 \\ 13/4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

· $A^TAx = A^Tb$

$$\underbrace{\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}}_{A^TA} \underbrace{\begin{bmatrix} c \\ b \\ a \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}}_{\substack{A^Tb \\ \shortparallel \\ d}}$$

NORMALNI SISTEM

• Ta sistem bomo sedaj rešili z razcepom Choleskega

$A^T A = V^T V$

$$V = \begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix}$$

$v_{22} = \sqrt{6-1} = \sqrt{5}$

$v_{32} = \frac{1}{\sqrt{5}} (8-3 = \sqrt{5}$

$v_{33} = \sqrt{18-5-9} = 2$

$\underbrace{V^T V x}_{y} = d$

1) $Vy = d$

$$\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

$y_1 = 4$

$4 + \sqrt{5} y_2 = 4 \longrightarrow y_2 = 0$

$12 + 2y_3 = 16 \longrightarrow y_3 = 2$

$\Longrightarrow y = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

2) $V^T x = y$

$$\begin{bmatrix} 2 & 1 & 3 \\ & \sqrt{5} & \sqrt{5} \\ & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$x_3 = 1$

$\sqrt{5} x_2 + \sqrt{5} = 0 \longrightarrow x_2 = -1$

$2x_1 - 1 + 3 = 4 \longrightarrow x_1 = 1$

$\Longrightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\Longrightarrow$ $c = 1$, $b = -1$, $a = 1$ $\Longrightarrow$ $\boxed{p(x) = 1 - x + x^2}$