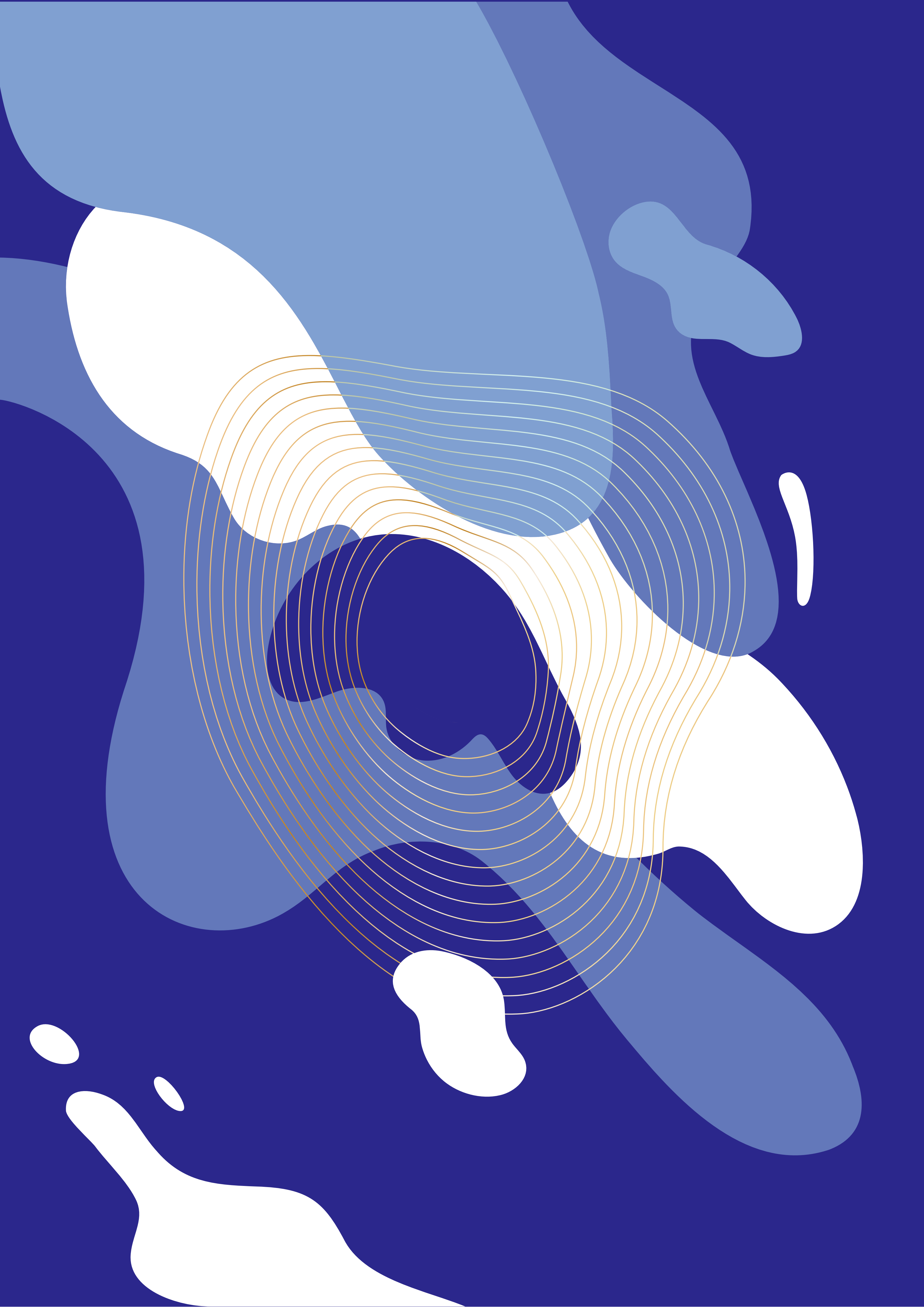
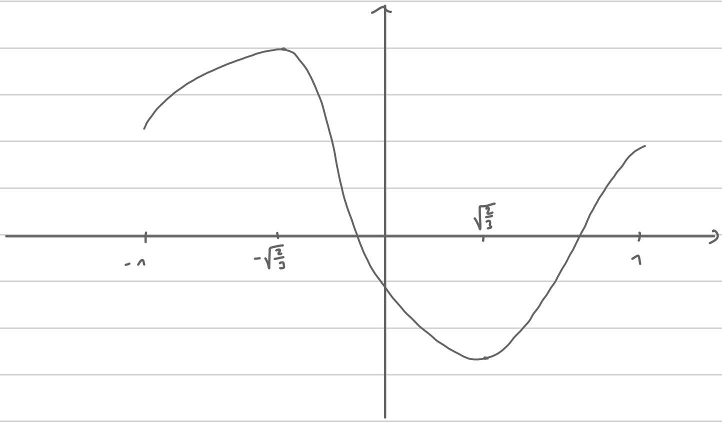


① Podana je funkcija $f(x) = x^3 - 2x + 4$ in interval $I = [-1, 1]$. Izračunaj $\|f\|_\infty$ in $\|f\|_2$ na I

$$\boxed{\|f\|_\infty = \max_{x \in I} |f(x)|}$$



$$f'(x) = 3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x_{1,2} = \pm\sqrt{2/3}$$

$$f(x_1 = -\sqrt{2/3}) = \left(\tfrac{2}{3}\right)^{3/2} + 2\sqrt{\tfrac{2}{3}} + 4 = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}} + 2 \cdot \sqrt{\tfrac{2}{3}} + 4 \doteq 5{,}09$$

$$f(x_2 = +\sqrt{2/3}) = \tfrac{2}{3}\sqrt{\tfrac{2}{3}} - 2\sqrt{\tfrac{2}{3}} + 4 \doteq 2{,}91$$

$$f(x_3 = -1) = -1 + 2 + 4 = 5$$

$$f(x_4) = 1 - 2 + 4 = 3$$

$$\Rightarrow \|f\|_\infty = f(x_1) \doteq 5{,}09$$

$$\boxed{\|f\|_2 = \sqrt{\langle f, f \rangle}} = \int_{-1}^{1} f^2(x)\,dx = \int_{-1}^{1} (x^3 - 2x + 4)^2\,dx = \int_{-1}^{1} (x^3 - 2x + 4)(x^3 - 2x + 4)\,dx =$$

$$= \int_{-1}^{1} x^6 - 2x^4 + 4x^3 - 2x^4 + 4x^2 - 8x + 4x^3 - 8x + 16\,dx =$$

$$= \int_{-1}^{1} x^6 - 4x^4 + 8x^3 + 4x^2 - 16x + 16\,dx =$$

$$= \frac{x^7}{7} - 4\frac{x^5}{5} + 8\frac{x^4}{4} + 4\frac{x^3}{3} + 16\frac{x^2}{2} + 16x \Big|_{-1}^{1} =$$

$$= \tfrac{1}{7} - \tfrac{4}{5} + 2 + \tfrac{4}{3} + 8 + 16 - \left(-\tfrac{1}{7} + \tfrac{4}{5} + 2 - \tfrac{4}{3} - 8 - 16\right) =$$

$$= \tfrac{2}{7} - \tfrac{8}{5} + \tfrac{8}{3} + 32$$

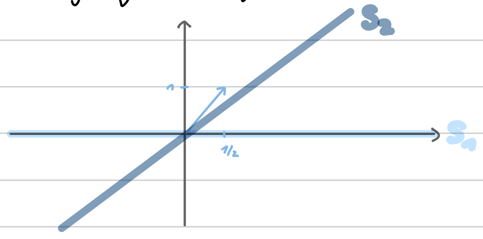
$$= \frac{3562}{105}$$

$$\xRightarrow{\sqrt{\ }} \|f\|_2 = 5{,}77$$

OPTIMALNI APROKSIMACIJSKI PROBLEM

Naj bo X vektorski prostor z normo $\|.\|$ in naj bo $S \subseteq X$. Za $f \in X$ rečemo $\tilde{f} \in S$, da je $\|f - \tilde{f}\| = \inf_{s \in S} \|f - s\|$

② Naj bo $X=\mathbb{R}^2$. Podan je vektor $f=\begin{bmatrix}1/2\\1\end{bmatrix}$. Poiščite vektorja $\tilde{f}_1$ in $\tilde{f}_2$ v prostorih $S_1=\mathcal{L}in\{\begin{bmatrix}1\\0\end{bmatrix}\}$ in $S_2=\mathcal{L}in\{\begin{bmatrix}1\\1\end{bmatrix}\}$, ki sta najbližja vektorju f v $\|\cdot\|_\infty$ in $\|\cdot\|_2$

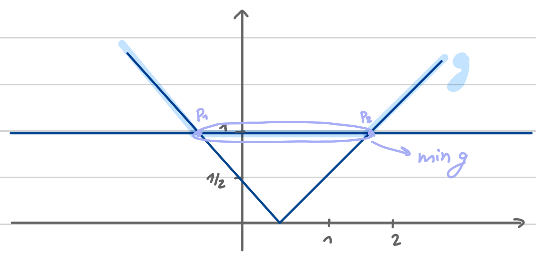


$$\|f-s_1\|_\infty = \left\|\begin{bmatrix}1/2\\1\end{bmatrix} - \alpha\begin{bmatrix}1\\0\end{bmatrix}\right\|_\infty$$
$$= \left\|\begin{bmatrix}1/2-\alpha\\1\end{bmatrix}\right\|_\infty$$
$$= \max\{|1/2-\alpha|, 1\}$$

Iščemo $\min \max\{|1/2-\alpha|, 1\}$

Označimo $g(\alpha)=\max\{|1/2-\alpha|, 1\}$ $\longrightarrow$

narišemo:



Izračunamo P_1 in P_2

$1/2-\alpha<1$

$\alpha>-1/2 \longrightarrow p_1=(-1/2, 1)$

$\alpha-1/2=1$

$\alpha=3/2 \longrightarrow p_2=(\frac{3}{2}, 1)$

$\Downarrow$

$\alpha\in[-1/2, 3/2]$

$\Longrightarrow$ $\tilde{f}_1$ v $\|\cdot\|_\infty$: $\alpha\cdot\begin{bmatrix}1\\0\end{bmatrix}$, $\alpha\in[-1/2, 3/2]$

• Poiščimo še $\tilde{f}_1$ v $\|\cdot\|_2$

$$\|f-s_1\|_2 = \left\|\begin{bmatrix}1/2-\alpha\\1\end{bmatrix}\right\|_2 = \sqrt{(1/2-\alpha)^2+1^2} = \sqrt{\tfrac{1}{4}-\alpha+\alpha^2+1} = \sqrt{\alpha^2-\alpha+\tfrac{5}{4}}$$

Označimo $g(\alpha)=\alpha^2-\alpha+\frac{5}{4}$

$g'(\alpha)=2\alpha-1=0 \longrightarrow \alpha=1/2$

$\Rightarrow$ $\tilde{f}_1$ v $\|\cdot\|_2$: $\begin{bmatrix}1/2\\0\end{bmatrix}$

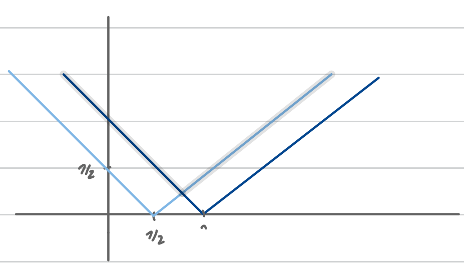
↳ Vzamemo 2. prostor $S_2=\mathcal{L}in\{\begin{bmatrix}1\\1\end{bmatrix}\}$

• $\tilde{f}_2$ v $\|\cdot\|_\infty$: $\|f-s_2\|_\infty = \left\|\begin{bmatrix}1/2\\1\end{bmatrix}-\alpha\begin{bmatrix}1\\1\end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix}1/2-\alpha\\1-\alpha\end{bmatrix}\right\|_\infty = \max\{|\frac{1}{2}-\alpha|, |1-\alpha|\}$

$\alpha-\frac{1}{2}=1-\alpha$

$2\alpha=3/2$

$\alpha=3/4$



• še $\tilde{f}_2$ v $\|\cdot\|_2$: $\|f-s_2\|_2 = \left\|\begin{bmatrix}1/2-\alpha\\1-\alpha\end{bmatrix}\right\|_2 = \sqrt{(\alpha^2-\alpha+\frac{1}{4})+1-2\alpha+\alpha^2} = \sqrt{2\alpha^2-3\alpha+5/4}$

$g(\alpha)=2\alpha^2-3\alpha+5/4$

$g'(\alpha)=4\alpha-3=0 \longrightarrow \alpha=3/4$

$\Rightarrow$ $\tilde{f}_2$ v $\|\cdot\|_2$: $\frac{3}{4}\begin{bmatrix}1\\1\end{bmatrix}$

Bernsteinovi bazni polinom:

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \quad i=0,1,\dots,n \quad x\in[0,1]$$

Bernsteinov polinom:

$$\mathcal{B}_n f(x) = \sum_{i=0}^{n} f\left(\tfrac{i}{n}\right) \mathcal{B}_i^n(x)$$

$$\|f - \mathcal{B}_n f\|_{\infty,[0,1]} \xrightarrow{n\to\infty} 0$$

③ a) Zapišite in narišite Bernsteinove bazne polinome za $n=0,1,2$

b) kakšne so vrednosti B.b.p. v točkah $x=0$ in $x=1$ za poljuben n?

a) $n=0$: $\mathcal{B}_0^0(x) = 1\cdot1\cdot1 = 1$

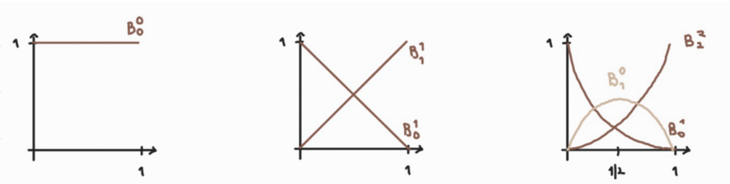
$n=1$: $\mathcal{B}_0^1(x) = 1\cdot1\cdot(1-x)^1 = (1-x)$

$\mathcal{B}_1^1(x) = x$

$n=2$: $\mathcal{B}_0^2(x) = \binom{2}{0}\cdot(1-x)^2 = (1-x)^2 = x^2-2x+1$

$\mathcal{B}_1^2(x) = 2x(1-x) = 2x-2x^2 = 2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}$

$\mathcal{B}_2^2(x) = x^2$



b) $\boxed{x=0}$: $\mathcal{B}_i^n(0) = \binom{n}{i} 0^i (1-0)^{n-i}$ $\boxed{0^0=1}$

→ $i=0$: $\mathcal{B}_0^n(0) = 1\cdot1\cdot1 = 1$

→ $0<i\leq n$: $\mathcal{B}_i^n(0) = 0$

$\boxed{x=1}$: $\mathcal{B}_i^n(1) = \binom{n}{i} 1^i (1-1)^{n-i}$

→ $i=n$: $\mathcal{B}_n^n(1) = 1$

→ $0\leq i<n$: $\mathcal{B}_i^n(1) = 0$

4. Zapiši $\mathcal{B}_2 f$ in $\mathcal{B}_2 p$ za:

   a) $f(x) = \frac{1}{3x+1}$

   b) $p(x) = 1$

In ju izrazi v potenčni bazi

a) $\mathcal{B}_2 f(x) = \sum_{i=0}^{2} \frac{1}{3(\frac{i}{2})+1} \cdot \mathcal{B}_i^2(x) = x^2 - 2x + 1 - \frac{4}{5}x^2 + \frac{4}{5}x + \frac{x^2}{4} = \frac{20-16+5}{20}x^2 - \frac{6}{5}x + 1 = \frac{9}{20}x^2 - \frac{6}{5}x + 1$

$\boxed{i=0}$ $\frac{1}{1} B_0^2(x) = (1-x)^2$

$\boxed{i=1}$ $\frac{2}{5} B_1^2(x) = \frac{4}{5}x(1-x)$

$\boxed{i=2}$ $\frac{1}{4} B_2^2(x) = \frac{x^2}{4}$

b) $B_2 p(x) = \sum_{i=0}^{2} 1 \cdot B_i^2(x) = x^2 - 2x + 1 - 2x^2 + 2x + x^2 = 1$

$\boxed{i=0}$: $B_0^2(x) = (1-x)^2$

$\boxed{i=1}$: $B_1^2(x) = 2x(1-x)$

$\boxed{i=2}$: $B_2^2(x) = x^2$

V splošnem: $p(x) = 1$

$B_n p(x) = \sum_{i=0}^{n} 1 \cdot B_i^n(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = (x + (1-x))^n = 1^n = 1$

$\sum_{i=0}^{r} \binom{r}{i} a^i b^{r-i} = (a+b)^r$

$\sum_{i=0}^{n} B_i^n(x) = 1 \qquad i = 0, \dots, n$

B_i^n tvorijo razčlenitev enote

28.2

PONOVITEV: $\mathcal{B}_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad x \in [0,1]$

$\mathcal{B}_n f = \sum_{i=0}^{n} f(\frac{i}{n}) \mathcal{B}_i^n(x)$

$\sum_{i=0}^{n} \mathcal{B}_i^n(x) = 1$

5. Naj bo $p_1(x) = x$. Pokaži, da velja $\mathcal{B}_n p_1 = p_1$

$$\mathcal{B}_n p_1 = \sum_{i=0}^{n} \overbrace{\frac{i}{n}}^{p_1(\frac{i}{n})} \overbrace{\underbrace{\binom{n}{i}}_{\frac{n!}{(n-i)!\,i!}} x^i (1-x)^{n-i}}^{\mathcal{B}_i^n(x)} = \sum_{i=0}^{n} \underbrace{\frac{i}{n} \cdot \frac{n!}{(n-i)!\,i!}}_{=\binom{n-1}{i-1}} x^i (1-x)^{n-i} = \sum_{i=0}^{n} \binom{n-1}{i-1} x^i (1-x)^{n-i} = x \cdot \sum_{i=1}^{n} \binom{n-1}{i-1} x^{i-1} (1-x)^{n-i} = x \cdot (x + (1-x))^{n-1} = x = p_1$$

$\mathcal{B}_i^{n-1} = \binom{n-1}{i} x^i (1-x)^{n-i-1}$

binomski izrek

APROKSIMACIJA PO METODI NAJMANJŠIH KVADRATOV

- imamo prostor $(X, \langle\cdot,\cdot\rangle)$, $f \in X$
- $f^* \in S$, $S \leq X$, $f^* = \sum_{i=1}^{n} \alpha_i f_i$ ← baza za S

  $\|f - f^*\|_2 = \min_{s \in S} \|f - s\|_2$
- Veljati mora: $\langle f - f^*, f_i \rangle = 0$ za $\forall i$:

$$\underbrace{\begin{bmatrix} \langle f_1,f_1\rangle & \langle f_1,f_2\rangle & \dots & \langle f_1,f_n\rangle \\ \langle f_2,f_1\rangle & \langle f_2,f_2\rangle & \dots & \langle f_2,f_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle f_n,f_1\rangle & \langle f_n,f_2\rangle & \dots & \langle f_n,f_n\rangle \end{bmatrix}}_{\text{GRAMOVA MATRIKA}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle f_1, f\rangle \\ \langle f_2, f\rangle \\ \vdots \\ \langle f_n, f\rangle \end{bmatrix}$$

① Poiščite polinom $p^* \in \mathbb{P}_1$, ki po m.n.k. najbolje aproksimira funkcijo $f(x) = e^x$ na intervalu $[-1,1]$. Za skalarni produkt vzemite

a) $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

b) $\langle f,g\rangle = \sum_{i=1}^{3} f(x_i)g(x_i)$

točke x_1, x_2, x_3 izberemo ekvidistančno $-1 \quad 0 \quad 1$

a) $p = \alpha_2 x + \alpha_1$

$f_1(x) = 1 \qquad f_2(x) = x$

$\langle f_1, f_1\rangle = \int_{-1}^{1} 1\,dx = 2$

$\langle f_1, f_2\rangle = \int_{-1}^{1} x\,dx = 0 \qquad \Longrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} = G$

$\langle f_2, f_2\rangle = \frac{2}{3}$

$\langle f_1, f\rangle = \int_{-1}^{1} e^x = e - \frac{1}{e}$

$\langle f_2, f\rangle = \int_{-1}^{1} x e^x = x e^x \Big|_{-1}^{1} - e^x \Big|_{-1}^{1} = e^1 + e^{-1} - (e - \frac{1}{e}) = \frac{2}{e}$

$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} e - \frac{1}{e} \\ \frac{1}{e} \end{bmatrix} \Longrightarrow \alpha_1 = \frac{e - \frac{1}{e}}{2} \Longrightarrow p^* = \frac{3}{2}x + \frac{e - \frac{1}{e}}{2}$

$\alpha_2 = \frac{3}{2} \cdot \frac{2}{e} = \frac{3}{e}$

b) $\langle f_1, f_1\rangle = \sum_{i=1}^{3} 1 = 3$

$\langle f_1, f_2\rangle = \sum_{i=1}^{n} 1 \cdot f_2(x) = f_2(x_1 = -1) + f_2(x_2 = 0) + f_2(x_3 = 1) = 0 \qquad \Longrightarrow G = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\langle f_2, f_2\rangle = \sum_{i=1}^{3} f_2(x_i) \cdot f_2(x_i) = \sum_{i=1}^{n} x_i^2 = (-1)^2 + 0^2 + 1^2 = 2$

$\langle f_1, f\rangle = \sum_{i=1}^{3} e^{x_i} = e^{-1} + \overset{e^0}{1} + e^1$

$\langle f_2, f\rangle = \sum_{i=1}^{3} f_2(x_i) e^{x_i} = -e^{-1} + 0 \cdot e^0 + 1 \cdot e^1$

$\Longrightarrow \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 + e + \frac{1}{e} \\ e - \frac{1}{e} \end{bmatrix} \Longrightarrow \alpha_1 = \frac{1 + e + \frac{1}{e}}{3} \Longrightarrow p^*(x) = \alpha_2 x + \alpha_1$

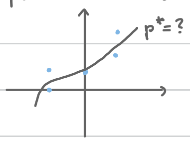
$\alpha_2 = \frac{e - \frac{1}{e}}{2}$

② Točke $(-1,0), (-1,1), (0,1), (1,2), (1,3)$ aproksimirajte po m.n.k. s:

a) Polinomom $p_2^* \in \mathbb{P}_2$

b) Polinomom $p_3^* \in \mathbb{P}_3$

a) $p_2^* = \alpha_1 + \alpha_2 x + \alpha_3 x^3$



Vzamemo $f_1(x)=1$, $f_2(x)=x$, $f_3(x)=x^2$

$\langle f,g\rangle = \sum_{i=1}^{5} f(x_i)\, g(x_i)$

$\langle f_1,f_1\rangle = \langle 1,1\rangle = \sum_{i=1}^{5} 1 = 5$

$\langle f_1,f_2\rangle = \langle 1,x\rangle = \sum_{i=1}^{5} 1(x_i)\cdot x(x_i) = 1\cdot(-1) + 1\cdot(-1) + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 = 0$

$\langle f_2,f_2\rangle = \langle x,x\rangle = (-1)^2 + (-1)^2 + 0^2 + 1^2 + 1^2 = 4$

$\langle f_1,f_3\rangle = \langle 1,x^2\rangle = 1\cdot(-1)^2 + 1\cdot(-1)^2 + 1\cdot 0^2 + 1\cdot 1^2 + 1\cdot 1^2 = 4$

$\langle f_2,f_3\rangle = \langle x,x^2\rangle = (-1)(-1)^2 + (-1)(-1)^2 + 0\cdot 0^2 + 1\cdot 1^2 + 1\cdot 1^2 = 0$

$\langle f_3,f_3\rangle = \langle x^2,x^2\rangle = (-1)^2\cdot(-1)^2 + (-1)^2(-1)^2 + 0^2 0^2 + 1^2 1^2 + 1^2\cdot 1^2 = 4$

$$\Longrightarrow G = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$\langle f,1\rangle = \sum_{i=1}^{5} f(x_i)\, 1(x_i) = 0\cdot 1 + 1\cdot 1 + 1\cdot 1 + 2\cdot 1 + 3\cdot 1 = 7$

$\langle f,x\rangle = 0(-1) + 1\cdot(-1) + 1\cdot 0 + 2\cdot 1 + 3\cdot 1 = 4$

$\langle f,x^2\rangle = 0\cdot(-1)^2 + 1(-1)^2 + 1\cdot 0^2 + 2\cdot 1^2 + 3\cdot 1^2 = 6$

$$\Longrightarrow \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix} \Longrightarrow \alpha_2 = 1$$

$5\alpha_1 + 4\alpha_3 = 7 \rightarrow \alpha_1 = 1$

$4\alpha_1 + 4\alpha_3 = 6 \rightarrow \alpha_3 = 1/2$

$$\Longrightarrow p_2(x) = 1 + x + \frac{x^2}{2}$$

b) OP: Prvi trije bazni polinomi ostanejo enaki, samo dodamo še $f_4(x) = x^3$, in tudi G nerabimo celo šeenkrat računat

$p_3^* = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$

$f_4(x) = x^3$

$\langle x,x^3\rangle = (-1)(-1)^3 + (-1)(-1)^3 + 0\cdot 0^3 + 1^3 + 1\cdot 1^3 = 4$

$\langle x^2,x^3\rangle = 0$

$\langle x^3,x^3\rangle = 4$

$\langle 1,x^3\rangle = 1\cdot(-1)^3 + 1\cdot(-1)^3 + 1\cdot 0^3 + 1\cdot 1^3 + 1\cdot 1^3 = 0$

$$\Longrightarrow G = \begin{bmatrix} 5 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}$$

$\langle f,x^3\rangle = 0\cdot(-1) + 1\cdot(-1)^3 + 1\cdot 0^3 + 2\cdot 1^3 + 3\cdot 1^3 = 4$

$$\Longrightarrow \begin{bmatrix} 5 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \\ 4 \end{bmatrix}$$

dva sistema enačb

$\longrightarrow \left.\begin{array}{l} 5\alpha_1 + 4\alpha_3 = 7 \\ 4\alpha_1 + 4\alpha_3 = 6 \end{array}\right] \Longrightarrow \begin{array}{l} \alpha_1 = 1 \\ \alpha_3 = 1/2 \end{array}$

$\longrightarrow \left.\begin{array}{l} 4\alpha_2 + 4\alpha_4 = 4 \\ 4\alpha_2 + 4\alpha_4 = 4 \end{array}\right] \Longrightarrow \begin{array}{l} \alpha_4 = 1 - \alpha_3 \\ \alpha_2 \in \mathbb{R} \end{array}$

$$\Longrightarrow p_3(x) = 1 + tx + \tfrac{1}{2}x^2 + (1-t)x^3, \quad t \in \mathbb{R}$$

3 Za funkcijo $f(x) = \sin^4 x$ na intervalu $[0, \frac{\pi}{2}]$ poišči element najboljše aproksimacije po zvezni m.n.k. iz prostora $S = \mathcal{L}in\{\overset{\varphi_1}{1}, \overset{\varphi_2}{\sin x}, \overset{\varphi_3}{\cos x}\}$

$$f^* = \alpha_1 \cdot 1 + \alpha_2 \cdot \sin x + \alpha_3 \cdot \cos x$$

$$\langle f, g\rangle = \int_0^{\pi/2} f \cdot g \, dx$$

$$\langle 1, 1\rangle = \frac{\pi}{2}$$

$$\langle 1, \sin x\rangle = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1$$

$$\langle \sin x, \sin x\rangle = \int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{1}{2} \frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1!} = \frac{1}{4}\pi$$

$$\boxed{\begin{aligned} &\int_0^{\pi/2} \sin^{2p-1} x \cdot \cos^{2q-1} x \, dx = \frac{1}{2} \beta(p, q) = \frac{1}{2} \frac{\Gamma(p) \cdot \Gamma(q)}{\Gamma(p-q)} \\ &\Gamma(n) = (n-1)! \quad n \in \mathbb{N} \\ &\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \\ &\Gamma(r+1) = r \cdot \Gamma(r) \end{aligned}}$$

$$\langle 1, \cos x\rangle = \int_0^{\pi/2} \cos x \, dx = \sin \frac{\pi}{2} = 1$$

$$\langle \sin x, \cos x\rangle = \int_0^{\pi/2} \sin x \cdot \cos x \, dx = \int_0^1 t \, dt = \frac{t^2}{2} \Big|_0^1 = \frac{1}{2}$$

$\sin x = t$
$\cos x \, dx = dt$

$$\langle \cos x, \cos x\rangle = \int_0^{\pi/2} \cos^2 x = \frac{1}{2} \beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4}$$

$$\langle \varphi_1, f\rangle = \langle 1, \sin^4 x\rangle = \int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{2} \beta\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \pi}{2!} = \frac{3}{16}\pi$$

$2p-1 = 4$
$p = 5/2$

$$\langle \varphi_2, f\rangle = \langle \sin x, \sin^4 x\rangle = \frac{1}{2} \beta\left(3, \frac{1}{2}\right) = \frac{8}{15}$$

$$\langle \varphi_3, f\rangle = \langle \cos x, \sin^4 x\rangle = \frac{1}{2} \beta\left(\frac{5}{2}, 1\right) = \frac{1}{5}$$

$$\Longrightarrow \begin{bmatrix} \frac{\pi}{2} & 1 & 1 \\ 1 & \frac{\pi}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{3\pi}{16} \\ \frac{8}{15} \\ \frac{1}{5} \end{bmatrix}$$

$$\Longrightarrow \boxed{\begin{aligned} \alpha_1 &\doteq 1{,}248 \\ \alpha_2 &\doteq -0{,}101 \\ \alpha_3 &\doteq -1{,}26 \end{aligned}}$$

rešitev povedala

(4) Točke $\overset{x_1,\, f(x_1)}{(1,0)}, \overset{x_2,\, f(x_2)}{(2,2)}, \overset{x_3,\, f(x_3)}{(1,2)}, (3,2), (1,1), (2,1), (2,0), (3,0), (3,1)$ aproksimiraj po diskretni m.n.k. z elementom f^* iz prostora $S = \mathcal{L}in\{1, e^x\}$

$$f^* = \alpha_1 + \alpha_2 e^x$$

$$\varphi_1(x) = 1, \quad \varphi_2(x) = e^x$$

$$\langle f, g\rangle = \sum_{i=1}^{9} f(x_i)\, g(x_i)$$

$$\langle \varphi_1, \varphi_1\rangle = \langle 1, 1\rangle = \sum_{i=1}^{9} 1 = 9$$

$$\langle \varphi_1, \varphi_2\rangle = \langle 1, e^x\rangle = \sum_{i=1}^{9} e^{x_i} = e^1 + e^2 + e^3 + e^1 + e^1 + e^2 + e^2 + e^3 + e^3 = 3e + 3e^2 + 3e^3$$

$$\langle \varphi_2, \varphi_2\rangle = \langle e^x, e^x\rangle = \sum_{i=1}^{9} e^{2x_i} = e\cdot e + e^2e^2 + e^3e^3 + ee + ee + e^2e^2 + e^2e^2 + e^3e^3 + e^3e^3 = 3e^2 + 3e^4 + 3e^6$$

$$\Longrightarrow G = \begin{bmatrix} 9 & 3e+3e^2+3e^3 \\ 3e+3e^2+3e^3 & 3e^2+3e^4+3e^6 \end{bmatrix}$$

$$\langle f, 1\rangle = \sum_{i=1}^{9} f(x_i) = 0+2+2+2+1+1+0+0+1 = 9$$

$$\langle f, e^x\rangle = \sum_{i=1}^{9} f(x_i) e^{x_i} = 0\cdot e + 2\cdot e^2 + 2\cdot e + 2\cdot e^3 + 1\cdot e + e^2 + 0 + 0 + e^3 = 3e + 3e^2 + 3e^3$$

$$\Longrightarrow \begin{bmatrix} 9 & 3e+3e^2+3e^3 \\ 3e+3e^2+3e^3 & 3e^2+3e^4+3e^6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3e+3e^2+3e^3 \end{bmatrix} \longrightarrow \text{opazimo trivialno rešitev } \begin{bmatrix} \alpha_1 = 1 \\ \alpha_2 = 0 \end{bmatrix}$$

$$\Longrightarrow f^* = 1$$

7.3

ALGORITEM (Gram-Schmidt)

Vhod: baza $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$

Izhod: ON baza $\{f_1, \dots, f_n\}$

```
for i = 1:n
    f_i = φ_i
end

for i = 1:n
    f_i = f_i / ||f_i||_2
    for j = i+1:n
        f_j = f_j − <f_j, f_i> f_i
    end
end
```

FORMULE ZA VSOTE

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{1}{30} n\cdot(n+1)(2n+1)(3n^2+3n-1)$$

⑤ Naj bo $\langle g,h\rangle = \sum_{i=1}^{4} g(i)\,h(i)$

a) Za skalarni produkt poišči ON bazo za $\mathbb{P}_2$

b) Poišči $p^* \in \mathbb{P}_2$, ki po MNK najbolje aproksimira $f(x) = \frac{4}{1+x}$

a) $\{1, x, x^2\}$, where $1 = f_1$, $x = f_2$, $x^2 = f_3$.

$\boxed{i=1}$ $f_1 = \frac{f_1}{\|f_1\|_2} = \frac{1}{\sqrt{\langle f_1,f_1\rangle}} = \frac{1}{\sqrt{\sum_{i=1}^{4} 1}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

$f_2 = f_2 - \langle f_2,f_1\rangle f_1 = x - \left(\sum_{i=1}^{4} x(i)\cdot \frac{1}{2}(i)\right)\cdot\frac{1}{2} = x - \frac{1}{4}\sum_{i=1}^{4} i = x - \frac{10}{4} = x - \frac{5}{2}$

$f_3 = f_3 - \langle f_3,f_1\rangle \cdot f_1 = x^2 - \left(\sum_{i=1}^{4} i^2\cdot\frac{1}{2}\right)\cdot\frac{1}{2} = x^2 - \frac{1}{4}(1+4+9+16) = x^2 - \frac{15}{2}$

$\Longrightarrow f = \left\{\frac{1}{2},\ x-\frac{5}{2},\ x^2-\frac{15}{2}\right\}$

$\boxed{i=2}$ $f_2 = \frac{f_2}{\|f_2\|_2} = \frac{x-\frac{5}{2}}{\sqrt{\langle f_2,f_2\rangle}} = \frac{x-\frac{5}{2}}{\sqrt{5}} = \frac{2\sqrt{5}x - 5\sqrt{5}}{10}$

$\langle f_2,f_2\rangle = \sum_{i=1}^{4}\left(i-\frac{5}{2}\right)^2 = \sum_{i=1}^{4} i^2 - 5i + \frac{25}{4} = 1-5+4-10+9-15+16-20+25 = 5$

$f_3 = f_3 - \langle f_3, f_2\rangle f_2 = x^2 - \frac{15}{2} - \left(\sum_{i=1}^{4}\left(i^2-\frac{15}{4}\right)\cdot\frac{\sqrt{5}(2i-5)}{10}\right)\frac{\sqrt{5}(2i-5)}{10} =$

$= x^2 - \frac{15}{2} - \frac{5(2x-5)}{2\cdot 10\cdot 10}\left(\sum_{i=1}^{4}\left(2i^3-5i^2-15i+\frac{75}{2}\right)\right) =$

uporabimo formule za vsote

$2\sum_{i=1}^{4} i^3 = 2\cdot\frac{4^2(4+1)^2}{4} = 200$

$-5\sum_{i=1}^{4} i^2 = -5\cdot 30 = -150$

$-15\sum_{i=1}^{4} i = -150$

$\frac{75}{2}\sum_{i=1}^{4} 4^2 = 150$

$= x^2 - \frac{15}{2} - \frac{2x-5}{20}\cdot 50 =$

$= x^2 - 5x + 5$

$\boxed{i=3}$ $f_3 = \frac{f_3}{\|f_3\|_2} = \frac{f_3}{\sqrt{\langle f_3,f_3\rangle}} = \frac{x^2-5x+5}{2}$

$\langle f_3,f_3\rangle = \sum_{i=1}^{4}(x^2-5x+5)^2 = \sum_{i=1}^{4}(x^4-10x^3+35x^2-50x+25) = 354-10\cdot 100+35\cdot 30-50\cdot 10+25\cdot 4 = 4$

$\Longrightarrow f = \left\{\frac{1}{2},\ \frac{\sqrt{5}}{10}(2x-5),\ \frac{x^2-5x+5}{2}\right\}$

b) postopek: $G\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} = \begin{bmatrix}\langle f,f_1\rangle\\ \vdots\\ \langle f,f_n\rangle\end{bmatrix} \Longrightarrow p^*(x) = \alpha_1\cdot f_1 + \alpha_2\cdot f_2 + \alpha_3\cdot f_3$

$\alpha_1 = \langle f,f_1\rangle = \sum_{i=1}^{4}\frac{4\cdot 2}{1+i}\cdot\frac{1}{2} = 2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) = 2\cdot\left(\frac{30+20+15+12}{60}\right) = \frac{77}{30}$

$\alpha_2 = \langle f,f_2\rangle = \sum_{i=1}^{4}\frac{4}{1+i}\cdot\frac{\sqrt{5}}{10}(2i-5) = \ldots = -\frac{59}{30\sqrt{5}}$

$\alpha_3 = \langle f,f_3\rangle = \sum_{i=1}^{4}\frac{4}{1+i}\cdot\frac{i^2-5i+5}{2} = \ldots = \frac{7}{30}$

ENAKOMERNA APROKSIMACIJA S POLINOMOM

- za $f \in C([a,b])$ iščemo polinom $p^* \in \mathbb{P}_n$ za katerega velja $\|f-p^*\|_{\infty,[a,b]} = \min_{p \in \mathbb{P}_n} \|f-p\|_{\infty,[a,b]}$
- ALGORITEM (Remesov postopek)
  - → Vhod: $f, [a,b], n$, množica $n+2$ točk $E=\{x_i \,;\, a \le x_0 < x_1 < \ldots < x_{n+1} \le b\}$
  - → Izhod: $p^* \in \mathbb{P}_n$
  - → for $k=0,1,\ldots$
    1) poišči polinom $p_k^* \in \mathbb{P}_n$, ki zadošča $f(x_i) - p_k^*(x_i) = (-1)^i \cdot m$ za $i=0,1,\ldots,n+1$
       [$n+2$ enačb za $\underbrace{n+2}_{\text{koef. } p_k,\ m}$ neznank]
    2) Poišči ekstrem residuala
       $r_k = f - p_k^*$, tj. poišči $u \in [a,b]$, da bo $|r_k(u)| = \|r_k\|_{\infty,[a,b]}$
    3) če je $|r_k(u)| = |m|$ končaj in vzemi p_k^*. Sicer zamenjaj točko v množici E z u tako, da se ohrani alterniranje residuala
  - → OP: če imamo za vhodni podatek še toleranco ε, potem v 3) točki gledaš če velja pogoj $(|r_k(u)| - |m|) < \varepsilon$

① Poišči $p^* \in \mathbb{P}_n$, ki je polinom najboljše enakomerne aproksimacije (p.n.e.a) za $f(x)=e^x$ na intervalu $[0,1]$. Za množico E vzemite $E = \{\underset{x_0}{0}, \underset{x_1}{\tfrac{1}{2}}, \underset{x_2}{1}\}$

1. KORAK ALGORITMA

$\boxed{k=0}$ $p_0^* = ax+b$

$i=0,1,2$

$i=0$: $f(x_0) - p_0^*(x_0) = (-1)^0 \cdot m$
$f(0) - (a \cdot 0 + b) = (-1) \cdot m$
$1-b=m$

$i=1$: $e^{\frac{1}{2}} - (a \cdot \frac{1}{2} + b) = -m$
$e^{\frac{1}{2}} - \frac{a}{2} - b = -m$

$i=2$: $e-a-b=m$

Dobili smo 3 enačbe za 3 neznanke, sedaj rešimo sistem

$$1-b=m$$
$$e^{\frac{1}{2}} - \frac{a}{2} - b = -m$$
$$e-a-b=m$$

$1-b = e-a-b$
$\boxed{a=e-1}$

$e^{\frac{1}{2}} - \frac{e-1}{2} - b = -m$
$e^{\frac{1}{2}} - \frac{e-1}{2} - b = b-1$
$2b = e^{\frac{1}{2}} - \frac{e}{2} + \frac{1}{2} + 1$
$\boxed{b = \frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} + \frac{3}{2}\right)}$

$m = 1 - \frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} + \frac{3}{2}\right) =$
$= -\frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2}\right) + \frac{1}{4} =$
$\boxed{= -\frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} - \frac{1}{2}\right)}$

Dobili smo torej $a = e-1$
$b = \frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} + \frac{3}{2}\right)$
$m = -\frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} - \frac{1}{2}\right)$

2. KORAK ALGORITMA

$r_0 = f - p_0^* = e^x - (e-1)x - \frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} + \frac{3}{2}\right)$

ekstrem:

$r_0' = e^x - e + 1 = 0$

$e^x = e - 1$

$u = x = \ln(e-1)$

po teoriji vemo: $|r_0(0)| = |f(0) - p_0^*(0)| = |m|$

$|r_0(1)| = |f(1) - p_0^*(1)| = |m|$

v tej nalogi je $|m| \doteq 0{,}1052$

$r(u) = e - 1 - (e-1)\ln(e-1) - \frac{1}{2}\left(e^{\frac{1}{2}} - \frac{e}{2} + \frac{3}{2}\right)$

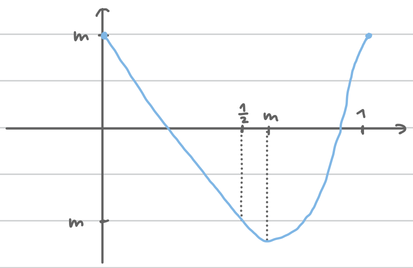
$|r(u)| = 0{,}1067$

$r(u) < 0$

$\Longrightarrow$ Ugotovili smo, da $|m| \neq |r_0(u)|$

3. KORAK ALGORITMA

Najprej narišimo $p_0^* = ax + b$



$E_1 = \{0, \ln(e-1), 1\}$

14.3

b) Izračunajmo sedaj je za $E_1 = \{\overset{x_0}{0}, \overset{x_1}{\ln(e-1)}, \overset{x_2}{1}\}$

$p(x) = ax + b$

Poglejmo naprej $f(x_i) - p(x_i) = (-1)^i \cdot m$

$\boxed{i=0}$: $e^0 - b = (-1)^0 \cdot m$

$1 - b = m \longrightarrow b = 1 - m$

$\boxed{i=1}$: $e^{\ln(e-1)} - a\ln(e-1) - b = (-1)^1 \cdot m$

$e - 1 - a \cdot \ln(e-1) - b = -m$

$\boxed{i=2}$: $e - a - b = m$

$e - a - b = 1 - b$

$\boxed{a = e - 1}$

$e - 1 - (e-1)\ln(e-1) - (1-m) = -m$

$e - 2 - (e-1)\ln(e-1) = -2m$

$\boxed{m = \frac{-e + 2 + (e-1)\ln(e-1)}{2}}$

$b = 1 - \frac{-e + 2 + (e-1)\ln(e-1)}{2}$

$\boxed{b = \frac{e - (e-1)\ln(e-1)}{2}}$

$\Longrightarrow r(x) = f(x) - p(x)$

$r(x) = e^x - (e-1)x - \frac{e - (e-1)\ln(e-1)}{2}$

$r'(x) = e^x - (e-1) = e^x - e + 1 = 0$

$e^x = e - 1 \quad /\ln$

$\ln(e^x) = \ln(e-1)$

$x = \ln(e-1) = x_2 = u$

$\Longrightarrow |r(\ln(e-1))| = |r(0)| = |r(1)| = |m|$

Še 3. korak samo preverimo za vsak slučaj:

$r(\ln(e-1)) = e^{\ln(e-1)} - (e-1)\ln(e-1) - \frac{e - (e-1)\ln(e-1)}{2}$

$= (e-1) - (e-1)\ln(e-1) - \frac{e - (e-1)\ln(e-1)}{2}$

$= \frac{e - 2 - (e-1)\ln(e-1)}{2}$

Torej velja $|r(u)| = |m|$

INTERPOLACIJA

Iščemo polinom, ki bo potekal skozi dane točke. Če imamo n+1 točk, obstaja enoličen polinom stopnje n, ki gre skozi te točke

LAGRANGEVA OBLIKA INT. POLINOMA

Lagrangev bazni polinom: $\boxed{\ell_{i,n}(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}}$ $\quad i = 0, 1, \ldots, n$

Interpolacijski polinom p, ki se z f ujema v x_i, $i = 0, 1, \ldots, n$: $\boxed{p(x) = \sum_{i=0}^{n} f(x_i) \cdot \ell_{i,n}(x)}$

① Določite polinom v Lagrangevi obliki, ki interpolira točke:

$\underset{x_0}{(1,2)}, \underset{x_1}{(2,0)}, \underset{x_2}{(4,-1)}, \underset{x_3}{(6,3)}$ in $\underset{x_4}{(7,5)}$

$$\ell_{0,4} = \prod_{j=1}^{4} \frac{x - x_j}{x_i - x_j} = \frac{(x-2)(x-4)(x-6)(x-7)}{(-1)(-3)(-5)(-6)} = \frac{(x-2)(x-4)(x-6)(x-7)}{90}$$

$$\ell_{1,4} = \frac{(x-1)(x-4)(x-6)(x-7)}{1 \cdot (-2) \cdot (-4) \cdot 1} = \frac{(x-1)(x-4)(x-6)(x-7)}{-40}$$

$$\ell_{2,4} = \frac{(x-1)(x-2)(x-6)(x-7)}{36}$$

$$\ell_{3,4} = \frac{(x-1)(x-2)(x-4)(x-7)}{-40}$$

$$\ell_{4,4} = \frac{(x-1)(x-2)(x-4)(x-6)}{90}$$

$$\Rightarrow p(x) = 2 \cdot \ell_{0,4} + 0 - 1 \cdot \ell_{2,4} + 3\ell_{3,4} + 5\ell_{4,4}$$

② Za funkcijo $f(x) = \cos(\frac{\pi}{4}x)$ poiščite Lagrangev interpolacijski polinom, ki se z f ujema v točkah iz množice $\{-1, 0, 2, 5\}$

$$\ell_{0,3} = \frac{x(x-2)(x-5)}{(-1)(-1-2)(-1-5)} = \frac{x(x-2)(x-5)}{-18}$$

$$\ell_{1,3} = \frac{(x+1)(x-2)(x-5)}{10}$$

$$\ell_{2,3} = \frac{(x+1)x(x-5)}{-18}$$

$$\ell_{3,3} = \frac{(x+1)x(x-2)}{90}$$

$$f(x_0) = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f(x_1) = \cos(0) = 1$$

$$f(x_2) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f(x_3) = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$p(x) = \frac{\sqrt{2}}{2} \cdot \ell_{0,3}(x) + \ell_{1,3}(x) + 0 - \frac{\sqrt{2}}{2}\ell_{4,3}(x)$$

③ Podana je funkcija $f(x) = \log x$ in ekvidistantne točke $x_i = x_0 + i \cdot h$, $i=0,1,2,3$, $x_0 > 0$

Določite vodilni koeficient interpolacijskega polinoma p, ki se z f ujema v točkah x_i, $i=0,1,2,3$

$$p(x) = \sum_{i=1}^{n} f(x_i) \cdot \frac{\prod_{j=1, i\neq j}^{n} (x-x_j)}{\prod_{j=1, i\neq j}^{n} (x_i - x_j)}$$

to ni relevantno za iskanje vodilnega koef

$$= \underbrace{\frac{\sum_{i=1}^{n} f(x_0)}{\prod_{j=1, i\neq j}^{n} (x_i-x_j)} \prod_{j=1, i\neq j}^{n} (x-x_j)}_{\text{polinom } n\text{-te stopnje}} + \underbrace{\frac{f(x_1)}{\cdots}}_{\text{polinom } n\text{-te stopnje}} \cdots$$

$\Longrightarrow$ Torej bo vodilni koeficient vsota vodilnih koeficientov za polinome $[a_1 \cdot x^n + b_1 \cdot x^n = (a_1+b_1)x^n]$

$$\frac{f(x_0)}{\prod_{j=0, j\neq 0}^{3} (x_0-x_j)} = \frac{\log(x_0)}{(x_0-(x_0+h))(x_0-(x_0+2h))(x_0-(x_0+3h))} = \frac{\log(x_0)}{(-h)(-2h)(-3h)} = \frac{\log(x_0)}{-6h^3}$$

$x_2 = x_0+2h$, $x_3 = x_0+3h$, $x_1 = x_0+h$

$$\frac{f(x_1)}{\prod_{j=0, j\neq 1}^{3} (x_1-x_j)} = \frac{\log(x_1)}{(x_0+h-(x_0))(x_0+h-(x_0+2h))(x_0+h-(x_0+3h))} = \frac{\log(x_1)}{h\cdot(-h)\cdot(-2h)} = \frac{\log(x_1)}{2h^3}$$

$$\frac{f(x_2)}{\prod_{j=0}^{3} (x_2-x_j)} = \frac{\log(x_2)}{(x_0+2h-x_0)(x_0+2h-(x_0+h))(x_0+2h-(x_0+3h))} = \frac{\log(x_2)}{(2h)(h)(-h)} = \frac{\log(x_2)}{-2h^3}$$

$$\frac{f(x_3)}{\prod_{j=0}^{3} (x_3-x_j)} = \frac{\log(x_3)}{(x_0+3h-x_0)(x_0+3h-(x_0+h))(x_0+3h-(x_0+2h))} = \frac{\log(x_3)}{(3h)(2h)(h)} = \frac{\log(x_3)}{6h^3}$$

$x_3 = x_0 + 3h$

Sedaj seštejmo, da dobimo vodilni koef:

$$\frac{\log(x_0)}{-6h^3} + \frac{\log(x_1)}{2h^3} + \frac{\log(x_2)}{-2h^3} + \frac{\log(x_3)}{6h^3} = \frac{\log\left(\frac{x_3}{x_0}\right)}{6h^3} + \frac{\log\left(\frac{x_1}{x_2}\right)}{2h^3} =$$

$$= \frac{\log\left(\frac{x_3}{x_0}\right) + \log\left(\frac{x_1}{x_2}\right)^3}{6h^3} = \frac{\log\left(\frac{x_3}{x_0}\cdot\left(\frac{x_1}{x_2}\right)^3\right)}{6h^3}$$

④ Preštej število operacij, ki jih potrebujemo za izračun vrednosti int. polinoma v Lagrangevi obliki

Vrednost pri $x = \tilde{x}$

$$p(x) = \sum_{i=0}^{n} f(x_i) \cdot \frac{\prod_{j=0, j\neq i}^{n} (x-x_j)}{\prod_{j=0, j\neq i}^{n} (x_i-x_j)}$$

$\prod_{j=0, j\neq i}^{n} (x-x_j) = \frac{\omega(x)}{x-x_i}$ ← to uvedemo, da pridemo iz $(n-1)$ produktov na 1 produkt

$\prod_{j=0, j\neq i}^{n} (x_i-x_j) = d_i$ — konstanta, saj smo to že izračunali ko smo naredili bazo

$$\ell_{i,n}(x) = \frac{\prod_{j=0, j\neq i}^{n} (x-x_i)}{d}$$

$$\omega(x) = (x-x_0)(x-x_1)\cdot\ldots\cdot(x-x_n)$$

Torej je $p(x) = \sum_{i=0}^{n} f(x_i) \cdot \frac{\omega(x)}{d_i(x-x_i)} = \omega(x) \cdot \sum_{i=0}^{n} \frac{f(x_i)}{d_i(x-x_i)}$

$\omega(x)$: neodvisno od i in bo potrebno izračunati samo enkrat

Če sedaj vstavimo $\tilde{x}$: $p(x) = \underbrace{\omega(\tilde{x})}_{2n+1} \underbrace{\sum_{i=0}^{n} \frac{f(x_i)}{d_i(\tilde{x}-x_i)}}_{3(n+1)}$

$d_i(\tilde{x}-x_i)$: $n+1$ operacij

$2n+1$: $(n+1)$ odštevanj, n množenj

$3(n+1)$:
- $n+1$: razlika
- $n+1$: $\times d_i$
- $n+1$: f deliš z dobljenim

Skupaj: $(3n+3)+(2n+1)+1 = \underline{\underline{5n+5}}$

$+1$: je ena operacija da zmnožimo $\omega(x)$ in vsoto

NEWTONOVA OBLIKA INTERPOLACIJSKEGA POLINOMA

- newtonovi bazni polinomi : $1, (x-x_0), (x-x_0)(x-x_1), \dots, (x-x_0)\dots(x-x_{n-1})$
- polinom : $p(x) = [x_0]f \cdot 1 + [x_0,x_1]f \cdot (x-x_0) + \dots + [x_0,x_1,\dots,x_n]f \cdot (x-x_0)\dots(x-x_{n-1})$
  $= \sum_{i=0}^{n} [x_0,x_1,\dots,x_i]f \ (x-x_0)\dots(x-x_{i-1})$
- deljene diference : $x_i \le x_{i+1} \le \dots \le x_{i+k}$ $\quad ([x_i]f = f(x_i))$

$$[x_i,\dots,x_{i+k}]f = \begin{cases} \dfrac{f^{(k)}(x_i)}{k} & ; x_i = \dots = x_{i+k} \quad \text{k+1 točk} \\[2ex] \dfrac{[x_{i+1},\dots,x_{i+k}]f - [x_i,\dots,x_{i+k-1}]f}{x_{i+k}-x_i} & ; x_i \ne x_{i+k} \end{cases}$$

① Podana je funkcija $f(x) = \frac{x}{1+x}$. Določite Newtonov int. polinom, za katerega velja $p(-\frac{1}{2}) = f(-\frac{1}{2})$, $p(0) = f(0)$, $p(1) = f(1)$, $p(2) = f(2)$ (x_0, x_1, x_2, x_3)

baza : $\{1, (x+\frac{1}{2}), (x+\frac{1}{2})(x-0), (x+\frac{1}{2})(x-0)(x-1)\}$
$\quad [x_0]f \quad [x_0,x_1]f \quad [x_0,x_1,x_2]f \quad [x_0,x_1,x_2,x_3]f$

	$[\cdot]f$	$[\cdot,\cdot]f$	$[\cdot,\cdot,\cdot]f$	$[\cdot,\cdot,\cdot,\cdot]f$
$-\frac{1}{2}$	-1	$\frac{0+1}{0+\frac{1}{2}} = 2$		
0	0	$\frac{\frac{1}{2}}{1-0} = \frac{1}{2}$	$\frac{\frac{1}{2}-2}{1+\frac{1}{2}} = -1$	$\frac{-\frac{1}{6}+1}{2+\frac{1}{2}} = \frac{1}{3}$
1	$\frac{1}{2}$	$\frac{\frac{1}{3}-\frac{1}{2}}{2-1} = \frac{1}{6}$	$\frac{\frac{1}{6}-\frac{1}{2}}{2-0} = -\frac{1}{6}$	
2	$\frac{1}{3}$			

$p(x) = -1 + 2(x+\frac{1}{2}) - (x+\frac{1}{2})x + \frac{1}{3}(x+\frac{1}{2})x(x-1)$

HORNERJEV ALGORITEM

$d_i = [x_0, x_1, \dots, x_i]f \quad , \quad i = 0,1,\dots,n$

```
vhod :  x0, x1, ..., xn
        d0, d1, ..., dn
        x
izhod : vrednost p(x)
vn = dn
for i = n-1 : -1 : 0
    vi = di + (x - xi) v(i+1)
end
p(x) = v0
```

② Interpolacijske točke : $(1,1), (2,5), (3,3), (4,8)$ s kubičnim polinomom p v Newtonovi obliki.
S Hornerjevim algoritmom izračunajte $p(0)$.

	$[x_i]f$	$[x_i,x_{i+1}]f$	$[x_i,x_{i+1},x_{i+2}]f$	
1	1	4		
2	5	$\frac{-2}{1} = -2$	$-\frac{6}{2} = -3$	$\frac{\frac{7}{2}+\frac{6}{2}}{3} = \frac{13}{6}$
3	3	5	$\frac{7}{2}$	
4	8			

$B = \{1, (x-1), (x-1)(x-2), (x-1)(x-2)(x-3)\}$

$p(x) = 1 + 4(x-1) - 3(x-1)(x-2) + \frac{13}{6}(x-1)(x-2)(x-3)$

		d_3	d_2	d_1	d_0
		$\frac{16}{3}$	-3	4	1
x	0		$\boxed{-\frac{13}{2}}$	$\boxed{19}$	-23
		$\frac{13}{6}$	$-\frac{19}{2}$	23	-22
$1, x, x^2$		v_3	$v_2 = d_2 + \square$		

$(x-x_2)v_3$ $(0-3)\frac{13}{6}$

$v_2(x-x_1)$ $-\frac{19}{2}(0-2)$

$p(0) = -22$

③ Dana je funkcija $f(x) = \frac{4}{1+x}$

a) Določite p v Newtonovi obliki za katerega velja

$p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$

$p(1) = f(1)$, $p'(1) = f'(1)$, $p''(1) = f''(1)$

$x_0, x_1, x_2, x_3, x_4, x_5$ $\boxed{n=5}$

$x_0 = x_1 = x_2 = 0$

$x_3 = x_4 = x_5 = 1$

$f'(x) = -4(1+x)^{-2}$

$f''(x) = 8(1+x)^{-3}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0	4	$x_0=x_1$: $[x_0,x_1]f = -\frac{4}{1!} = 4$				
0	4	$[x_0,x_1]f = 4$	$[x_0,x_1,x_2]f = \frac{f''(0)}{2} = \frac{8}{2} = 4$	$\neq$ $[x_0,x_1,x_2,x_3] = \frac{-2}{1} = -2$		
0	4		$[x_1,x_2,x_3]f = \frac{-2+4}{1-0} = \frac{2}{1} = 2$		$\frac{1}{1} = 1$	
1	2	$-\frac{2}{1} = -2$		$-\frac{1}{1} = -1$		$-\frac{1}{2}$
1	2	-1 $[x_3,x_4]f = \frac{f'(x_3)}{1} = \frac{-1}{1} = 1$	$[x_2,x_3,x_4]f = \frac{1}{1} = 1$	$\frac{-\frac{1}{2}}{1} = -\frac{1}{2}$	$\frac{1}{2}$	
1	2	-1 $[x_4,x_5]f = 1$	$[x_3,x_4,x_5]f = \frac{1}{2!} = \frac{1}{2}$			

$p(x) = 4 - 4x + 4x^2 - 2x^3 + x^3(x-1) - \frac{1}{2}x^3(x-1)^2$

b) Ocenite napako $\|f-p\|_{\infty,[0,1]}$

$f(x) - p(x) = \omega(x) \underbrace{[x_0, x_1, \ldots, x_n, x]f}$

obstaja ξ_x, da je $\frac{f^{(n+1)}(\xi_x)}{(n+1)!}$, $\xi_x \in [x_0, x_n]$

$\omega(x) = \prod_{i=0}^{n}(x - x_i)$

$\|f-p\|_{\infty,[0,1]} = \max_{x\in[0,1]} |f(x) - p(x)| = \max_{x\in[0,1]} \left|\omega(x) \cdot \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\right| = \left|\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\right| \underbrace{\max_{x\in[0,1]} |\omega(x)|}_{\le \|\omega\|_{\infty,[0,1]}} \le \frac{1}{(n+1)!} \|f^{(n+1)}\|_{\infty,[0,1]} \|\omega\|_{\infty,[0,1]}$

$f(x) = \frac{4}{1+x}$

$f'(x) = -\frac{4}{(1+x)^2}$

$f''(x) = \frac{4\cdot 2}{(1+x)^3}$

$f^{(3)}(x) = -\frac{4\cdot 2\cdot 3}{(1+x)^4}$

$f^{(4)}(x) = \frac{4\cdot 2\cdot 3\cdot 4}{(1+x)^5}$

$f^{(5)}(x) = -\frac{4\cdot 2\cdot 3\cdot 4\cdot 5}{(1+x)^6}$

$f^{(6)}(x) = \frac{4\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{(1+x)^7} = g(x)$

$f^{(7)}(x) = 0 \; : -\frac{4\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{(1+x)^8} = 0$

$\omega(x) = \prod_{i=0}^{n}(x - x_i) = (x-x_0)(x-x_1)\ldots(x-x_n) \xrightarrow{n=5} = (x-0)(x-0)(x-0)(x-1)(x-1)(x-1) = x^3(x-1)^3$

$\omega'(x) = 3x^2(x-1)^3 + 3x^3(x-1)^2$

$\omega'(x) = 0 : \; 3x^2(x-1)^2(x-1+x) = 0$

$3x^2(x-1)^2(2x-1) = 0$

$\hookrightarrow x_{1,2} = 0 \rightarrow \omega(0) = 0$

$x_{2,3} = 1 \rightarrow \omega(1) = 0$

$x_{4,5} = \frac{1}{2} \rightarrow \omega(\frac{1}{2}) = \frac{1}{8}(-\frac{1}{2})^3 = -\frac{1}{64}$ $\Longrightarrow$ $\underline{\underline{\|\omega\|_{\infty,[0,1]} = \frac{1}{64}}}$

$g(0) = \frac{4\cdot 6!}{(1+0)^7} = 4\cdot 6!$

$g(1) = \frac{4\cdot 6!}{(1+1)^7} = \frac{4\cdot 6!}{2^7}$

1. Dana je funkcija $f(x) = \frac{3}{x+1} + x$

a) Določite p v Newtonovi obliki, za katerega velja, da funkcija f interpolira dvojno v $x=0$ in $x=2$ ter enojno v $x=\frac{1}{2}$

$$x=0 \downarrow \; f(0)=p(0),\; f'(0)=p'(0) \qquad x=2 \downarrow \; f(2)=p(2),\; f'(2)=p'(2) \qquad x=\tfrac{1}{2} \downarrow \; f(\tfrac{1}{2})=p(\tfrac{1}{2})$$

5 točk $\longrightarrow$ 4 bo stopnja polinoma

0	3				
		-2			
0	3		2		
		$\frac{\frac{5}{2}-3}{\frac{1}{2}} = -1$		$-\frac{2}{3}$	
$\frac{1}{2}$	$\frac{5}{2}$		1		$\frac{2}{9}$
		$\frac{3-\frac{5}{2}}{2-\frac{1}{2}} = \frac{1}{3}$		$-\frac{2}{9}$	
2	3		$\frac{2}{9}$		
		$\frac{2}{3}$			
2	3				

$$f'(x) = \frac{-3}{(x+1)^2} + 1$$

$$p(x) = 3 - 2x + 2x^2$$

$$\|f-p\|_{\infty,[0,\frac{1}{2}]} \leq \frac{1}{(n+1)!}\, \|\omega\|_{\infty,[0,\frac{1}{2}]} \cdot \|f^{(n+1)}\|_{\infty,[0,\frac{1}{2}]}$$

$\rightarrow$ st. interpolacij polinoma $\rightarrow$ v našem primeru 2

$$\omega(x) = \prod_{i=0}^{n} (x-x_i) = x^2(x-\tfrac{1}{2}) = x^3 - \tfrac{1}{2}x^2$$

$$\omega'(x) = 3x^2 - x = 0$$

$$x(3x-1) = 0$$

$\hookrightarrow$ $x_1 = 0 \longrightarrow \omega(0) = 0$ ✗

$x_2 = \frac{1}{3} \longrightarrow \omega(\frac{1}{3}) = \frac{1}{9}(-\frac{1}{6}) = -\frac{1}{54}$ ✓

$$f''(x) = \frac{6}{(x+1)^3}$$

$$f'''(x) = -\frac{18}{(x+1)^4}$$

$$f^{(4)}(x) = \frac{72}{(x+1)^5} \longrightarrow \text{nima ničel}$$

] max doseže v $x=0$ $f'''(0) = -18$

$$\|f-p\|_{\infty,[0,\frac{1}{2}]} \leq \frac{1}{6} \cdot \frac{1}{54} \cdot 18 = \frac{3}{54} = \frac{1}{18} \doteq 0{,}056$$

Kaj pa residual?

$$r(x) = f(x) - p(x) = \frac{3}{(1+x)} + 3x - 2x^2 - 3$$

$$r'(x) = -\frac{3}{(1+x)^2} + 3 - 4x = 0 \quad / (1+x)^2 \quad x \neq -1$$

$$-3 + 3(1+x)^2 - 4x(1+x)^2 = 0$$

$$-3 + 3 + 6x + 3x^2 - 4x - 8x^2 - 4x^3 = 0$$

$$2x - 5x^2 - 4x^3 = 0$$

$$x(2 - 5x - 4x^2) = 0$$

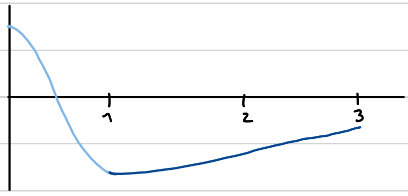
$\hookrightarrow$ $x_1 = 0$

$$x_{2,3} = \frac{-5 \pm \sqrt{57}}{8} = \begin{cases} x_2 = 0{,}32 \in [0, \frac{1}{2}] & ✓ \\ x_3 = -1{,}57 & ✗ \end{cases}$$

$$\Longrightarrow \underline{\underline{r(x_2) = 0{,}028}}$$

② Določite kubični zlepek $s \in \mathcal{C}^1([0,3])$ sestavljen iz dveh polinomov $p_1 : [0,1] \to \mathbb{R}$, $p_2 : [1,3] \to \mathbb{R}$, ki interpolirata

	x_i	$f(x_i)$	$f'(x_i)$
p_1	0	1	0
p_1, p_2	1	-1	-1
p_2	3	$-\frac{1}{2}$	$\frac{1}{2}$



p_1				
0	1	0	-2	3
0	1	-2	1	
1	-1	-1		
1	-1			

p_1: stopnje 3

x_0 : $p_1(0) = 1$

x_1 : $p_1'(0) = 0$

x_2 : $p_1(1) = -1$

x_3 : $p_1'(1) = -1$

$\Rightarrow p_1(x) = 1 - 2x^2 + 3x^2(x-1)$

p_2				
1	-1	-1	$\frac{5}{8}$	$-\frac{1}{4}$
1	-1	$\frac{1}{4}$	$\frac{1}{8}$	
3	$-\frac{1}{2}$	$\frac{1}{2}$		
3	$-\frac{1}{2}$			

$\Rightarrow p_2(x) = -1 - (x-1) + \frac{5}{8}(x-1)^2 - \frac{1}{4}(x-1)^2(x-3)$

NUMERIČNO ODVAJANJE

Diferenčne formule $x_i = x_0 + ih$ (EKUIDISTANTNE)

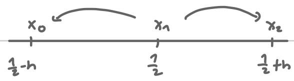
$f'(x_0) = \frac{f(x_1) - f(x_0)}{h} - \frac{h}{2} f''(\xi)$ $\quad \xi \in [x_0, x_1]$ enostranska diferenca

$f'(x_1) = \frac{f(x_2) - f(x_0)}{2h} - \frac{h}{6} f'''(\xi)$ $\quad$ simetrična diferenca za 1. odvod

$f''(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{3h} - \frac{h^2}{12} f^{(4)}(\xi)$ $\quad \xi \in [x_0, x_2]$ simetrična diferenca za 2. odvod

③ $f(x)=\frac{1}{x+1}$. Izračunajte enostransko in obe simetrični diferenci v točki $\tilde{x}=\frac{1}{2}$ za poljuben $h\in[0,1]$. Rezultate primerjajte z $f'(\frac{1}{2})$ in $f''(\frac{1}{2})$ v limiti $h\to 0$.

$$f'(\tfrac{1}{2})=\frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}=\frac{\frac{1}{\frac{3}{2}+h}-\frac{2}{3}}{h}=\frac{\frac{2}{3+2h}-\frac{2}{3}}{h}=\frac{\frac{6-6-4h}{(3+2h)3}}{h}=\frac{-4}{3(3+2h)}$$



$$f'(\tfrac{1}{2})\approx\frac{f(\frac{1}{2}+h)-f(\frac{1}{2}-h)}{2h}=\frac{\frac{2}{3+2h}-\frac{2}{3-2h}}{2h}=\frac{\frac{3-2h-3-2h}{(3+2h)(3-2h)}}{h}=\frac{-4}{(3+2h)(3-2h)}$$

$$f''(\tfrac{1}{2})=\frac{f(\frac{1}{2}-h)-2f(\frac{1}{2})+f(\frac{1}{2}+h)}{h^2}=\ldots=\frac{16}{3(3-2h)(3+2h)}$$

- PRIMERJAVA

$f'(x)=-\frac{1}{(x+1)^2}$ $\qquad$ $f''(x)=\frac{2}{(1+x)^3}$

$f'(\frac{1}{2})=-\frac{1}{(\frac{3}{2})^2}=-\frac{4}{9}$ $\qquad$ $f''(\frac{1}{2})=\frac{16}{27}$

1) $\lim\limits_{h\to 0} -\frac{4}{3}\frac{1}{(3+2h)}=-\frac{4}{3}\cdot\frac{1}{3}=-\frac{4}{9}$ ✓

2) $\lim\limits_{h\to 0} -\frac{4}{(3-2h)(3+2h)}=-\frac{4}{9}$ ✓

3) $\lim\limits_{h\to 0} \frac{16}{3(3-2h)(3+2h)}=\frac{16}{27}$ ✓

$\Longrightarrow$ Vse se ujema

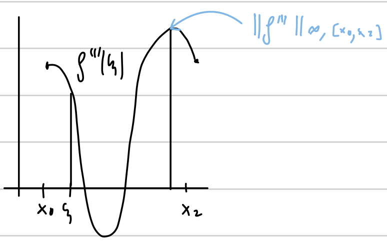
4.4

① Z uporabo simetričnih diferenc izračunajte približka za prvi in drugi odvod funkcije $f(x)=e^{2x}$ v točki $\tilde{x}=0$ za $h=\frac{1}{10}$. Ocenite napaki in primerjajte dobljene vrednosti s točnimi.

$x_0 = -\frac{1}{10}$, $x_1 = 0$, $x_2 = \frac{1}{10}$ $\qquad$ Vedno računamo v srednji točki

$$f'(x_1)=\frac{f(\frac{1}{10})-f(-\frac{1}{10})}{\frac{2}{10}}=\frac{e^{\frac{1}{5}}-e^{-\frac{1}{5}}}{\frac{2}{10}}=5(e^{\frac{1}{5}}-e^{-\frac{1}{5}})\approx 2.01336$$

$$f''(x_1)=\frac{f(-\frac{1}{10})-2f(0)+f(\frac{1}{10})}{\frac{1}{100}}=\frac{e^{-\frac{1}{5}}-2+e^{\frac{1}{5}}}{\frac{1}{100}}\approx 4.01335$$



$f(x)=e^{2x}$, $f'(x)=2e^{2x}$, $f''(x)=4e^{2x}$, $f'''(x)=8e^{2x}$, $f^{(4)}=16e^{2x}$

$$\left|\frac{h^2}{6}f'''(\xi)\right|\le\frac{\frac{1}{100}}{6}\|f'''\|_{\infty,[x_0,x_2]}=\frac{1}{600}\|8e^{2x}\|_{\infty,[-1/10,1/10]}=\frac{1}{600}8e^{\frac{1}{5}}\doteq 0.0163$$

$$\left|\frac{h^2}{12}f^{(4)}(\xi)\right|\le\frac{\frac{1}{100}}{12}\|f^{(4)}\|_{\infty,[x_0,x_2]}=\frac{1}{1200}\|16e^{2x}\|_{\infty,[-1/10,1/10]}=\frac{1}{1200}16e^{\frac{1}{5}}\doteq 0{,}0163$$

$f'(0)=2\cdot e^{2\cdot 0}=2$ $\qquad$ napaka: 0.01336

$f''(0)=4e^{2\cdot 0}=4$ $\qquad$ napaka: 0.01335

$\Longrightarrow$ Ocenili smo, da so napake ≤ 0.0163

2. *Metoda nedoločenih koeficientov.*

a) Preko metode nedoločenih koeficientov izpeljite formulo za aproksimacijo odvoda funkcije f oblike

$$f'(x_2) = Af(x_0) + Bf(x_1) + Cf(x_2) + Df(x_3) + Ef(x_4) + Ff^{(m)}(\xi),$$

kjer so $x_i = x_0 + ih$ ekvidistantne točke.

- želimo, da je pravilo točno za polinome čim višjih stopenj
- Premaknjene potence: $(x-x_2)^i$ $i=0,1,2,\dots$ — vrednost v katerih računamo
- m: najnižja stopnja polinoma, za katero pravilo ni točno

$\boxed{i=0}$ $f(x) = (x-x_2)^0 = 1$

$f'(x) = 0$

$f'(x_2) = 0 = A\cdot 1 + B + C + D + E$

$\boxed{i=1}$ $f(x) = (x-x_2)^1 = x - x_2$

$f'(x) = 1$

$f(x_0) = (x_0 - x_2) = x_0 - x_0 - 2h = -2h$

$f'(x_2) = 1 = A(-2h) + B(-h) + C\cdot 0 + D\cdot h + E\cdot 2h$

$\boxed{i=2}$ $f(x) = (x-x_2)^2 = x^2 - 2xx_2 + x_2^2$

$f'(x) = 2x - 2x_2$

$f'(x_2) = 0 = A(-2h)^2 + B(-h)^2 + C0 + Dh^2 + E(2h)^2$

$\boxed{i=3}$ $f(x) = (x-x_2)^3$

$f'(x) = 3(x-x_2)^2$

$f'(x_2) = 0 = A(-2h)^3 + B(-h)^3 + C\cdot 0 + Dh^3 + E(2h)^3$

$\boxed{i=4}$ $f(x) = (x-x_2)^4$

$f'(x) = 4(x-x_2)^3$

$f'(x_2) = 0 = A(-2h)^4 + B(-h)^4 + C0 + Dh^4 + E(2h)^4$

$$\begin{aligned}
0 &= A + B + C + D + E \\
1 &= -2hA - hB + \quad + hD + 2hE \\
0 &= 4h^2A + h^2B + \quad + h^2D + 4h^2E \\
0 &= -8h^3A - h^3B + \quad + h^3D + 8h^3E \\
0 &= 16h^4A + h^4B + \quad + h^4D + 16h^4E
\end{aligned}$$

$$\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} \\ 2h \\ + \\ 2h \\ + \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix}
\begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2h & -h & 0 & h & 2h & 1 \\ 0 & -h^2 & 0 & 3h^2 & 8h^2 & 2h \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 & 0 \\ 0 & -h^4 & 0 & 3h^4 & 32h^4 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & \frac{1}{h} \\ 0 & -1 & 0 & 3 & 8 & \frac{2}{h} \\ 0 & 3 & 0 & -3 & 0 & -\frac{4}{h} \\ 0 & 0 & 0 & 0 & 24 & -\frac{2}{h} \end{array}\right] \to E = \frac{1}{12h}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & \frac{1}{h} \\ 0 & -1 & 0 & 3 & 8 & \frac{2}{h} \\ 0 & 0 & 0 & 6 & 24 & \frac{2}{h} \end{array}\right] \quad D = \frac{2}{3}h$$

$$\Longrightarrow \quad E = -\frac{1}{12h}, \quad D = \frac{2}{3h}, \quad B = -\frac{2}{3h}, \quad A = \frac{1}{12h}, \quad C = 0$$

$\cdot\ f'(x_2) = \frac{1}{12h} f(x_0) - \frac{2}{3h} f(x_1) + \frac{2}{3h} f(x_3) - \frac{1}{12h} f(x_4)$

$f(x) = (x-x_2)^5$

$f'(x) = 5(x-x_2)^4$

$f'(x_2) = 0 \overset{?}{=} \frac{1}{12h}(-2h)^5 - \frac{2}{3h}(-h)^5 + \frac{2}{3h}h^5 - \frac{1}{12h}(2h)^5 = -\frac{4}{3}h^4 + \frac{2}{3}h^4 + \frac{2}{3}h^4 - \frac{4}{3}h^4 = -\frac{4}{3}h^4 \neq 0$

$\Downarrow$

$m=5$

$f^{(5)}(x) = 5!$

$0 = -4h^4 + F f^{(5)}(\xi)$

$\Longrightarrow 0 = -4h^4 + F\,5! \quad \Longrightarrow F = \frac{4h^4}{5!} = \frac{h^4}{30}$

rešitev: $\frac{1}{12h}\hat{f}(x_0) - \frac{2}{3h}\hat{f}(x_1) + \frac{2}{3h}\hat{f}(x_3) - \frac{1}{12h}\hat{f}(x_4) + \frac{h^4}{30} f^{(5)}(\xi)$

b) Za $f(x) = \frac{1}{x+1}$ na intervalu $[0,1]$ poiščite optimalen h, da bo skupna napaka (napaka metode + napaka aritmetike) čim manjša. Predpostavite, da za napako aritmetike velja $|f(x_i) - \hat{f}(x_i)| \leq 5 \cdot 10^{-6}$, kjer z $\hat{f}$ označimo numerično izračunane vrednosti f.

- napaka metode: Dm (pada, ko h pada)
- napaka aritmetike: Da (raste, ko h pada)

$\cdot\ \boxed{Dm}: \left|\frac{h^4}{30} f^{(5)}(\xi)\right| \leq \frac{h^4}{30} \|f^{(5)}\|_{\infty,[0,1]} \leq 4h^4$ $\qquad \boxed{Dm \leq 4h^4}$

$f'(x) = -(1+x)^{-2}$

$f''(x) = 2(1+x)^{-3}$

$f'''(x) = -6(1+x)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5}$

$f^{(5)}(x) = -120(1+x)^{-6}$

$f^{(6)}(x) = 720(1+x) \longrightarrow$ ni ekstrema $\Longrightarrow$ ekstrem je v robnih točkah

$|f^{(5)}(0)| = 120 \qquad |f^{(5)}(1)| = \frac{120}{2^6} \qquad \|f^{(5)}\|_\infty = 120$

$\cdot\ \boxed{Da} = |f'(x_2) - \hat{f}'(x_2)|$

$Da = \Big|\frac{1}{12h}\underbrace{(f(x_0) - \hat{f}(x_0))}_{\leq 5\cdot 10^{-6}} - \frac{2}{3h}(f(x_1) - \hat{f}(x_1)) + \frac{2}{3h}(f(x_3) - \hat{f}(x_3)) - \frac{1}{12h}(f(x_4) - \hat{f}(x_4))\Big|$

$\leq \frac{1}{12h} 5\cdot 10^{-6} + \frac{2}{3h} 5\cdot 10^{-6} + \frac{2}{3h} 5\cdot 10^{-6} + \frac{1}{12h} 5\cdot 10^{-6}$

$= 5\cdot 10^{-6} \frac{3}{2h}$

$\boxed{Da \leq \frac{3}{2h} \cdot 5\cdot 10^{-6}}$

$\cdot\ D = Dm + Da = 4h^4 + \frac{3}{2h} \cdot 5\cdot 10^{-6}$ (želimo optimizirati to napako)

$D' = 16h^3 - \frac{15}{2h^2} \cdot 10^{-6} = 0$

$32h^5 - 15\cdot 10^{-6} = 0$

$h = \sqrt[5]{\frac{15\cdot 10^{-6}}{32}} \doteq 0{,}05422$

NUMERIČNO INTEGRIRANJE

Iščemo pravilo oblike : $\int_a^b f(x)dx = \sum_{i=0}^{n} \underset{\text{uteži}}{A_i} - \underset{\text{vozli}}{f_i(x)} + \underset{\text{napaka}}{Rf}$

NEWTON-COTESOVA PRAVILA:

- vozli so ekvidistantni : $x_i = x_0 + i \cdot h$, $x_0 = a$, $h = \frac{b-a}{n}$, $i = 0, 1, \ldots, n$
- uteži določimo preko int. polinoma ali preko metode nedoločenih koeficientov

→ zaprt tip : upoštevamo tudi točki x_0 in x_n

→ odprt tip : ne upoštevamo krajišč

TRAPEZNO : $\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\left(f(x_0) + f(x_1)\right) - \frac{h^3}{12} f''(\xi)$ (zaprto NC pravilo na dveh točkah) [red = 1]

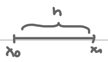
$m+1$

SIMPSONOVO : $\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{3}\left(f(x_0) + h f(x_1) + f(x_2)\right) - \frac{h^5}{90} f^{(4)}(\xi)$ (zaprto NC pravilo na treh točkah) [red = 3]

red pravila je enak m, če je pravilo točno za vse polinome stopnje $\leq m$

1. Izračunajte približek za integral
$$\int_0^{\frac{\pi}{4}} \sin x dx$$
z uporabo trapeznega in Simpsonovega pravila. Ocenite napaki pravil in primerjajte dobljene rezultate s točnimi.

- TRAPEZNO : $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $f(x) = \sin x$



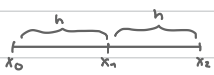
$$\int_0^{\pi/4} \sin x\, dx = \frac{\left.\frac{\pi/4 - 0}{1}\right]^{h}}{2}\left(\sin 0 \cdot \sin\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{10} \doteq 0,27768$$

$$|Rf| = \left|\frac{h^3}{12} f''(\xi)\right| = \left|\frac{\pi^3}{4^3 \cdot 12}(-\sin\xi)\right| \leq \left|\frac{\pi^3}{4^3 \cdot 12} \underbrace{\frac{\sqrt{2}}{2}}_{\sin(\frac{\pi}{4})}\right| \doteq 0,02855$$

$\|f''\|_{\infty,[0,\pi/4]}$

$f''(x) = -\sin x$

- SIMPSONOVO : $x_0 = 0$, $x_2 = \frac{\pi}{4}$, $h = \frac{\pi}{8}$, $x_1 = \frac{\pi}{8}$



$$\int_0^{\pi/4} \sin x\, dx = \frac{\pi}{24}\left(\sin 0 + 4\sin\frac{\pi}{8} + \sin\frac{\pi}{4}\right) \doteq 0,29293$$

$$|Rf| = \left|\frac{h^5}{90} f^{(3)}(\xi)\right| = \left|\frac{\pi^5}{90 \cdot 8^5}\sin(\xi)\right| \leq \left|\frac{\pi \cdot 5}{90 - 8^5} \cdot \frac{\sqrt{2}}{2}\right| \doteq \underbrace{7,3 \cdot 10^{-5}}$$

resnična napaka mora biti manjša od tega

$$\int_0^{\pi/4} \sin x\, dx = -\cos\Big|_0^{\pi/4} = -\frac{\sqrt{2}}{2} + 1 \doteq 0,29289$$

2. Izpeljite zaprto Newton-Cotesovo pravilo na 4 točkah.

a) Preko Lagrangeevega interpolacijskega polinoma.

b) Preko metode nedoločenih koeficientov.

$x_0 \quad x_1 \quad x_2 \quad x_3 \qquad n=3$

m = red

$$\int_{x_0}^{x_3} f(x)\,dx = A f(x_0) + B f(x_1) + C f(x_2) + D f(x_3) + E f^{(m+1)}(\xi)$$

a) $f \approx p \implies \int f \approx \int p$

$$\ell_{i,3}(x) = \prod_{\substack{i=0 \\ j\neq i}}^{3} \frac{x-x_j}{x_i-x_j}$$

$$\int_{x_0}^{x_3} p(x) = \int_{x_0}^{x_3} \sum_{i=0}^{3} f(x_i)\cdot \ell_{i,3}(x) = \int_{x_0}^{x_3} \left(f(x_0)\cdot \ell_{0,3}(x) + f(x_1)\cdot \ell_{1,3}(x) + f(x_2)\,\ell_{2,3}(x) + f(x_3)\,\ell_{3,3}(x) \right) dx$$

$$= f(x_0) \int_{x_0}^{x_3} \ell_{0,3}(x)\,dx + f(x_1)\int_{x_0}^{x_3} \ell_{1,3}(x)\,dx + f(x_2)\int_{x_0}^{x_3} \ell_{2,3}(x)\,dx + f(x_3)\int_{x_0}^{x_3} \ell_{3,3}(x)\,dx$$

- $$\int_{x_0}^{x_3} \ell_{0,3}(x) = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{\underbrace{(x_0-x_1)}_{-h}\underbrace{(x_0-x_2)}_{-2h}\underbrace{(x_0-x_3)}_{-3h}}\,dx = -\frac{1}{6h^3}\int_0^3 (x_0+th-(x_0+h))(x_0+th-(x_0+2h))(x_0+th-(x_0+3h))h\,dt =$$

ekvidistantne točke

$x = x_0 + th$, $dx = h\,dt$; $x = x_0 \to t=0$, $x = x_3 \to t=3$

$$= -\frac{1}{6h^3}\int_0^3 (h(t-1))(h(t-2))(h(t-3))h\,dt =$$

$$= -\frac{1}{6h^3}h^4 \int_0^3 (t-1)(t-2)(t-3)\,dt =$$

$$= -\frac{h}{6}\int_0^3 (t^3-6t^2+11t-6)\,dt =$$

$$= -\frac{h}{6}\left(\frac{t^4}{4} - 6\frac{t^3}{3} + 11\frac{t^2}{2} - 6t\right)\Big|_0^3 =$$

$$= -\frac{h}{6}\left(\frac{81}{4} - 2\cdot 27 + \frac{99}{2} - 18\right) =$$

$$\boxed{= \frac{3h}{8} = A}$$

- $$\int_{x_0}^{x_3} \ell_{1,3}(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_0)(x-x_2)(x-x_3)}{\underbrace{(x_1-x_0)}_{h}\underbrace{(x_1-x_2)}_{-h}\underbrace{(x_1-x_3)}_{-2h}}\,dx = \frac{1}{2h^3}\int_0^3 (th)(th-2h)(th-3h)h\,dt =$$

$x = x_0 + th$, $dx = h\,dt$

$$= \frac{h}{2}\int_0^3 (t^3-5t^2+6t)\,dt =$$

$$= \frac{h}{2}\left(\frac{t^4}{4} - 5\frac{t^3}{3} + 6\frac{t^2}{2}\right)\Big|_0^3 =$$

$$= \frac{h}{2}\left(\frac{81}{4} - \frac{5\cdot 27}{3} + 6\frac{9}{2}\right) =$$

$$= \boxed{\frac{9h}{8} = B}$$

- $$\int_{x_0}^{x_3} \ell_{2,3}(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_0)(x-x_1)(x-x_3)}{\underbrace{(x_2-x_0)}_{2h}\underbrace{(x_2-x_1)}_{h}\underbrace{(x_2-x_3)}_{-h}}\,dx =$$

$x = x_0 + th$, $dx = h\,dt$

$$= -\frac{1}{2h^3}\int_0^3 (th)(th-h)(th-3h)h\,dt =$$

$$= -\frac{h}{2}\int_0^3 t^3-4t^2+3t\,dt =$$

$$= -\frac{h}{2}\left(\frac{t^4}{4} - 4\frac{t^3}{3} + 3\frac{t^2}{2}\right)\Big|_0^3 =$$

$$= -\frac{h}{2}\left(\frac{81}{4} + 4\frac{27}{3} + 3\frac{9}{2}\right)$$

$$= \boxed{\frac{9h}{8} = C}$$

- $$\int_{x_0}^{x_3} \ell_{3,3}(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,dx = \boxed{\frac{3h}{8} = D}$$

[ker je simetrično]

b) $1,\ x_1-x_0,\ (x-x_0)^2, \dots$

- $f(x)=1 \qquad (k=0)$
- $f(x)=x-x_0 \qquad (k=1)$
- $f(x)=(x-x_0)^2 \qquad (k=2)$
- $f(x)=(x-x_0)^3 \qquad (k=3)$

$$\int_{x_0}^{x_3}(x-x_0)^k\,dx=\frac{(x-x_0)^{k+1}}{k+1}\Bigg|_{x_0}^{x_3}=\frac{(x_3-x_0)^{k+1}}{k+1}=\frac{(3h)^{k+1}}{k+1}$$

- $f(x)=1 \xrightarrow{k=0} 3h=A+B+C+D$
- $f(x)=x-x_0 \xrightarrow{k=1} \frac{9h^2}{2}=\underset{f(x_0)}{A\cdot 0}+\underset{f(x_1)}{B\cdot h}+\underset{f(x_2)}{C\cdot 2h}+\underset{f(x_3)}{D\cdot 3h}$ — točke so ekvidistantne
- $f(x)=(x-x_0)^2 \xrightarrow{k=2} \frac{27h^3}{3}=9h^3=A\cdot 0+B\cdot h^2+C\cdot(2h)^2+D(3h)^2$
- $f(x)=(x-x_0)^3 \xrightarrow{k=3} \frac{81h^4}{4}=A\cdot 0+B\cdot h^3+C(2h)^3+D(3h)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & h & 2h & 3h & \frac{9h^2}{2} \\ 0 & h^2 & 4h^2 & 9h^2 & 9h^3 \\ 0 & h^3 & 8h^3 & 27h^3 & \frac{81h^3}{4} \end{array}\right] \implies \begin{array}{l} A=D=\frac{3h}{8} \\ B=C=\frac{9h}{8}\end{array}$$

4 OCENIMO NAPAKO: zanima nas, če je približek za integral za funkcijo $f(x)=(x-x_0)^4$ $(k=4)$ še vedno točno

$$\frac{(3h)^5}{5}\overset{?}{=}\frac{3h}{8}\cdot 0+\frac{9h}{8}\cdot h^4+\frac{9h}{8}(2h)^4+\frac{3h}{8}(3h)^4=\frac{44h^5 9}{8}$$

$$\frac{(3h)^5}{5}\neq\frac{44h^5\cdot 9}{8} \implies \text{red}=3$$

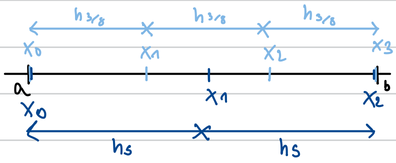
napaka:
$$\frac{(3h)^5}{5}=\frac{44\cdot 9}{8}h^5+E\cdot f^{(4)}(\xi)$$
$$E=\frac{\frac{(3h)^5}{5}-\frac{44\cdot 9}{8}h^5}{4!}=-\frac{3h^5}{80}$$

- $\frac{3}{8}$ pravilo: $\int_{x_0}^{x_3}f(x)\,dx=\frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)\underbrace{-\frac{3h^5}{80}f^{4}(\xi)}_{R_{3/8}}$

- Simsonovo pravilo: $\int_{x_0}^{x_2}f(x)\,dx=\frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)\underbrace{-\frac{h^6}{90}f^{(4)}(\xi)}_{R_S}$

18.4

1. Primerjajte red in napako 3/8 pravila s Simpsonovim pravilom, če obe pravili uporabimo za integriranje na istem intervalu $[a, b]$.



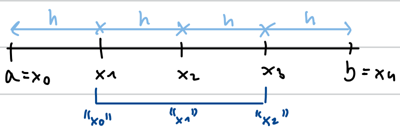
$h_S = \frac{b-a}{2} \longrightarrow |R_S| = \left|\frac{(b-a)^5 f^{(4)}(\xi)}{90 \cdot 2^5}\right| \leq \frac{(b-a)^5}{90 \cdot 2^5} \|f^{(4)}\|_{\infty,[a,b]} = \frac{1}{2880} (b-a)^5 \|f^{(4)}\|_{\infty,[a,b]}$

$h_{3/8} = \frac{b-a}{3} \longrightarrow |R_{3/8}| = \left|\frac{(b-a)^5 f^{(4)}(\xi)}{80 \cdot 3^4}\right| \leq \frac{(b-a)^5}{80 \cdot 3^4} \|f^{(4)}\|_{\infty,[a,b]} = \frac{1}{6480} (b-a)^5 \|f^{(4)}\|_{\infty,[a,b]}$

ugotovitev: pričakujemo $|R_{3/8}| < |R_S|$

Red je 3 za obe pravili, vendar pa imamo pri 3/8 eno točko več.

2. Izpeljite odprto Newton-Cotesovo pravilo na 3 točkah.



3 točke

$$\boxed{\int_{a\,(x_0)}^{b\,(x_n)} f(x)\,dx = \sum_{\cancel{i=0},\ i=1}^{n-1} A_i f(x_i) + Rf}$$

$$\int_{x_0}^{x_4} f(x)dx = Af(x_1) + Bf(x_2) + Cf(x_3) + Rf = \frac{8h}{3} f(x_1) - \frac{4h}{3} f(x_2) + \frac{8h}{3} f(x_3) + Rf$$

$$\ell_{i,n}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$\int_{x_0}^{x_4} f(x)dx \approx \int_{x_0}^{x_4} p(x)\,dx = \int_{x_0}^{x_4} \sum_{i=1}^{n-1} f(x_i)\ell_{i,n}(x)\,dx = \int_{x_0}^{x_4} \Big(\underbrace{f(x_1)}_{\text{"}x_0\text{"}} \ell_{0,2}(x) + \underbrace{f(x_2)}_{\text{"}x_1\text{"}} \ell_{1,2}(x) + \underbrace{f(x_3)}_{\text{"}x_2\text{"}} \ell_{2,2}(x)\Big)dx$$

- $A = \int_{x_0}^{x_4} \ell_{0,2}(x)dx = \int_{x_0}^{x_4} \frac{(x-x_2)(x-x_3)}{\underbrace{(x_1-x_2)}_{=-h}\underbrace{(x_1-x_3)}_{=-2h}} dx = \int_{x_0}^{x_4} \frac{(x-x_2)(x-x_3)}{2h^2} dx = \int_0^4 \frac{h(t-2)\,h(t-3)}{2h^2} h\,dt = \frac{h}{2}\left(\frac{t^3}{3} - 5\frac{t^2}{2} + 6t \Big|_0^4\right) = \frac{h}{2}\left(\frac{64}{3} - 40 + 24\right) = \frac{16h}{6} = \frac{8h}{3}$

$x = x_0 + th$
$dx = h\,dt$

- $B = \int_{x_0}^{x_4} \ell_{1,2}(x)dx = \int_{x_0}^{x_4} \frac{(x-x_1)(x-x_3)}{\underbrace{(x_2-x_1)}_{=h}\underbrace{(x_2-x_3)}_{=-h}} dx = \int_{x_0}^{x_4} \frac{(x-x_1)(x-x_3)}{-h^2} dx = \int_0^4 \frac{(t-1)h \cdot h(t-3)}{-h^2} h\,dt = -h\int_0^4 (t^2-4t+3)dt = -h\left(\frac{64}{3} - 4\frac{16}{2} + 3\cdot 4\right) = -\frac{4h}{3}$

- $C = \int_{x_0}^{x_4} \ell_{2,2}(x)dx = \int_{x_0}^{x_4} \frac{(x-x_1)(x-x_2)}{\underbrace{(x_3-x_1)}_{=2h}\underbrace{(x_3-x_2)}_{=h}} dx = \int_{x_0}^{x_4} \frac{(x-x_1)(x-x_2)}{2h^2} dx = \int_0^4 \frac{h^2(t-1)(t-2)}{2h^2} h\,dt = \frac{h}{2}\left(\frac{64}{3} - 3\frac{16}{2} + 2\cdot 4\right) = \frac{16h}{6}$

- RED?

→ $f(x) = 1$
→ $f(x) = (x-x_0)$
→ $f(x) = (x-x_0)^2$] $m \geq 2$
→ $f(x) = (x-x_0)^3$
→ $f(x) = (x-x_0)^4$

→ 3. red?

$\int_{x_0}^{x_4} (x-x_0)^3 dx = \frac{(x-x_0)^4}{4}\Big|_{x_0}^{x_4} = \frac{(x_4-x_0)^4}{4} = 64h^4$

$\frac{8h}{3}(\underbrace{x_1-x_0}_{=h})^3 - \frac{4h}{3}(\underbrace{x_2-x_0}_{=2h})^3 + \frac{8h}{3}(\underbrace{x_3-x_0}_{=3h})^2 = \frac{8h}{3}(h^3+3^3h^3) - \frac{4h}{3}\cdot 2^3h^3 = \frac{192h^4}{3} = 64h^4$

$\Longrightarrow$ ker je enako je red vsaj 3

→ 4. red?

$\int_{x_0}^{x_4} (x-x_0)^4 dx = \frac{(x-x_0)^5}{5}\Big|_{x_0}^{x_4} = \frac{(x_4-x_0)^5}{5} = \frac{4^5h^5}{5}$

$\frac{8h}{3}(x_1-x_0)^4 - \frac{4h}{3}(x_2-x_0)^4 + \frac{8h}{3}(x_3-x_0)^4 = \frac{8h}{3}(h^4+3^4h^4) - \frac{4h}{3}\cdot 2^4h^4 = \frac{592h^5}{3} \neq \frac{4^5h^5}{5}$

$\Longrightarrow$ ni red 4 $\Longrightarrow \boxed{m=3}$

- NAPAKA?

$((x-x_0)^4)^{(4)} = 4! = 24$

$\underbrace{\int_{x_0}^{x_4} f(x)dx}_{=\frac{4^5h^5}{5}} = Af(x_1) + Bf(x_2) + Cf(x_3) + Rf = \underbrace{\frac{8h}{3}f(x_1) - \frac{4h}{3}f(x_2) + \frac{8h}{3}f(x_3)}_{=\frac{592h^5}{3}} + \frac{14h^5}{15} f^{(4)}(\xi)$

$+ D \cdot 4!$

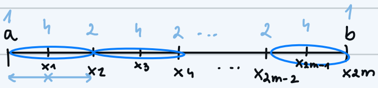
$\Longrightarrow D = \left(\frac{4^5h^5}{5} - \frac{592h^5}{3}\right)\cdot\frac{1}{24} = \frac{14h^5}{45}$

SESTAVLJENA PRAVILA

↳ Sestavljeno trapezno pravilo :

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(x_0) + 2\cdot\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) - \frac{h^2}{2}(b-a)f''(\mu) \quad ; \quad h = \frac{b-a}{m}$$

↳ Sestavljeno Simpsonovo pravilo :



$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(x_0) + 4\sum_{i=0}^{m-1} f(x_{2i+1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right) - \frac{h^4}{180}(b-a)f^{(4)}(\mu) \quad ; \quad h = \frac{b-a}{2m}$$

3. Izračunajte integral

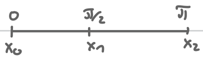
$$\int_0^\pi \sin x \; dx$$

z uporabo sestavljenega trapeznega in sestavljenega Simpsonovega pravila za $m = 2$ in $m = 4$. Ocenite napake dobljenih približkov.

① TRAPEZNO PRAVILO

- $\boxed{m=2}$

$h = \frac{\pi}{2}$



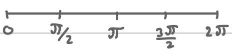
$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{4}(0+2\cdot 1+0) = \frac{\pi}{2}$$

$$Rf = \frac{\pi^2}{4\cdot 12}\cdot\pi\,(-\sin(\eta)) \quad , \eta\in[0,\pi]$$

$$|Rf| \le \frac{\pi^3}{48} \approx 0{,}636$$

- $\boxed{m=4}$

$h = \frac{\pi}{4}$



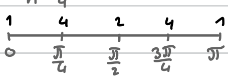
$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{8}\left(0+2\cdot\left(\frac{\sqrt2}{2}+1+\frac{\sqrt2}{2}\right)+0\right) = \frac{\pi(2\sqrt2+2)}{8} = \frac{\pi(\sqrt2+1)}{4}$$

$$|Rf| = \left|\frac{\pi^2}{16\cdot 12}\cdot\pi\right| = \frac{\pi^3}{192} \approx 0{,}161$$

② SIMPSONOVO PRAVILO

- $\boxed{m=2}$

$h = \frac{\pi}{4}$

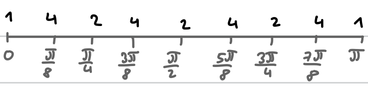


$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{12}\left(0+4\frac{\sqrt2}{2}+2\cdot 1+4\frac{\sqrt2}{2}+0\right) = \frac{\pi(2\sqrt2+1)}{6}$$

$$|Rf| = \left|\frac{\pi^4}{4^4\cdot 180}\cdot\pi\right| = \frac{\pi^5}{4^4\cdot 180} \approx 0{,}0060$$

- $\boxed{m=4}$

$h = \frac{\pi}{8}$



$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{24}\left(0+4\sin\frac{\pi}{8}+2\cdot\frac{\sqrt2}{2}+4\sin\frac{3\pi}{8}+2\cdot 1+4\sin\frac{5\pi}{8}+2\frac{\sqrt2}{2}+4\sin\frac{7\pi}{8}+0\right) \approx 2.00027$$

$$|Rf| \le \left|\frac{\pi^4}{8^4\cdot 180}\cdot\pi\right| \approx 4.2\cdot 10^{-4}$$

25.4

Od zadnjič: $I=\int_0^{\pi}\sin x\,dx$

- → sestavljeno trapezno:
  - → $m=2$: $h=\frac{\pi}{2}$, $\underbrace{I=\frac{\pi}{2}}_{T_{\frac{\pi}{2}}}$, $|Rf|\leq 0.646$
  - → $m=4$: $h=\frac{\pi}{4}$, $\underbrace{I=\frac{\pi}{4}(\sqrt{2}+1)\approx 1.896}_{T_{\frac{\pi}{4}}}$, $|Rf|\leq 0.161$
- → sestavljeno simpsonovo:
  - → $m=2$: $h=\frac{\pi}{4}$, $\underbrace{I=\frac{\pi}{6}(2\sqrt{2}+1)\approx 2.00456}_{S_{\frac{\pi}{4}}}$, $|Rf|\leq 0.0066$
  - → $m=4$: $h=\frac{\pi}{8}$, $\underbrace{I\approx 2.00027}_{S_{\frac{\pi}{8}}}$, $|Rf|\leq 4.2\cdot 10^{-6}$

RICHARDSONOVA EKSTRAPOLACIJA

F_h ... približek za integral funkcije f pri razmiku h. Iz F_h in $F_{\frac{h}{2}}$ določimo natančnejši približek:

$$If=\frac{2^pF_{\frac{h}{2}}-F_h}{2^p-1}\qquad (\text{trapezno } p=2,\ \text{simpson}: p=4)$$

Oceni za napako → približka $F_{\frac{h}{2}}=\dfrac{F_{\frac{h}{2}}-F_h}{2^p-1}$

→ približka $F_h=2^p\cdot\dfrac{F_{\frac{h}{2}}-F_h}{2^p-1}$

ROMBERGOVA METODA

T_hf ... sestavljeno trapezno pravilo pri razmiku h

$$T^{(1)}_{\frac{h}{2}}f=\frac{4\cdot T_{\frac{h}{2}}f-T_hf}{3}$$

$$T^{(r)}_{\frac{h}{2^r}}=\frac{2^{2r}\,T^{(r-1)}_{\frac{h}{2^r}}f-T^{(r-1)}_{\underbrace{\frac{h}{2^{r-1}}}_{\frac{h}{2^r}\cdot 2}}f}{2^{2r}-1}\qquad r=2,3,\dots$$

① Naj bo $I=\int_0^{\pi}\sin x\,dx$

a) Z Rombergovo metodo izračunajte $T^{(2)}_{\frac{\pi}{4}}$ za I

b) Naj S_h označuje približek za I s sestavljenim Simpsonovim pravilom pri razmiku h. Z uporabo Richardsonove ekstrapolacije ocenite napaki približkov $S_{\frac{\pi}{4}}$ in $S_{\frac{\pi}{8}}$ in ju primerjajte z ocenami iz prejšnje naloge. Nato iz $S_{\frac{\pi}{4}}$ in $S_{\frac{\pi}{8}}$ izračunajte natančnejši približek.

a) $k=2$

$$T^{(2)}_{\frac{h}{4}}=\frac{2^4\cdot T^{(1)}_{\frac{h}{4}}f-T^{(1)}_{\frac{h}{2}}f}{2^4-1}$$

$$T^{(2)}_{\frac{\pi}{4}}=\frac{2^4\cdot \overbrace{T^{(1)}_{\frac{\pi}{4}}f}^{*}-\overbrace{T^{(1)}_{2\cdot\frac{\pi}{4}}f}^{**}}{2^4-1}$$

$(*)$ $T^{(1)}_{\frac{\pi}{4}}f=\dfrac{4\cdot T_{\frac{\pi}{4}}f-T_{\frac{\pi}{2}}f}{3}=\dfrac{4\cdot\frac{\pi}{4}(\sqrt{2}+1)-\frac{\pi}{2}}{3}=\dfrac{2\sqrt{2}\pi+\pi}{}$

Torej sedaj potrebujemo $T_{\frac{\pi}{4}}f$ in $T_{\frac{\pi}{2}}f$, ki smo jih izračunali v prejšnji nalogi: $T_{\frac{\pi}{2}}f=\frac{\pi}{2}$ $\quad T_{\frac{\pi}{4}}f=\frac{\pi}{4}(\sqrt{2}+1)$

$(**)$ $T^{(1)}_{\frac{\pi}{2}}f=\dfrac{4\cdot T_{\frac{\pi}{2}}f-\underbrace{T_{\pi}f}_{\text{osnovno trapezno pravilo}}}{3}=\dfrac{4\cdot\frac{\pi}{2}-0}{3}=\dfrac{2\pi}{3}$

$$\Rightarrow T^{(2)}_{\frac{\pi}{4}}=\frac{2^4\cdot\frac{(2\sqrt{2}+1)}{6}-\frac{2\pi}{3}}{2^4-1}=\frac{2^4\,\pi\,(2\sqrt{2}+1)-4\pi}{6\cdot 15}=\frac{16\pi\,2\sqrt{2}+12\pi}{90}=\frac{4\pi\,(8\sqrt{2}+3)\cdot 2}{90\ \ 45}=\frac{2\pi\,(3\sqrt{2}+3)}{4}\approx 1{,}9986$$

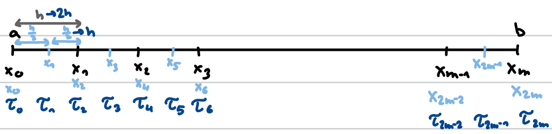
b) $|Rf|$: $\left|S_{\frac{\pi}{4}}\right|:\left|2^p\,\dfrac{S_{\frac{\pi}{8}}-S_{\frac{\pi}{4}}}{2^p-1}\right| \underset{p=4}{=} \left|16\cdot\dfrac{S_{\frac{\pi}{8}}-S_{\frac{\pi}{4}}}{15}\right|=\left|16\cdot\dfrac{2.00027-2.00456}{15}\right|\approx 0.00458$

$|Rf|$: $\left|S_{\frac{\pi}{8}}\right|=\left|\dfrac{S_{\frac{\pi}{8}}-S_{\frac{\pi}{4}}}{15}\right|\approx 0.000286=2{,}8\cdot 10^{-4}$ $\underset{\text{boljša ocena kot v simpsonu}}{<}$ $4{,}2\cdot 10^{-4}$ ✓

$If=\dfrac{16\cdot S_{\frac{\pi}{8}}-S_{\frac{\pi}{4}}}{15}=\dfrac{16\cdot 2{,}00027-2{,}00456}{15}\approx 1{,}\bar{9}$ $\quad$ [res smo bližje dvojki, kot kadarkoli prej]

$$Tr: \int_a^b f(x)\,dx = \frac{h}{2}\left(f(x_0) + 2\cdot\sum_{i=1}^{m-1} f(x_i) + f(x_m)\right)$$
$$Sim: \int_a^b f(x)\,dx = \frac{h}{3}\left(f(x_0) + 4\sum_{i=0}^{m-1} f(x_{2i+1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right)$$

② Dokažite, da Richardsonova ekstrapolacija sestavljenega trapeznega pravila ustreza sestavljenemu simpsonovemu pravilu



vzamemo vsaki drugi τ

$$T_{\underset{2h}{h}} = \frac{2h}{2}\left(f(\tau_0) + 2\cdot\sum_{i=1}^{m-1} f(\tau_{2i}) + f(\tau_{2m})\right)$$
$$T_{\underset{h}{h/2}} = \frac{h}{2}\left(f(\tau_0) + 2\cdot\sum_{i=1}^{2m-1} f(\tau_i) + f(\tau_{2m})\right)$$

$$\frac{2^2\cdot T_{\frac{h}{2}} - T_h}{3} = \frac{4\cdot\frac{h}{2}\left(f(\tau_0) + 2\left(f(\tau_1)+f(\tau_2)+\ldots+f(\tau_{2m-1})\right) + f(\tau_{2m})\right) - h\left(f(\tau_0) + 2\left(f(\tau_2)+f(\tau_4)+\ldots+f(\tau_{2m-2})\right)+f(\tau_{2m})\right)}{3}$$
$$= \frac{hf(\tau_0) + 4h\left(f(\tau_1)+f(\tau_3)+\ldots+f(\tau_{2m-1})\right) + 2h\left(f(\tau_2)+f(\tau_4)+\ldots+f(\tau_{2m-2})\right) + hf(\tau_{2m})}{3}$$
$$= \frac{h}{3}\left(f(\tau_0) + 4\sum_{i=0}^{m-1} f(\tau_{2i+1}) + 2\cdot\sum_{i=1}^{m-1} f(\tau_{2i}) + f(\tau_m)\right)$$

③ Izpeljite sestavljeno pravilo za N-C odprto pravilo na 1 točki:

$$\int_{x_0}^{x_2} f(x)\,dx = \underbrace{2h\,f(x_1) + \frac{h^3}{3}f''(\xi)}\qquad \xi\in[x_0,x_2]$$

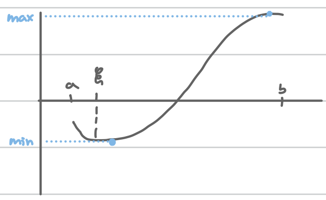
$$h = \frac{b-a}{2m}\ \downarrow\ m = \frac{b-a}{2h},\qquad \xi_i\in[x_i, x_{i+1}]$$

$$\int_{x_0}^{x_{2m}} f(x)\,dx = \sum_{i=1}^{m}\underbrace{\int_{x_{2i}}^{x_{2i+2}} f(x)\,dx} = \sum_{i=0}^{m-1}\left(2h\,f(x_{2i+1}) + \frac{h^3}{3}f''(\xi_i)\right)$$

$$= \sum_{i=0}^{m-1} 2hf(x_{2i+1}) + \sum_{i=0}^{m-1}\frac{h^3}{3}f''(\xi_i)\ *\qquad \left(\sum_{i=0}^{m-1} f''(\xi_i) = f''(\xi_1) + f''(\xi_2) + \ldots + f''(\xi_{m-1})\right)$$

ocena $*$: $\min\limits_{x\in[a,b]}(f''(x)) \le f''(\xi_i) \le \max\limits_{x\in[a,b]}(f''(x))\quad /\cdot m$

$$m\cdot\min_{x\in[a,b]}(f''(x)) \le \underbrace{\sum_{i=0}^{m-1} f''(\xi_i)}_{\substack{\| \\ m\cdot f''(\mu) \\ \text{za } \mu\in[a,b]}} \le m\cdot\max_{x\in[a,b]}(f''(x))$$



$$\Longrightarrow\ = \sum_{i=0}^{m-1} 2hf(x_{2i+1}) + \underbrace{\frac{h^3}{3}\cdot m\cdot f''(\mu)}$$
$$= \frac{(b-a)^3}{2^3 m^3\cdot 3}\, m\cdot f''(\mu)$$
$$= \frac{h^2}{6}(b-a)\,f''(\mu)$$

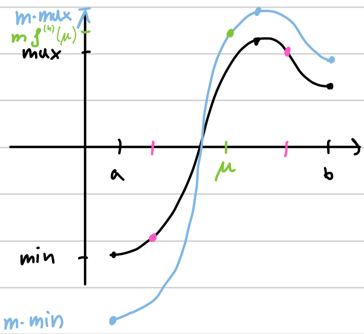
1. Izpeljite sestavljeno pravilo za 3/8 pravilo.

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(f(x_0) + 3\left(f(x_1) + f(x_2)\right) + f(x_3)\right) - \frac{3h^5}{80} f^{(4)}(\xi)$$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{m-1} \int_{x_{3i}}^{x_{3i+3}} f(x)\,dx = \sum_{i=0}^{m-1} \frac{3h}{8}\left(f(x_{3i}) + 3\left(f(x_{3i+1}) + f(x_{3i+2})\right) + f(x_{3i+3})\right) - \frac{3h^5}{8} f^{(4)}(\xi_i)) =$$

$\xi_i \in [x_{3i}, x_{3i+3}]$

$$= \frac{3h}{8}\left(f(x_0) + 3\sum_{i=0}^{m-1}\left(f(x_{3i+1}) + f(x_{3i+2})\right) + 2\sum_{i=0}^{m-2} f(x_{3i+3}) + f(x_{3m})\right) - \underbrace{\sum_{i=0}^{m-1} \frac{3h^5}{80} f^{(4)}(\xi_i)}_{*}$$



$$\min\{f^{(4)}(x) \;;\; x \in [a,b]\} \le f^{(4)}(\xi_i) \le \max\{f^{(4)}(x) \;;\; x \in [a,b]\}$$

$$m \cdot \min_{x \in [a,b]} f^{(4)}(x) \le \sum_{i=0}^{m-1} f^{(4)}(\xi_i) \le \sum_{i=0}^{m-1} f^{(4)}(\xi_i) \le m \cdot \max_{x \in [a,b]} f^{(4)}(x)$$

$\exists \mu$, da je $\sum_{i=0}^{m-1} f^{(4)}(\xi_i) = m \cdot f^{(4)}(\mu)$ $\quad \mu \in [a,b]$

$$* = \frac{3h^5}{80} m \cdot f^{(4)}(\mu) = \frac{3h^5}{80} \cdot \frac{b-a}{3h} f^{(4)}(\mu) = \frac{h^4(b-a)}{80} f^{(4)}(\mu)$$

GAUSSOVA INTEGRACIJSKA PRAVILA

Pravilo je oblike: $\int_a^b f(x)dx = \sum_{i=0}^{n} A_i f(x_i) + Rf$

Vozle x_i določimo iz pogojev: $w \perp 1,\ w \perp x, \dots, w \perp x^n$, kjer je $\omega(x) = (x-x_0)(x-x_1)\dots(x-x_n)$, glede na skalarni produkt $\int_a^b f(x)g(x)dx$

ko določimo vozle, je določitev uteži A_i linearen problem, določimo jih preko MNK ali preko int. polinoma.

IZREK: Za $f \in \mathcal{C}^{2n+2}([a,b])$ je napaka Rf oblike: $Rf : \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b \omega^2(x)dx$

2. Določite Gaussovo integracijsko pravilo oblike

$$\int_{-1}^{1} f(x)dx = Af(x_0) + Bf(x_1) + Cf(x_2) + Rf.$$

$n=2$, $\omega(x) = (x-x_0)(x-x_1)(x-x_2)$

- $\omega \perp 1 \implies \langle \omega, 1 \rangle = 0$

$$\langle \omega, 1 \rangle = \int_{-1}^{1} (x-x_0)(x-x_1)(x-x_2)dx = \int_{-1}^{1} (x^2 - xx_1 + xx_0 + x_0x_0)(x-x_2)dx$$
$$= \int_{-1}^{1} (x^3 + x^2x_2 - x^2x_1 + xx_1x_2 - x^2x_0 + xx_0x_2 + xx_0x_1 - x_0x_1x_2)dx$$
$$= \int_{-1}^{1} x^3 - x^2(x_2+x_1+x_0) + x(x_1x_2 + x_0x_2 + x_0x_1) - x_0x_1x_2 \,)dx$$
$$= \frac{x^4}{4} - \frac{x^3}{3}\underbrace{(x_0+x_1+x_2)}_{a} + \frac{x^2}{2}\underbrace{(x_0x_1 + x_1x_2 + x_2x_0)}_{b} - x \cdot \underbrace{x_0x_1x_2}_{c} \Big|_{-1}^{1}$$
$$= \frac{1}{4} - \frac{1}{3}a + \frac{1}{2}b - c - \frac{1}{4} - \frac{1}{3}a - \frac{1}{2}b - c = -\frac{2}{3}a - 2c$$

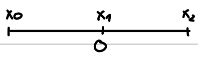
- $\omega \perp x \implies \langle \omega, x \rangle = 0 \implies \langle \omega, x \rangle = \int_{-1}^{1} (x^4 - x^3(x_0+x_1+x_2) + x^2(x_0x_1 + x_1x_2 + x_0x_2) - x \cdot x_0x_1x_2)dx$

$$= \frac{x^5}{5} - \frac{x^4}{4}a + \frac{x^3}{3}b - \frac{x^2}{2}a \Big|_{-1}^{1}$$
$$= \frac{2}{5} + \frac{2}{3}b$$

- $\omega \perp x^2 \implies \langle \omega, x^2 \rangle = 0 \implies \langle \omega, x^2 \rangle = \int_{-1}^{1} (x^5 - x^4(x_0+x_1+x_2) + x^3(x_0x_1 + x_1x_2 + x_0x_2) - x^2 \cdot x_0x_1x_2)dx$

$$= \frac{x^6}{6} - \frac{x^5}{5}a + \frac{x^4}{4}b - \frac{x^3}{3}c \Big|_{-1}^{1}$$
$$= -\frac{2}{5}a - \frac{2}{3}c$$

$$\left.\begin{aligned} &-\tfrac{2}{3}(x_0+x_1+x_2) - 2x_0x_1x_2 = 0 \\ &\tfrac{2}{5} + \tfrac{2}{3}(x_0x_1 + x_1x_2 + x_0x_2) = 0 \\ &-\tfrac{2}{5}(x_0+x_1+x_2) - \tfrac{2}{3}x_0x_1x_2 = 0 \end{aligned}\right] \implies -\frac{8}{3}x_0x_1x_2 = 0$$



- $x_1 = 0$
- $\frac{2}{5} + \frac{2}{3}x_0x_2 = -\frac{3}{5} \to x_0^2 = \frac{3}{5} \to x_0 = \pm\sqrt{\frac{3}{5}}$
- $x_0 = -x_2 \to x_2 = \sqrt{\frac{3}{5}}$

- <u>PREKO INT. POLINOMA</u>

$A = \int_{-1}^{1} \ell_{0,2}(x)$ $\quad \ell_{0,2} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-\sqrt{3/5})}{-\sqrt{3/5}\,(-2\sqrt{3/5})} = \frac{x^2 - x\sqrt{3/5}}{6/5} \implies \int_{-1}^{1} \frac{5}{6}(x^2 - x\sqrt{3/5}) = \frac{5}{6}\left(\frac{x^3}{3} - \frac{x^2}{2}\sqrt{3/5}\right)\Big|_{-1}^{1} = \frac{5}{9} = A$

$B = \int_{-1}^{1} \ell_{1,2}(x)$ $\quad \ell_{1,2}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+\sqrt{3/5})(x-\sqrt{3/5})}{\sqrt{3/5}\,(-\sqrt{3/5})} = \frac{x^2 - 3/5}{-3/5} \implies \int_{-1}^{1} -\frac{5}{3}(x^2 - \frac{3}{5}) = -\frac{5}{3}\left(\frac{x^3}{3} - \frac{3}{5}x\right)\Big|_{-1}^{1} = \frac{8}{9} = B$

$C = \int_{-1}^{1} \ell_{2,2}(x)$ $\quad \ell_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \dots$ $\quad \frac{5}{9} = C$

- napaka : $Rf = \frac{f^{(6)}(\xi)}{6!} \int_{-1}^{1} \omega^2(x)dx$

$\omega(x) = (x-x_0)(x-x_1)(x-x_2) = (x+\sqrt{3/5})\,x\,(x-\sqrt{3/5}) = x^3 - \frac{3}{5}x$ $\quad \omega^2(x) = x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2$ $\quad \int_{-1}^{1}\omega^2(x) = \frac{8}{175}$

$\implies Rf = \frac{f^{(6)}(\xi)}{6!} \cdot \frac{8}{175} = \frac{f^{(6)}(\xi)}{15750}$

- pravilo: $\boxed{\int_{-1}^{1} f(x)dx = \frac{5}{9}f(-\sqrt{3/5}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{3/5}) + \frac{f^{(6)}(\xi)}{15750}}$

NUMERIČNO REŠEVANJE NAVADNIH DE

Rešujemo: $y' = f(x,y)$, $y(a) = y_a$

$f : [a,b] \times \mathbb{R} \to \mathbb{R}$

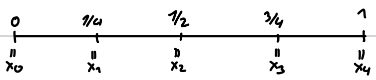
$y_n \approx y(x_n)$
$y_1 \approx y(x_1)$

- **EULERJEVA METODA**: $\to$ $y_{n+1} = y_n + h \cdot f(x_n, y_n)$ — eksplicitna
  $\to$ $y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$ — implicitna

- **TRAPEZNA METODA**: $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right)$ — implicitna

- Pri implicitnih metodah je potrebno rešiti (nelinearno) enačbo. To lahko naredimo:
  $\to$ direktno
  $\to$ z neko numerično metodo, npr. navadno iteracijo: $\alpha := y_{n+1}$, $\alpha^{(0)} = y_n$, $r = 0,1,2,\ldots$, $\alpha^{(r+1)} = g(\alpha^{(r)})$

1. Začetni problem $y' = x + y - 1$ na $[0,1]$ rešujemo z eksplicitno in implicitno Eulerjevo metodo. S korakom $h = \frac{1}{4}$ določite numerične približke y_1, y_2, y_3 in y_4 pri začetnem pogoju $y_0 = 1$.



$f(x,y) = x + y - 1$

- **EKSPLICITNA METODA**

$$y_1 = y_0 + \tfrac{1}{4} f(\underset{0}{x_0}, \underset{1}{y_0}) = 1 + \tfrac{1}{4}(0 + 1 - 1) = 1$$

$$y_2 = y_1 + \tfrac{1}{4} f(x_1, y_1) = 1 + \tfrac{1}{4}\left(\tfrac{1}{4} + 1 - 1\right) = \tfrac{17}{16} \doteq 1{,}0623$$

$$y_3 = y_2 + \tfrac{1}{4} f(x_2, y_2) = \tfrac{17}{16} + \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{17}{16} - 1\right) = \tfrac{77}{64} \doteq 1{,}2031$$

$$y_4 = y_3 + \tfrac{1}{4} f(x_3, y_3) = \tfrac{77}{64} + \tfrac{1}{4}\left(\tfrac{3}{4} + \tfrac{77}{64} - 1\right) = \tfrac{369}{256} \doteq 1{,}4414$$

- **IMPLICITNA METODA**

$$y_1 = y_0 + \tfrac{1}{4} f(x_1, y_1) = y_0 + \tfrac{1}{4}(x_1 + y_1 - 1)$$

$$y_{n+1} = y_n + \tfrac{1}{4} f(x_{n+1} + y_{n+1} - 1) = y_n + \tfrac{1}{4} x_{n+1} + \tfrac{1}{4} y_{n+1}$$

$$\tfrac{3}{4} y_{n+1} = y_n + \tfrac{1}{4} x_{n+1} - 1$$

$$\boxed{y_{n+1} = \tfrac{4}{3} y_n + \tfrac{1}{3} x_{n+1} - \tfrac{4}{3}}$$

$$y_1 = \frac{4y_0 + x_1 - 1}{3} = \frac{4 + \frac{1}{4} - 1}{3} = \frac{13}{12} \doteq 1{,}0833$$

$$y_2 = \frac{4y_1 + x_2 - 1}{3} = \frac{\frac{13}{3} + \frac{1}{2} - 1}{3} = \frac{23}{18} \doteq 1{,}2778$$

$$y_3 = \frac{4y_2 + x_3 - 1}{3} = \frac{46/9 + 3/4 - 1}{3} = \frac{175}{108} \doteq 1{,}6204$$

$$y_4 = \frac{4y_3 + x_4 - 1}{3} = \frac{4 \cdot 175/108 + 1 - 1}{3} = \frac{175}{81} = 2{,}1605$$

- **TOČNA REŠITEV**

$y(x) = e^x - x$

$y'(x) = e^x - 1$

$y' = x + y - 1$

$e^x - 1 = x + e^x - x - 1$

$y(x_1) = e^{1/4} - \frac{1}{4} \doteq 1{,}034$

$y(x_2) = \sqrt{e} - \frac{1}{2} \doteq 1{,}1487$

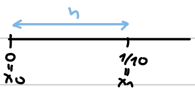
$y(x_3) = e^{3/4} - \frac{3}{4} \doteq 1{,}3670$

$y(x_4) = e - 1 \doteq 1{,}7183$

- KOMENTAR: z eksplicitno metodo dobimo boljše približke (v splošnem ni res)

2. Diferenicalno enačbo $y' = 2x^2 - y^2$ z začetnim pogojem $y_0 = 0$ v $x_0 = 0$ rešujemo s trapezno metodo. S korakom $h = \frac{1}{10}$ določite numerični približek za y_1. Uporabite navadno iteracijo, kjer naredite 2 koraka.

približek za $y(x_1) = y(\frac{1}{10})$



$$y_1 = y_0 + \frac{1}{20}\left(f(x_0,y_0) + f(x_1,y_1)\right) = 0 + \frac{1}{20}\left(0 + 2\frac{1}{100} - y_1^2\right) = \underbrace{\frac{1}{1000} - \frac{1}{20}y_1^2}$$

iteracijska funkcija: $g(z) = \frac{1}{1000} - \frac{1}{20}z^2$

y_1 je njena negibna točka $g(y_1) = y_1$

$\alpha = y_1$

$\alpha^{(0)} = y_0 = 0$

$\alpha^{(1)} = g(\alpha^{(0)}) = \frac{1}{1000} - 0 = 0{,}001$

$\alpha^{(2)} = g(\alpha^{(1)}) = \frac{1}{1000} - \frac{1}{20} \cdot \frac{1}{(1000)^2} = 0{,}0009995 = y_1$

A-STABILNOST

A-stabilnost metode testiramo tako, da preverimo, ali se numerični približki y_n za računanje posameznega začetnega problema $y' = \lambda \cdot y$, $y(0) = y_0 \neq 0$, $\lambda < 0$, ko gre $n \to \infty$ ujemajo z vrednostmi točne rešitve, ko gre $x \to \infty$.

Torej: $\lim\limits_{n\to\infty} y_n = \lim\limits_{x\to\infty} y(x)$

4. Testirajte A-stabilnost trapezne metode.

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n,y_n) + f(x_{n+1},y_{n+1})\right)$$

$$y_1 = y_0 + \frac{h}{2}(\lambda y_0 + \lambda y_1)$$

$$y_1 - \frac{h}{2}y_1 = y_0 + \frac{h}{2}\lambda y_0$$

$$y_1 = \frac{y_0(1 + \frac{h}{2}\lambda)}{(1 - \frac{h\lambda}{2})}$$

$$y_2 = \frac{y_1(1 + \frac{h\lambda}{2})}{(1 - \frac{h\lambda}{2})} = \frac{y_0(1 + \frac{h\lambda}{2})^2}{(1 - \frac{h\lambda}{2})^2}$$

$$\Longrightarrow y_n = \frac{y_0(1 + \frac{h\lambda}{2})^n}{(1 - \frac{h\lambda}{2})^n} = y_0\left(\frac{2 + \lambda h}{2 - h\lambda}\right)^n$$

- $-1 < \frac{2 + h\lambda}{2 - h\lambda} < 1 \quad / \cdot (2 - h\lambda) > 0$

$h\lambda - 2 < \underline{2 + h\lambda < 2 - h\lambda}$

$2h\lambda < 0 \quad / \cdot \frac{1}{2\lambda}, \ \lambda < 0$

$h > 0$

$\Downarrow$

$\boxed{h \in (0, \infty)}$ ok ✓

3. Testirajte A-stabilnost eksplicitne in implicitne Eulerjeve metode.

Torej določite splošni formuli za približek y_n in določite tiste h, za katere velja $\lim\limits_{n\to\infty} y_n = \lim\limits_{x\to\infty} y(x)$

$f(x,y) = \lambda y$

$$\left.\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) = y_0 + h\lambda y_0 = y_0(1+h\lambda) \\ y_2 &= y_1 + h\lambda y_1 = y_0(1+h\lambda)^2 \end{aligned}\right] \Rightarrow y_n = y_0(1+h\lambda)^n$$

$y' = \lambda y \qquad \frac{dy}{dx} = \lambda y \qquad \frac{dy}{dy} = \lambda dx \quad //$

$\ln|y| = \lambda x + C$

$y = e^{\lambda x + C}$

$y(x) = D e^{\lambda x}$

$y_0 = y(0) = D$

$\Rightarrow y(x) = y_0 \cdot e^{\overset{<0}{\lambda x}} \xrightarrow{x\to\infty} 0$

- $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} y_0(1+h\lambda)^n \overset{?}{=} 0$

$-1 < 1+h\lambda < 1$

$-2 < h\lambda < 0$

- $-2 < h\lambda \ /:\lambda$ • $h\lambda < 0 \ /:\lambda$ $\boxed{\lambda<0}$

$\frac{-2}{\lambda} > h$ $h > 0$

$\Downarrow$

$\boxed{h \in (0, -\frac{2}{\lambda})}$

$\overset{\vee}{0}$, saj $\lambda < 0$

- $y_1 = y_0 + h f(x_1, y_1) = y_0 + h\lambda y_1$

$y_1(1-h\lambda) = y_0 \longrightarrow y_1 = \frac{y_0}{1-h\lambda}$

$y_2 = y_1 + h\lambda \longrightarrow y_2 = \frac{y_1}{(1-h\lambda)} = \frac{y_0}{(1-h\lambda)^2}$

$\Rightarrow y_n = \frac{y_0}{(1-h\lambda)^n}$

- $\lim\limits_{n\to\infty} \frac{y_0}{(1-h\lambda)^n} \overset{?}{=} 0$

$1 - h\lambda > 1$

$-h\lambda > 0$

$-h < 0 \Rightarrow h > 0 \longrightarrow \boxed{h \in (0,\infty)}$ implicitno

RUNGE-KUTTA METODA

$k_i = h f(x_n + \alpha_i h, y_n + \sum_{j=1}^{s} \beta_{ij} k_j) \qquad i = 1, 2, \dots, s$

$y_{n+1} = y_n + \sum_{i=1}^{s} \gamma_i k_i$

Koeficienti $\alpha_i, \beta_{ij}, \gamma_i$ so podani z Butcherjevo shemo:

$$\begin{array}{c|cccc} \alpha_1 & \beta_{11} & \beta_{12} & \dots & \beta_{1s} \\ \alpha_2 & \beta_{21} & \beta_{22} & \dots & \beta_{2s} \\ \vdots & & & & \\ \alpha_s & \beta_{s1} & \beta_{2s} & \dots & \beta_{ss} \\ \hline & \gamma_1 & \dots & & \gamma_s \end{array}$$

Heunova metoda je podana kot:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

16.5

① Začetni problem $y' = x + y - 1$ na $[0,1]$ rešujemo s Heunovo metodo. S korakom $h = 1/4$ določite y_1, y_2, y_3 in y_4 pri $y_0 = 1$.

a) Izračunajte k_1, k_2

$\alpha_1 = 0$	$\beta_{11} = 0$	$\beta_{12} = 0$	$\gamma_1 = 1/2$
$\alpha_2 = 1$	$\beta_{21} = 1$	$\beta_{22} = 0$	$\gamma_2 = 1/2$

$$\Rightarrow k_1 = \frac{1}{4}\left(f(x_n, y_n)\right) = \frac{1}{4}(x_n + y_n - 1)$$

$$\Rightarrow k_2 = \frac{1}{4}\left(f(x_n + 1 \cdot \frac{1}{4}, y_n + 1 \cdot k_1)\right)$$
$$= \frac{1}{4}(x_n + \frac{1}{4} + y_n + k_1 - 1)$$
$$= \frac{1}{4}(x_n + \frac{1}{4} y_n - 1 + \frac{x_n}{4} + \frac{y_n}{4} - \frac{1}{4})$$
$$= \frac{1}{4}\left(\frac{5}{4} x_n + \frac{5}{4} y_n - 1\right)$$
$$= \frac{5}{16}\left(x_n + y_n - \frac{4}{5}\right)$$

b) Izračunajte numerične približke y_1, y_2, y_3, y_4 pri začetnem pogoju $y_0 = 1$ v točki $x_0 = 0$

$\gamma_1 = 1/2 \quad \gamma_2 = 1/2$

$$y_1 = y_0 + \sum_{i=1}^{2} \gamma_i k_i = y_0 + \gamma_1 k_1 + \gamma_2 k_2 = 1 + \frac{1}{2} \, \frac{1}{4}(x_0 + y_0 - 1) + \frac{1}{2} \cdot \frac{5}{16}\left(x_0 + y_0 - \frac{4}{5}\right) =$$
$$= 1 + \frac{1}{8}(0 + 1 - 1) + \frac{5}{32}\left(0 + 1 - \frac{4}{5}\right)$$
$$= \frac{33}{32}$$

$$\boxed{y_{n+1} = y_n + \gamma_1 k_1 + \gamma_2 k_2 = y_n + \frac{1}{8}(x_n + y_n - 1) + \frac{5}{32}\left(x_n + y_n - \frac{4}{5}\right) = \frac{41}{32} y_n + \frac{9}{32} x_n - \frac{1}{4}}$$

$$y_2 = \frac{41}{32} \cdot \frac{33}{32} + \frac{9}{32} \cdot \frac{1}{4} - \frac{1}{4} = \frac{1164}{1024}$$
$$y_3 = \frac{41}{32} \cdot \frac{1169}{1024} + \frac{9}{32} \cdot \frac{1}{2} - \frac{1}{4} \doteq 1{,}3533$$
$$y_4 = \frac{41}{32} \cdot y_3 + \frac{9}{32} \cdot \frac{3}{4} - \frac{1}{4} \doteq 1{,}69485$$

② Metoda RK3 je podana z Butcherjevo shemo

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1/2 & 1/2 & 0 & \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 4/6 & 1/6 \end{array}$$

Določite približek y_1 ($\approx y(x_1)$) za rešitev začetnega problema $y' = -2xy$ pri začetnem pogoju $y_0 = 1$ v točki $x_0 = 0$, pri razmiku $h = 1/10$

$f(x,y) = -2xy$

$$k_1 = \frac{1}{10} f(x_n + 0 \cdot \frac{1}{10}, y_n + 0) = \frac{1}{10}(-2 x_n y_n) = -\frac{1}{5} x_n y_n = -\frac{1}{5} x_0 y_0 = 0$$
$$k_2 = \frac{1}{10} f(x_0 + \frac{1}{2} \cdot \frac{1}{10}, y_0 + \frac{1}{2} \cdot 0) = \frac{1}{10}(-2)(x_0 + \frac{1}{20}) y_0 = -\frac{1}{5} \cdot \frac{1}{20} = -\frac{1}{100}$$
$$k_3 = \frac{1}{10} f(x_0 + \frac{1}{10}), y_0 - k_1 + 2k_2) = -\frac{1}{5}\left(\frac{1}{10}\right) \cdot \left(1 - 0 - \frac{1}{50}\right) = -\frac{49}{2500}$$

$$y_1 = y_0 + 1/6\, k_1 + 4/6\, k_2 + 1/6\, k_3 = 1 + 1/6 \cdot 0 + \frac{4}{6}\left(-\frac{1}{100}\right) + \frac{1}{6}\left(-\frac{49}{2500}\right) \doteq \underline{\underline{0{,}99007}}$$

· točna rešitev: $y = e^{-x^2}$

$$y\left(\frac{1}{10}\right) = 0{,}99005$$

SISTEMI DIFERENCIALNIH ENAČB

Začetni problem: $y_1' = f_1(x, y_1, y_2, \dots, y_d)$; $y_1(a) = y_{1,a}$

$y_2' = f_2(x, y_1, y_2, \dots, y_d)$; $y_2(a) = y_{2,a}$

$\vdots$

$y_d' = f_d(x, y_1, y_2, \dots, y_d)$; $y_d(a) = y_{d,a}$

Sistem zapišemo v vektorski obliki: $Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_d(x) \end{bmatrix}$, $Y : [a,b] \rightarrow \mathbb{R}^d$

$F : \underset{x\in}{[a,b]} \times \underset{Y\in}{\mathbb{R}^d} \rightarrow \mathbb{R}^d$

$$F\left(x, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) = F(x, Y) = \begin{bmatrix} f_1(x, y_1, \dots, y_d) \\ f_2(x, y_1, \dots, y_d) \\ \vdots \\ f_d(x, y_1, \dots, y_d) \end{bmatrix}$$

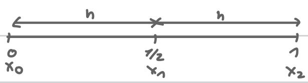
Začetni problem zapišemo kot: $Y' = F(x, Y)$, $Y(a) = Y_a = \begin{bmatrix} y_{1,a} \\ y_{2,a} \\ \vdots \\ y_{d,a} \end{bmatrix}$

③ Podan je sistem DE $y' = \underbrace{y + z}_{f_1(x,y,z)}$, $z' = \underbrace{4y + z}_{f_2(x,y,z)}$ z začetnima pogojema $y_0 = 8$ in $z_0 = 12$ v $x_0 = 0$

Numerične približke sistemu iščemo z dvema različnima metodama pri koraku $h = 1/2$

a) Z uporabo dveh korakov eksplicitne Eulerjeve metode določite približke za $y(1/2)$, $z(1/2)$, $y(1)$ in $z(1)$

$Y = \begin{bmatrix} y \\ z \end{bmatrix}$ $\quad F(x, Y) = F\left(x, \begin{bmatrix} y \\ z \end{bmatrix}\right) = \begin{bmatrix} y + z \\ 4y + z \end{bmatrix} \begin{matrix} \leftarrow f_1 \\ \leftarrow f_2 \end{matrix}$



$\boxed{Y_{n+1} = Y_n + h f(x_n, Y_n)}$

$Y_n = \begin{bmatrix} y_n \\ z_n \end{bmatrix}$ $\quad Y_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$

$$Y_1 = Y_0 + h F(x_0, Y_0) = \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \frac{1}{2} F\left(0, \begin{bmatrix} 8 \\ 12 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 20 \\ 44 \end{bmatrix} = \begin{bmatrix} 18 \\ 34 \end{bmatrix}$$

$y_1 = y(x_1) = y(1/2) = 18$

$z_1 = z(x_1) = z(1/2) = 34$

$$Y_2 = Y_1 + h F(x_1, Y_1) = \begin{bmatrix} 18 \\ 34 \end{bmatrix} + \frac{1}{2} F\left(\frac{1}{2}, \begin{bmatrix} 18 \\ 34 \end{bmatrix}\right) = \begin{bmatrix} 18 \\ 34 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 18 + 34 \\ 4 \cdot 18 + 34 \end{bmatrix} = \begin{bmatrix} 44 \\ 87 \end{bmatrix} \begin{matrix} \rightarrow y_2 = 44 \\ \rightarrow z_2 = 87 \end{matrix}$$

$y_2 \approx y(x_2) = y(1) \approx 44$

$z_2 \approx z(x_2) = z(1) \approx 87$

b) Približka za $y(1/2)$ in $z(1/2)$ določite še z uporabo modificirane (izboljšane) Eulerjeve metode:

$$\begin{array}{c|cc} 0 & 0 & \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

$$k_1 = h f(x_0, Y_0) = \frac{1}{2} F(x_0, Y_0) = \frac{1}{2} F\left(x_0, \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} y_0 + z_0 \\ 4y_0 + z_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 20 \\ 44 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \end{bmatrix}$$

$$k_2 = h F\left(x_0 + \frac{1}{2} h, Y_0 + \frac{1}{2} k_1 + 0 \cdot k_2\right) = \frac{1}{2} F\left(\frac{1}{4}, \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 10 \\ 22 \end{bmatrix}\right) = \frac{1}{2} F\left(\frac{1}{4}, \begin{bmatrix} 13 \\ 23 \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} 13 + 23 \\ 4 \cdot 13 + 23 \end{bmatrix} = \begin{bmatrix} 18 \\ 75/2 \end{bmatrix}$$

$$Y_1 = Y_0 + 0 k_1 + 1 k_2 = \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 18 \\ 75/2 \end{bmatrix} = \begin{bmatrix} 26 \\ 99/2 \end{bmatrix}$$

$\rightarrow y(1/2) \approx y_1 = 26$

$z(1/2) \approx z_1 = 99/2$

23.5

1. Sistem diferencialnih enačb

$$y' = -z, \quad y(0) = 1,$$
$$z' = y, \quad z(0) = 0,$$

rešujemo s trapezno metodo. Izpeljite eksplicitno formulo za Y_{n+1} in nato z razmikom $h = \frac{1}{10}$ določite Y_1.

trapezna metoda: $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right)$

$$Y_{n+1} = Y_n + \frac{h}{2}\left(F(x_n, Y_n) + F(x_{n+1}, Y_{n+1})\right) \qquad Y = \begin{bmatrix} y \\ z \end{bmatrix} \qquad F\left(x, \begin{bmatrix} y \\ z \end{bmatrix}\right) = \begin{bmatrix} -z \\ y \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ z_n \end{bmatrix} + \frac{h}{2}\left(\begin{bmatrix} -z_n \\ y_n \end{bmatrix} + \begin{bmatrix} -z_{n+1} \\ y_{n+1} \end{bmatrix}\right)$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} - \frac{h}{2}\begin{bmatrix} -z_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ z_n \end{bmatrix} + \frac{h}{2}\begin{bmatrix} -z_n \\ y_n \end{bmatrix} \qquad Y_{n+1} = \underline{\quad} \cdot Y_n + \underline{\quad}$$

$$\begin{bmatrix} -z_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix}$$

$a = 0, \ b = -1, \ c = 1, \ d = 0$

$$\underbrace{\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix}}_{Y_{n+1}} - \frac{h}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix}}_{Y_{n+1}} = \underbrace{\begin{bmatrix} y_n \\ z_n \end{bmatrix}}_{Y_n} + \frac{h}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} y_n \\ z_n \end{bmatrix}}_{Y_n}$$

$$\left(I - \frac{h}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \left(I - \frac{h}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} y_n \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & h/2 \\ -h/2 & 1 \end{bmatrix} \xrightarrow{\text{inverz}} \frac{1}{1 + h^2/4}\begin{bmatrix} 1 & -h/2 \\ h/2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{4+h^2} & \frac{-2h}{4+h^2} \\ \frac{2h}{4+h^2} & \frac{4}{4+h^2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & h/2 \\ -h/2 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h/2 \\ -h/2 & 1 \end{bmatrix}\begin{bmatrix} y_n \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \frac{1}{1 + h^2/4}\begin{bmatrix} 1 & -h/2 \\ h/2 & 1 \end{bmatrix}^2\begin{bmatrix} y_n \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \frac{1}{4+h^2}\begin{bmatrix} 4-h^2 & -4h \\ 4h & -h^2+4 \end{bmatrix}\begin{bmatrix} y_n \\ z_n \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \frac{1}{4 + \frac{1}{100}}\begin{bmatrix} 4 - \frac{1}{100} & -4/10 \\ 4/10 & 4 - \frac{1}{100} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{100}{401}\begin{bmatrix} \frac{399}{100} \\ \frac{4}{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{399}{401} \\ \frac{40}{401} \end{bmatrix}$$

DIFERENCIALNE ENAČBE VIŠJEGA REDA

Začetni problem reda p: $y^{(p)} = f(x, y, y', \dots, y^{(p-1)})$

$$y(a) = y_{a,0}$$
$$y'(a) = y_{a,1}$$
$$\vdots$$
$$y^{(p-1)}(a) = y_{a,p-1}$$

Problem prevedemo na sistem DE 1.reda tako, da uvedemo nove spremenljivke (neznane funkcije) $z_1, z_2, \dots, z_{p-1}$:

$$z_1 := y'$$
$$z_2 := z_1' = y''$$
$$\vdots$$
$$z_{p-1} := y^{(p-1)}$$
$$z_p := y^{(p)} = f(x, y, z_1, z_2, \dots, z_{p-1})$$

Sistem je oblike $Y' = \begin{bmatrix} y' \\ z_1' \\ \vdots \\ z_{p-1}' \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ f(x,y,z_1,\dots,z_{p-1}) \end{bmatrix} = F(x,Y)$ $\qquad \begin{bmatrix} y(a) \\ z_1(a) \\ \vdots \\ z_{p-1}(a) \end{bmatrix} = \begin{bmatrix} y_{a,0} \\ y_{a,1} \\ \\ y_{a,p-1} \end{bmatrix}$

2. Podana je diferencialna enačba 2.reda

$$y'' = y'y^2 - y, \quad y(0) = 1, \quad y'(0) = 0.$$

Enačbo prevedite na sistem enačb 1.reda ter z modificirano Eulerjevo metodo pri razmiku $h = \frac{1}{10}$ izračunajte približka Y_1 in Y_2.

nova funkcija: $z = y'$

$$Y = \begin{bmatrix} y \\ z \end{bmatrix} \qquad F(x,Y) = \begin{bmatrix} z \\ zy^2 - y \end{bmatrix}$$

sistem: $y' = z$

$$z' = z \cdot y^2 - y$$

Modificirana Eulerjeva metoda:

$$\begin{array}{r|cc} k_1 \rightarrow 0 & 0 & \\ k_2 \rightarrow 1/2 & 1/2 & 0 \\ \hline Y_{n+1} \longrightarrow & 0 & 1 \\ & k_1 & k_2 \end{array}$$

<u>1.KORAK</u>: $k_1 = h \cdot F(x_0, Y_0) = \frac{1}{10} F(x_0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \frac{1}{10} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{10} \end{bmatrix}$

$$k_2 = h \cdot F(x_0 + \tfrac{h}{2}, Y_0 + \tfrac{1}{2} k_1) = \frac{1}{10} F\left(\frac{1}{20}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ -1/10 \end{bmatrix}\right) = \frac{1}{10} F\left(\frac{1}{20}, \begin{bmatrix} 1 \\ -1/20 \end{bmatrix}\right) = \frac{1}{10} \begin{bmatrix} -1/20 \\ -21/20 \end{bmatrix} = \begin{bmatrix} -1/200 \\ -21/200 \end{bmatrix}$$

$$Y_1 = Y_0 + 0 \cdot k_1 + k_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/200 \\ -21/200 \end{bmatrix} = \begin{bmatrix} 0{,}995 \\ -0{,}105 \end{bmatrix}$$

<u>2.KORAK</u>: $k_1 = hF(x_1, Y_1) = \frac{1}{10} F(x_1, \begin{bmatrix} 0{,}995 \\ -0{,}105 \end{bmatrix}) = \frac{1}{10} \begin{bmatrix} -0{,}105 \\ -0{,}105\,(0{,}995)^2 - 0{,}995 \end{bmatrix} = \begin{bmatrix} -0{,}0105 \\ -0{,}1099 \end{bmatrix}$

$$k_2 = \tfrac{1}{10} F(x_1 + \tfrac{h}{2}, Y_1 + \tfrac{1}{2} k_1) = \frac{1}{10} F\left(x_1 + \frac{1}{10}, \begin{bmatrix} 0{,}995 \\ -0{,}105 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -0{,}0105 \\ -0{,}1099 \end{bmatrix}\right) = \frac{1}{10} F\left(x_1 + \frac{h}{2}, \begin{bmatrix} 0{,}9898 \\ 0{,}16 \end{bmatrix}\right) = \frac{1}{10} \begin{bmatrix} -0{,}16 \\ -0{,}16(0{,}9898)^2 - 0{,}9898 \end{bmatrix} = \begin{bmatrix} -0{,}016 \\ -0{,}1147 \end{bmatrix}$$

$$Y_2 = Y_1 + 0 \cdot k_1 + k_2 = \begin{bmatrix} 0{,}995 \\ -0{,}105 \end{bmatrix} + \begin{bmatrix} -0{,}016 \\ -0{,}1147 \end{bmatrix} = \begin{bmatrix} 0{,}979 \\ -0{,}2197 \end{bmatrix}$$

NUMERIČNO RAČUNANJE LASTNIH VREDNOSTI

↳ Za matriko $A \in \mathbb{R}^{n \times n}$ iščemo dominantno [$\max|\lambda|$] lastno vrednost in pripadajoči lastni vektor s potenčno metodo

↳ ALGORITEM:

- Uhod: z_0 (= začetni približek za dominantni lastni vektor), matrika A
- for $k = 0, 1, 2, \dots, n$

  $y_{k+1} = A z_k$

  $z_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}$

  end
- Izhod: $x = z_n$

↳ Pripadajočo lastno vrednost izračunamo kot: $\lambda = \frac{x^T A x}{x^T x}$

3. S pomočjo enega koraka potenčne metode določite približek za dominantno lastno vrednost matrike

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}$$

z začetnim vektorjem $z_0 = \frac{1}{\sqrt{2}}[1 \quad 1]^T$.

Kolikšna je točna vrednost dominantne lastne vrednosti?

$\boxed{k=0}$: $y_1 = A z_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

$$z_1 = \frac{y_1}{\|y_1\|} = \frac{\frac{1}{\sqrt{2}} \begin{bmatrix} 9 \\ 5 \end{bmatrix}}{\sqrt{53}} = \frac{1}{\sqrt{106}} \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$x_1 = z_1$

$$\lambda = \frac{x^T A x}{\underbrace{x^T x}_{=1}} = \frac{\frac{1}{106} [9 \; 5] \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix}}{\underbrace{\frac{1}{106} [9 \; 5] \begin{bmatrix} 9 \\ 5 \end{bmatrix}}_{=1}} = \frac{1}{106} [27+5 \quad 54+20] \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \frac{1}{106} [32 \quad 74] \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \frac{658}{106} \approx 6.2$$

[$\|x\|^2 = 1 \Rightarrow x^T x = 1$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 6 = 12 - 7\lambda + \lambda^2 - 6 = (\lambda-6)(\lambda-1) \longrightarrow \begin{matrix} \lambda_1 = 6 \\ \lambda_2 = 1 \end{matrix}$$