Notes on Morse Theory
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Review of classical Morse Theory. (PART I)

M. Morse, Calculus of variations in the large (1934).

$$\text{Energy functional}: \Omega(M;p,q) \longrightarrow \mathbb{R}, \quad \gamma \longrightarrow \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 \, dt$$

Geometry (analysis)
(Riem-metric, curvature etc.)

* "critical pts": geodesics on M.
* "degeneration at critical pts": Jacobi fields
* "index at (non-degenerate) a critical pt": accumulations of degenerations.

$\updownarrow$

Topology
(Homology, homotopy gps etc.)

* CW cells
* dim of a cells

$\rightsquigarrow$ CW approximation of $\Omega(M;p,q)$.

$$\Omega(M;p,q) \simeq \Omega^*(M;p,q)$$

The gist of classical Morse Theory:

- Using smooth functions on M to tell the topological information of M.

Ref : · J. Milnor, Morse Theory (1963).

· J. Milnor, Lectures on the h-cobordism theorem (1965).

1 Classical approach

* The mfds occur in the following are all assumed to be closed.

Def 1.1 · Given a C^∞-mfd M and a $f \in C^\infty(M, \mathbb{R})$, a critical pt of f is a $p \in M$ such that $df(p) = 0$.

· At each critical pt $p \in M$, one can define a symmetric $\binom{0}{2}$-tensor $H(f,p)$:
$$T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \tilde{v}_p \tilde{w} f$$
, where $\tilde{v}, \tilde{w}$ are two arbitrary local fields around p such that $\tilde{v}_p = v$, $\tilde{w}_p = w$.

Rmk : One can check that $H(f,p)$ is well-defined and symmetric ($\tilde{v}_p \tilde{w} f - \tilde{w}_p \tilde{v} f = [\tilde{v}, \tilde{w}]_p f = 0$).

Given a local chart (x^i) centered at p, we have:
$$H(f,p) \sim \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{ij}.$$

Def 1.2 A $f \in C^\infty(M, \mathbb{R})$ is called a Morse func if all critical pts of f are non-degenerate, i.e. $\det H(f,p) \neq 0$ for all critical $p \in M$.

Rmk : We denote by $\mathrm{Crit}(f)$ the set of all critical pts of f.

Lem 1.3 (Lemma of Morse, local rigidity of non-degenerate crit pt)

Let $p \in Crit(f)$. $\exists$ local coord (x^i) centered at p such that $f = f(0) - (x^1)^2 - \cdots - (x^k)^2 + (x^{k+1})^2 + \cdots + (x^n)^2$, where $k = ind(f,p)$, the index of $H(f,p)$ at p. $\boxtimes$

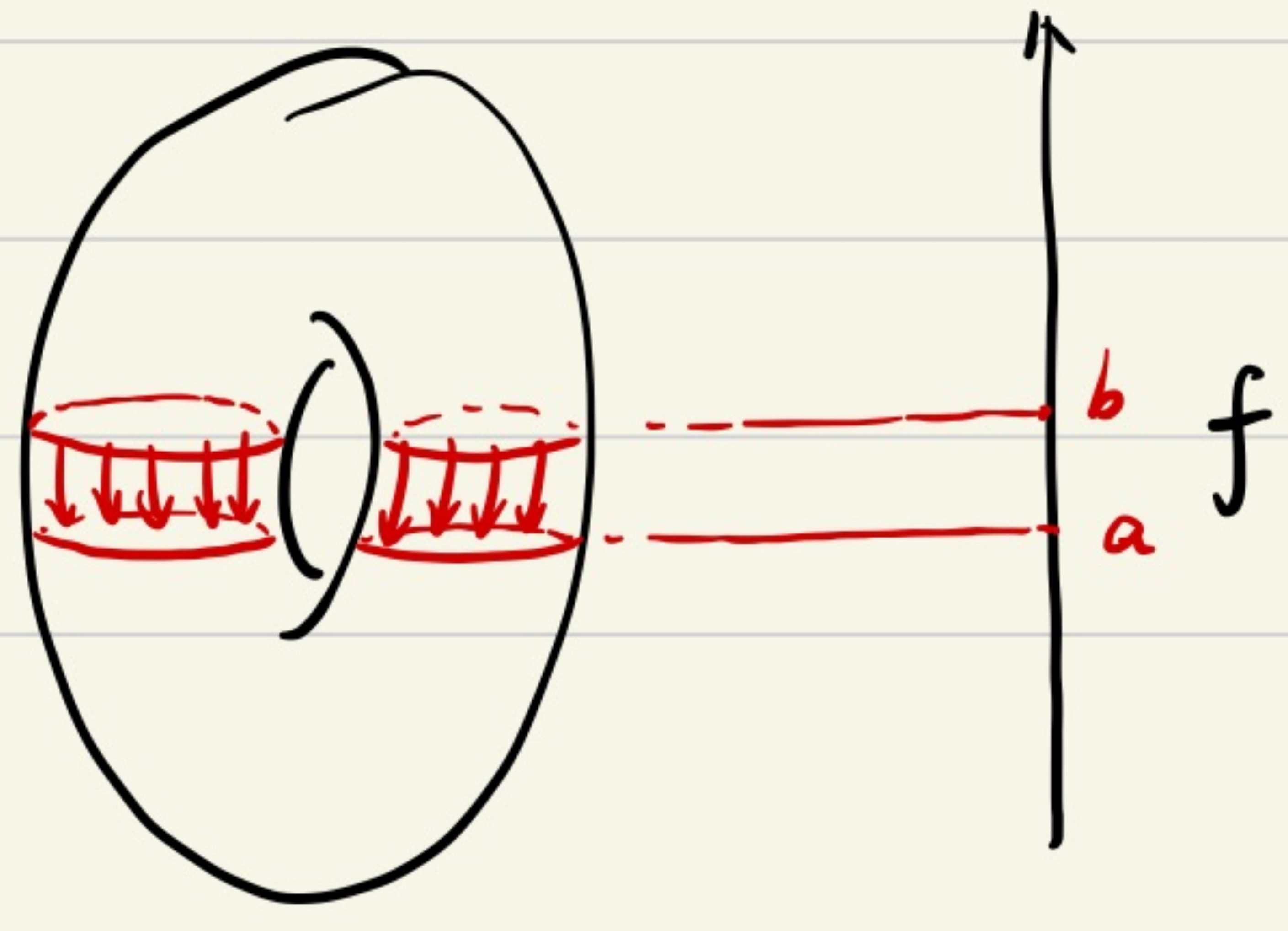
Cor 1.4 $Crit(f)$ of a Morse func is a discrete closed subset of M. $\boxtimes$

Rmk: Alternatively, consider $\nabla f \in \mathfrak{X}(M)$, then $Crit(f)$ equals the zero of ∇f, which is discrete by the Inverse Map Thm.

- Handles decomposition (homotopy version).

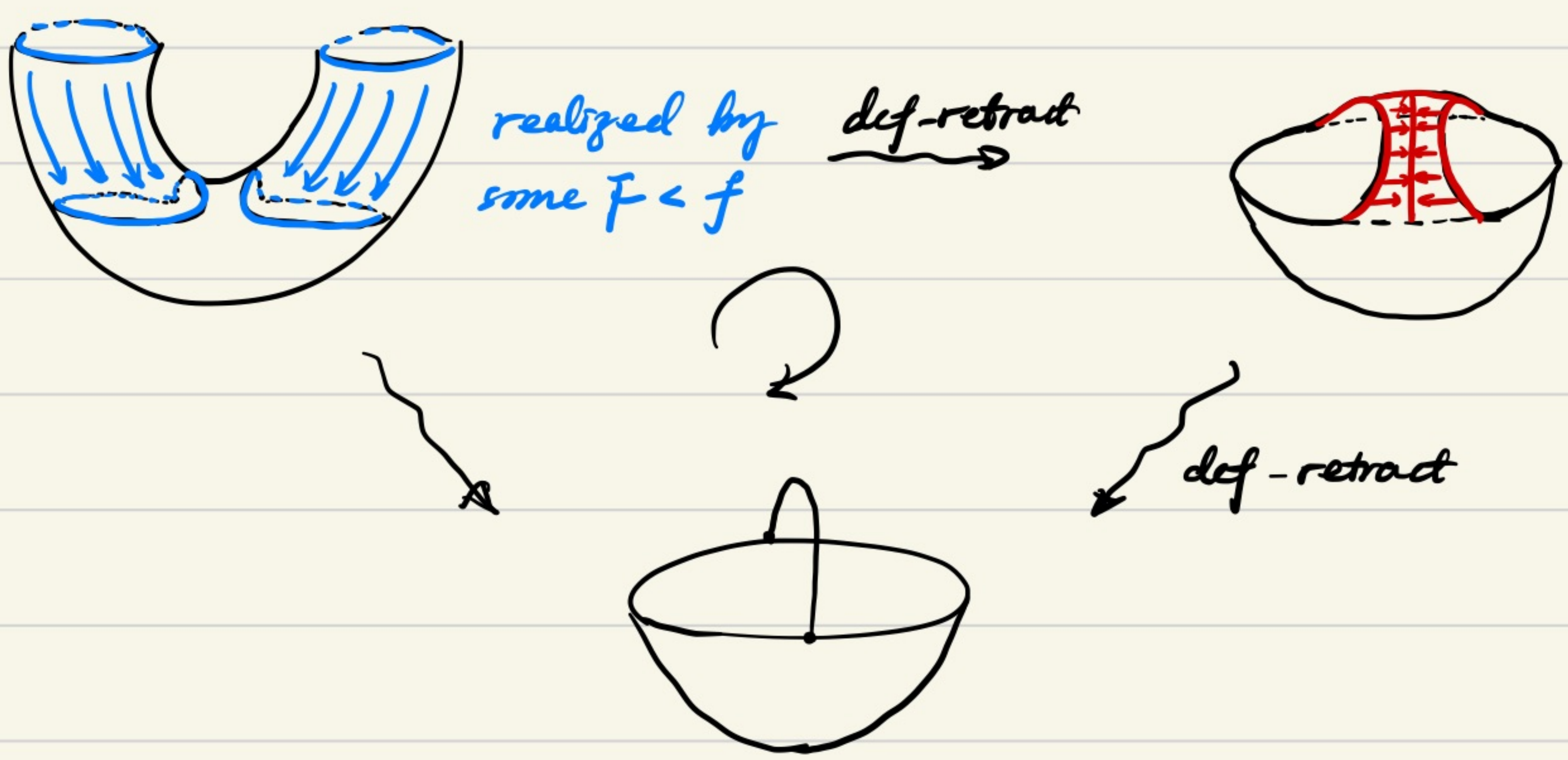
$M_a \triangleq f^{-1}(-\infty, a]$.

Thm 1.4 Suppose $f^{-1}[a,b] \cap Crit(f) = \phi$. Then $M^a \overset{\text{diffeo}}{\approx} M^b$ and $M^a \hookrightarrow M^b$ is a deformation retract. $\boxtimes$



Thm 1.5 Suppose $p \in \mathrm{Crit}(f)$, non-degenerate and that $f^{-1}[c-\varepsilon, c+\varepsilon] \cap \mathrm{Crit}(f) = \{p\}$ $(c = f(p))$ for some $\varepsilon > 0$.
Then for some sufficiently small ε, $M^{c+\varepsilon} \simeq_h M^{c-\varepsilon} \cup_{\partial \mathbb{D}^k} \mathbb{D}^k$,
where $k = \mathrm{ind}(f, p)$.

Idea of proof:



$\square$

Thm 1.6 Suppose f is a Morse func on M, then M is homophic to a CW-complex M', with one cell of dim k for each $p \in \mathrm{Crit}(f)$ with $\mathrm{ind}(f,p) = k$.

Idea of proof:

$$\begin{tikzcd}
M^{a_1} \arrow[d] & \subset M^{a_2} \arrow[d] & \subset \cdots \arrow[r, "\text{homotopy}", "\text{direct limit}"'] & M \arrow[d, "\simeq"] \\
K_1 & \subset K_2 & \subset \cdots \arrow[r] & \varinjlim K_n
\end{tikzcd}$$

$\square$

Cor 1.7 (Morse Inequality)

$\forall k \geq 0: \quad c_k - c_{k-1} + c_{k-2} - \cdots \pm c_0 \geq b_k(M) - b_k(M) + \cdots \pm b_0(M)$,

where $b_k(M)$ is the k-th Betti number of M. $\blacksquare$

Applications:

- f Morse func on $T^2 \Rightarrow \# \operatorname{Crit}(f) \geq 4$.
- Poincaré – Hopf Theorem:

  $\forall X \in \mathfrak{X}(M)$, with non-degenerate zeros $\Rightarrow \operatorname{Ind}(X) = \chi(M)$.

Prop 1.8 A generic $f \in C^\infty(M, \mathbb{R})$ is Morse.

Approach 1: $M \hookrightarrow \mathbb{R}^n$, $d(p, \cdot)^2$, $p \in \mathbb{R}^n$ fixed

Approach 2: Apply implicit func thm (Banach ver.) on:

$$\begin{tikzcd} C^k(M, \mathbb{R}) \times T^*M \arrow[d, shift right] \\ C^k(M, \mathbb{R}) \times M \arrow[u, shift right, "\psi : (f, x) \mapsto df(x)"'] \end{tikzcd}$$

$\blacksquare$

[Ref]:
- Austin & Braam's paper
- Hutchings's notes.

2. A newer approach: gradient flow lines

Gist of this part:

$$\text{Metric} \rightsquigarrow \text{Flow lines} \rightsquigarrow \begin{cases} \text{Descending mfd} \\ \text{Ascending mfd} \end{cases}$$

(natural oriented)

$\xrightarrow[\text{condition}]{\text{Morse-Smale}}$ CW-structure & degree of cells attaching.

$H_{Morse} \xrightarrow{\cong} H_{\Delta}$

Given (M, g) and a Morse func f on M. Let $V = -\text{grad} f = -(df)^{\#}$ and denote by ψ_t the flow generated by V, called the negative gradient flow of (M, g, f). It is easy to show that each $\psi_t(x)$ tends to some pt of $\text{Crit}(f)$ as $t \to \pm\infty$. (since f here serves as a natural Liapunov func of ψ_t).

<u>Def 2.1</u>: If $p \in \text{Crit}(f)$, we define the descending mfd (i.e. unstable...) by $\mathcal{D}(p) \triangleq \{x \in M \mid \psi_t(x) \xrightarrow{t \to -\infty} p\}$, and the ascending mfd (i.e. stable mfd) by $\mathcal{A}(p) \triangleq \{x \in M \mid \psi_t(x) \xrightarrow{t \to +\infty} p\}$.

FACT:

- If p is a non-degenerate crit pt, then $\mathcal{D}(p)$ is an embedded open disk in M, with $\dim \mathcal{D}(p) = \operatorname{ind}(p)$, and $T_p\mathcal{D}(p) \subset T_pM$ is the negative eigenspace of $H(f,p)$
- Similarly, $\mathcal{A}(p)$ is an embedded open disk in M with $\dim \mathcal{A}(p) = \dim M - \operatorname{ind}(p)$.

Def 2.2 We call (f, g) is Morse - Smale if $\mathcal{D}(p) \pitchfork \mathcal{A}(q)$ for $\forall\, p, q \in \operatorname{Crit}(f)$.

Prop 2.3: Given a Morse func f, for generic g, (f,g) is Morse - Smale. (See you later).

From now on, we assume f to be Morse - Smale.

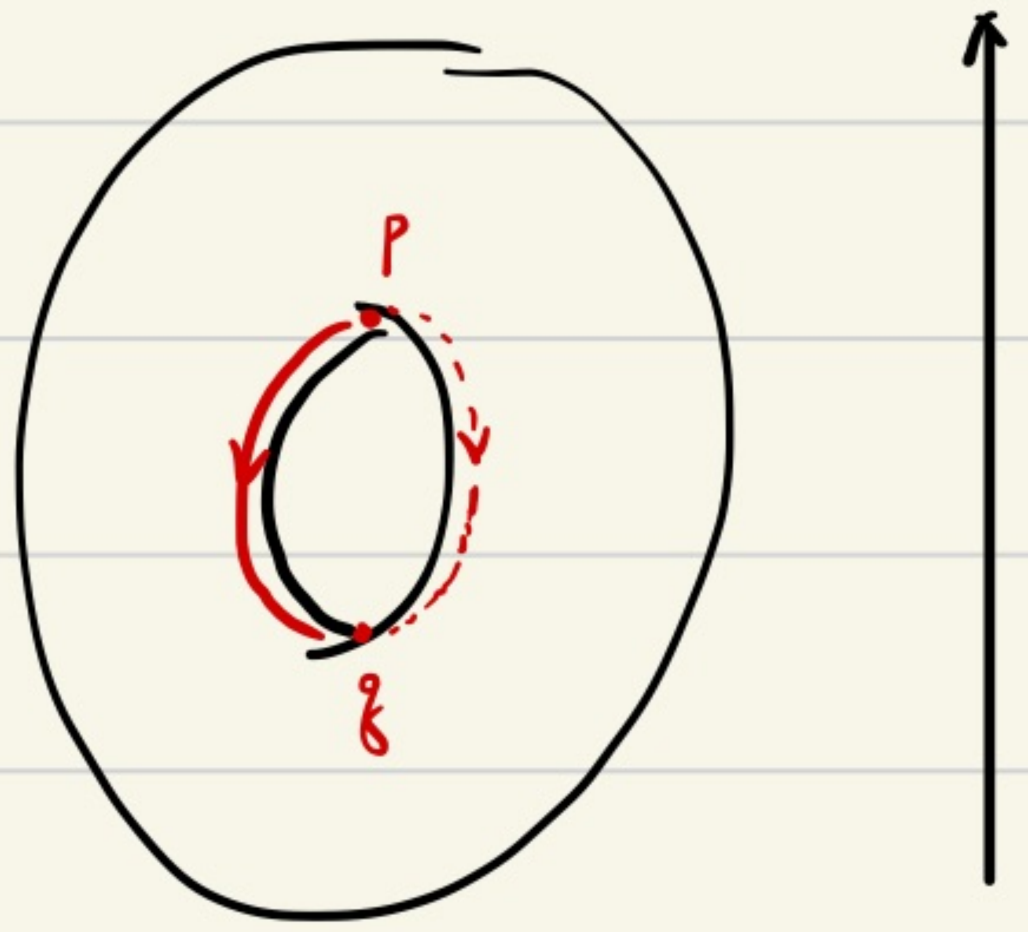
Def 2.4: Moduli space of flow lines from p to q:

$$\mathcal{M}(p,q) = \mathcal{D}(p) \cap \mathcal{A}(q) \,/\, \mathbb{R} \quad \text{(admit a natural smooth str.)}$$

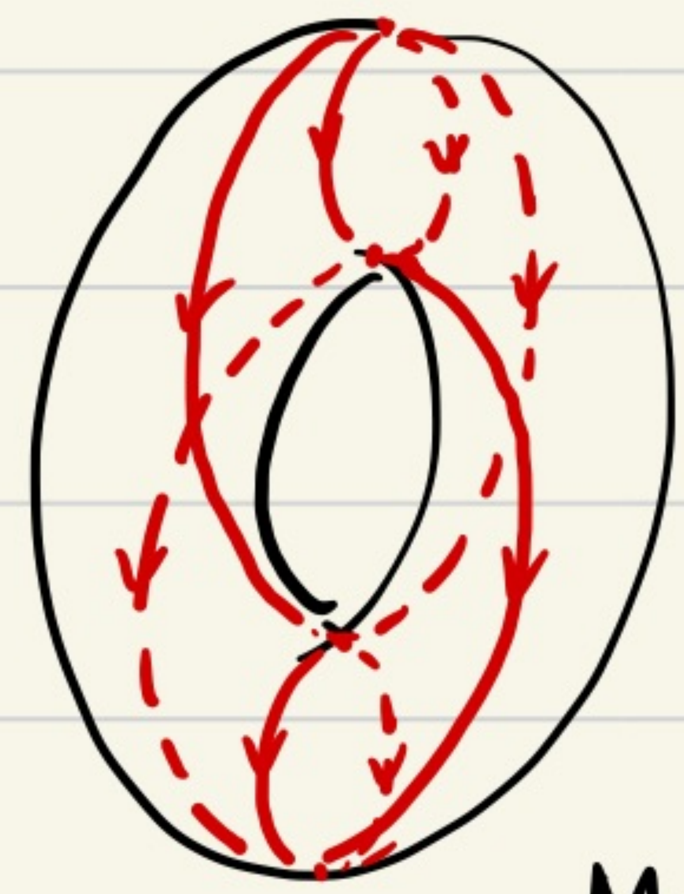
Rmk $\dim \mathcal{M}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$ (except the case $p=q$, when $\mathbb{R}$ action is trivial)

Examples

$\hookrightarrow \mathbb{R}^3$

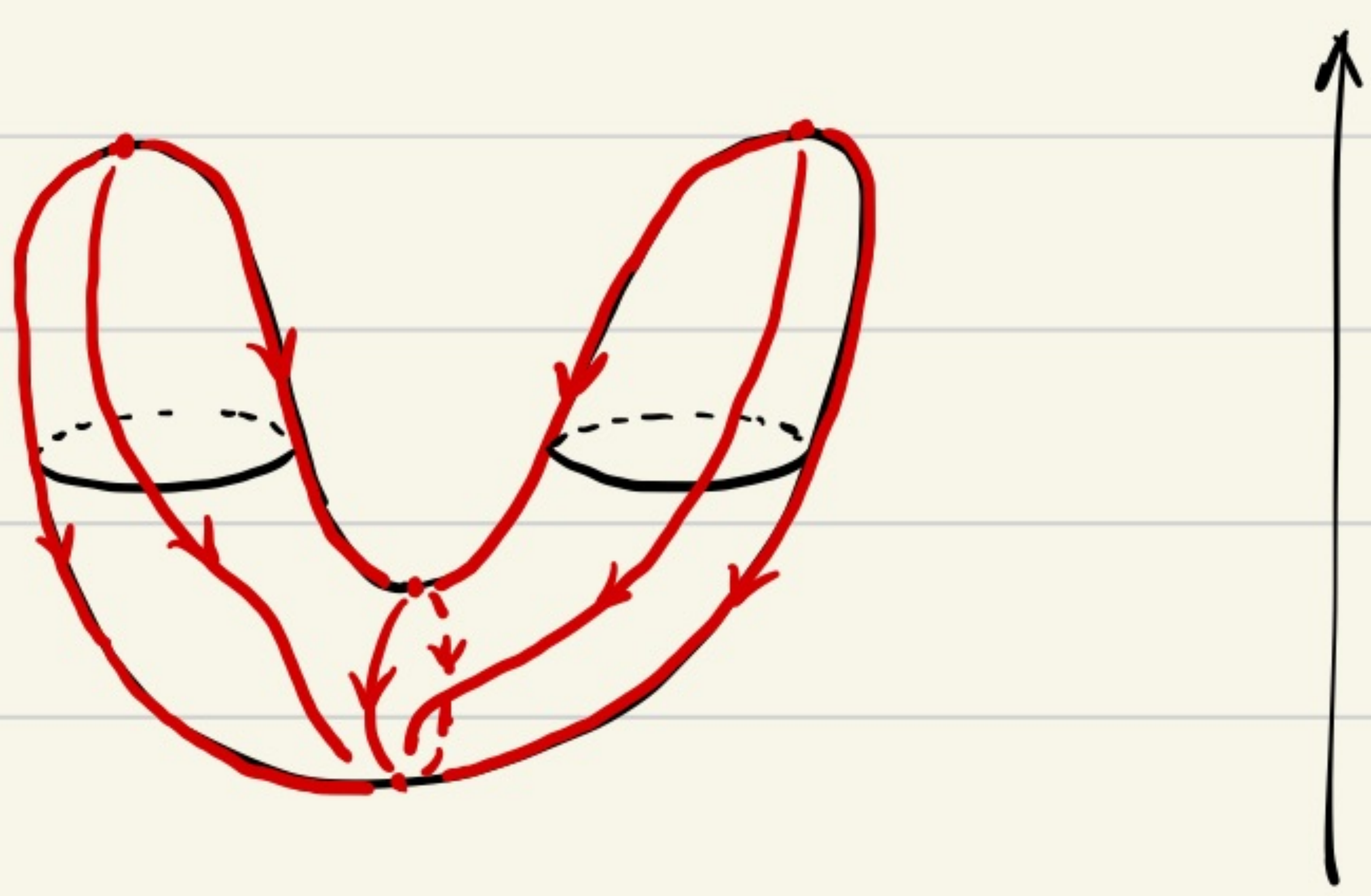
- 

f is not Morse-Smale $\xrightarrow[\text{metric}]{\text{perturb}}$



Morse Smale ✓

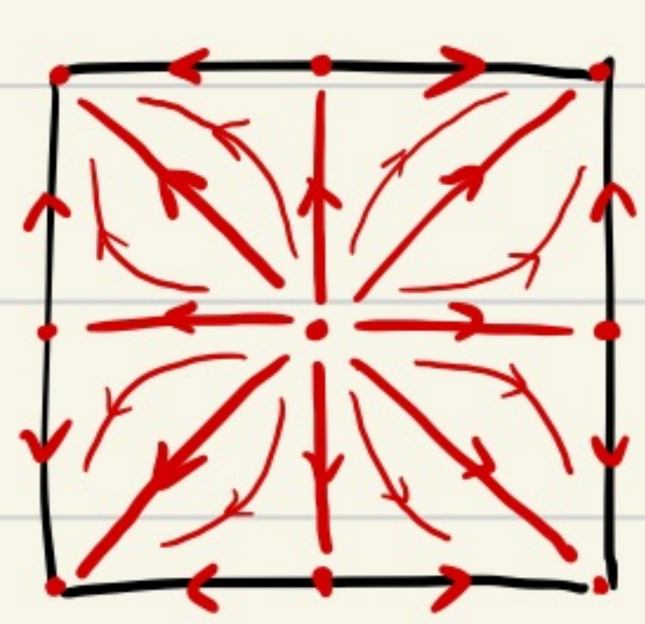
The metric induced by embedding

- 

Morse-Smale ✓

- $f: [-1,1]^2 \longrightarrow \mathbb{R}$, $(x_1, x_2) \longmapsto \frac{1}{4}\sum_{i=1}^{2}(x_i+1)^2(x_i-1)^2$

$$-\nabla f = -\left((x_1^2-1)x_1,\ (x_2^2-1)x_2\right), \quad J(-\nabla f) = \begin{pmatrix} 1-3x_1^2 & 0 \\ 0 & 1-3x_2^2 \end{pmatrix}$$



Morse-Smale ✓

Q: What's the natural compactification of $\mathcal{D}(p)$?

Once this has been established, we will have a natural CW-rep of M (provided a (f,g) Morse-Smale pair)!

Thm 2.5: (Compactification of $\mathcal{D}(p)$)

Given a $p \in \mathrm{Crit}(f)$, there is a natural compactification to a smooth mfd with corners (i.e. locally looks diffeomorphically like $\mathbb{R}^{n-k} \times [0,\infty)^k$) $\overline{\mathcal{D}(p)}$, whose codimension k stratum is:

$$\overline{\mathcal{D}(p)}_k = \bigcup_{\substack{q_1,\dots,q_k \in \mathrm{Crit}(f) \\ p,q_1,\dots,q_k \text{ distinct}}} \mathcal{M}(p,q_1) \times \mathcal{M}(q_1,q_2) \times \cdots \times \mathcal{M}(q_{k-1},q_k) \times \mathcal{D}(q_k)$$

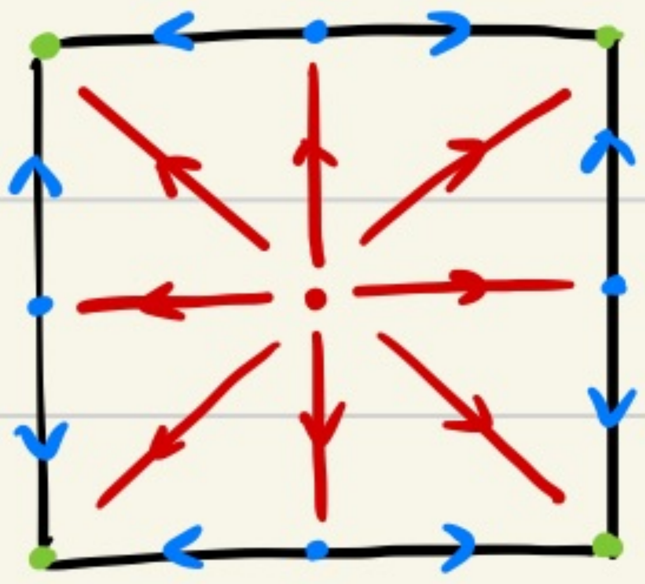
In particular, $\partial\overline{\mathcal{D}(p)} = \overline{\mathcal{D}(p)}_1 = \bigcup_{\substack{q \in \mathrm{Crit}(f) \\ q \neq p}} \mathcal{M}(p,q) \times \mathcal{D}(q)$.

The maps $\overline{\mathcal{D}(p)}_k$ given by projection to $\mathcal{D}(q_k) \subset M$ patch together to a almost smooth map: $e: \overline{\mathcal{D}(p)} \longrightarrow M$ extending the inclusion $\mathcal{D}(p) \hookrightarrow M$. $\boxtimes$

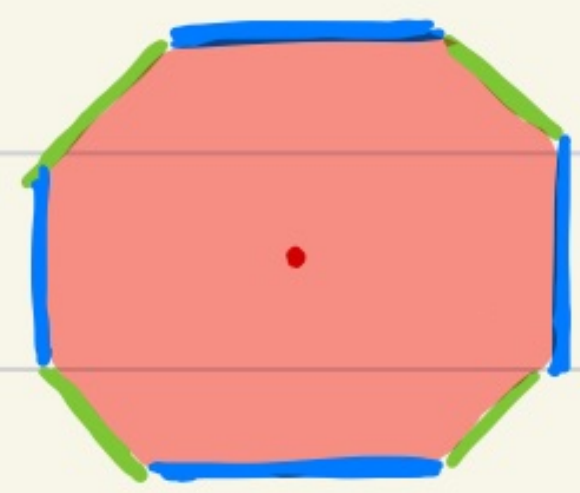
Examples:

- $f : [-1,1]^n \longrightarrow \mathbb{R}$, $(x_i) \longmapsto \frac{1}{4}\sum_{i=1}^{n} (x_i-1)^2(x_i+1)^2$

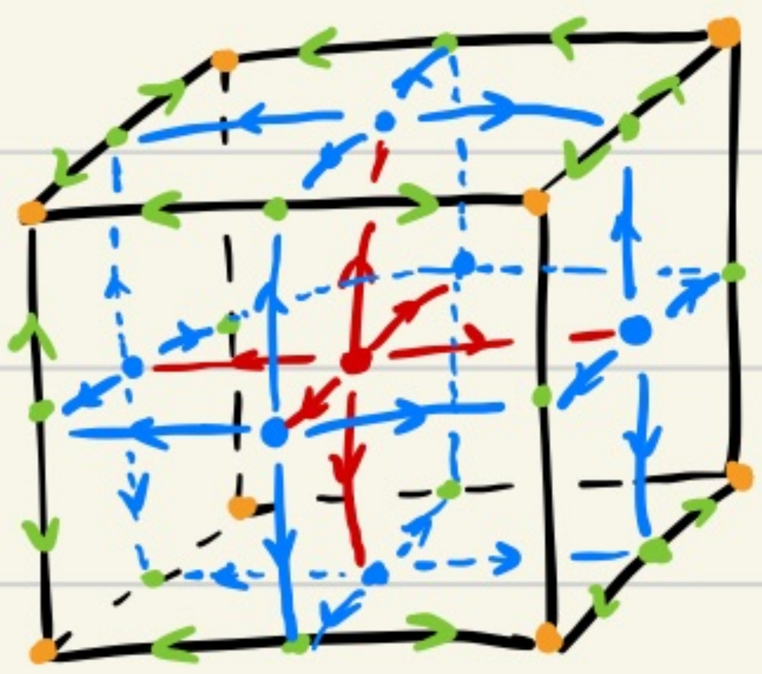
① $n=2$:



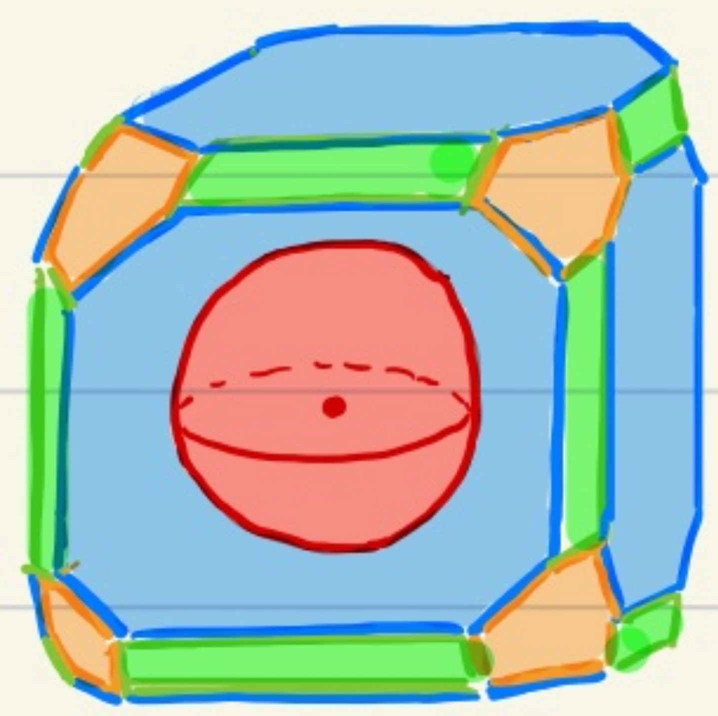
$\rightsquigarrow \overline{\mathcal{D}(\cdot)}$:



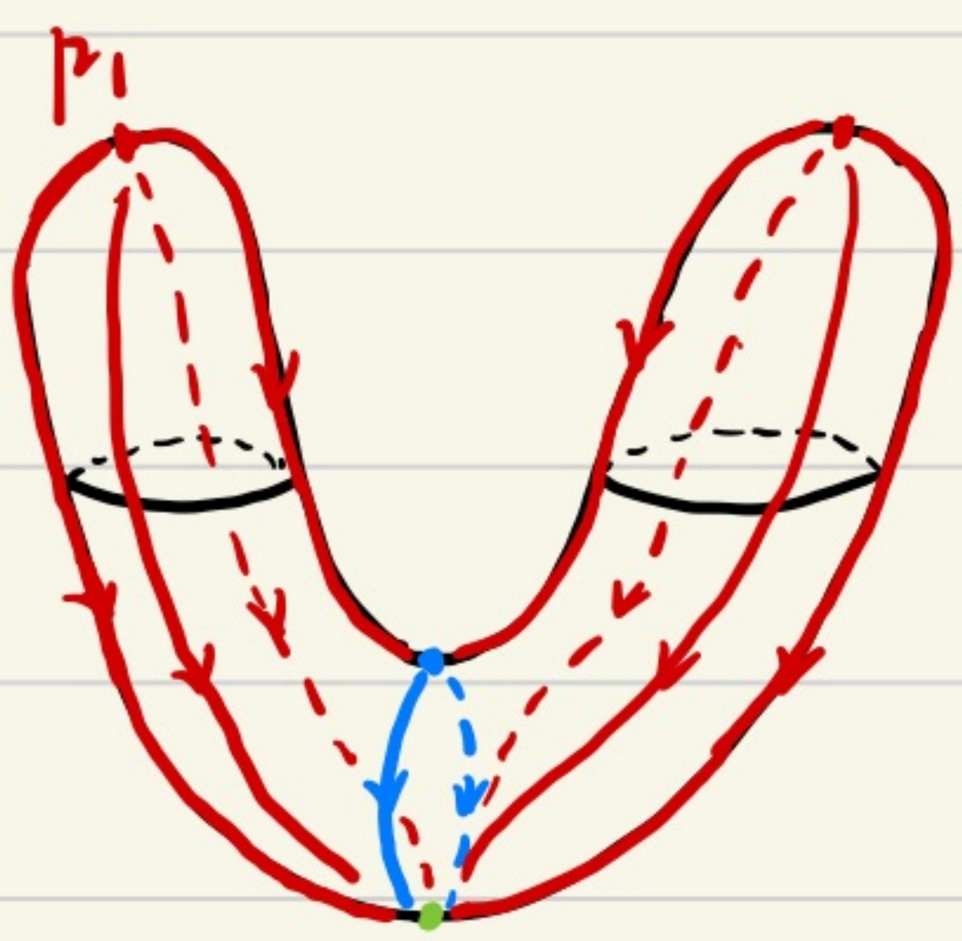
② $n=3$:



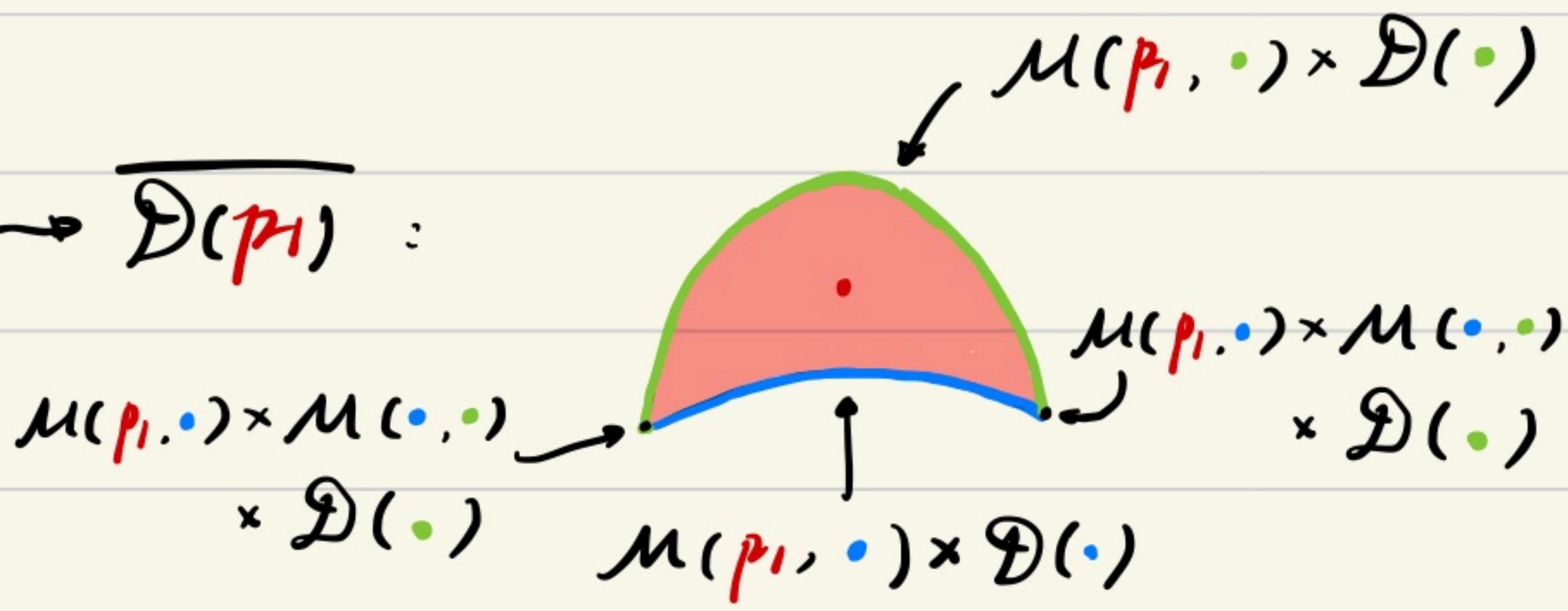
$\rightsquigarrow \overline{\mathcal{D}(\cdot)}$:



fully truncated n-cube.

- 

$\rightsquigarrow \overline{\mathcal{D}(p_1)}$:



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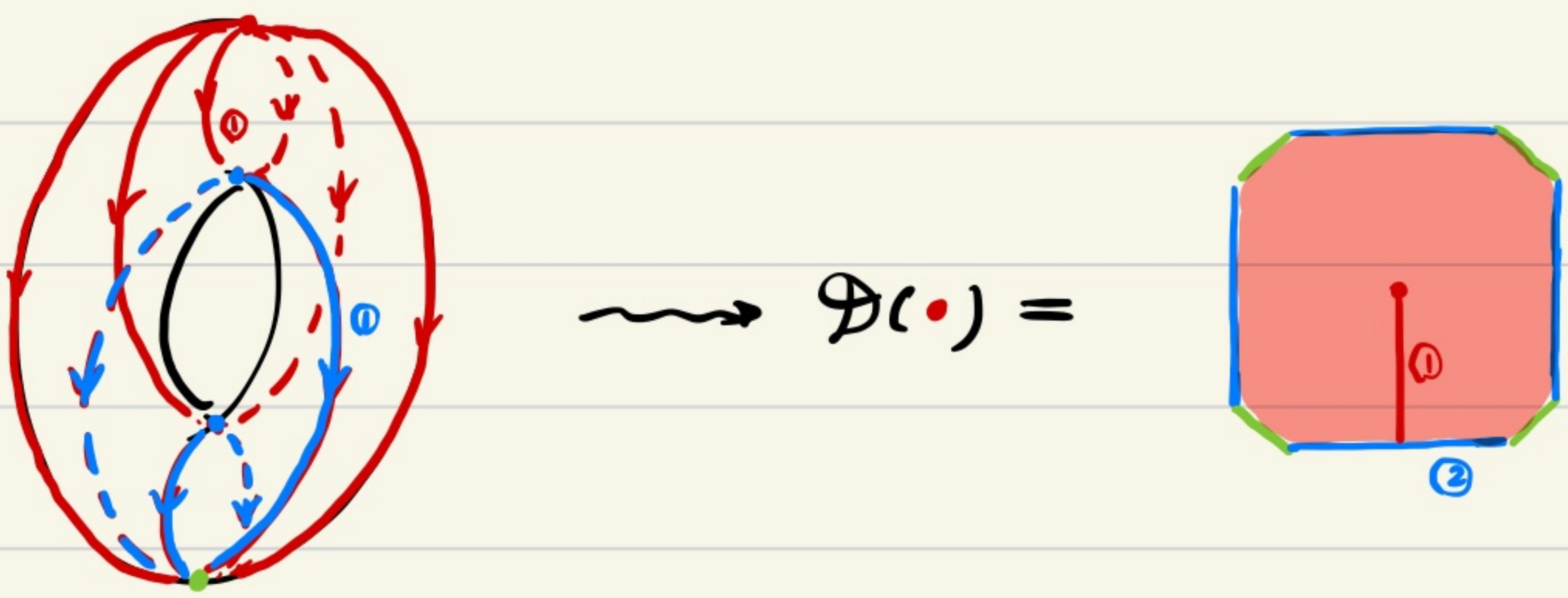


Rmk:

• One can show in general that $\overline{\mathcal{D}(p)}$ is homeomorphic to a closed ball of dim ind(p). Hence $e: \overline{\mathcal{D}(p)} \longrightarrow M$'s give M the str. of a CW complex. This gives us a way to calculate the cellular (co)homology of M once we find an efficient method to calculate the degree of characteristic maps. The following discussion serves exactly for this purpose, and it turns out that gradient flow provides a canonical way to deal with the orientations bewteen $\mathcal{D}(p)$'s and hence settles the problem of degrees calculations. A further abstraction of this process then generates H_{Morse}!

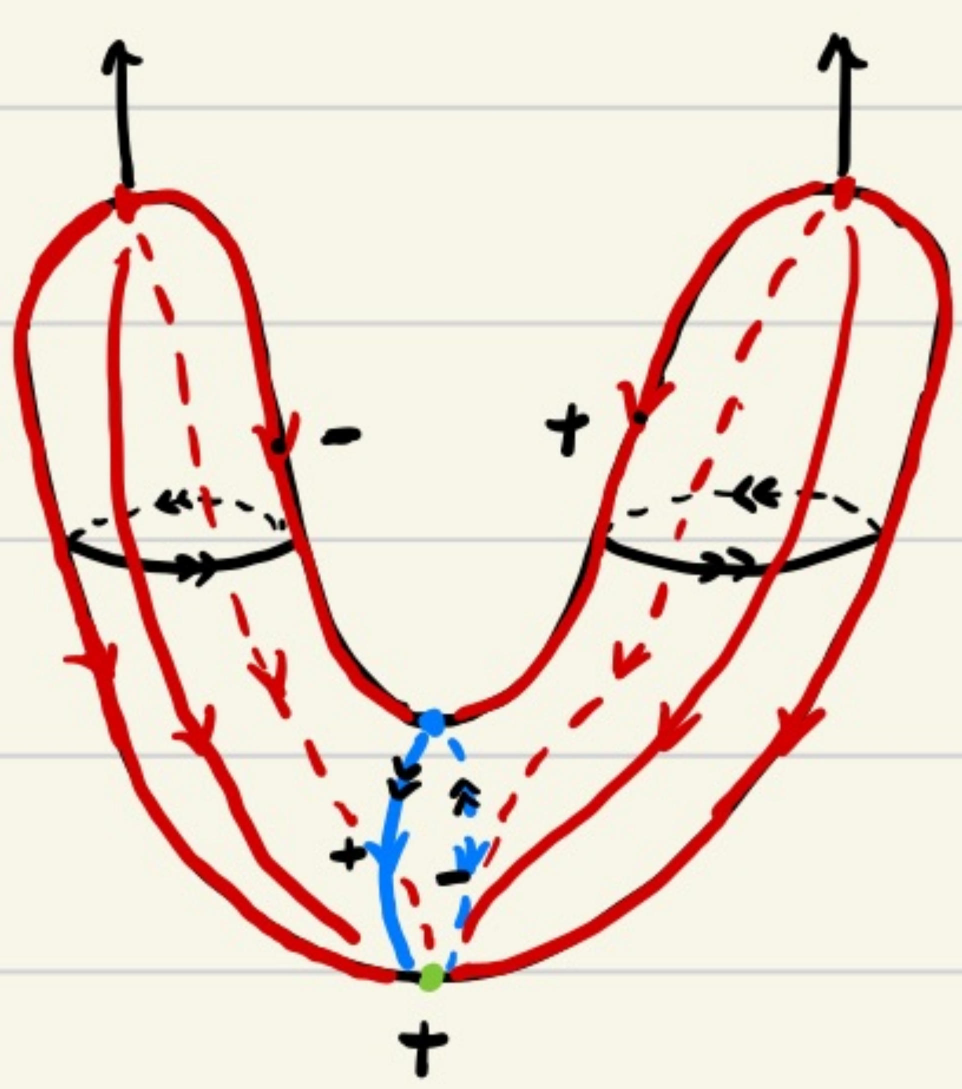
Conventions on Orientations:

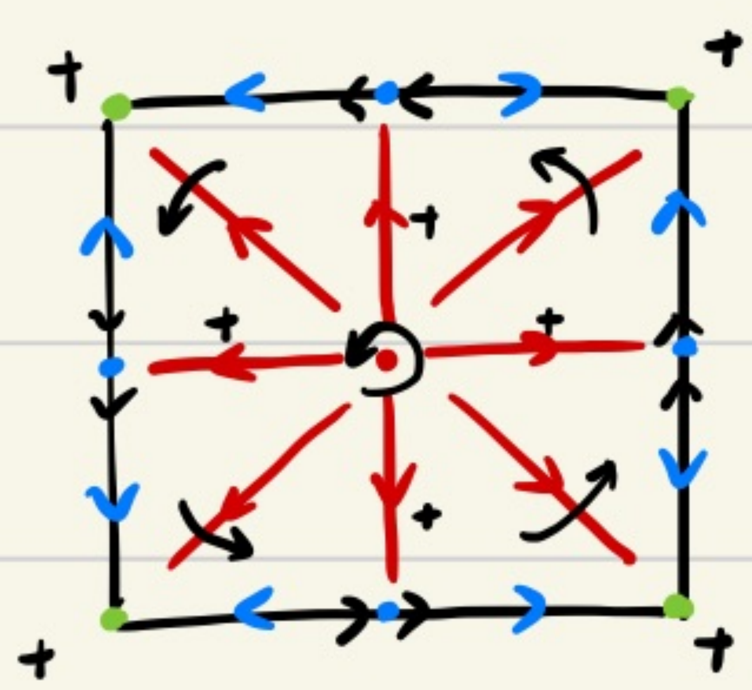
For each $p \in \mathrm{Crit}(f)$, choose an orientation of $\mathcal{D}(p)$. Define the orientation of each $\mathcal{M}(p,q)$ such that:

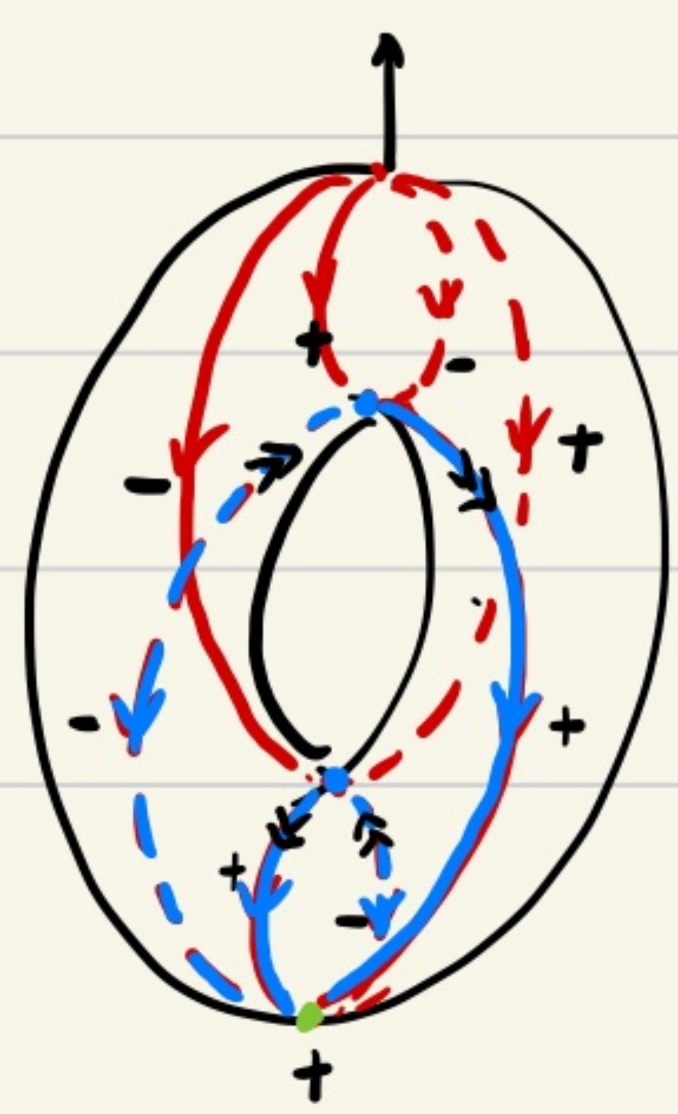
$$[\mathcal{D}(p)] = [\gamma][\mathcal{M}(p,q)][\mathcal{D}(q)]$$

- $[\partial \mathcal{D}(p)] = [\mathcal{M}(p,q)][\mathcal{D}(q)]$, since $[\gamma]$ always represents the outward normal orientation of $\mathcal{D}(p)$.
- $[\mathcal{D}(p) \cap \mathcal{A}(q)] = [\gamma][\mathcal{M}(p,q)]$, hence the complementary ort of $\mathcal{D}(p) \cap \mathcal{A}(q)$ in $\mathcal{D}(p)$ coincides with the complementary ort of $\mathcal{A}(q)$ in M

Examples:







Thm 2.6 (Compactification of $\mathcal{M}(p,q)$)

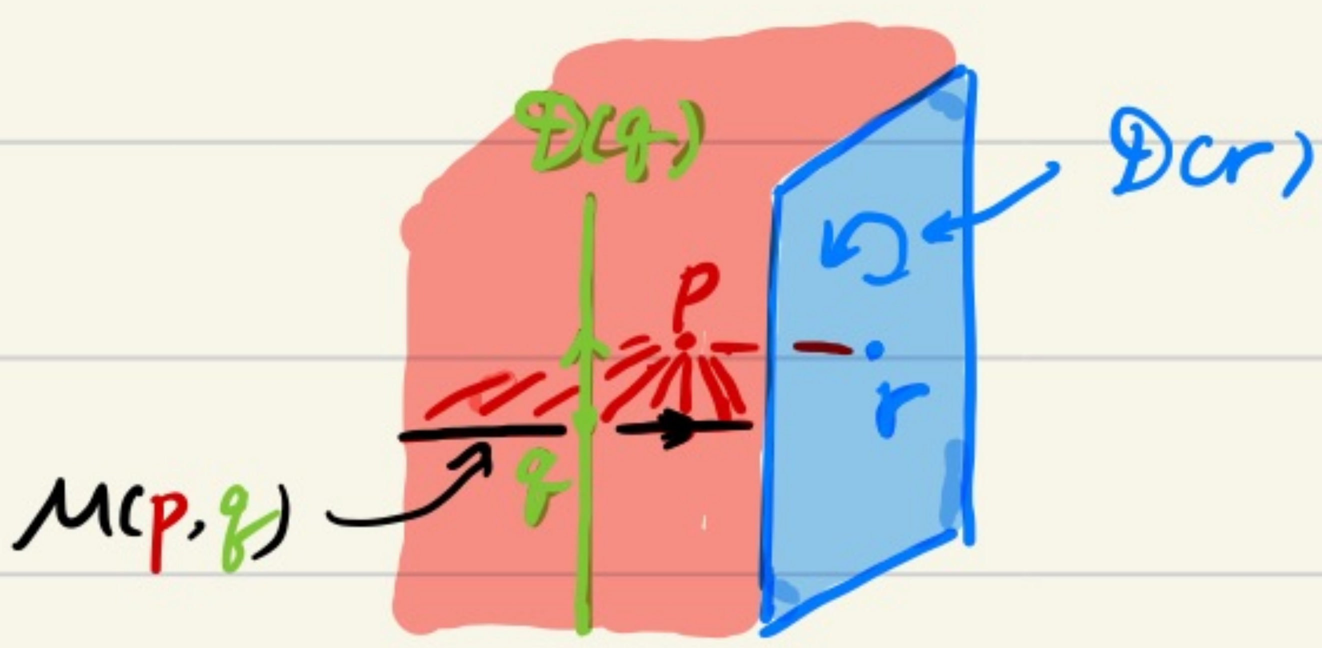
Given $p, q \in \mathrm{Crit}(f)$, $\mathcal{M}(p,q)$ has a natural compactification to a smooth mfd with corners $\overline{\mathcal{M}(p,q)}$, whose codimension k stratum is

$$\overline{\mathcal{M}(p,q)}_k = \bigcup_{\substack{r_1,\dots,r_k \in \mathrm{Crit}(f) \\ p, r_1, \dots, r_k, q \text{ distinct}}} \mathcal{M}(p,r_1) \times \mathcal{M}(r_1,r_2) \times \cdots \times \mathcal{M}(r_{k-1},r_k) \times \mathcal{M}(r_k,q)$$

In particular, $\mathcal{M}(p,q)$ is compact when $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$.
When $k=1$, as oriented mfds we have:

$$\partial \overline{\mathcal{M}(p,q)} = \overline{\mathcal{M}(p,q)}_1 = \bigcup_{\substack{r \in \mathrm{Crit}(f) \\ p, r, q \text{ distinct}}} (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)} \mathcal{M}(p,r) \times \mathcal{M}(r,q).$$

<u>Explanation on the orientations:</u>



$$[\mathcal{D}(p)] = [\gamma(p \to q)][\mathcal{M}(p,q)][\mathcal{D}(q)]$$

$$\Rightarrow [\partial \mathcal{D}(p)] = [\mathcal{M}(p,q)][\mathcal{D}(q)] = [\gamma(q \to r)][\partial \overline{\mathcal{M}(p,q)}][\mathcal{D}(q)];$$

Similarly:

$$[\partial \mathcal{D}(p)] = [\mathcal{M}(p,r)][\mathcal{D}(r)] = [\mathcal{M}(p,r)][\gamma(r \to q)][\mathcal{M}(r,q)][\mathcal{D}(q)]$$

$$= (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)} [\gamma(q \to r)][\mathcal{M}(p,r)][\mathcal{M}(r,q)][\mathcal{D}(q)]$$

$$\Longrightarrow [\partial \overline{\mathcal{M}(p,q)}] = (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)} [\mathcal{M}(p,r)][\mathcal{M}(r,q)]$$

$\blacksquare$

Def (also as a Prop) 2.7: (Morse Homology) Given (f,g) M-S pair.

We define the Morse complex (C_*, ∂) & (C^*, δ) as follows.

Let $\mathrm{Crit}_i(f)$ denote the set of index i crit pt of f.

- $C^i(f,g) = C_i(f,g) \triangleq \mathbb{Z}\,\mathrm{Crit}_i(f)$

- $\partial : C_i \to C_{i-1}, \quad p \longmapsto \sum_{q \in \mathrm{Crit}_{i-1}(f)} \# \mathcal{M}(p,q) \cdot q$

  $\delta : C_i \longrightarrow C_{i+1}, \quad p \longmapsto \sum_{r \in \mathrm{Crit}_{i+1}(f)} \# \mathcal{M}(r,p) \cdot r$

  Here we count $\#$ with the signs given the orientations.

$$\langle \partial^2(p), l \rangle = \sum_{q \in \mathrm{Crit}_{i-1}(f)} \langle \partial p, q \rangle \langle \partial q, l \rangle$$

$$= \sum_q \# \mathcal{M}(p,q) \times \# \mathcal{M}(q,l)$$

$$= -\# \partial \overline{\mathcal{M}(p,l)} = 0. \rightsquigarrow (C_*, \partial) \text{ is a complex.}$$

Similarly, (C^*, δ) is also a complex.

- The Morse homology (cohomology) grps $H_*^M(f,g)$ ($H_M^*(f,g)$) is defined to be the homology grps of (C_*, ∂) ((C^*, δ)).

Thm 2.8. There exists natural isomorphisms:

$$H_*^M(C_*, \partial) \approx H_*(M;\mathbb{Z}), \quad H_M^*(C^*, \delta) \approx H^*(M;\mathbb{Z})$$

Pf: The preceding discussions have shown:

$$(f, g) \rightsquigarrow \text{natural CW str. on } M$$

$$\Big\downarrow$$

$$H_*(M;\mathbb{Z}) \approx H_*^{CW}(M;\mathbb{Z}) \approx H_*^M(C_*, \partial)$$

this is obvious with
our conventions on orientations!

Alternative proof using currents and de-Rham theory can be found in Hutchings notes and the discussions on Morse-Bott Theory in Austin & Braam's paper. ☒

Rmk As examples, one can check the isomorphism on the preceding Morse-Smale funs on $\mathbb{S}^2$ and $\mathbb{T}^2$.

Cor 2.9 (Poincaré Duality) $H^*(M;\mathbb{Z}) \cong H_{n-*}(M;\mathbb{Z})$. ☒