§ Morse Bott Theory

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I. Foundamental settings:

- (B, g) a closed dim $= n$ mfd (not necessarily ort)

- $B \xrightarrow{f} \mathbb{R}$ a Morse-Bott func

  Def:
  - △ Crit(f) consists of sub-mfd
  - △ For each crit mfd S, $\forall p \in S$, $\mathrm{Hess}(f, g)|_{\nu_p(S)}$ is non-degenerate

Notations: For S_i, the crit mfd with index i,

$\mathcal{U}_i \triangleq$ the unstable (descending) mfd of S_i

$\mathcal{S}_i \triangleq$ the stable mfd of S_i.

- Smale conditions

  Def:
  - ▵ For all i, j, $\mathcal{U}_i^s \pitchfork \mathcal{S}_j$
  - ▵ All S_i & $\nu^-(S_i)$ are oriented

In what follows, we always assume that

$B \xrightarrow{f} \mathbb{R}^1$ satisfies the above conds!

Def: $\mathcal{M}(S_i, S_j) \triangleq \mathcal{U}_i \cap S_j$

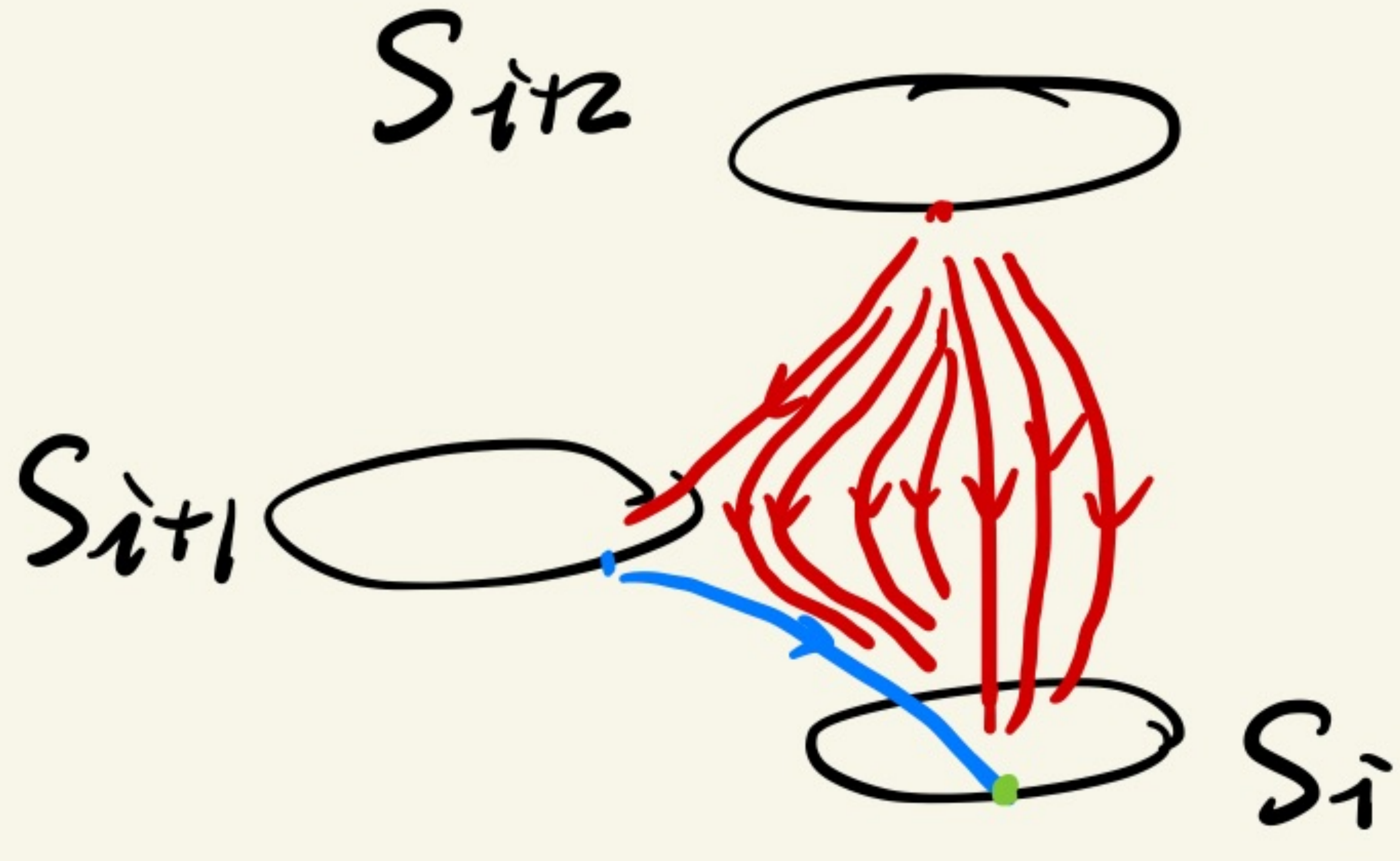
$\widetilde{\mathcal{M}}(S_i, S_j) \triangleq \mathcal{M}(S_i, S_j)/\mathbb{R}^1$

Rmk:

- $$\begin{array}{c} S_i \\ \uparrow u^i_j \\ \widetilde{\mathcal{M}}(S_i, S_j) \\ \downarrow l^i_j \\ S_j \end{array}$$ are fiber bundles

- The dim of the fiber of $\widetilde{\mathcal{M}}(S_i, S_j) \xrightarrow{u^i_j} S_i$ is $i-j-1$:

(compatible with Morse-Smale thy)



Def: $X_{i_0,i_1,\dots,i_m} = (\ell^{i_m}_{i_{m-1}})^* \cdots (\ell^{i_1}_{i_0})^* (\widetilde{\mathcal{M}}(S_{i_1},S_{i_0}))$

$$Y_{i_0,\dots,i_m} = (\ell^{i_m}_{i_0})^* \cdots (\ell^{i_1}_{i_0})^* \mathcal{U}_{i_0}$$

Lem 3.3 (Compactification)

- $\partial \overline{\mathcal{M}(S_i,S_j)} = \bigcup_{j=i_0<\dots<i_m=i} X_{i_0,i_1,\dots,i_m}$

- $\partial \overline{\mathcal{U}_i} = \bigcup_{i_0<\dots<i_m=i} Y_{i_0,\dots,i_m}$

In particular:

- $\partial_1 \overline{\widetilde{\mathcal{M}}(S_{i+k},S_i)} = \bigcup_{0<l<k} X_{i,i+l,i+k}$

$$[\partial_1 \overline{\widetilde{\mathcal{M}}(S_{i+k},S_i)}]^{x} = (-1)^{k-l-1} [\widetilde{\mathcal{M}}(i+k,i+l)]^{x} [\widetilde{\mathcal{M}}(i+l,i)]^{y}$$

- $\partial_1 \overline{\mathcal{U}_k} = \bigcup_{l<k} Y_{l,k}$

$$[\partial_1 \overline{\mathcal{U}_k}]^{x} = (-1)^{k-l-1} [\widetilde{\mathcal{M}}(k,l)]^{x} [\mathcal{U}_l]^{y}$$

Rmk: The ortations convention in [AB]:

$$[\mathcal{U}_k]^x = [\tilde{\mathcal{M}}(k,\ell)]^x\,[\nabla f]\,[\mathcal{U}_\ell]^y$$

II. Austin-Braam complex

$$\Omega^{i,j} \triangleq \Omega^j(S_i).$$

$$\partial_r : \Omega^j(S_i) \longrightarrow \Omega^{j-r+1}(S_{i+r}) : \begin{cases} d & r=0 \\ (-1)^j (u^{i+r}_i)_* (\ell^{i+r}_i)^* \omega \end{cases}$$

$$\begin{tikzcd}
\Omega^2(S_0) \arrow[r, "\partial_1"] & \Omega^2(S_1) & \\
\Omega^1(S_0) \arrow[u, "\partial_0"] \arrow[r, "\partial_1"] \arrow[drr, dashed, "\partial_2"] & \Omega^1(S_1) \arrow[u, "\partial_0"'] & \\
\Omega^0(S_0) \arrow[u, "\partial_0"] \arrow[r, "\partial_1"] & \Omega^0(S_1) \arrow[u, "\partial_0"'] \arrow[r, "\partial_1"] & \Omega^0(S_2)
\end{tikzcd}$$

$$\partial \triangleq \sum_r \partial_r$$

Prop: $\sum_{\ell=0}^{k} \partial_{k-\ell}\partial_\ell = 0$, $\forall k$. Hence $\partial^2 = 0$.

Idea of pf: $\pi_*(d\omega) = d\pi_*(\omega) + (-1)^{j-d+1}(\pi_\partial)_*(\omega|_{\partial E})$

for $\omega \in \Omega^j(E)$, where d = fiber dim

Rmk: $(\Omega^{i,j}, d=\partial_0, \delta=\partial_1)$ is in generally not a double complex:

$$\sum_{\ell=0}^{2}\partial_\ell\partial_{2-\ell}=0 \Longrightarrow \delta^2\neq 0.$$

Def: $\left(C^k \triangleq \bigoplus_{i+j=k}\Omega^{i,j}, \partial\right)$ with the obvious filtration defined by Morse-Bott index is the so-called Austin-Braam complex.

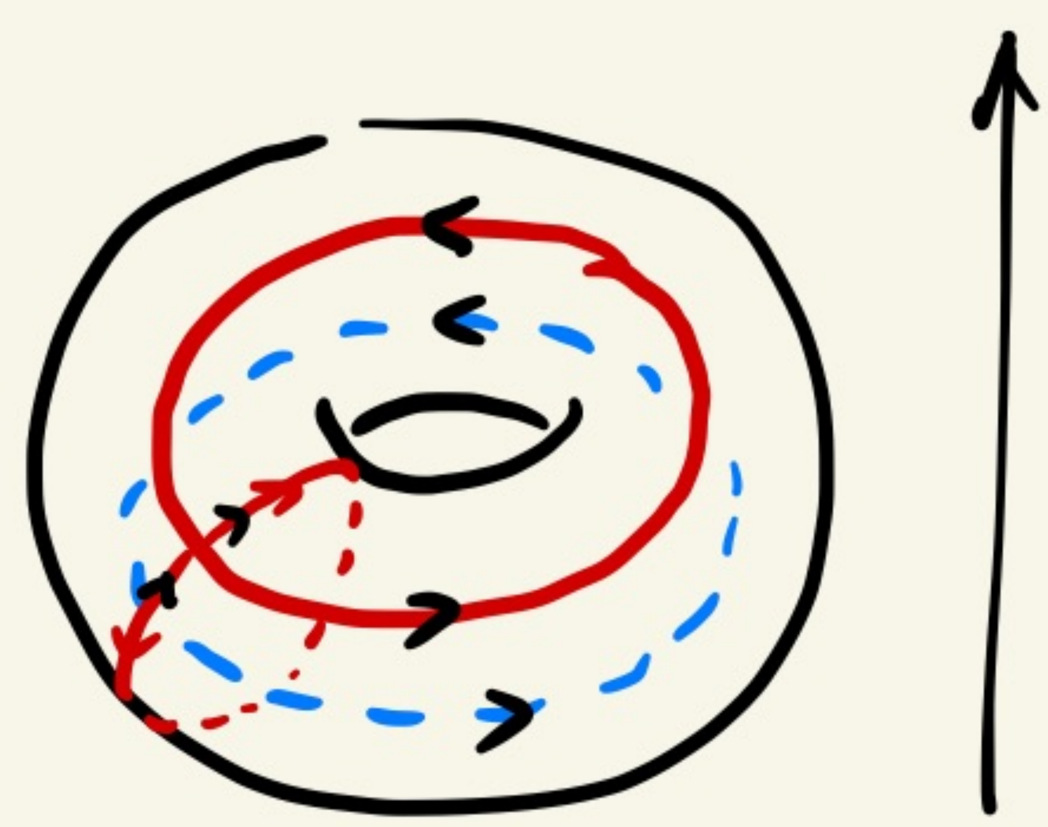
Example

1. exterior form

E_1 term

$$\begin{array}{c|cc} & \mathbb{R} & \mathbb{R} \\ & \mathbb{R} & \mathbb{R} \\ \hline & & \end{array}$$

index



Note that $\partial_1=0 \Longrightarrow E_1=E_2=\cdots=E_\infty$

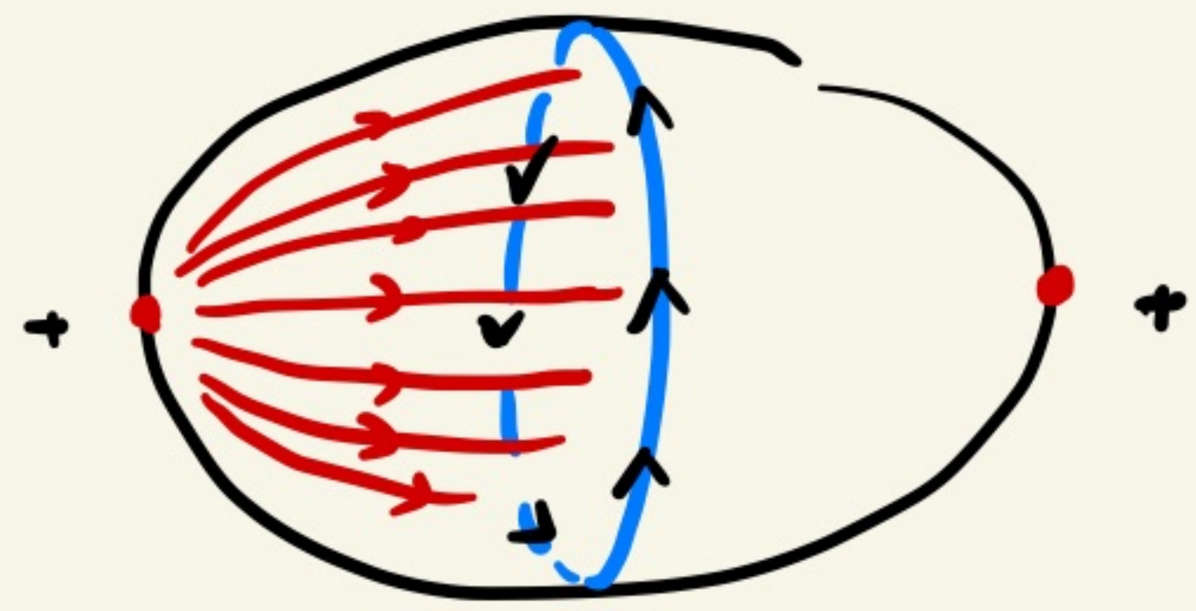
$\approx H^*(C,\partial)$

$$\Longrightarrow H^*(C,\partial)\approx H^*_{dR}(\mathbb{T}^2) \cong \begin{cases} 0 & \text{others.} \\ \mathbb{R} & *=0,2 \\ \mathbb{R}^2 & *=1 \end{cases}$$

2. $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\} \xrightarrow{f} \mathbb{R}^1$

$$(x,y,z) \longmapsto z^2 .$$

$E_1 = E_2$:
$$\begin{tikzcd} \mathbb{R} \arrow[drr, "d_2"] & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R}^2 \end{tikzcd}$$



$E_3 = \cdots = E_\infty$:
$$\begin{array}{|ccc} 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R} \\ \hline \end{array} \Longrightarrow H^*(C,\partial) \approx H^*_{dR}(S^2)$$

III Main Results

Thm 3.1 $H^*(C^*,\partial) \approx H^*_{dR}(B)$

From now on, we assume f to be strickly self-indexing ($f(S_i)=i$) and let $B_k = f^{-1}(k-\frac{1}{2}, +\infty)$

Rmk: • B_k^c contains all S_i for $i<k$ as well as $\mathcal{U}_i$ while containing no crit mfds of higher index.

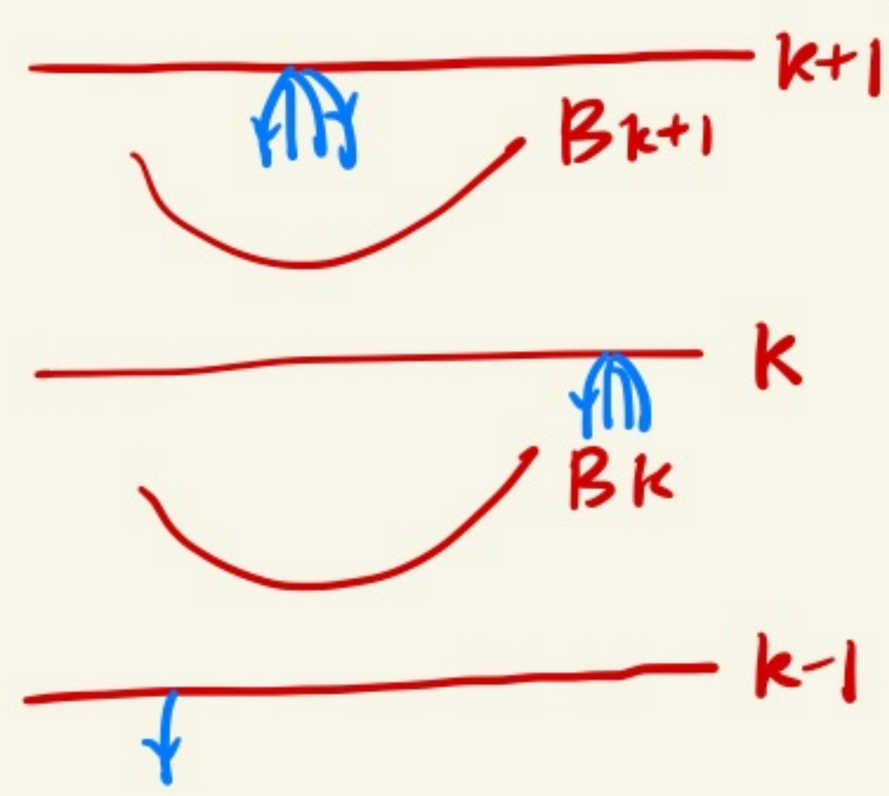
- In the general case where f is not strickly self-indexing, it is possible, using transversality cond to define sets $\{B_k\}$ also having the above property.
- $(\Omega^*(B), d)$ is now filtered by:

$$\Omega^j(B) = \Omega^j_c(B_{-1}) = \Omega^j_c(B_0) \supset \cdots \supset \Omega^j_c(B_n) \supset 0$$

<u>Def</u>: Denote by C'^* the filtered de-Rham complex mentioned above, and define: $\Phi = \oplus \Phi_k : C'^j \to C^j$ where $\Phi_k(\omega) = (u^k)_*(\omega|_{\mathcal{U}_k}) \in \Omega^{j-k}(S_k)$.

<u>Lem 3.6</u>: Φ is a map of filtered complexes.

<u>Pf</u>: It is easy to see that Φ preserves filtration.



It remains to prove that $\Phi \circ d = \partial \circ \Phi$.

$$\begin{tikzcd} & \Omega^{j-l}(S_l) \arrow[dr, "\partial_{k-l}"] & \\ \Omega^{j}(B) \arrow[ur, "\Phi_l"] \arrow[r, "d"] & \Omega^{j+1}(B) \arrow[r, "\Phi_k"] & \Omega^{j+1-k}(S_k) \end{tikzcd}$$

We need to show that $\Phi_k(d\omega) = \sum_{l=0}^{k} \partial_{k-l} \circ \Phi_l$

$$\Phi_k(d\omega) = u^k_*(d\omega)$$

$$= d u^k_*(\omega) + (-1)^{j-k+1} (u^k_\partial)_*(\omega)$$

Recall that $[\partial \overline{\mathcal{U}_k}]^x = (-1)^{k-l+1} [\widehat{\mathcal{M}}(k,l)]^x [\overline{\mathcal{U}_l}]^y$

$$\Longrightarrow \Phi_k(d\omega) = d u^k_*(\omega) + \sum_{l<k} (-1)^{j-l} (u^k_l)_* (l^k_l)^* (u^l)_*(\omega)$$

$$= \Big(\partial_0 \Phi_k + \sum_{l<k} \partial_{k-l} \Phi_l \Big)(\omega)$$

$$\Longrightarrow \Phi_k \circ d = \sum_{l=0}^{k} \partial_{k-l} \circ \Phi_l \, . \qquad \blacksquare$$

Lem 3.7 Let $f : K^1 \to K^2$ be a chain map of filtered complexes. If f induces an isomorphism of the E_1 term of the associated spectral sequences, then f induces isomorphism on homology.

$\blacksquare$

Thm 3.8 Φ induces an isomorphism of filtered cohomology gps; hence:

$$\text{image}\left(H^p(C^*_k, \partial) \to H^p(C^*, \partial)\right) \cong \text{image}\left(H^p(B, N_k) \longrightarrow H^p(B)\right)$$

$H^p_c(B_k) \cong H^p(B, N_k)$, where $N_k = B_k^c$

Sketch of pf: By the preceding lemma, it is sufficient to prove that Φ induces isomorphism between to E_1 terms.

Recall that $GC^p_k = \Omega^{p-k}(S_k)$

$$GC'^p_k = \Omega^p_c(B_k) / \Omega^p_c(B_{k+1})$$

Hence on E_1 terms, Φ induces:

$$\Phi_1 : H^p(GC'^p_k) \longrightarrow H^{p-k}(S_k)$$
$$[\omega] \longmapsto (\mathcal{U}^k)_*(\omega|_{\mathcal{U}_k})$$

To see that Φ_1 is actually an isomorphism, we need to characterize $H^p(GC'^p_k)$ in a more geometric insight:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^p_c(B_{k+1}) \arrow[r] \arrow[d] & \Omega^p_c(B_k) \arrow[r] \arrow[d] & GC'^p_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta^p(B, N_{k+1}) \arrow[r] & \Delta^p(B, N_k) \arrow[r] & \Delta^p(N_{k+1}, N_k) \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
\arrow[r] & H^p_c(B_{k+1}) \arrow[r] \arrow[d, "\approx"'] & H^p_c(B_k) \arrow[r] \arrow[d, "\approx"'] & H^p(GC'_k) \arrow[r] \arrow[d, "\approx"'] & {} \\
\arrow[r] & H^p(B, N_{k+1}) \arrow[r] & H^p(B, N_k) \arrow[r] & H^p(N_{k+1}, N_k) \arrow[r] & {}
\end{tikzcd}$$

$$\Longrightarrow \quad H^p(GC'_k) \xrightarrow{\approx} H^p(N_{k+1}, N_k)$$

<u>FACT</u> : $F_k \triangleq B_k \cap \mathcal{U}_k \approx D(\nu^-(S_k))$

$$
\begin{tikzcd}
\Longrightarrow & H^p(G C_k') \arrow[r, "\approx"] \arrow[d, "\approx"'] \arrow[dd, bend right=60, "\Phi_1"'] & H^p(N_{k+1}, N_k) \arrow[r, "\approx"] \arrow[d] & H^p(F_k \cup N_k, N_k) \arrow[dl, "\approx"', "\text{excision}"] \\
& H^p_c(F_k) \arrow[r, "\approx"] \arrow[d, "(u^k)_*"', "\text{Thom iso}"] & H^p(F_k, \partial F_k) & \\
& H^{p-k}(S_k) & &
\end{tikzcd}
$$

N_{k+1} can deform retract to $F_k \cup N_k$. (this can be viewed as a handle attachment in a fiber version!)

∎

<u>Cor 3.9</u> There is a polynomial $Q(t)$ with ≥ 0 coefficients such that:

$$
\underbrace{\sum_{i,j} \dim H^j(S_i; \mathbb{R}) t^{i+j}}_{E_1 \text{ term}} = \underbrace{\sum_k \dim H^k(B; \mathbb{R}) t^k}_{E_\infty \text{ term}} + (1+t) Q(t).
$$

<u>Rmk</u>: A Morse-Bott func is called perfect if $Q(t) = 0$.

More examples:

1. $\mathbb{C}P^n = S^{2n+1}/S^1 = \{[z_0 : \cdots : z_n] \mid \sum_i |z_i|^2 = 1\}$

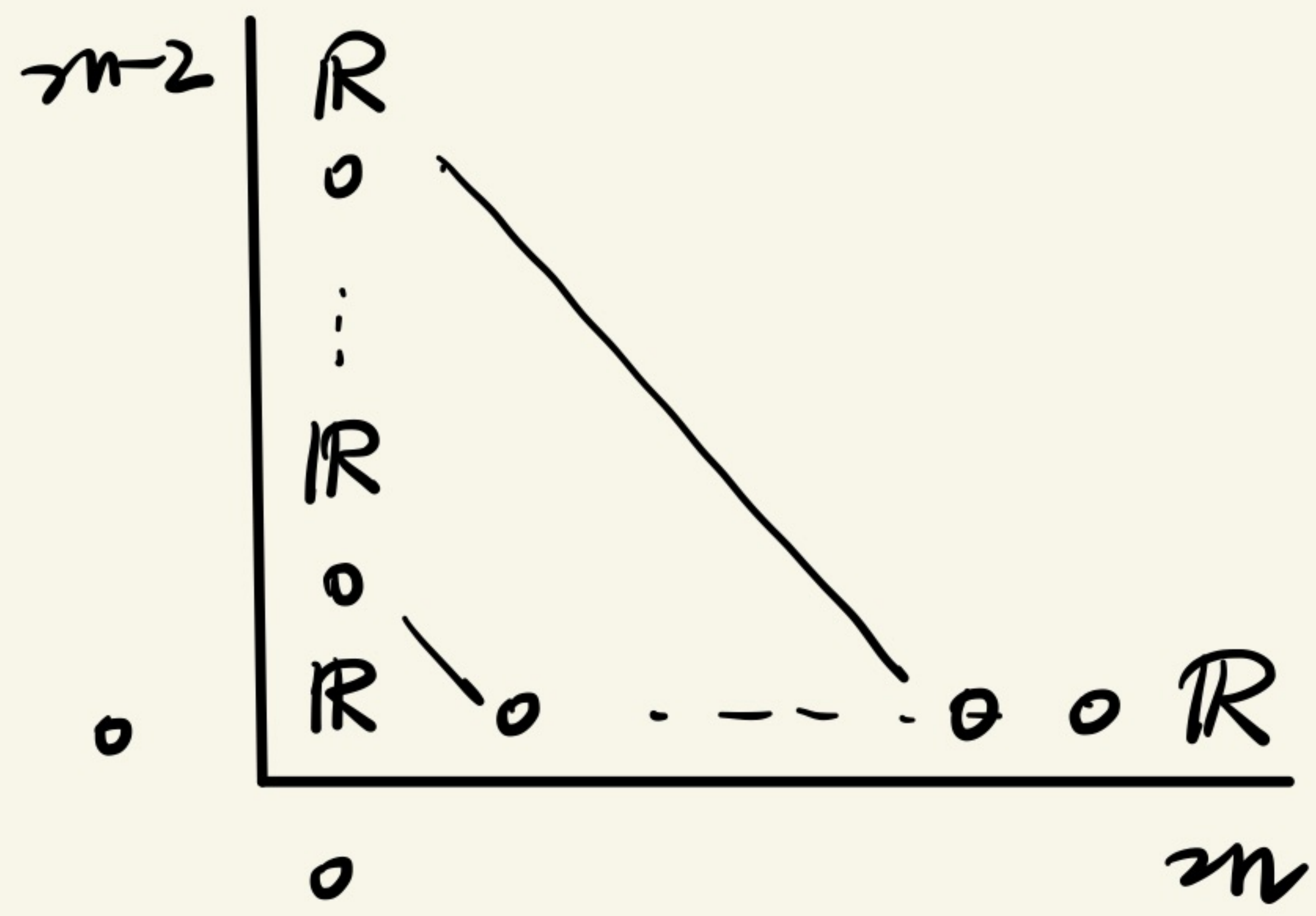
Define $\mathbb{C}P^n \xrightarrow{f} \mathbb{R}^1$

$[z_i] \longmapsto |z_n|^2$

There are exactly two crit mfds of f:

$S_0 = \{[z_i] \mid z_n = 0\}$, $S_{2n} = \{[0 : \cdots : 0 : 1]\}$

E_1:



$d_1 = d_2 = \cdots = 0 \Rightarrow E_1 \approx E_\infty \approx H^*_{dR}(\mathbb{C}P^n)$.

2. $F \to E \xrightarrow{\pi} B$, and let $f: B \to \mathbb{R}$ be a Morse func, then $\tilde{f} = f \circ \pi$ is a Morse-Bott func on E. In those good cases, Smale cond on f induces Smale cond on $\tilde{f}$ (e.g. π is a Riemannian submersion). In this case, the ABSS of $\tilde{f}$ has E_2 term isomorphic to $\check{H}^*(B, \mathcal{H})$. In particular, if B is simply connected, then $E_2 \approx H^*(F) \otimes H^*(B)$

Rmk: In fact, ABSS $\approx$ Leray SS in this case

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[r, "\pi"] & S^2 \arrow[d, "h"] \\ & & \mathbb{R}^1 \end{tikzcd}$$

height func

$$E_1 = E_2: \quad \begin{array}{ccc} \mathbb{R} & 0 & \mathbb{R} \\ \mathbb{R} & 0 & \mathbb{R} \end{array} \quad d_2$$

$$E_3 = E_\infty: \quad \begin{array}{ccc} 0 & 0 & \mathbb{R} \\ \mathbb{R} & 0 & 0 \end{array}$$

3. Equivariant cohomology of G-space X:

Let $EG \to BG$ be ~~the~~ the classifying bundle for G.

Def: $H_G(X) \triangleq H(X_G \triangleq EG \times_G X)$

And hence $H_G(X)$ can be calculated by AB-complex.

e.g. $G = SU(2)$, $BG = \mathbb{HP}^\infty \xrightarrow{f} \mathbb{R}^1$ a perfect Morse func whose crit pts all have index divisible by 4.

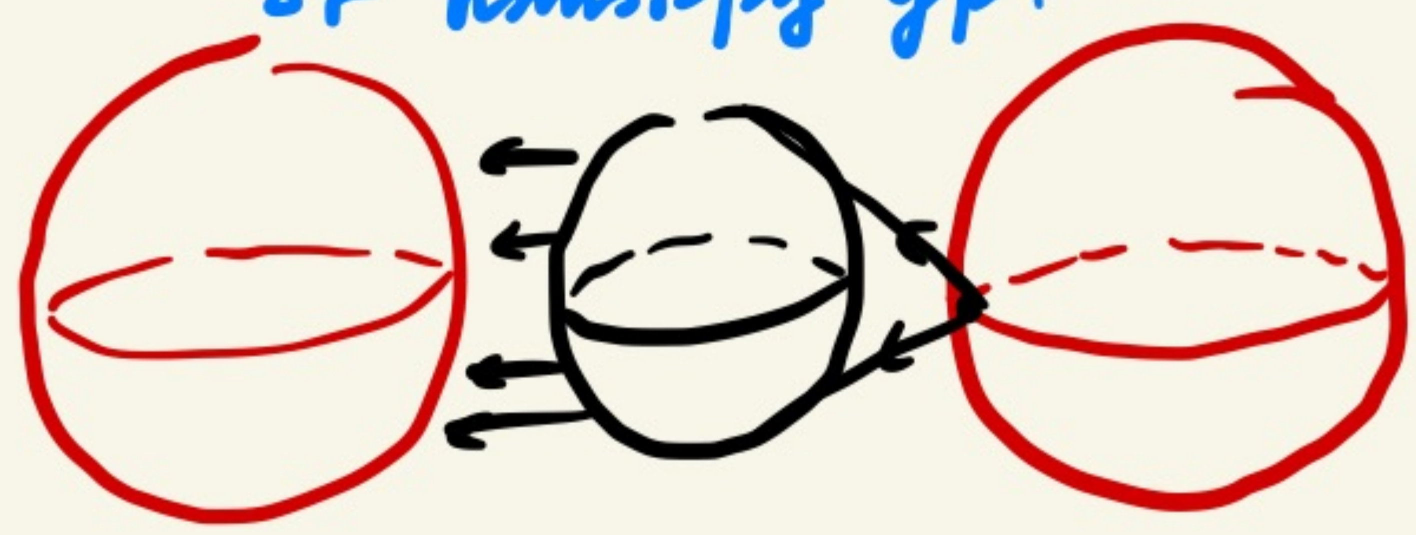
? • The only non-zero component of the boundary operator are ∂_0 & ∂_4.

? • ∂_4 can be described as follows: $S^7 \to S^4$

reasonable: either by Euler class or homotopy gp.

$\overline{\mathcal{M}}(X_{i+4}, X_i) = X \times SU(2)$

$u_i^{i+4} = pr_X$, $l_i^{i+4}(x,g) = g \cdot x$

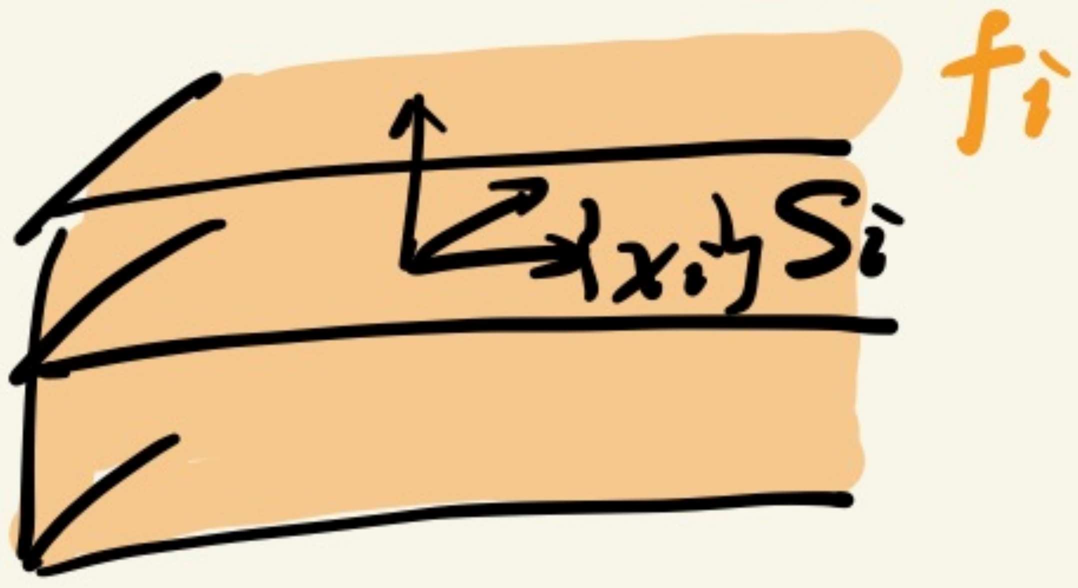


$\Longrightarrow \int_M \partial_4(\omega) = \int_{G\cdot M} \omega$

IV. Perturbation of Morse-Bott func.

$B \xrightarrow{f} \mathbb{R}^1$ Morse-Bott $\rightsquigarrow f_\varepsilon = f + \varepsilon \sum_i \rho_i f_i$

For sufficiently small ε, $\mathrm{Crit}(f_\varepsilon) = \bigcup_i \mathrm{Crit}(f_i)$



horizontal

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{pmatrix} \frac{\partial^2 f_i}{\partial\ \partial} & 0 \\ 0 & \frac{\partial^2 f}{\partial\ \partial} \end{pmatrix}$$

vertical

$\forall p \in \mathrm{Crit}(f_i)$, $\mathrm{ind} f_\varepsilon(p) = i + \mathrm{ind} f_i(p)$

Prop 5.1 For small ε, the map $F : (C^*, \partial) \to (C^*_\varepsilon, \partial^*_\varepsilon)$ defined by $F(\omega) = \left(\int_{\mathcal{U}^i_\alpha} \omega\right)\alpha$ where $\omega \in \Omega^j(S_i)$ is a chain morphism inducing an isomorphism on cohomology. $\boxed{}$

Inspect $\mathcal{M}_\varepsilon(\beta, p)$ (dim $= j+1$)

① $F(\partial_k \omega) = \sum \# \widetilde{\mathcal{M}}_\varepsilon(\beta, \alpha)\beta$

② F induces isomorphism on E_1 terms