CS 101 Fall 2023 - Quiz 9

12/18/2023 - 20 Minutes **Name:** **ID number:**

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1. (2 points) Notes of discussion

I promise that I will complete this QUIZ independently and will not use any electronic products or paper-based materials during the QUIZ, nor will I communicate with other students during this QUIZ.

True or False: I have read and understood the notes. $\checkmark$ True ◯ False

2. (8 points) True or False

Determine whether the following statements are true or false.

(a)	(b)	(c)	(d)
F	T	F	T

(a) (2') A DAG has a unique topological sorting if it has only one source.

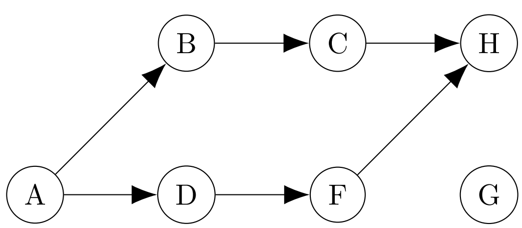
(b) (2') Topological sort can be modified to detect whether a directed graph has a cycle in $O(|V|+|E|)$ time.

(c) (2') A directed graph with n vertices and $n-1$ edges must have a topological sorting.

(d) (2') Greedy algorithms select the locally optimal choice at each step, hoping that these local choices will lead to a global optimum.

3. (5 points) Topological sort counting

Consider the graph below.



(a) (5') How many topological sort results are there for the graph (show your calculate process)?

Solution: $\frac{4!}{2!\times 2!} \times 7 = 42$.
Consider the left side's sub-graph and G separately. $\frac{4!}{2!\times 2!}$ is the number of all permutations of $\{B,C,D,F\}$, 7 is the number of choices where G can be placed.

4. (10 points) Coin changing

Recall the greedy algorithm we use solving the coin changing problem. Suppose we need to make change for n, and the available coins are in the denominations that are powers of c, i.e., the denominations are $c^0, c^1, \ldots, c^k$ for some integers $c > 1$ and $k \geq 1$. Show that the greedy algorithm always yields an optimal solution.

Solution: The greedy algorithm always uses one coin of the largest denomination such that $c^t \leq n$ first, and then solve the sub-problem to make change for $n - c^t$. Then we will show that the optimal solution always uses one coin c^t.

Denote x_i as the number of coins c^i used in the optimal solution, then $n = \sum_{i=0}^{t} x_i c^i$, and we will prove that $x_t > 0$.

Note that $\forall i \in [0, t-1], x_i \leq c - 1$ in the optimal solution, because if at least c coins of c^i are used, then we can replace them with one c^{i+1} coin to get a better solution. Then

$$
\begin{aligned}
x_t c^t &= n - \sum_{i=0}^{t-1} x_i c^i \\
&\geq n - (c-1) \sum_{i=0}^{t-1} c^i \\
&= n - (c-1) \frac{c^t - 1}{c - 1} \\
&= n - c^t + 1 \\
x_t c^t + c^t &\geq n + 1 \\
x_t + 1 &\geq \frac{n+1}{c^t} > \frac{n}{c^t} \geq 1 \\
x_t &> 0
\end{aligned}
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