CS 101 Fall 2023 - Quiz 7

12/11/2023 - 25 Minutes **Name:** **ID number:**

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1. (6 points) True or False

Note: You should write down your answers in the box below.

(a)	(b)	(c)	(d)	(e)	(f)
F	T	T	F	F	T

(a) (1') A directed graph with n vertices has $n-1$ edges could be strongly connected. ($n > 1$)

(b) (1') A union tree generated by a union-by-height DSU of height h has $\Omega(2^h)$ nodes.

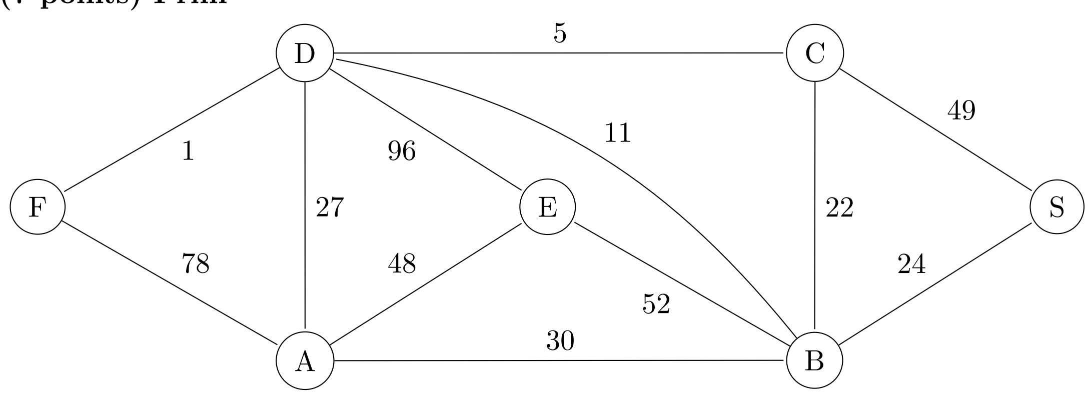
(c) (1') If we use BFS(breadth-first search) to find a path from u to v on an unweighted graph, this path will be the shortest path between u and v.

(d) (1') If we consider the disjoint set time complexity, the time complexity of Kruskal's algorithm is $O(|E|\alpha(|V|)\log|V|)$.

(e) (1') The maximum weight edge in a graph will not be in any MST.

(f) (1') Let G be a weighted undirected graph with positive weights where edge e has weight w_e for all $e \in E$. And G' is a copy of G except that every edge e has weight w_e^2. Then any MST in G is also a MST in G'.

2. (7 points) Prim



(a) (3') Use Prim's algorithm to find the Minimal Spanning Tree of the graph. You should select S as the root node. Write the visit order of all nodes.

Solution: S→B→D→F→C→A→E

(b) (4') Let's modify the weights of some edges. Please give out the maximum and minimum weight of the edges given that won't change the MST. (You can write $+\infty$ or $-\infty$ if there is no maximum or minimum weight. We choose nodes by lexicographical order if we come across the same weight.)

- AD: Maximum: 30 Minimum: $-\infty$
- DE: Maximum: $+\infty$ Minimum: 48

3. (2 points) Fill in the blanks

(a) (2') Assume we have an undirected graph $G = (V, E)$ with 2 connected components. If $|V| = 2n$, the maximum of $|E|$ is $\underline{(2n-1)(n-1)}$ and the minimum of $|E|$ is $\underline{\quad 2n-2 \quad}$.

4. (7 points) kTrees

Given a complete undirected graph K_n on n vertices, every pair of distinct vertices i, j is connected with a **positive** weight w_{ij}.

The first k vertices $v_1, \cdots, v_k$ are chosen to serve as roots of k trees. You need to put other $n-k$ vertices $v_{k+1}, \cdots, v_n$ to form k trees $\{T_1, \cdots, T_k\}$, where each T_i is rooted at v_i.

The cost is defined as $\sum_{i=1}^{k} W(T_i)$, where $W(T_i)$ is the sum of the weights of the edges in T_i. You need to design an efficient algorithm to find a solution of minimum cost.

(a) (4') Describe an algorithm to find a solution of minimum cost, you don't need to prove its correctness in this question.

Solution:

1. Define a new vertex s.
2. Connect s and v_i for $1 \leq i \leq k$, where the weight is 0.
3. Run Kruskal's algorithm on the new graph $G' = (V + s, E')$ (or run Prim starting at s), which is an alternative solution of minimum cost.

(b) (3') Analyze the time complexity of your algorithm. You **need** to use $k = O(n)$ to simplify your answer to $O(f(n))$.

Solution:

- New vertex: $\Theta(1)$.
- Connect k edges: $O(n)$.
- MST: $|E| = n(n-1)/2 + k$, $O(n^2 \log n)$

Therefore, the time complexity is $O(n^2 \log n)$.