Data Representation as Low Rank Matrix Factorization

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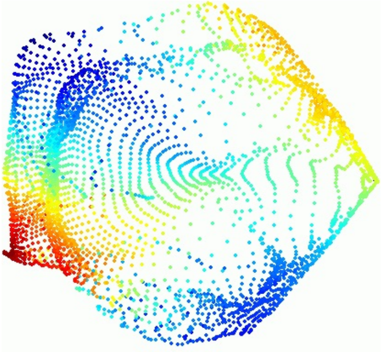
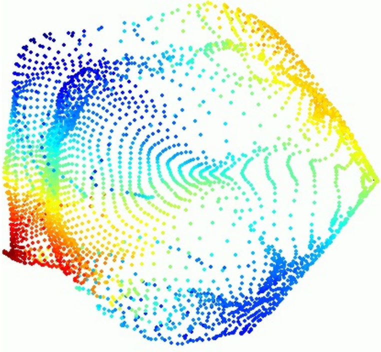
Points in $\mathbb{R}^m$

In many contexts in data science and linear algebra, we have lots of points in a high-dimensional space.



Denote these $x_i \in \mathbb{R}^m$ for $i = 1 \ldots n$.

How do we cluster x_i?



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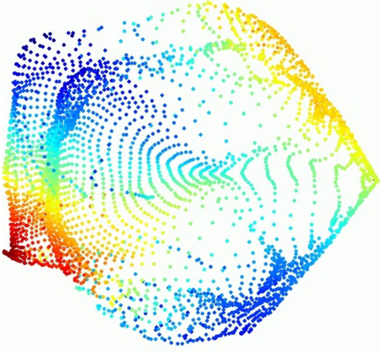

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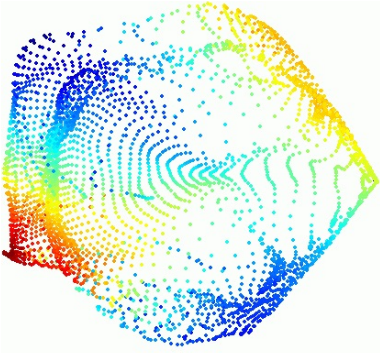
How do we cluster x_i? How do we perform dimensionality reduction? How do we visualize them?



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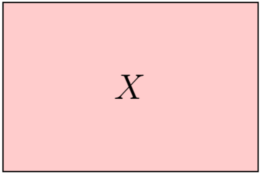
Representation and Factorization

We can *approximate* $X \in \mathbb{R}^{n \times m}$.



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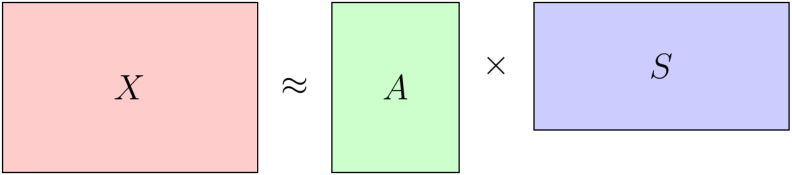

with $\text{rank}(AS) = k << min(m, n)$.

Low Dimensional Interpretation

Our original high-dimensional points are linear combinations of "basis elements'.'

$$X_i = \sum_{j=1}^{k} A_{i,j} S_j$$

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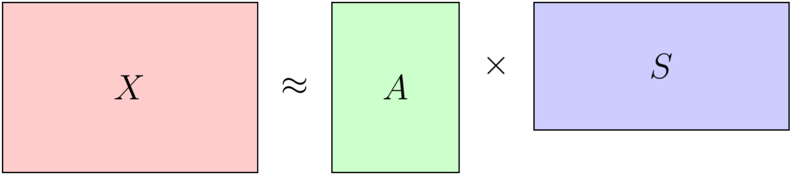

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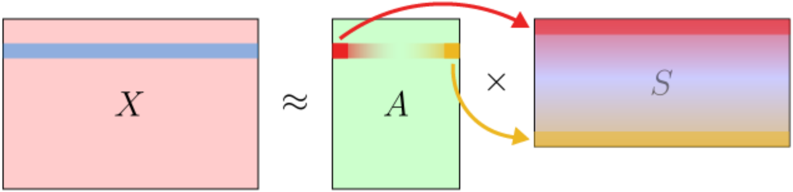

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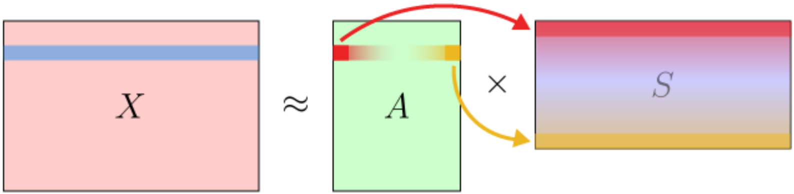

Find factorization by solving optimization problem

$$\min_{A,S} ||X - AS||$$

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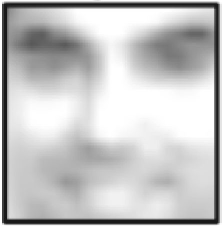

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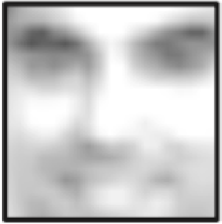

Take a database of 2,429 faces to get $X \in \mathbb{R}^{2429 \times 361}$

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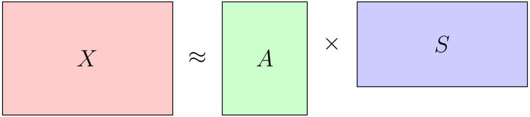
Before: Find any A and S such that $||X - AS||$ is minimized.



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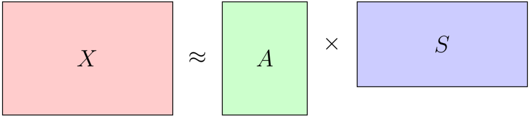

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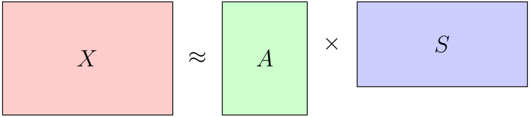

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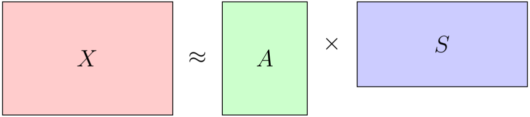

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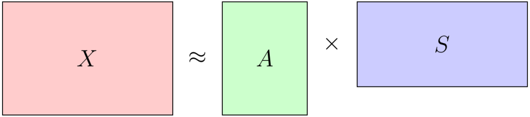

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For each, find A and S subject to constraints that minimize

$$||X - AS||$$

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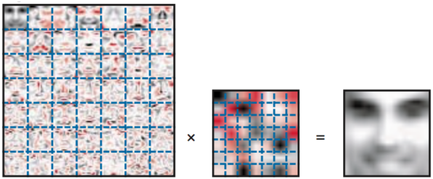
1: PCA 

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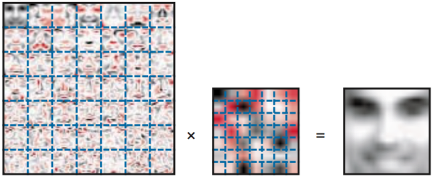
1: PCA  "eigenfaces"

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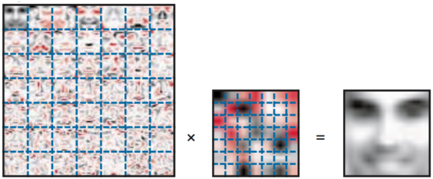
1: PCA  “eigenfaces”

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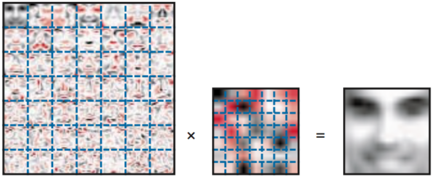

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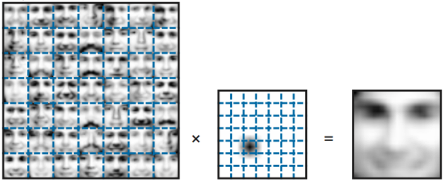

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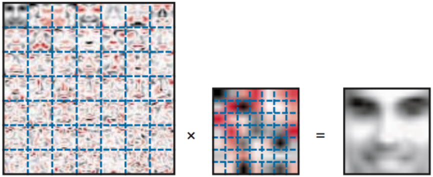

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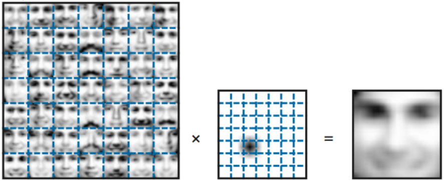

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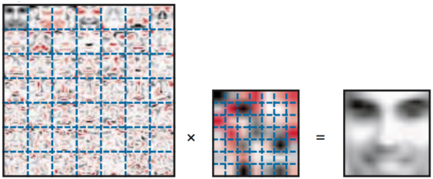

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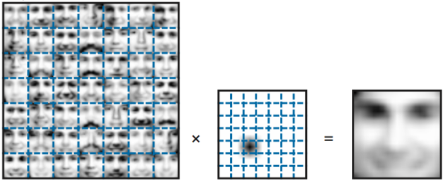

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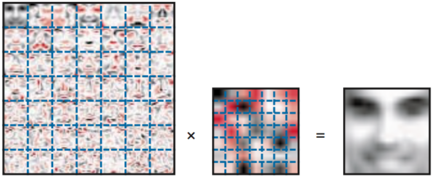

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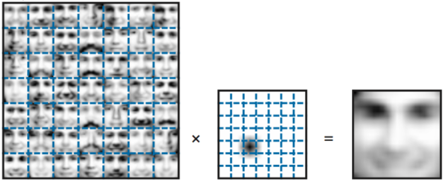

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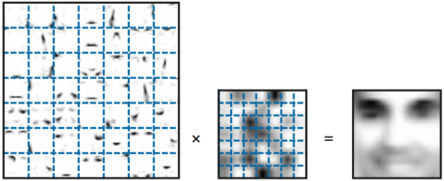

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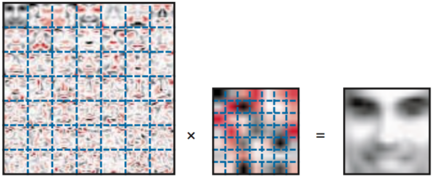
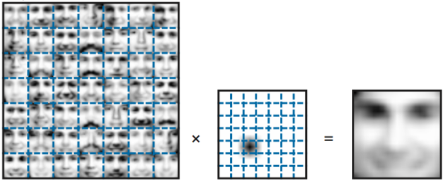
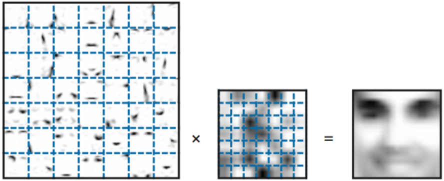
1: PCA  "eigenfaces"

2: VQ  "protofaces"

3: NMF  parts-based

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which is what our brain does when recognizing images!