刘兴沛

The following discussions can be divided into two parts:

I) $\Lambda = \mathbb{Z}$ $\longrightarrow$ Euler class, Euler number,
Poincaré-Hopf thm etc.

II) $\Lambda = \mathbb{Z}/2$ $\longrightarrow$ Stiefel Whitney classes

PART I : $\Lambda = \mathbb{Z}$

$E^k \longrightarrow M$ with M closed C^∞-mfd, E orientable and oriented.

$$\begin{tikzcd}
u \in H^k(E, E-M) \arrow[r] & H^k(E) \ni u' \arrow[d, "\cong"', "i^*"] \\
H^0(M) \arrow[u, "\phi", "\cong"'] \arrow[r, "\cup e"] & H^k(M) \ni e \\
1 \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}$$

$e(E)$ is defined to be the Euler class of $E^k \to M$ w.r.t. the chosen orientation.

The very interesting case is when M^n is closed oriented C^∞-mfd where E^n is chosen to be TM, the tangent bundle of M. In this case, a natural question arises:

$\boxed{e(TM)([M]) = \,?\ \text{where } [M] \text{ is the fundamental class w.r.t. the orientation used to define } e(TM)}$

To answer this question, we need the following observation which is quite intuitive:

Prop: $E^k \longrightarrow M^n$ an oriented bundle over oriented closed M.
Then $u \in H^k_c(E)$ is the Poincaré duality of $[M]$ in E,
i.e. $\mu_M \frown u = [M] \in H_n(E)$, where $\mu_M \in H_{n+k}(E, E-M)$ is the fundamental class w.r.t. the orientation on E (as mfd) determined by those of $E \to M$ and M.

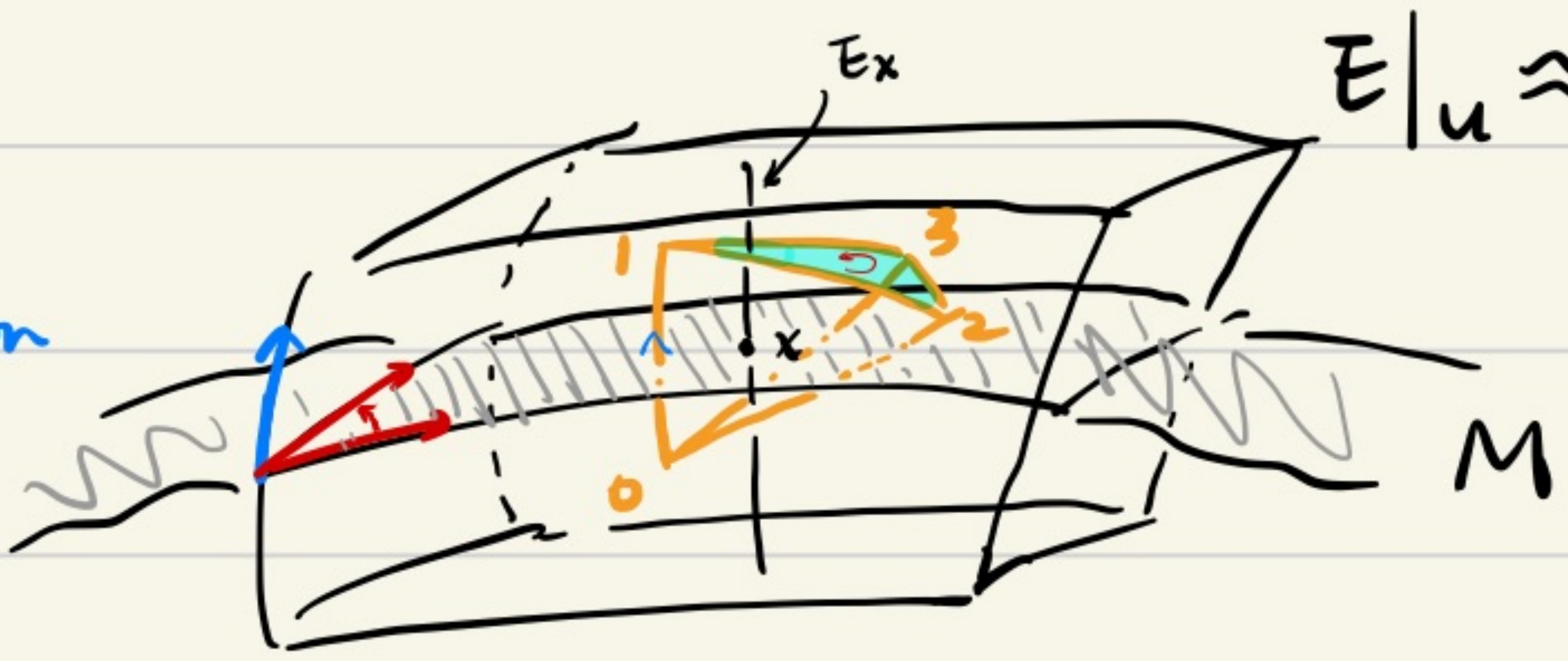
Idea of pf:

It is sufficient to prove that $p_*(\mu_M \frown u) = [M] \in H_n(M)$

$\Longleftarrow$ $p_*(\mu_M \frown u) \in H_n(M, M-\{x\}) \longleftarrow H_n(M)$ equals the prescribed orientation on M. Intuitively, this can be seen from the def of Thom class (remember how can we understand it as a general function on E!), cap product and the following picture:

$E|_U \approx U \times \mathbb{R}^k$ (in this picture, $k=1$).

- fiber orientation
- base orientation
- total bundle orientation



To be more precise, we can argue in this way:

let $\varphi \in H^n(M, M-\{x\})$ be the δ-func dual to the orientation on M. Then $p^*(\varphi) \in H^n(E, E-E_x)$.

$\Rightarrow u \smile p^*(\varphi) \in H^n(E, E-\{x\})$. Let $[\alpha] \in H_{n+k}(E, E-\{x\})$ be the bundle orientation at x, represented by an emebbed standard simplex α (see the simplex in the last picture colored in •). Then:

$$u \smile p^*(\varphi)(\mu_M) = u \smile p^*(\varphi)(\mu_x) = p^*(\varphi)(\mu_x \frown u) = 1$$

$$\begin{aligned} &\parallel \\ &p^*(\varphi)(\mu_M \frown u) = \varphi(p_*(\mu_M \frown u)) \end{aligned}$$

$\Rightarrow p_*(\mu_M \frown u) = [M]$ in $H_n(M) \Rightarrow \mu_M \frown u = [M]$ in $H_n(E)$.

$\blacksquare$

We can now answer our question:

$e(TM)([M]) = ?$ where $[M]$ is the fundamental class w.r.t. the orientation used to define $e(TM)$.

Prop: $E^k \longrightarrow M^n$ a smooth oriented bundle over oriented closed C^∞-mfd M, with $k \leq n$. Suppose $M \xrightarrow{\iota} E$ a smooth section such that $\iota \pitchfork E_0$ ($E_0 \cong M$ the zero section), then $e(E) = \iota^*(u')$. Additionally, it is dual to the fundamental class $[\iota^{-1}(E_0)]$ in M, where the orientation of $\iota^{-1}(E_0)$ is induced by that of M, $E \to M$ and ι.

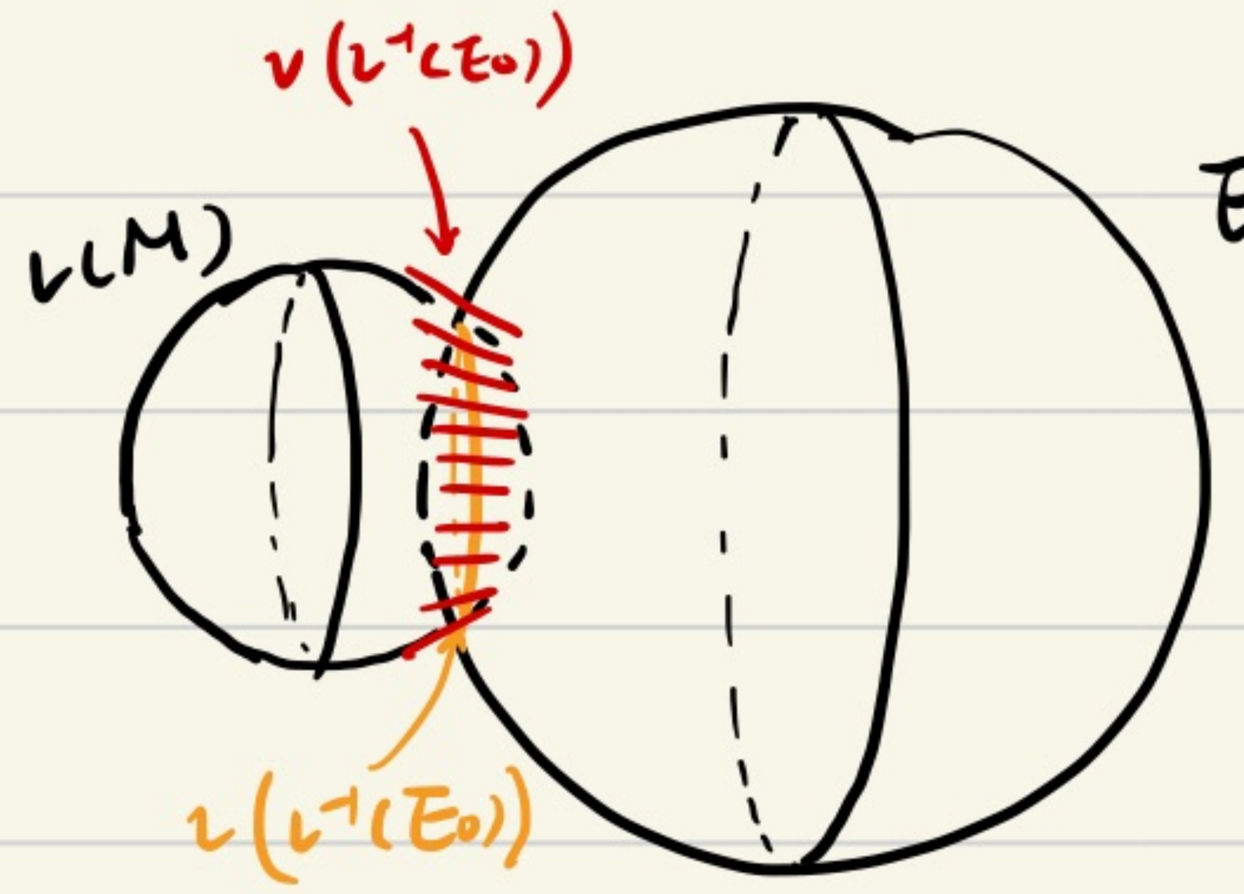
Rmk:

Such ι always exists, which can be proved by Sard's thm.

Pf of Prop:

Since $\iota \simeq_h i_0$ ($i_0 : M \xrightarrow{\approx} E_0 \hookrightarrow E$), it is clear that $\iota^*(u') = i_0^*(u') = e(E) \in H^k(M)$. On the other hand, the transversality $\iota \pitchfork E_0$ induces the following pull-back diagram:

$$\begin{tikzcd} M & \nu(\iota^{-1}(E_0)) \arrow[l, hook'] \arrow[r, "\tilde{\iota}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ & \iota^{-1}(E_0) \arrow[r, "\iota"] & M \approx E_0 \end{tikzcd}$$



and the orientation on $\iota^{-1}(E_0)$ is determined so that bundle map $\nu(\iota^{-1}(E_0)) \xrightarrow{\tilde{\iota}} E$ preserves orientation. It follows that $\tilde{\iota}^*(u)$ is the corresponding Thom class of $\nu(\iota^{-1}(E_0)) \to \iota^{-1}(E_0)$, and

$$\begin{array}{ccc} H^k(M, M - \iota^{-1}(E_0)) & \longrightarrow & H^k(M) \\ \Psi & & \Psi \\ \tilde{\iota}^*(u) & \longmapsto & \tilde{\iota}^*(u') = e(E) \end{array}$$

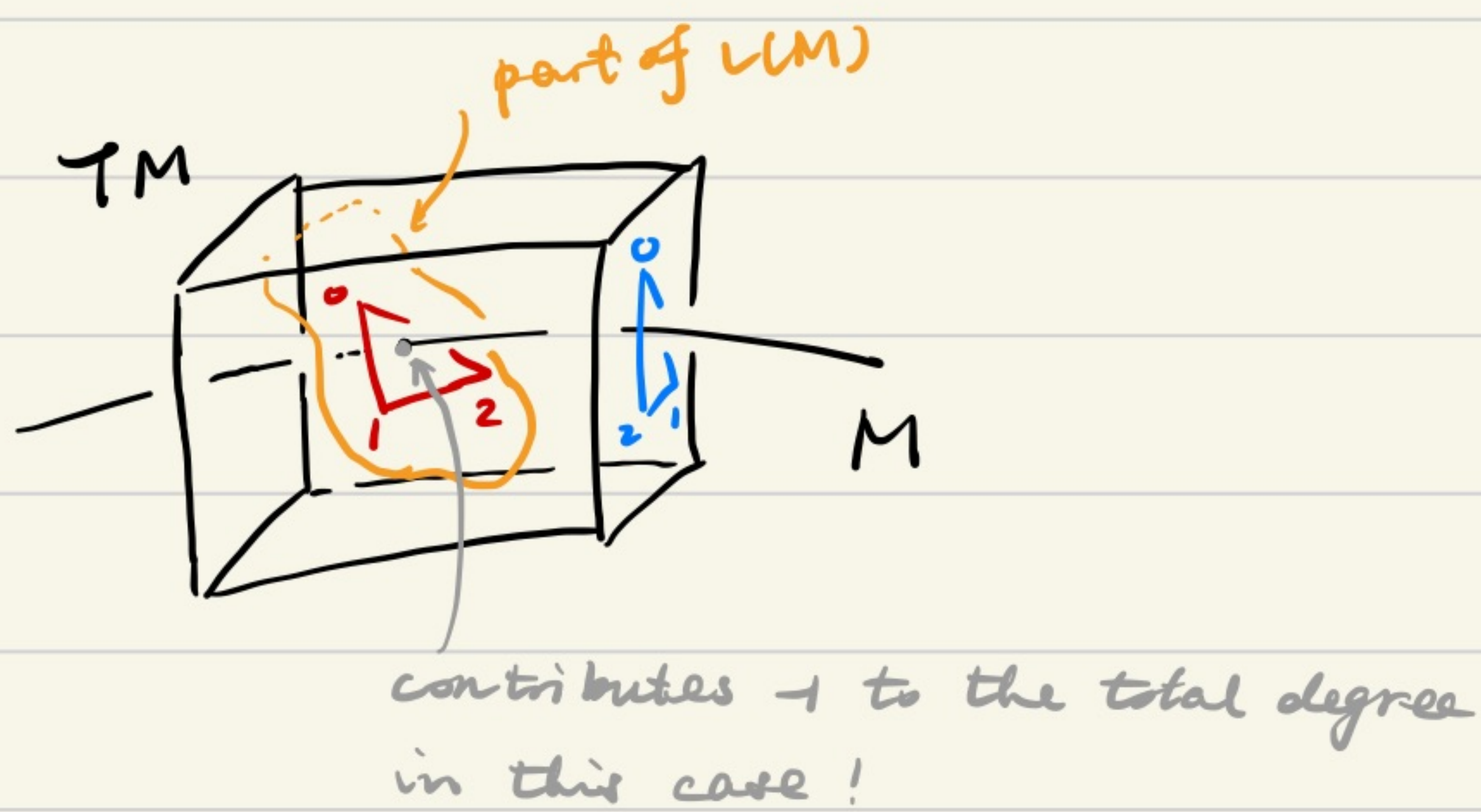
Therefore, by the previous prop, $e(E) = \tilde{\iota}^*(u)$ is the Poincaré dual of $[\iota^{-1}(E_0)]$ in M. ∎

We now apply the preceding proposition to TM to see what the $e(TM)([M])$ is :

<u>Prop</u>: Let $\iota : M \longrightarrow TM$ be a vector field which is transverse to the zero section. In other word, ι only has non-degenerate zero pts. Then $e(TM)([M])$ equals the total degrees of ι which, by Poincaré-Hopf thm is equal to the Euler number $\chi(M)$.

<u>Idea of proof</u>:

This follows immediately from the last prop above, or one can see the following picture to get the intuition :



∎

<u>Rmk</u> :

Poincaré-Hopf Thm has an elegant proof using Morse function, see Milnor's Morse Theory for reference.

There is another way to consider our original question.
Euler class determined by Thom class, and the latter appear naturally when a smooth embedding occurs. So if we can embedd M into some familiar exterior space such that the corresponding normal bundle is exactly the TM itself (up to a bundle isomorphism), then it might be possible to calculate $e(TM)([M])$ by the homology results of this outer space.
This idea does work when one regards the diagonal embedding $M \xrightarrow{\Delta} M \times M$, and then the Kunneth formula, Poincaré - duality and properties of slant product can help us calculate $e(TM)([M])$, see Milnor's C.C.

PART II : $\Lambda = \mathbb{Z}/2$. $E^k \longrightarrow M$.

Now Thom class can be defined on any vector bundle, orientable or not since we are using $\mathbb{Z}/2$ as coefficient. Similar to what we have done in PART I, we have diagram:

$$\begin{tikzcd}
u \ni H^k(E, E-M) \arrow[r] & H^k(E) \ni u' \arrow[d, "i^*"] \\
1 \ni H^0(M) \arrow[u] \arrow[r, "\smile w_k"] & H^k(M) \ni w_k(E^k)
\end{tikzcd}$$

2 Homotopy direct limit

Consider a topological space X and a sequence of $X_0 \subset X_1 \subset \cdots$ of subspaces. Let $X_\Sigma \triangleq X_0 \times [0,1] \cup X_1 \times [1,2] \cup X_2 \times [2,3] \cup \cdots$, endowed with the subspace topology of $X \times \mathbb{R}^1$.

DEF: We call X is the homotopy direct limit of $\{X_i\}_{i \geq 0}$ if the projection $p: X_\Sigma \to X$ defined by $p(x,t) = x$, is a homotopy equivalence.

Examples:

① Suppose that X is a paracompact space, and that each pt of X lies in the interior of some X_i. Using partition of unity, one can show that X is the homotopy direct limit of $\{X_i\}$.

② Let X be a CW complex, and let the X_i be subcomplexes with union X. Then X is the homotopy direct limit of $\{X_i\}$ by Whitehead thm.

③ Let $X = [0,1]$ and $X_i = \{0\} \cup [\frac{1}{i}, 1]$. Then X is not the homotopy direct limit of $\{X_i\}_{i \geq 1}$, for X_Σ is not path-conn.

Thm: Suppose X is the direct limit of $\{X_i\}$ and Y is the homotopy direct limit of $\{Y_i\}$. Let $f: X \to Y$ be a map which carries each X_i into Y_i by a homotopy equiv. Then f itself is a homotopy equiv.

Pf:

$$\begin{tikzcd} X_\Sigma \arrow[r, "\tilde f"] \arrow[d, "p", "\simeq"'] \arrow[dr, phantom, "\circlearrowright"] & Y_\Sigma \arrow[d, "p", "\simeq"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\Longrightarrow$ $\tilde f$ is a homotopy equiv iff f is a homotopy equiv.

Hence it is sufficient to prove that $\tilde f$ is a equiv under the assumptions. Recall a proposition proved in Hatcher's [AT]:

Prop 0.21:

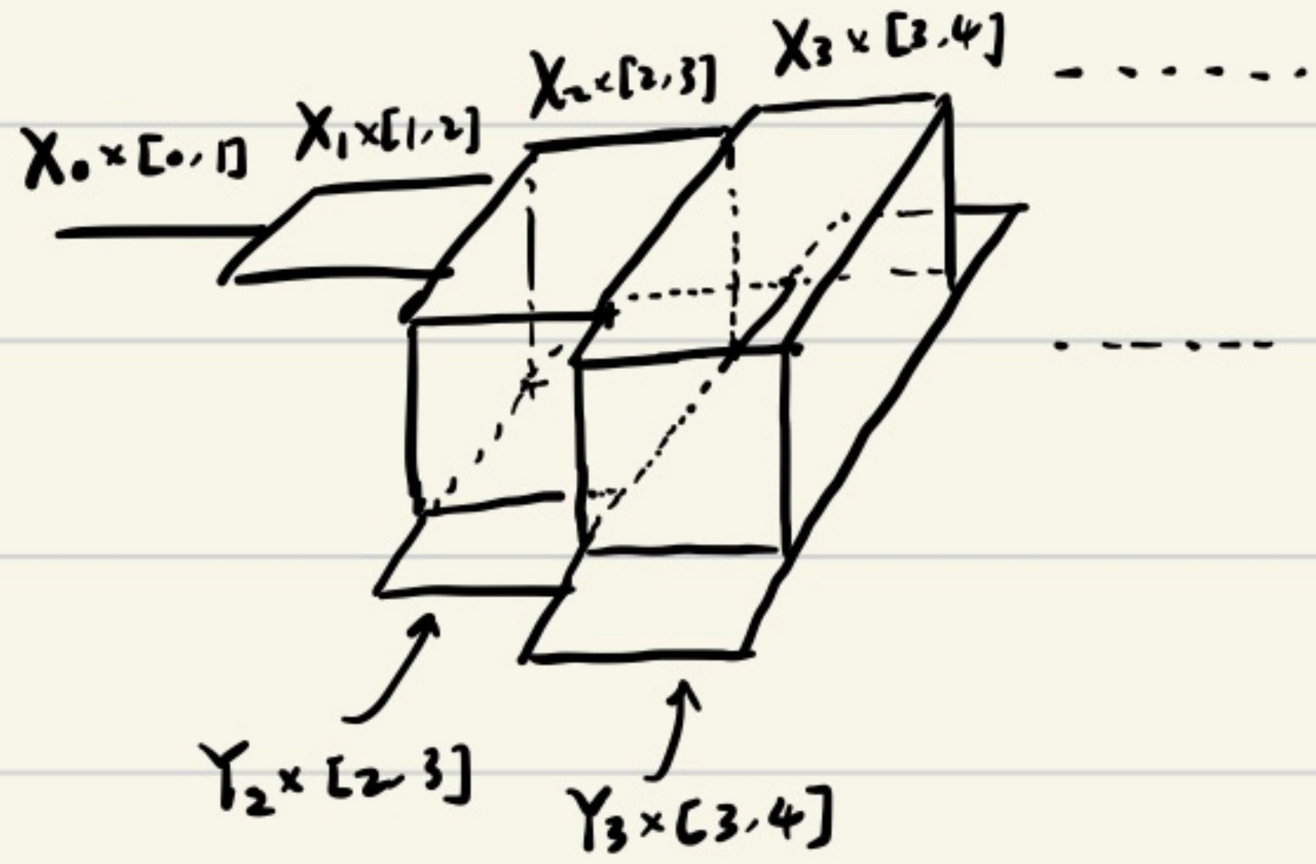
A map $f: X \to Y$ is a homotopy equiv iff X is a deformation retract of the mapping cylinder M_f. $\blacksquare$

So it remains to prove that $M_{\tilde f}$ can deformation retract to X_Σ, assuming that $M_{f_i \triangleq f|_X}$ can deformation retract to X_i via r_t^i.

Step 1: Let $\tilde f_i \triangleq \tilde f|_{X_i \times [i, i+1] \cup X_{i+1} [i+1, i+2] \cup \cdots}$, and $\hat M_i \triangleq M_{\tilde f_i} \cup X_\Sigma$

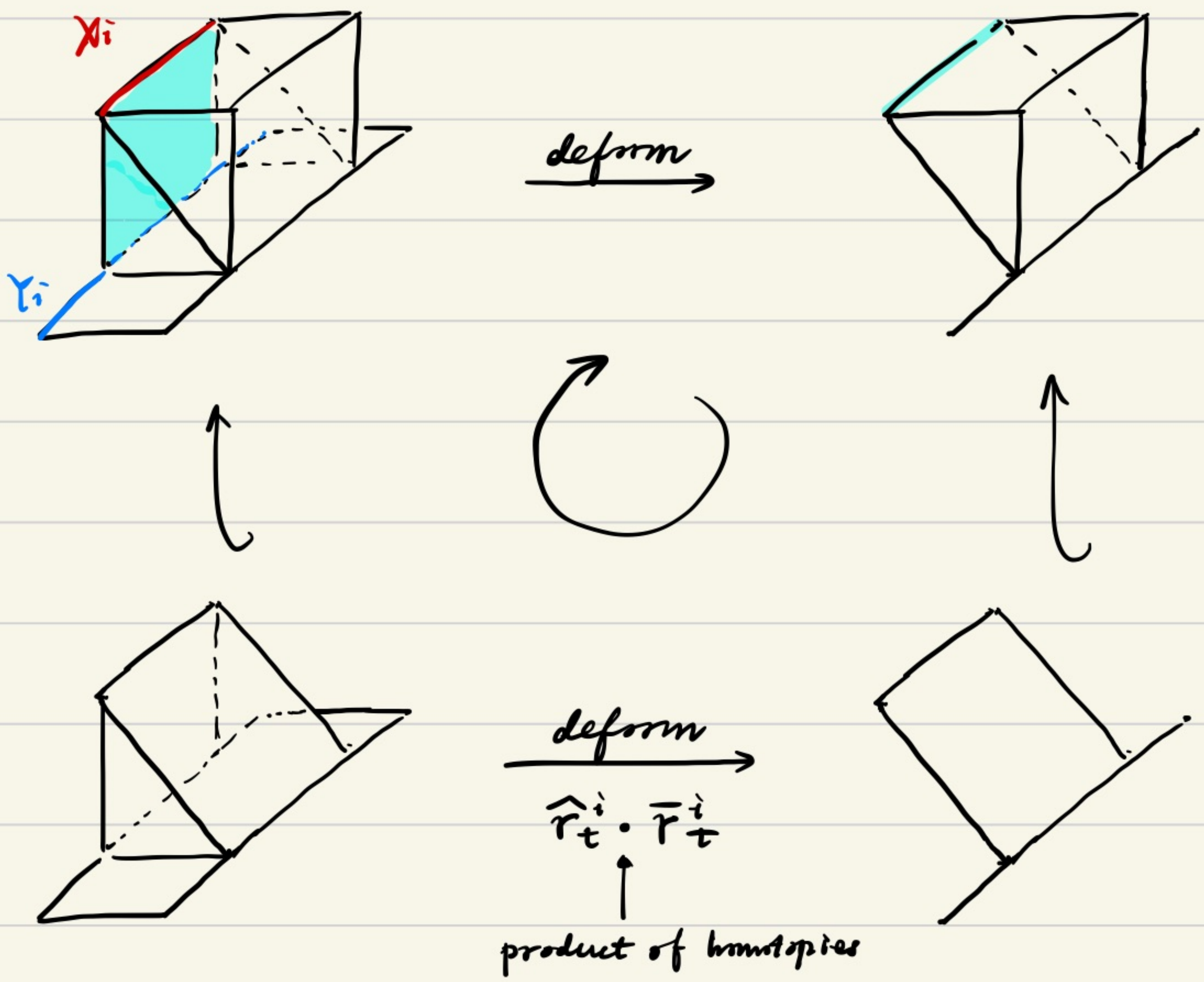
e.g.

$\widehat{M}_3 =$



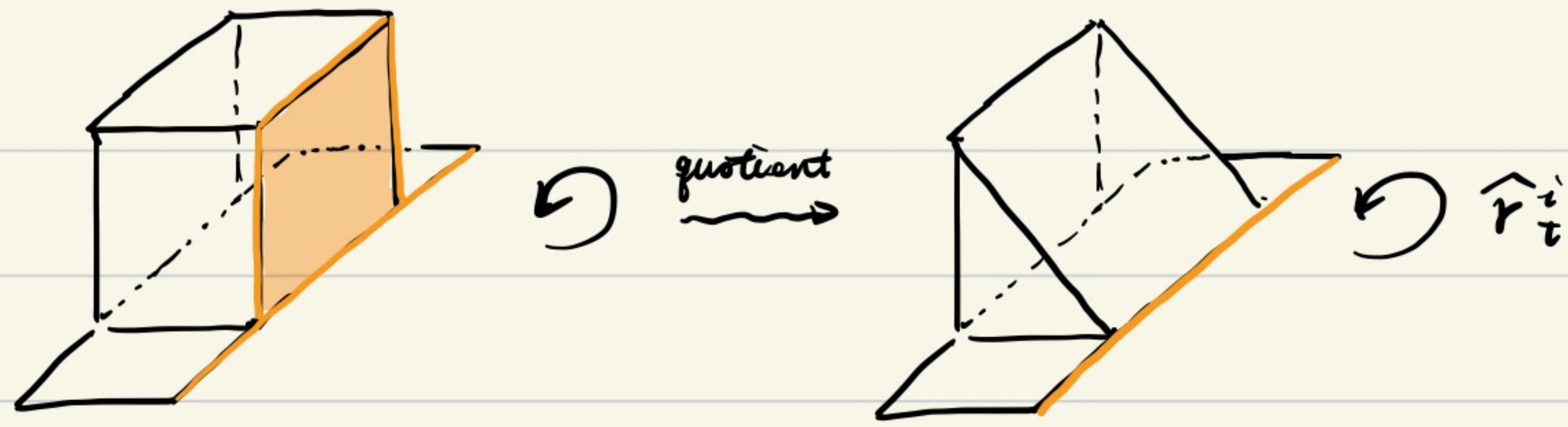
We claim that $\widehat{M}_i$ can deformation retract ~~to~~ $\widehat{M}_{i+1}$ such that $M_{f_i} \times \{i\}$ goes into $X_i \times \{i\}$:

a)



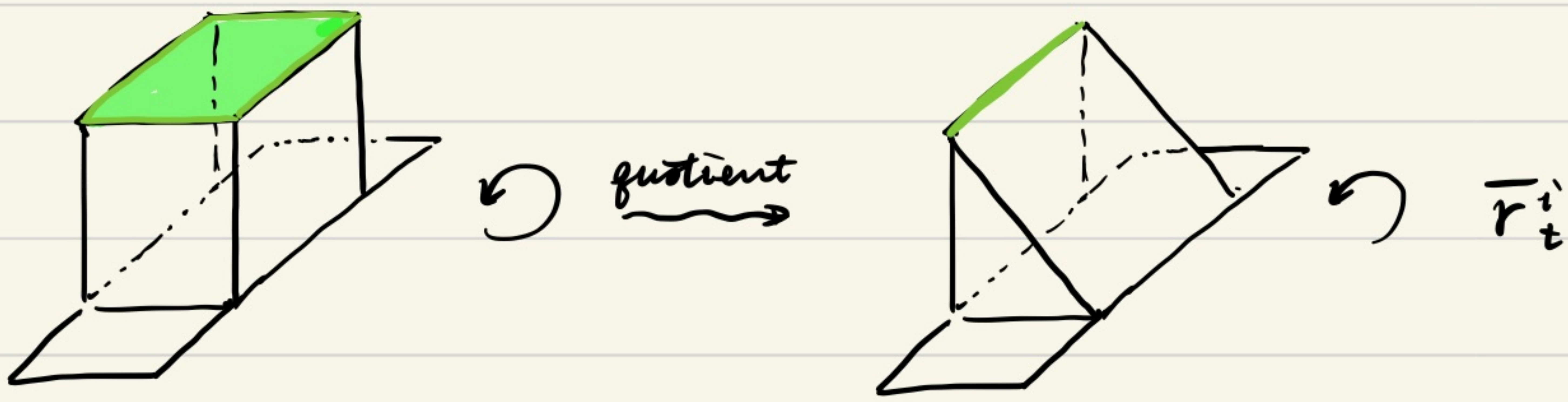
where $\hat{r}^i_t$ is induced by: $M_{f_i} \times [i, i+1] \longrightarrow M_{f_i} \times [i, i+1]$

$$(x, s) \longmapsto (r^i_{(i+1-s)t}, s)$$

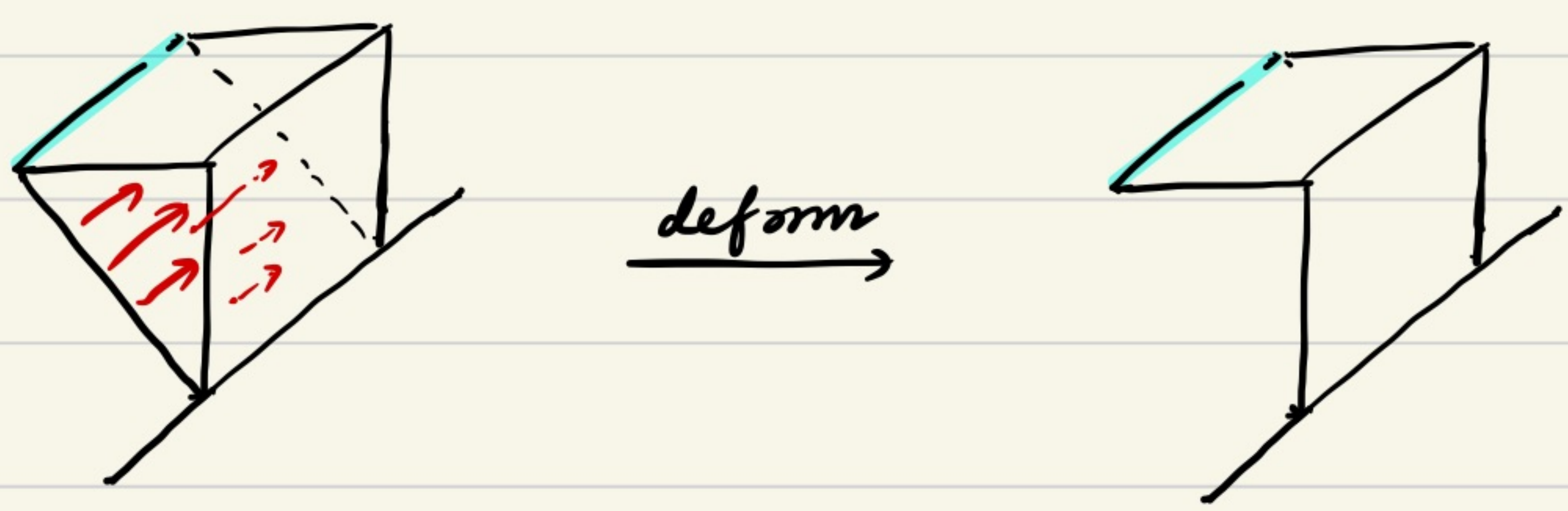


and $\bar{r}^i_t$ is induced by $M_{f_i} \times [i, i+1] \longrightarrow M_{f_i} \times [i, i+1]$

$$(x, s) \longmapsto (x, (1-t)s + t(i+1))$$



b)



$\Rightarrow$ The product of the deformation a) & b) gives us the expected deformation from $\widetilde{M}_i$ to $\widetilde{M}_{i+1}$, we denote it by $\tilde{r}^i_t$.

Step 2: To obtain a deformation from M_γ to X_Σ, we can conduct each $\hat{r}^i_t$ in time interval $[1-\frac{1}{2^i}, 1-\frac{1}{2^{i+1}}]$. This completes the proof. ▨