Data Representation as Low Rank Matrix Factorization

Ziv Epstein
ziv.epstein@pomona.edu

Pomona College
Advised by Blake Hunter

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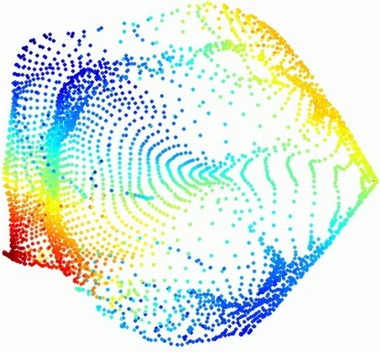
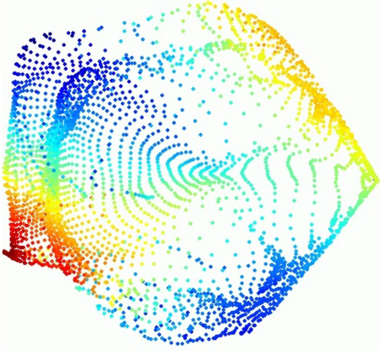
Points in $\mathbb{R}^m$

In many contexts in data science and linear algebra, we have lots of points in a high-dimensional space.



Denote these $x_i \in \mathbb{R}^m$ for $i = 1 \ldots n$.

How do we cluster x_i?



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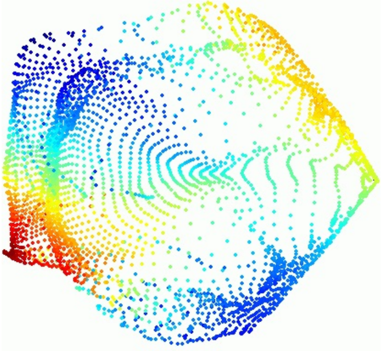
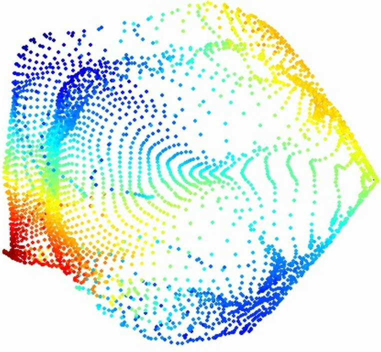
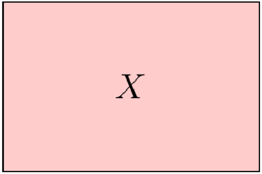


How do we cluster x_i? How do we perform dimensionality reduction? How do we visualize them?



Representation and Factorization

We can *represent* $X \in \mathbb{R}^{n \times m}$.



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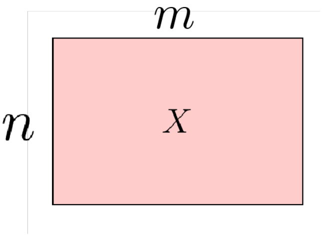

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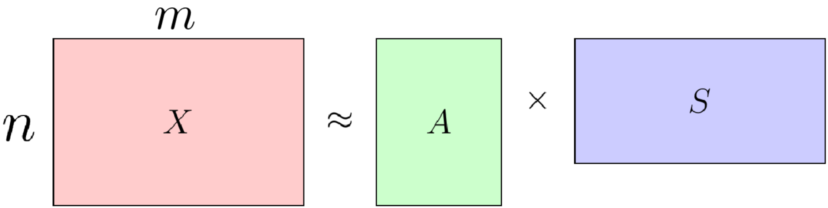

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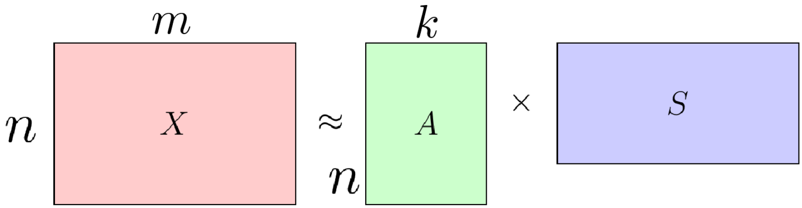

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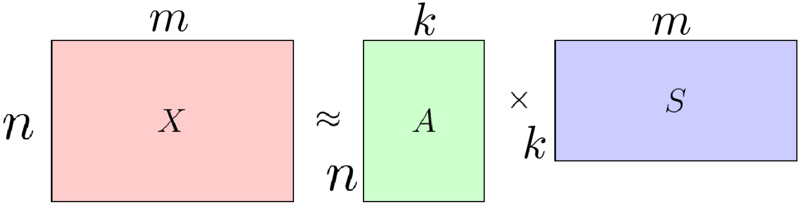

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$$x_i = \sum_{j=1}^{k} A_{i,j} S_{j,\_}$$

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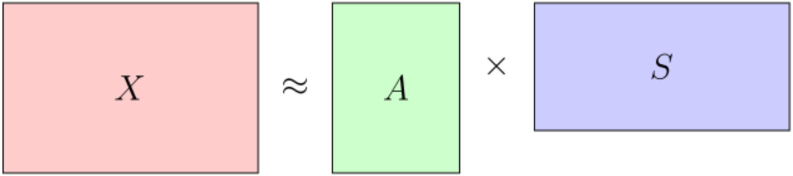

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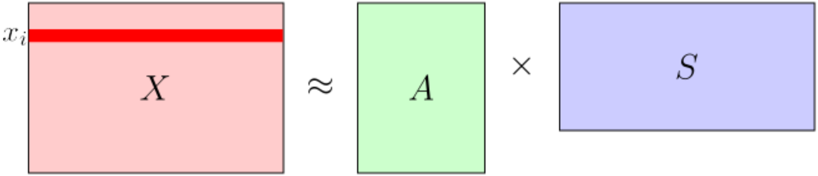

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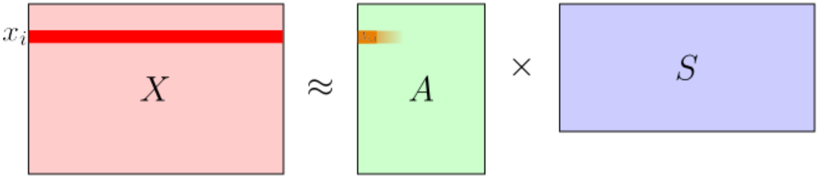

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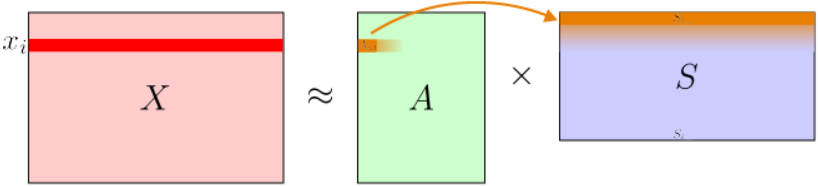

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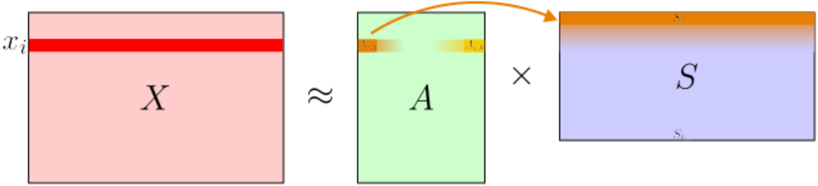

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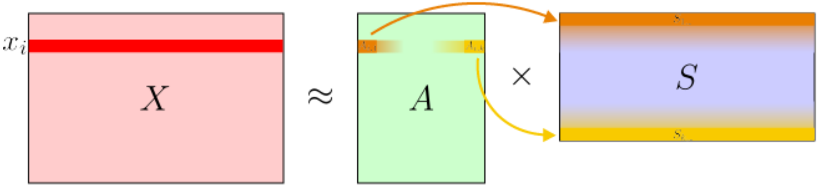

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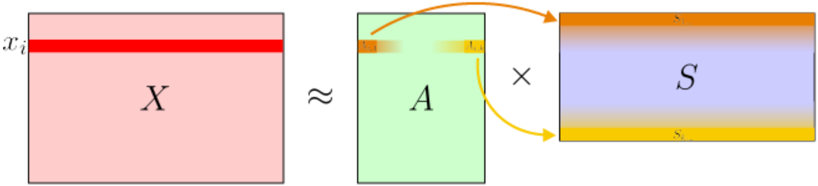

Find factorization by solving optimization problem

$$\min_{A,S} ||X - AS||$$

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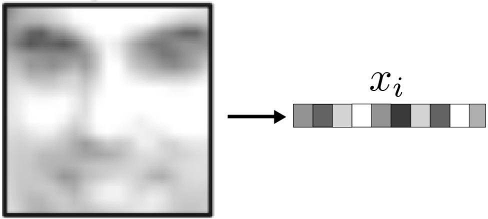

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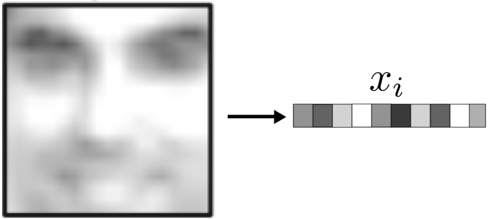

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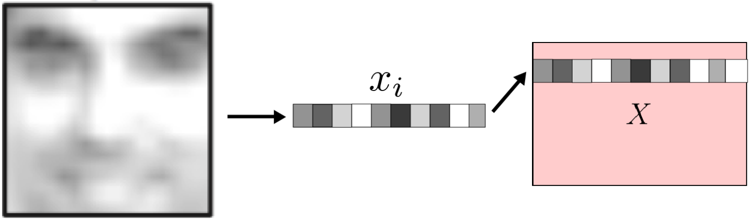

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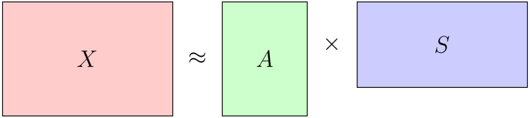

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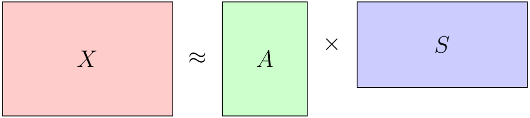

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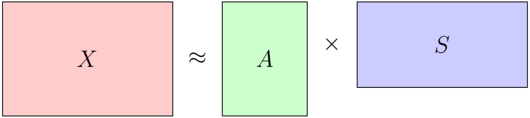

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For each, find A and S subject to constraints that minimize

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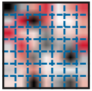
1: PCA  = 

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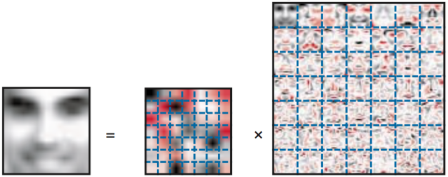
1: PCA 

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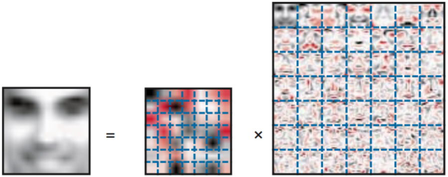
1: PCA  "eigenfaces"

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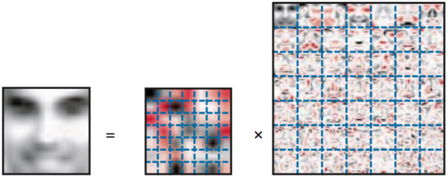
1: PCA  “eigenfaces”

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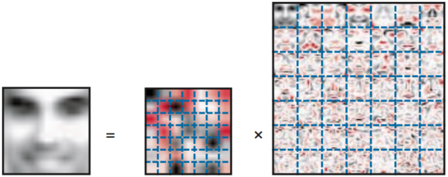
1: PCA  "eigenfaces"

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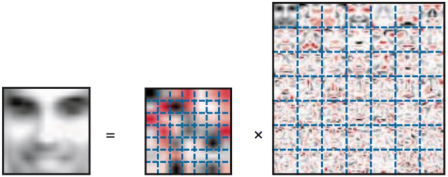
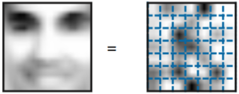
1: PCA  “eigenfaces”

1: NMF 

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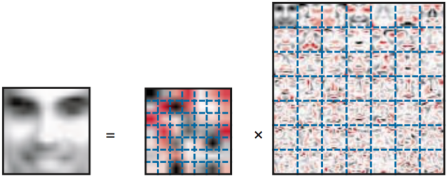
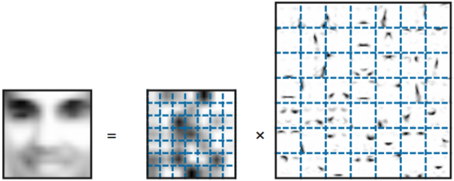
1: PCA  "eigenfaces"

1: NMF 

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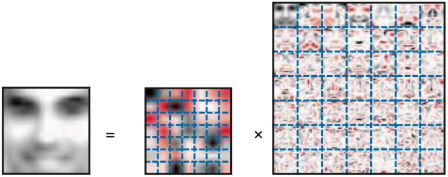
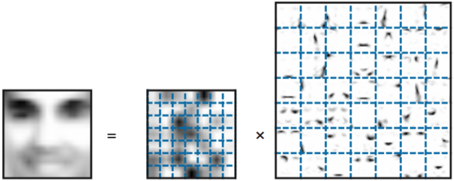
1: PCA  “eigenfaces”

1: NMF  parts based

NMF learns an additive, parts based model

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which is what our brain does when recognizing images!

Recognition-by-parts

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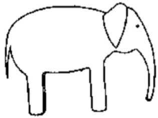
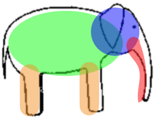
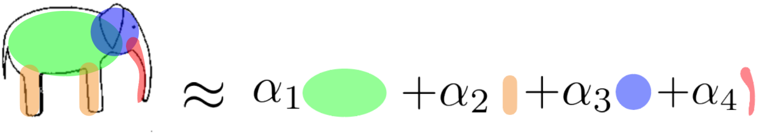
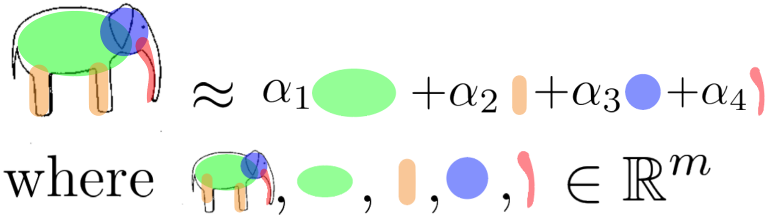
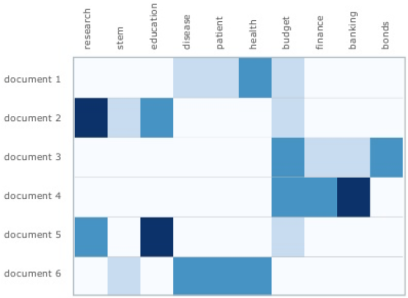


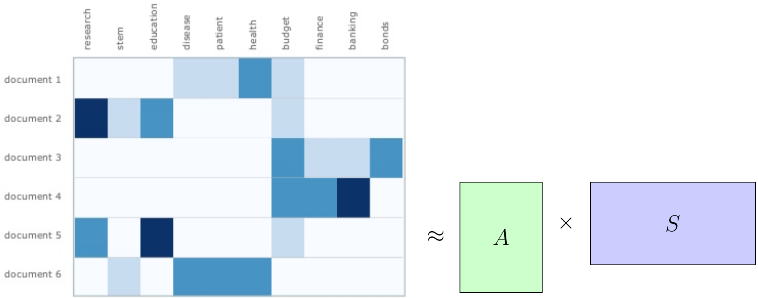



≈ α1 +α2 +α3 +α4


$\approx \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
where , , , , $\in \mathbb{R}^m$

Topic Modelling





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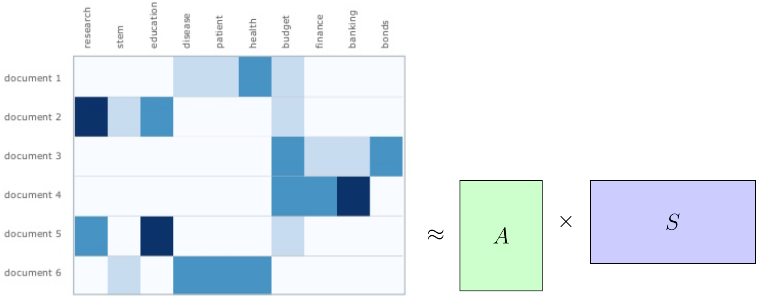

Perform with corpus of news documents with $k = 10$

Topic Modelling

topic 1	topic 2	topic 3	topic 4	topic 5	topic 6	topic 7	topic 8	topic 9	topic 10
reds	twins	jays	report	banks	china	percent	watt	truth	invasion
cincinnati	minnesota	blue	hhs	loans	economic	revenue	freddie	opinions	allied
pirates	runs	toronto	exchanges	collateral	reforms	quarter	mae	reason	troops
hit	game	hit	consumers	abs	growth	billion	fannie	certain	normandy
season	said	game	plans	ecb	beijing	million	mac	nature	german
cueto	innings	said	cost	bank	economy	company	senate	objects	british
latos	inning	gibbons	health	lending	said	share	republican	TRUE	germans
pittsburgh	indians	run	costs	small	li	cents	nomination	men	landing
game	run	bautista	healthcare	european	year	year	panel	god	0
bruce	plouffe	rockies	premiums	assets	urged	sales	committee	thought	beaches
run	hit	davis	insurance	loan	fiscal	shares	obama	mind	france
left	sox	right	individual	businesses	ministry	rose	sec	order	eisenhower
inning	left	runs	reform	euro	speed	said	fhfa	knowledge	divisions
arizona	white	single	states	funds	spending	earnings	nominee	thoughts	beach
innings	tigers	reyes	lower	backed	premier	trading	government	doubt	allies
games	dozier	rays	affordable	credit	local	ebay	democrat	sciences	operation
got	cleveland	johnson	month	smes	policy	sandisk	carolina	ought	june
said	hits	left	silver	firms	sources	profit	demarco	life	1944
second	detroit	kawasaki	lowest	rbs	banks	stock	housing	principles	day
homer	season	walked	administratio	sme	government	fell	likely	hear	forces

One final application

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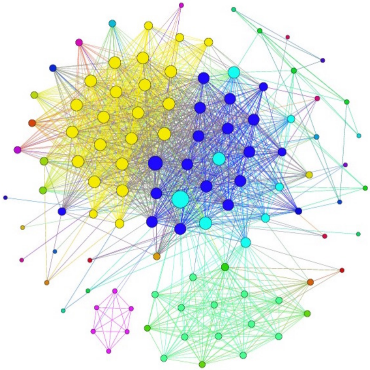
Graph clustering!



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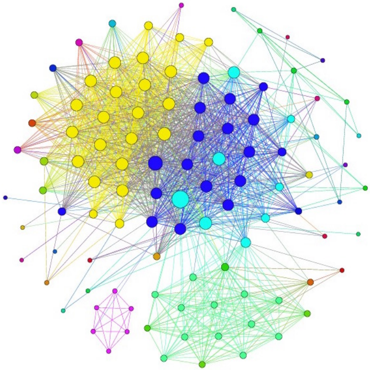

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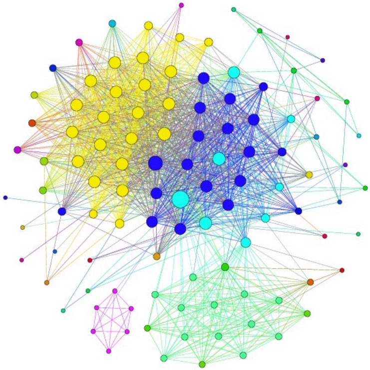

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$$\text{connectivity}(G, C) = \frac{\#\{(i,j) \in E | (i,j) \in C\} + \#\{(i,j) \notin E | (i,j) \notin C\}}{|V|^2}$$

The clustering C can be formulated as a membership matrix $M \in \{0,1\}^{n \times k}$ with $M_{ir} = 1$ if $i \in C_r$ and M_{ir} if $i \notin C_r$.

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Graph Clustering

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&= 1 - \frac{||X - MM^T||_F^2}{|V|^2}
\end{aligned}$$

Finding a clustering C that maximizes the connectivity is equivalent to the minimization problem

$$\cancel{\min_{M \in \{0,1\}^{n \times k}} ||X - MM^T||_F^2}$$

$$\min_{M \in \mathbb{R}_+^{n \times k}} ||X - MM^T||_F^2$$

The clustering C can be formulated as a membership matrix $M \in \{0,1\}^{n\times k}$ with $M_{ir} = 1$ if $i \in C_r$ and M_{ir} if $i \notin C_r$.

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where M is a weak membership matrix. That is, the larger $M_{i,j}$, the stronger membership of vertex i in cluster j.

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Thanks!