The University of Sydney

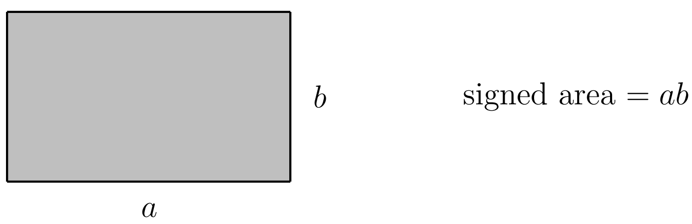
MOOC Introduction to Calculus

Notes for 'Equations and inequalities'

Important Ideas and Useful Facts:

(i) Positive, negative, nonnegative and nonpositive real numbers: A real number x is called *positive* if it appears to the right of zero on the real number line, in which case we write $a > 0$, and *negative* if it appears to the left of zero, in which case we write $a < 0$. We say that x is *nonnegative* if it is positive or zero, in which case we write $a \geq 0$, and *nonpositive* if it is negative or zero, in which case we write $a \leq 0$.

(ii) Multiplication and (signed) areas of rectangles: If a and b are real numbers then the *product* of a and b is the result of multiplying a by b, denoted typically by $a \times b$ or ab (juxtaposition). The product ab may be represented, geometrically, by the (signed) area of a rectangle with side lengths a and b. If both a and b are nonnegative then this is just the usual (nonnegative) area.



If one of a and b is positive and the other negative then the area ab of the rectangle is regarded as negative.

If both a and b are negative then the area ab is regarded as positive.

The formal rigorous explanation for this last fact is based on laws, or axioms, of arithmetic, and involves an advanced argument in abstract algebra that one typically sees in second or third year university mathematics.

Intuitively, one can just think of negatives of negatives being positive. It wouldn't make sense for the negative of a negative number to remain negative. This is mirrored in common speech when one uses double negatives: for example, to say "I didn't not say that!" is a complicated way of saying "I said that!" (though its use can help to emphasise the positive in a heated argument.) (The speech analogy also can break down, as some languages, including some colloquial English, use a double negative in certain turns of phrase to reinforce a single negative.)

For example, $2 \times 2 = 4$ and

$$2 \times (-2) \ = \ (-2) \times 2 \ = \ -4 \ ,$$

but

$$(-2) \times (-2) \ = \ -(2 \times (-2)) \ = \ -(-4) \ = \ 4 \ .$$

(iii) Factorisation of zero: If a and b are real numbers then the product ab is zero if and only if $a = 0$ or $b = 0$ (including the possibility that $a = b = 0$).

This follows from the representation of the product ab as the (signed) area of a rectangle with side length a and b. If the area vanishes, then at least one of the side lengths must vanish.

This fact about the factorisation of zero is very useful for solving equations. For example, if x is a real number such that

$$(x - 1)(x - 2) = 0$$

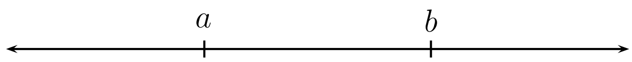
then either $x - 1 = 0$ or $x - 2 = 0$, so that either $x = 1$ or $x = 2$.

This fact clearly generalises to more than two factors. For example, if

$$(x - 4)(x + 1)(x + 3) = 0$$

then either $x - 4 = 0$, $x + 1 = 0$ or $x + 3 = 0$, so that either $x = 4$, $x = -1$ or $x = -3$.

(iv) Inequalities: Let $a, b \in \mathbb{R}$. We write $a < b$ if a appears to the left of b on the real number line, which is equivalent to the difference $b - a$ being positive (and the difference $a - b$ being negative).



In this case we say that a *is less than* b and b *is greater than* a, and also write $b > a$.

We write $a \leq b$ if $a = b$ or $a < b$, and write $a \geq b$ if $a = b$ or $a > b$.

For example,

$$-10 \;<\; -5 \;<\; -0.1 \;<\; -0.01 \;<\; 0 \;<\; 0.5 \;<\; 1 \;<\; 5\,.$$

Note that $-10 < 5$ even though you might think of -10 as a "larger" number than 5 in terms of "size".

(Later, we introduce the *magnitude* or *absolute value* of a real number, which captures the idea of "size" precisely. The word "larger" is ambiguous, and has to be read in context in mathematics.)

(v) Properties of inequalities: Let $a, b, c \in \mathbb{R}$.

(a) If $a < b$ and $c \leq d$ then $a + c < b + d$, so that, in particular, $a + c < b + c$.

(b) It follows also, from the previous part, that if $a < b$ then $a - c < b - c$.

(c) If $a < b$ and $c > 0$ then $ac < bc$ and $\dfrac{a}{c} < \dfrac{b}{c}$.

(d) If $a < b$ and $c < 0$ then $ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$, so that, in particular, $-a > -b$.

(e) If $0 < a < b$ then $\frac{1}{a} > \frac{1}{b} > 0$.

Variations of these properties hold using $>$, $\leq$ and $\geq$.

Examples and proofs:

1. Suppose that $2x+1<7$. By part (b) of (v),

$$2x \;=\; (2x+1)-1 \;<\; 7-1 \;=\; 6\;,$$

so that, by part (c) of (v)

$$x \;=\; \frac{2x}{2} \;<\; \frac{6}{2} \;=\; 3\;.$$

Hence $x<3$.

2. Suppose that $1-x<3x+6$. By part (a) of (v),

$$1 \;=\; (1-x)+x \;<\; (3x+6)+x \;=\; 4x+6\;,$$

so that, by part (b) of (v)

$$-5 \;=\; 1-6 \;<\; (4x+6)-6 \;=\; 4x\;,$$

so that

$$4x \;>\; -5\;.$$

Hence, by a variation of part (c) of (v),

$$x \;>\; -\frac{5}{4}\;.$$

Alternatively, from $1-x<3x+6$, one could deduce, by part (d) of (v), that

$$x-1 \;=\; -(1-x) \;>\; -(3x+6) \;=\; -3x-6\;,$$

so that, by part (a) of (v),

$$4x-1 \;=\; (x-1)+3x \;>\; (-3x-6)+3x \;=\; -6\;,$$

and therefore, also,

$$4x \;=\; (4x-1)+1 \;>\; -6+1 \;=\; -5\;.$$

Hence, by part (c) of (v),

$$x \;=\; \frac{4x}{4} \;>\; \frac{-5}{4} \;=\; -\frac{5}{4}\;,$$

yielding the same answer as before.

3. Here is a more challenging example. Suppose that we want to find all real numbers x such that

$$0 \;<\; \frac{1}{x-1} \;<\; \frac{2}{x+1}\;.$$

By part (e) of (v), reciprocating the positive fractions, this is equivalent to

$$x-1 \;=\; \frac{x-1}{1} \;>\; \frac{x+1}{2} \;>\; 0\;.$$

We may separate this out into two inequalities:

$$x-1 \;>\; \frac{x+1}{2} \qquad \text{and} \qquad \frac{x+1}{2} > 0\;.$$

From the first of these we get that $2x - 2 > x + 1$, so that $x > 3$. From the second we get that $x + 1 > 0$, so that $x > -1$. Since $-1 < 3$, we conclude, more simply, that

$$x > 3 ,$$

that is, x is a real number to the right of 3 on the real number line.

4. Suppose that we want to find all real numbers x such that

$$(x + 1)(x - 2) < 0 ,$$

that is, all real numbers x such that the product of $x + 1$ with $x - 2$ is negative. This can only occur if $x + 1$ is positive and $x - 2$ is negative, or if $x + 1$ is negative and $x - 2$ is positive.

However, if $x + 1$ is negative then $x + 1 < 0$, so that $x < -1$, and it follows that $x - 2 < -1 - 2 = -3 < 0$, so that $x - 2$ cannot possibly be positive.

Hence we conclude that $x + 1$ is positive and $x - 2$ is negative, that is

$$x + 1 > 0 \qquad \text{and} \qquad x - 2 < 0 ,$$

so that $x > -1$ and $x < 2$, that is,

$$-1 < x < 2 .$$

Hence, x is a real number to the right of -1 and to the left of 2 on the real number line.

5. We verify the claims of (v). Suppose that $a < b$ and $c \leq d$. Then $b - a > 0$ and $d - c \geq 0$. Hence

$$(b + d) - (a + c) = (b - a) + (d - c) > 0 ,$$

since adding a number greater than or equal to zero to a positive number always produces a positive number. This shows that $a + c < b + d$, and proves the first part of (v)(a). The second part of (v)(a) is immediate, because $c \leq c$. Part (b) of (v) is also immediate, because $-c \leq -c$.

We now verify part (c) of (v). Suppose that $a < b$ and $c > 0$. Then

$$bc - ac = (b - a)c > 0 ,$$

since $b - a > 0$ and a product of positive numbers is positive. This shows that $ac < bc$, proving the first part of (c). The second part follows immediately because, also, $\frac{1}{c} > 0$.

We now verify part (d) of (v). Suppose that $a < b$ and $c < 0$. Then

$$ac - bc = (a - b)c > 0 ,$$

since $a - b < 0$ and a product of negative numbers is positive. This shows that $ac > bc$, proving the first part of (d). The second part follows immediately because, also, $\frac{1}{c} < 0$. The third part is a special case by taking $c = -1$.

Suppose finally that $0 < a < b$. Then, certainly $b > 0$, so that $\frac{1}{b} > 0$. Also

$$\frac{1}{a} - \frac{1}{b} = \frac{b - a}{ab} > 0 ,$$

since $b - a > 0$, as $a < b$, and $ab > 0$ since $a > 0$ and $b > 0$, and a quotient of positive numbers is positive. This shows that

$$\frac{1}{a} > \frac{1}{b} > 0 ,$$

proving part (e), completing the verification of each of the claims of (v).