Visualizing Scissors Congruence

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Abstract

Consider two simple polygons with equal area. The Wallace–Bolyai–Gerwein theorem states that these polygons are scissors congruent: they can be dissected into finitely many congruent polygonal pieces. We present an interactive application that visualizes the constructive proof of the WBG theorem.

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1 Introduction

At the dawn of the 19th century, William Wallace [1] posed the following question:

> Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such, that those which constitute the one figure are respectively identical with those which constitute the other?

This sparked an active area of research, which culminated in several mathematicians, Wallace [1], Farkas Bolyai [2] and Paul Gerwein [3], each independently discovering the following theorem.

► **Theorem 1** (Wallace–Bolyai–Gerwein). *Any two simple polygons of equal area are scissors congruent – i..e they can be dissected into a finite number of congruent polygonal pieces.*

David Hilbert himself recognized the importance of this theorem, as he included it as "Theorem 30" in his *The Foundatations of Geometry* [4]. Furthermore, he included the three-dimensional generalization of Wallace's question as number three of his famous 23 problems: Given any two polyhedra of equal volume, can they be dissected into finitely many congruent tetrahedra? [5] This problem was solved by Hilbert's own student Max Dehn, who showed that unlike in the 2D case, the answer is "no."

The beauty of the original proof is that it is constructive: it describes an actual algorithm for constructing the polygonal pieces. To gain a deeper appreciation for this result, we built a interactive application that visualizes the algorithm in an intuitive and didactic manner.

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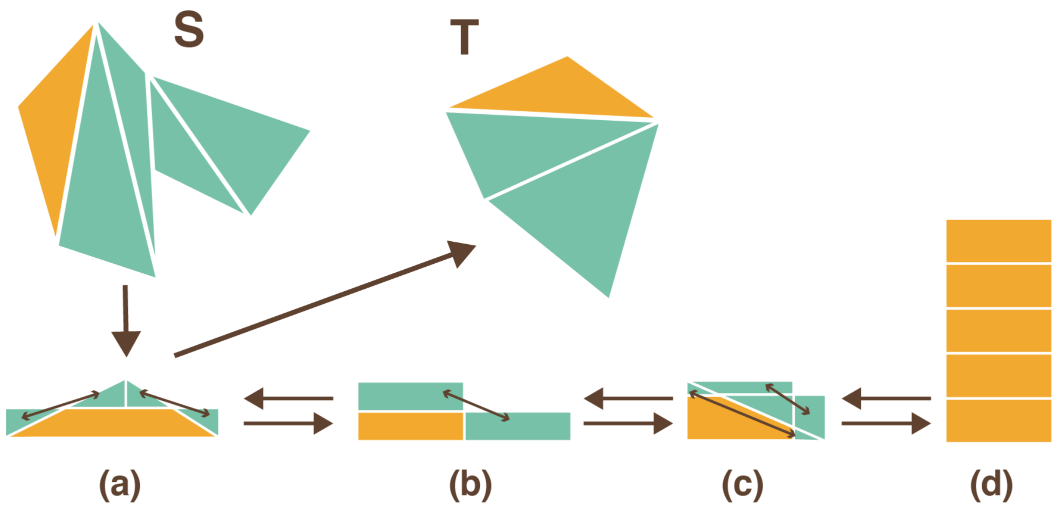

Figure 1 Visual repersentation of the algorithm. The right arrows show the procedure for converting polygon S to a stack and the left arrows show the reverse direction of converting the stack to polygon T. (a) shows the rigid transformation of the triangle to a rectangle, (b) shows the stacking procedure necessary for step three of the algorithm, (c) shows the equidecomposition of a rectnagle to another rectangle of fixed width and (d) represents the stack.

2 Algorithm

Indeed, the original proof demonstrates that any two simple polygons of equal area are "scissors congruent." We extend this notion to a slightly different result. We restate the theorem in a different manner that is more suited for visualization.

▶ **Corollary 2.** *Given two simple polygons of equal area S and T, there exists a finite sequence of cuts, rigid transformations that when applied to S result in T.*

This restatement of the theorem motivates a constructive proof that can be formulated with an algorithm to rigidly transform S to T:

1. Compute some triangulation S.
2. For each triangle in the trangulation, equidecompose that triangle into a rectangle.
3. For each rectangle generated above, equidecompose that rectangle into a rectangle of some fixed width w.
4. Stack all fixed width rectangles generated above into a single rectangle.
5. Perform the above steps in reverse order to equidecompose the rectangle into some triangulation of T.

Figure 1 shows the visual interpretation of each of these geometric procedures. For a more in-depth description and analysis of the procedure outlined above, see Kavanagh 2015.

3 Implementation

The visualization application is a client-side HTML5/JavaScript application run purely through the browser and can be found at `http://dmsm.github.io/scissors-congruence/`.

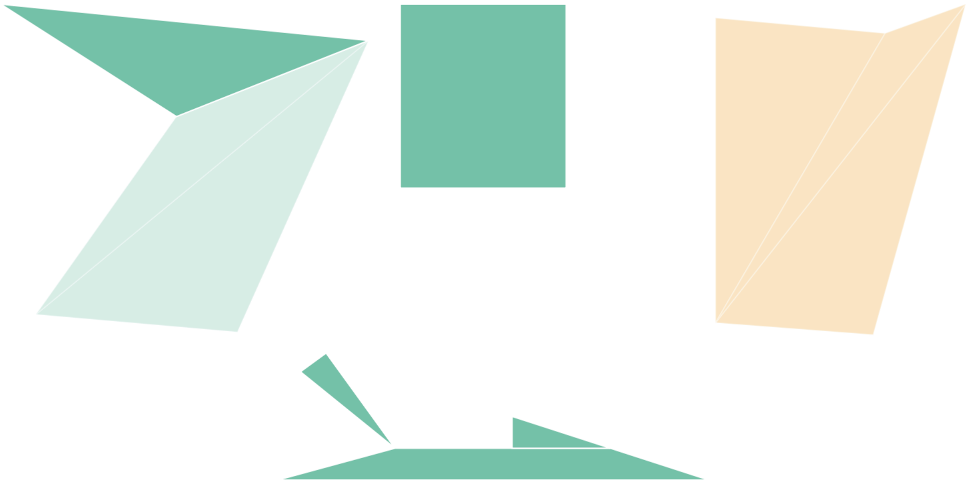

Figure 2 Screenshot of the application in step 2 of the algorithm: a triangle of the triangulation of the intial polygon (in sea foam green) is being equidecomposed into a rectangle and will eventually be stacked (in the middle) on its way to forming the triangulation of the terminal polygon (in apricot orange).

The interface allows the user to input their own intial and terminal polygons, and then the implementation of the above algorithm will rigidly transform their initial polygon to their terminal polygon (see Figure 2). The application takes advantage of the JavaScript libraries jQuery, Two.js, PolyK.js and Math.js in order to render and manipulate the polygons in a fast and modular way.

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