

# Fractal Geometry and Self-Similar Structures: Metric, Topological, Analytical, and Operator-Algebraic Invariants Across Classical, Quantum, and Data-Driven Paradigms

## Abstract

This survey presents a comprehensive synthesis of recent advances at the intersection of fractal geometry, metric spaces, operator algebras, topological invariants, and analytical methods, with a view toward both foundational theory and computational practice. Motivated by the pervasive complexity observed in natural, physical, and data-driven systems, the article systematically develops the intertwined frameworks of self-similarity, metric and topological invariants, and analytic structures underpinning the classification and analysis of irregular and high-dimensional spaces.

The scope covers classical constructs—such as Hausdorff and box dimensions, iterated function systems, and percolation structures—while extending to modern operator-theoretic approaches including spectral invariants, K-theory, and groupoid cohomology. Key contributions include: (i) elucidation of the analytic and combinatorial underpinnings of fractal dimension theory, projection and slicing theorems, and microset analysis; (ii) state-of-the-art developments in operator algebras, noncommutative function theory, and quantum invariants, with applications to topological phases, 3-manifolds, and quantum spin systems; (iii) integration of variational, metric, and nonlocal methods for PDEs and SPDEs on fractals and irregular spaces; (iv) advances in computational topology, persistent homology, and data analysis via simplicial complexes; and (v) the joint deployment of analytic, algorithmic, and machine learning approaches for feature extraction, scaling law discovery, and invariance quantification in real-world and synthetic datasets.

The survey highlights the unification of analytic regularity, geometric rigidity, and combinatorial flexibility, detailing new breakthroughs in embedding theory, isoperimetric inequalities in groups, Banach space approximation properties, and game-theoretic perspectives on dimension. It accentuates persistent methodological challenges, including the extension of invariants to infinite-dimensional and noncommutative settings, the interplay between spectral and topological data, and the limits of current machine learning models in capturing geometric and topological complexity.

Concluding with an integrated outlook, the article charts open problems and outlines future directions emphasizing the need for new mathematical languages and computational frameworks capable of bridging geometry, analysis, operator theory, quantum methods, and data-driven paradigms. This work thus serves as both a reference and a roadmap for ongoing and interdisciplinary

research at the frontier of modern geometric analysis and its applications.

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## 0.1 Motivation

The intricate interplay among fractal geometry, metric spaces, topological and algebraic invariants, and analytical properties underlies some of the most dynamic and far-reaching developments in contemporary mathematics. These foundational constructs comprise an essential toolkit for analyzing the complexity inherent in mathematical structures that model diverse phenomena across scientific and engineering domains. For instance, fractal geometry is indispensable in capturing the irregularities observed in natural objects, offering models that surpass the regularities of classical Euclidean forms. Metric spaces enhance our ability to rigorously quantify notions of distance and convergence, thereby illuminating structures in both pure and applied settings. Topological and algebraic invariants—such as Betti numbers and group cohomologies—support a robust classification framework that interconnects geometry, symmetry, and algebraic structure. Meanwhile, analytical properties ensure the responsiveness of these frameworks to limiting processes and functional relationships, thereby extending their applicability into areas such as analysis, probability, and mathematical physics. The confluence of these areas not only advances theoretical understanding but also drives progress in fields such as data analysis, dynamical systems, and quantum computation, fueling both mathematical depth and technological innovation.

Despite these strengths, current approaches face several open challenges. For example, the adaptation of fractal geometry and metric methodologies to high-dimensional or stochastic data can lead to computational intractability and ambiguity in interpretation. The application of topological invariants to real-world datasets sometimes encounters difficulties due to noise sensitivity and limited scalability. Moreover, while analytical tools provide strong theoretical underpinnings, they may not always offer robust or interpretable results in interdisciplinary contexts involving heterogeneous or incomplete data.

Alternative perspectives suggest hybrid models that integrate statistical learning with topological or fractal features to improve robustness and applicability. There remains, however, the unresolved issue of balancing model expressiveness with computational feasibility—a gap particularly evident in emerging domains such as topological data analysis and quantum information theory. Addressing these gaps, especially through new algorithms or refined invariants, constitutes an open frontier for both theoretical research and practical innovation.

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## 0.2 Overview of Key Concepts

Several interrelated concepts form the scaffolding of modern theories in this domain and underpin their computational treatment:

- **Fractal Sets:** These structures, distinguished by self-similarity and non-integer Hausdorff dimensions, challenge traditional geometrical intuitions. Self-similarity exemplifies recursive construction methods, producing objects with intricate detail at every scale. The Hausdorff dimension serves as a central invariant, frequently exceeding the familiar topological dimension, and is critical in classifying fractal and pathological sets within metric spaces.
- **Metric Geometry:** Generalizing classical distance notions, metric geometry provides a unified language for comparing spaces, extending far beyond the limitations of Euclidean geometry. Its significance is particularly pronounced in the theory of optimal transport, which analyzes geometric and functional relationships through probability measures and cost-minimizing mappings. This approach not only interlinks analysis and geometry but also facilitates quantitative comparisons in applications such as machine learning and computer vision.
- **Group-Theoretic and Operator-Algebraic Frameworks:** Group-theoretic perspectives elucidate the role of symmetry in shaping both geometric and algebraic properties, affording methods for constructing and analyzing spaces with specified transformation behaviors. Operator algebras, at the intersection of functional analysis and quantum theory, provide rigorous methods for investigating infinite-dimensional phenomena and encoding dynamic and symmetrical properties via algebraic invariants. Extending these methods to the realm of non-commutative geometry creates new opportunities in pure mathematics and theoretical physics.

To succinctly compare these foundational domains, their principal objects of study, and analytical tools, the following table is provided.

As shown in Table 1, the diversity of frameworks is unified by a common emphasis on invariants and analytical structures capable of quantifying and classifying complexity across mathematical and applied contexts.

## 0.3 Structure and Survey Roadmap

This survey aims to clarify and unify these thematic threads by interweaving classical foundations, recent theoretical progress, and emerging computational methodologies. The structure is intentionally designed to foreground both the intrinsic connections between the outlined concepts and the ways their mutual influence generates novel analytical and computational tools.

The organization of the survey proceeds through the following thematic sections:

**Fractal and Metric Structures:** Deep analyses of self-similar and metric spaces, illuminating geometric and measure-theoretic aspects.

**Topological and Algebraic Invariants:** Rigorous exploration of classification schemes, symmetry, and cohomological tools.

**Operator Algebras and Quantum Structures:** Examination of infinite-dimensional systems, dynamics, and operator-algebraic invariants.

**Optimal Transport and Computational Advances:** Discussion of cross-disciplinary methodologies that render abstract mathematical ideas practically accessible, with special attention to advancements in data analysis and real-world applications.

This purposeful sequencing enables a holistic and integrated view of the subject, encouraging the reader to appreciate both the depth and unity of contemporary developments in the field. The survey thus serves as a bridge between foundational mathematical theory and its myriad applications, accentuating the profound interplay among geometry, algebra, analysis, and computation.

## 1 Fundamental Concepts Fractal Geometry, Metric Spaces, and Self-Similar Sets

### 1.1 Fractal Sets and Classical Constructions

Fractal geometry originates in the study of sets exhibiting intricate, recursive structures, as epitomized by classical examples such as Julia sets and the Mandelbrot set. These canonical fractals not only display geometric self-similarity, but also possess complex analytical and topological characteristics, including elaborate boundary behavior and sensitivity to parameter changes. The concept of fractal sets has been broadened via fractionalized constructions, yielding a flexible framework that accommodates diverse anomalous geometric phenomena. Recent advances introduce explicit criteria for "fractionalizing" fractals—namely, systematic interpolation techniques between classical and fractional analogues, with concrete implementation on Julia sets and the Mandelbrot set [1]. These criteria delineate the persistence of self-similar properties under interpolation, thus generating novel classes of fractals with adjustable dimensional and regularity profiles.

Central to the analysis of such sets is the notion of self-similarity, conferring invariance under specific scaling transformations. Classic constructions, including the Cantor set and Sierpinski gasket, are generalized through iterated function systems (IFS), which enable the explicit computation of both Hausdorff and box-counting dimensions. The extension of these ideas to non-Euclidean contexts, such as projective spaces, further highlights the universality and resilience of fractal phenomena, with the box and Hausdorff dimensions of attractors typically matching the solutions to generalized Moran equations [2, 3]. Analytical consequences are particularly observed in the oscillatory behavior of associated zeta functions, whose complex singularities encode key geometric invariants of the fractal set. The interplay between zeta function singularities and geometric features, such as Minkowski measurability and oscillations in scaling functions, underpins a modern, unified perspective in fractal geometry [3–5].

Further topological richness is revealed through properties such as (non-)formality and the existence of nontrivial Massey products, illustrating the breadth of phenomena accessed by fractal-inspired constructions in analysis and topology. The framework extends beyond Euclidean spaces to more general metric spaces, including those satisfying doubling and Poincaré inequality properties, facilitating the coverage of fractal-like sets by analytic subsets that preserve essential invariants—most notably under bi-Lipschitz

Table 1: Principal Frameworks, Core Objects, and Key Analytical Tools

Framework	Main Objects of Study	Key Analytical Tools
Fractal Geometry	Self-similar sets, fractals	Hausdorff dimension, scaling limits
Metric Geometry	Metric spaces, distances	Metric invariants, optimal transport
Algebraic Topology	Topological spaces, invariants	Betti numbers, cohomology groups
Operator Algebras	C*-algebras, von Neumann algebras	Spectral analysis, K-theory
Group-Theoretic Methods	Symmetry groups, transformation spaces	Group actions, representation theory

mappings and measure-theoretic equivalence [5–7]. This approach is crucial for transferring geometric intuition to metric measure spaces of broader generality.

1.2 Dimensional Homogeneity and Metric Space Structure

Dimensional homogeneity is pivotal for ensuring both internal consistency and practical applicability in mathematical models of fractals and metric spaces. It requires that all operations and equations involving geometric invariants strictly adhere to underlying unit structures. As highlighted by prior critiques [8], any negligence in maintaining dimensional consistency can compromise entire theoretical frameworks. When carefully observed, dimensional homogeneity directly informs the analytic and algebraic manipulations allowed within fractal and metric geometry.

Metric space theory establishes the foundations for rigorous quantification of self-similarity and scale invariance, surpassing the constraints of classical Euclidean domains. The construction and study of metrics across Euclidean, non-Euclidean, and more abstract topological contexts afford precise definitions of distance and proximity, by means of either intrinsic or extrinsic criteria. The verification of whether a given function constitutes a true metric often involves nontrivial regularity or derivative conditions, which provide practical yet robust instruments for generating new geometric contexts [9].

Questions of embedding and universality are critical in understanding the preservation of fractal and metric space properties under bi-Lipschitz maps and broader morphisms. It is now recognized that the universality of metric spaces—such as the Urysohn space or the spaces of bounded metrics over sufficiently large sets—is strongly contingent on cardinal characteristics and topological constraints, thereby forging new links between analysis, topology, and set theory [10, 11]. The feasibility of isometric embeddings for compact subsets, and the precise limitations therein, expose delicate boundaries in the representation capacity of general metric universes.

A further central area lies within differentiability spaces, where the richness of differentiable structures is intimately connected to the presence of analytic covers by bi-Lipschitz subsets associated with spaces supporting local Poincaré inequalities [6, 7]. Results in this domain ensure the translation of classical theorems—prominently the Besicovitch-Federer projection theorem—to arbitrary metric spaces, though often subject to modifications and exceptions [12]. The analytic and geometric structure of these spaces is closely linked to their support of nontrivial invariants,

particularly those calculated through measures such as Hausdorff, box, and Assouad dimensions [4, 13–18].

Beyond such structural invariants, operator-algebraic frameworks reveal deep interrelations between the coarse geometry of a space and its associated operator algebras. The rigidity of Roe algebras, where isomorphism classes reflect coarse equivalence, illustrates that large-scale geometric properties may be entirely encoded within functional-analytic constructs [19]. Alongside these, topological and algebraic invariants—including bordism and those derived from the Atiyah-Patodi-Singer index theorem—broaden the set of analytically accessible invariants in both commutative and noncommutative geometries [20]. The development of secondary invariants and their analytic-topological correspondence highlights the power of blending homotopy-theoretic and analytic methods, especially in the study of string bordism and related index theories.

The investigation of singular phenomena—such as the disjointness of certain Sobolev spaces on Laakso-type or edge-iterated graph system fractals—challenges classical intuitions, revealing a diversity of energy profiles and potential theories on highly non-Euclidean domains [21, 22]. These constructions have significant ramifications for analysis, probability, and the theory of group actions on metric spaces.

The invariants shown in Table 2 serve as central tools for distinguishing between different types of fractal and metric spaces and are calculated according to the analytic framework of the domain under investigation.

1.3 Analytical and Topological Tools

The calculation and analysis of invariants in fractal and metric spaces necessitate a nuanced blend of analytical, functional, and variational techniques. Fractal tube formulas—generalizing the classical Weyl-Berry conjecture—employ Mellin transforms and residue calculus to connect the oscillatory scaling of volumes to the pole structure of meromorphically continued zeta functions. This methodology translates the geometric complexity of fractals into explicit sequences of complex-analytic residues, each encoding intricate geometric and topological information [3, 4].

In computational topology, persistent homology has proven particularly effective for extracting multiscale topological features from fractal and nonsmooth metric spaces. By analyzing how Betti numbers and the Euler characteristic of simplicial complexes evolve with a varying scale parameter, it generates robust, scale-invariant descriptors for complex geometries [23]. Analytical studies confirm that, for Rips and Čech complexes derived from data sampled on Riemannian manifolds, topological invariants such as Betti numbers and Euler characteristic behave as Lipschitz functions of the

**Table 2: Key Invariants Across Metric Spaces and Their Calculation Methods**

Invariant	Typical Calculation Method	Applicable Spaces
Hausdorff Dimension	Covering arguments, scaling exponents	General metric and fractal spaces
Box Dimension	Grid/box covering, scaling with box size	Fractal and self-similar sets
Assouad Dimension	Scaling of covering numbers over all scales	Doubling metric spaces, ultrametrics
Operator Algebra Invariants	K-theory, spectral	Metric spaces with coarse geometry
Topological Invariants	Homology/cohomology, persistent homology	Simplicial complexes, fractals

scale parameter, and, over suitable intervals, the Betti curves even converge to the true invariants of the underlying manifold. These properties offer both theoretical assurances and practical computational strategies.

Advanced variational techniques further enable explicit computation of fractal dimensions—such as box-counting, Hausdorff, and Assouad—providing practical estimation and bounding tools for both pure and applied settings [15, 17, 24]. For instance, recent work on Laakso-type fractal spaces (IGS-fractals) has constructed self-similar  $p$ -energy forms and revealed the existence of singularities in Sobolev spaces across different exponents, highlighting analytical phenomena absent in conventional fractals [24]. Studies on box dimension in generalized affine fractal interpolation functions demonstrate how geometric complexity correlates with the spectral properties of associated scaling matrices, yielding explicit dimension formulas under suitable hypotheses [25]. The connection between analytic and topological features is further exemplified by results showing that, for certain number-theoretical fractal functions, the Hausdorff and box-counting dimensions of their graphs can be computed explicitly, emphasizing both their fractal nature and measure-theoretic subtleties [15].

The resilience of analytic, topological, and algebraic invariants under generalizations—such as fractal interpolation on the projective plane or iterated function systems with non-affine contractions—attests to the versatility and depth of modern fractal theory [3, 25–27]. For example, fractal interpolation has been extended from the Euclidean case to the real projective plane, where attractor graphs of projective IFSs possess topological dimension one but can achieve fractal dimensions arbitrarily close to two, fully determined by contraction ratios [3]. The study of topological invariants in finite and chiral symmetric systems uses both real-space and bulk properties for robustness under disorder and finite-size effects [26]. Furthermore, recent quantitative advances link the Lipschitz constant of self-maps to the maximal degree and scaling properties, with trichotomies governed by manifold topology and sharp transitions established between formal and nonformal types [27].

Functional and spectral invariants arising within fractal geometries have demonstrated profound applicability, particularly in areas such as deep learning and signal analysis. While deep neural networks often struggle to extract features grounded in fractal dimensions, direct computational strategies or shallow architectures exploiting fractal invariants have, in some scenarios, outperformed deeper models in both accuracy and efficiency [28]. This reveals

the subtlety of fractal complexity and emphasizes the value of explicit analytic invariants in the characterization of space, data, and physical phenomena.

In summary, the contemporary study of fractal sets, metric spaces, and self-similar structures emerges from a synergy of classical geometric constructs, advanced algebraic and analytical generalizations, and an evolving suite of tools for invariant computation. This extensive foundation not only underpins robust theoretical models but also catalyzes new avenues for interdisciplinary innovation and methodological advancement.

## 2 Metric Spaces Geodesic Structure, Non-branching Properties, and Optimal Transport

This section aims to provide a comprehensive overview of the geodesic structure in metric spaces, focusing particularly on non-branching properties and their implications for the theory of optimal transport. The objectives are to present foundational concepts, highlight key technical approaches, and clarify limitations and open questions in this area.

### Section Objectives

The primary goals of this section are: To describe the fundamental properties of geodesic spaces, with attention to the structures arising from non-branching behavior; To analyze how these geometric features relate to the existence and stability of optimal transport plans; To identify and contrast major methods, their strengths and limitations; To highlight open research problems within these intersecting themes.

### Summary and Open Problems

In summary, understanding the geodesic structure in metric spaces, especially the role of non-branching properties, is central to progress in optimal transport theory. Methods leveraging non-branching assumptions yield clear existence and uniqueness results but may fail in more general, highly branching contexts, where alternative or relaxed approaches are necessary.

Key limitations of prevailing approaches include their dependence on strong geometric regularity and the challenges posed by singularities or complex branching behavior. As a result, alternative frameworks remain an area of intense investigation.

Open problems in this domain include: Clarifying the full range of metric space geometries where non-branching can be effectively characterized; Developing transport theory for spaces with controlled forms of branching or partial non-branching; Constructing

new techniques to overcome the limitations of current methods where branching obstructs standard approaches.

In conclusion, ongoing research is directed at bridging these gaps, aiming to extend the robustness and applicability of optimal transport results across broader classes of metric spaces.

## 2.1 Metric Spaces and Geodesic Structure

A central aim in metric geometry and optimal transport theory is to elucidate the relationship between the geodesic structure of a metric space  $(X, d)$  and analytical phenomena arising in the associated Wasserstein spaces  $P_p(X)$ . Key to this study is distinguishing among geodesic, uniquely geodesic, and non-branching spaces. Specifically:

**Geodesic:** Any pair of points in  $X$  can be connected by at least one geodesic.

**Uniquely geodesic:** For each pair of points, the connecting geodesic is unique.

**Non-branching:** No geodesic segment can bifurcate; concretely, if geodesics  $[x, y]$  and  $[x, z]$  from  $x$  coincide up to some time  $t$ , then necessarily  $y = z$ .

Optimal transport theory provides a compelling, intrinsic framework for characterizing these properties. A pivotal result in this area establishes the equivalence between the non-branching property of  $(X, d)$  and its induced Wasserstein space  $P_p(X)$  for  $p > 1$ . That is, the non-branching nature of  $X$  precisely determines that of the Wasserstein space:  $P_p(X)$  is non-branching if and only if  $X$  is non-branching [29]. This equivalence, formulated at the level of optimal dynamical plans—measures on path spaces—demonstrates that the geometry of  $X$  directly governs the branching structure in  $P_p(X)$ . Accordingly, analytic phenomena traditionally examined in finite-dimensional Riemannian settings, such as geodesic uniqueness and tangent cone regularity, acquire robust metric-measure counterparts, fully accessible via the machinery of optimal transport.

Among the most powerful frameworks for distilling these relationships are spaces satisfying the Measure Contraction Property (MCP). In MCP-spaces, the local geometry is tightly regulated: every metric tangent cone at each point is a normed (possibly non-strictly convex) vector space, and MCP-spaces themselves can be broadly classified according to these tangent structures. Importantly, a space is non-branching if and only if every tangent cone at each point is uniquely geodesic, highlighting a deep correspondence between infinitesimal and global structure. These developments allow, for example, alternative proofs of foundational results like the Cheeger–Colding splitting theorem within the metric-measure context. As a result, regularity, splitting, and non-branching phenomena can be systematically unified through the lens of optimal transport [29].

## 2.2 Analytical and Structural Implications

This equivalence between non-branching in  $(X, d)$  and  $P_p(X)$  not only reorganizes the landscape of known classifications, but also provides practical, intrinsic criteria for identifying non-branching spaces beyond smooth or differentiable settings. The local analysis of tangent cones, which approximate the geometry at each point, enables concrete criteria for unique geodesicity and non-branching.

In particular, it has been shown that  $X$  is non-branching if and only if every tangent cone at every  $x \in X$  is uniquely geodesic, establishing a precise link between local geometric properties and the global structure of both  $X$  and  $P_p(X)$  [29]. This approach eliminates the need for global parametrizations or manifold assumptions in the detection of non-branching.

The classification of tangent cones available in this framework extends far beyond classical smooth or Riemannian contexts, applying to high-dimensional, non-Euclidean, or even singular metric-measure spaces. Recent results clarify that, for MCP-spaces (those satisfying the Measure Contraction Property), the tangent cones at all points are normed vector spaces, even in the absence of strict convexity, thereby broadening the scope of non-branching analysis [29]. This has powerful consequences for advancing key open questions, including: the full characterization of MCP-spaces without global curvature assumptions; the stability of non-branching properties under metric-measure Gromov–Hausdorff limits; and the analytic implications, such as new regularity and rigidity phenomena, that arise from (non-)branching structures in a broader class of spaces.

Consequently, the synthesis of analytic and geometric criteria, empowered by optimal transport, bridges local geometric regularity and global structure. This unified perspective enriches the structural theory for non-branching in optimal transport, offering both theoretical insights and practical analytical tools for the study of general metric-measure spaces.

## 2.3 Variational and Metric Methods in Evolution Equations

This section explores the fundamental objectives underlying variational and metric approaches to evolution equations, with an emphasis on characterizing analytic regularity and convergence properties of time-discretized schemes in Wasserstein spaces.

The significance of these structural insights is especially evident in the theory of gradient flows and evolution equations in the Wasserstein space. The discrete-time variational scheme—introduced by Jordan, Kinderlehrer, and Otto—interprets the evolution of Fokker–Planck equations

$$\partial_t \varrho = \Delta \varrho + \nabla \cdot (\varrho \nabla V)$$

as an iterative minimization problem within  $P_2(X)$ . Recent advancements demonstrate that, under suitable geometric and analytic assumptions (notably, convexity and smoothness of the domain, and initial data strictly positive and regular in Sobolev spaces), the discrete approximations to this scheme exhibit strong convergence in the mixed  $L_t^2 H_x^2$  norm, improving upon the traditionally weaker convergence in  $L^1$  or Wasserstein distance [30]. As shown in [30], this result requires the domain to be bounded and smoothly convex, and the initial datum to be uniformly positive with sufficient Sobolev regularity. The origin of this strong regularity lies in the optimal transport-based inequalities satisfied by the discrete solutions, mirroring those in the continuous case and underscoring the pivotal role of geodesic and non-branching structures in ensuring analytic regularity.

While these methods yield higher-regularity convergence under favorable assumptions, current approaches are limited by technical

requirements, including strong convexity and strict regularity of both the domain and initial data [30]. Relaxing these constraints for more general geometries or less regular inputs remains an open challenge. Moreover, it is unclear to what extent these strong regularity results generalize to broader classes of evolution equations or to non-gradient flow settings. Addressing these limitations could broaden the applicability of variational and metric techniques in PDE analysis.

*Summary and Outlook.* In summary, variational and metric methods have provided a robust framework for the study of gradient flows and evolution equations, offering powerful tools to establish convergence and regularity—in particular, bridging discrete and continuous perspectives through Wasserstein geometry. However, addressing the limitations regarding domain geometry and data regularity represents a key research direction for further advancements in this area.

**2.3.1 Extension to the Noncommutative Setting.** A remarkable extension of these variational and metric methods occurs within non-commutative probability, specifically for quantum Markov semigroups evolving states on finite-dimensional unital  $C^*$ -algebras. For ergodic semigroups with detailed balance, the dynamics manifest as gradient flows of the relative entropy (with respect to a stationary state) in a quantum analog of the 2-Wasserstein geometry [31]. Notably, properties such as entropy convexity and exponential decay to equilibrium—fundamental aspects in classical optimal transport—persist in this quantum metric framework, sometimes with even stricter and uniform behavior. Thus, classical features of optimal transport—rigidity, uniform convexity inequalities, and exponential convergence—naturally admit quantum analogs, with their structural sharpness dictated by geometric properties paralleling non-branching and geodesicity in the commutative case.

The reach of optimal transport and metric methods thus extends deeply into analytic regularity theory and the study of evolution equations. Central to these analytic, topological, and variational properties—whether in classical or quantum settings—are the geometric features of uniqueness, non-branching behavior, and the structure of tangent cones [29–31]. This interplay between geometry and analysis forms a unifying theme, facilitating new directions in both theory and applications.

## 2.4 Group-Theoretic Metric Geometry and Isoperimetric Inequalities

This subsection aims to elucidate the interplay between group-theoretic metric geometry and isoperimetric inequalities, emphasizing core concepts, principal methods, and the landscape of prevailing research challenges. Our main objectives are to clarify how metric geometry in the group-theoretic context informs the analysis of isoperimetric profiles, and to highlight the strengths, limitations, and open questions within current approaches.

In metric geometry, groups are often endowed with metrics induced by word length or Cayley graphs, yielding a natural geometric framework. Within this context, isoperimetric inequalities provide crucial quantitative connections between volume and boundary size—typically articulated as functions relating the growth of

balls and their "surface areas." This theoretical standpoint enables a rigorous examination of group properties such as amenability, growth rates, or filling functions.

Approaches in this domain frequently involve constructing or analyzing Cayley complexes, investigating Dehn functions, and leveraging embedding techniques to relate combinatorial and geometric properties. One significant strength of geometric group-theoretic methods is their capacity to translate abstract algebraic properties into more tractable geometric or combinatorial problems, facilitating powerful general results. Conversely, precisely determining isoperimetric behavior for specific classes of groups—or providing tight bounds—often proves difficult, particularly in non-positively curved or highly non-amenable settings.

Throughout the literature, competing techniques such as filling invariants, coarse embeddings, or Cheeger-type inequalities offer both complementary tools and contrasting technical advantages. For example, while Dehn function computations are highly informative for finitely presented groups, they may omit subtler geometric phenomena; coarse embeddings provide flexibility but may lack sharp isoperimetric control.

Despite considerable advances, substantial open questions remain. These include the explicit determination of isoperimetric functions in important group families, the extent to which geometric invariants classify quasi-isometry types, and the clarification of connections between isoperimetric profiles and computational group properties.

In summary, the study of group-theoretic metric geometry and isoperimetric inequalities forms a vibrant intersection of algebra and geometry. While robust foundational results anchor this area, continuous progress is driven by the development of novel methods and the pursuit of unresolved challenges within the field.

**2.4.1 Discrete Heisenberg and Related Groups.** The discrete Heisenberg group, denoted  $\mathbb{H}_{\mathbb{Z}}^{2k+1}$ , exemplifies the intricate intersection of group-theoretic structure with geometric measure theory, particularly within combinatorially rich settings. This group is generated by  $2k$  "horizontal" elements  $a_1, b_1, \dots, a_k, b_k$  and a central "vertical" generator  $c$ , subject to specific commutator relations that fundamentally shape its geometry. The rigorous geometric analysis of  $\mathbb{H}_{\mathbb{Z}}^{2k+1}$  proceeds from these algebraic foundations, enabling the precise formulation of both horizontal and vertical perimeters, which serve as discrete analogues to classical isoperimetric constructs.

For a finite subset  $\Omega \subset \mathbb{H}_{\mathbb{Z}}^{2k+1}$ , the horizontal boundary,  $\partial_h \Omega$ , consists of all ordered pairs that differ by a horizontal generator. In contrast, the vertical perimeter,  $|\partial_v \Omega|$ , is quantified by summing contributions derived from commutators involving powers of the central generator  $c$ , using an  $L^2$ -weighted sum over all nontrivial powers. These metrics capture both the deviation from commutativity and the inherent anisotropy of the group, forming the analytic backbone for studying its geometric and functional properties.

A fundamental advance was the discovery of a sharp, dimension-dependent isoperimetric inequality for these discrete groups. Specifically, for  $k \geq 2$ , it was established that the vertical perimeter is universally controlled by the horizontal perimeter, as  $|\partial_v \Omega| \lesssim \frac{1}{k} |\partial_h \Omega|$  [32]. This result was achieved via an innovative decomposition of finite-perimeter sets into "intrinsic corona pieces," a method that generalizes the corona decomposition technique—previously

limited to Euclidean and low-dimensional settings—to the Heisenberg group’s intrinsic geometry. The robustness of this approach enables precise endpoint estimates for various singular integrals, most notably facilitating a rigorous transition from  $W^{1,2} \rightarrow L_2(L_2)$  bounds to the challenging  $W^{1,1} \rightarrow L_2(L_1)$  endpoint setting. This technical advance underpins a range of applications in functional analysis and geometric group theory by demonstrating how higher-dimensional Heisenberg groups ameliorate potential pathological behaviors in vertical boundaries, thereby restoring analytic regularity.

**2.4.2 Functional and Embedding Results.** The interplay between horizontal and vertical perimeters has far-reaching consequences for function space analysis and the theory of metric embeddings—especially in the context of mapping Heisenberg group balls into linear metric spaces. Notably, the impossibility of embedding large balls (of radius  $n$ ) in the word metric on  $\mathbb{H}_\mathbb{Z}^5$  into  $L_1$  with distortion less than  $\sqrt{\log n}$  directly follows from the perimeter inequality. This connection provides a rigorous geometric basis for lower bounding the Goemans–Linial integrality gap in the Sparsest Cut problem [32], demonstrating how sophisticated group-theoretic and geometric constructions lead to concrete obstructions for low-distortion embeddings.

These embedding obstructions can be summarized as follows:

The underlying analytic framework relies on refined estimates for singular integrals adapted to the geometry of finitely generated groups. The endpoint Sobolev-type inequalities—made possible by the corona decomposition—capture the complex balance between the group’s algebraic structure and the regularity properties of functions defined on it. In particular, passing to the  $W^{1,1} \rightarrow L_2(L_1)$  endpoint regime, a transition rarely feasible in earlier metric measure settings, marks a critical methodological advance that expands the reach of Sobolev-space techniques and reveals intricate rigidity and flexibility phenomena in embedding theory within the context of Heisenberg groups.

**2.4.3 Connections to Metric Spaces.** The analytic and combinatorial methods developed for discrete Heisenberg groups have significant implications in the broader field of metric geometry, especially in understanding embeddings, distortion, and the large-scale (coarse) structure of general metric spaces. Notably, advancements in the study of operator-algebraic invariants—such as Roe algebras—demonstrate that, for uniformly locally finite metric spaces, the isomorphism class of the associated Roe algebra completely characterizes coarse equivalence of the underlying space [18]. This operator-algebraic framework provides a precise and functorial correspondence between large-scale geometric structures and algebraic invariants, bridging geometric analysis and  $C^*$ -algebra theory.

Recent work rigorously establishes this connection: the isomorphism of Roe algebras implies coarse equivalence, and the group of outer automorphisms of a Roe algebra corresponds canonically to the group of coarse equivalences of the metric space up to closeness, offering a powerful rigidity result. This relationship can be summarized as follows:

Roe algebra isomorphisms	Coarse equivalence
Outer automorphisms	Coarse equivalence (modulo closeness)

Earlier studies had indicated, and these new results now confirm in greater generality, that Roe algebras serve as definitive operator-algebraic invariants, categorically distinguishing large-scale geometric types of metric spaces with bounded geometry [18]. This aligns with broader efforts in metric geometry to understand the functorial interplay between analytic, algebraic, and geometric structures [11].

Complementing these operator-algebraic developments, topological and combinatorial advances—in particular, investigations into tiling periodicity and cyclotomic phenomena in group-structured spaces—have merged analytical techniques (such as singular integral analysis and corona decompositions) with discrete and group-theoretic methods. This synergy has yielded powerful tools for controlling metric invariants, isoperimetric inequalities, and extremal properties related to embeddings and perimeter control [32].

In summary, these results collectively illuminate a unifying paradigm: isoperimetric inequalities and perimeter control in discrete groups interact intimately with the analytic and geometric theory of metric spaces, while operator-algebraic invariants such as Roe algebras provide functorial, canonical classifications of their large-scale structure [11, 18, 32]. These ongoing developments continue to enrich our understanding of the deep relationships among algebraic, geometric, and analytic properties of groups and metric spaces.

### 3 Tiling Theory and Structural Decomposition in Discrete Groups

#### Objectives and Scope

This section aims to systematically examine the fundamental connections between tiling theory and the structural decomposition of discrete groups. We seek to elucidate the principal guiding questions: How do tiling properties manifest within discrete group settings, and what frameworks enable systematic decomposition? We also aim to compare the efficacy and limitations of existing theoretical approaches, thereby highlighting key open research directions within this field.

#### Overview and Technical Approaches

Tiling theory in the context of discrete groups encompasses the study of how sets of group elements can cover the entire group through translations without overlaps. Central constructs include tileability, periodicity, and the identification of aperiodic or minimal tile sets. Structural decomposition examines how groups can be partitioned into simpler or regularly structured subcomponents—such as coset decompositions, subgroup chains, or more general algebraic frameworks. The interplay between group action, group embedding, and decomposition often reveals critical insights into the algebraic and combinatorial properties of both the group and the tiling system.

Standard technical approaches involve direct construction of tiling sets, identification of subgroup or coset structures that facilitate decomposition, and analysis of translation actions. Alternative frameworks sometimes employ spectral, dynamical, or ergodic-theoretical methods to address questions of density, completeness, or uniqueness in tilings. Each methodology brings characteristic

**Table 3: Representative lower bounds for embedding finite metric spaces or group balls into  $L_1$  with low distortion. See Section 4.2 for details.**

Group / Space	Target Metric Space	Lower Bound on Distortion
$\mathbb{H}_{\mathbb{Z}}^5$ (ball of radius $n$ )	$L_1$	$\sqrt{\log n}$
$\mathbb{Z}^d$ , $d \geq 2$ (ball of radius $n$ )	$L_1$	$O(1)$ (embeddable)
General finite metric space (size $n$ )	$L_1$	$O(\log n)$

strengths; for example, algebraic techniques often yield constructive results, while analytic ones can establish broad necessary or sufficient conditions. However, drawbacks may include limited applicability to non-amenable groups, challenges in achieving explicit construction, or difficulties generalizing results across different group classes.

### Current Limitations and Open Problems

While tiling theory and group decomposition have advanced significantly, several notable limitations persist. Many results are restricted to abelian or amenable groups, and a general theory accommodating arbitrary discrete groups remains elusive. Explicit construction of aperiodic tilings or minimal decomposition frameworks in complex or non-amenable groups is still an open challenge. Additionally, the relationships between different decomposition strategies, and the impact of group embeddings on tiling possibilities, are only partially understood.

Key unresolved questions include whether minimal aperiodic tile sets exist for wider classes of groups, how group-theoretic invariants constrain possible decompositions, and what new phenomena arise in the interaction of tiling and group embedding across group extensions or direct products. Addressing these open problems continues to be a central focus of current research.

### Summary and Outlook

In summary, tiling theory and structural decomposition within discrete groups offer a vibrant and multifaceted research landscape, grounded in deep algebraic and combinatorial questions. This section has outlined main research objectives, contrasted technical approaches and their limitations, and highlighted prominent gaps and open questions for further investigation. Understanding these interrelated themes is crucial for advancing both the theoretical foundations and applied methodologies in discrete mathematics and group theory.

#### 3.1 Integer and Group Tilings: Cyclotomic and Periodic Approaches

The study of tilings within discrete groups, especially among the integers and cyclic groups, has achieved notable progress by merging combinatorial constructs with advanced harmonic analytic methodologies. A foundational problem in this area is the characterization of finite sets  $A \subset \mathbb{Z}$  for which there exists a set  $B$  such that their translates yield a partition of a finite cyclic group, i.e.,  $A \oplus B = \mathbb{Z}_M$ . This tiling property has been classically approached using combinatorial group theory, with significant contributions such as Newman's exponential bound  $M \leq 2^{\max(A) - \min(A)}$  on the

minimal period. While these results provide constraints on possible periods, they do not reveal the finer internal structure of the tiles nor address the complexity introduced by periods with rich prime factorization.

Recent developments have bridged these gaps by integrating combinatorial decomposition strategies with harmonic analysis. Notably, the conceptual introduction of the box product and multi-scale cuboid constructions has substantially enhanced the capacity to reveal latent symmetrical and product structures within tiling sets. By leveraging these, alongside the theory of saturating sets, researchers have achieved a systematic decomposition of complex tiling sets into more elementary and regular substructures. This decomposition process is particularly effective for periods that are divisible by several distinct odd primes, where traditional subgroup reduction techniques become ineffectual due to the lack of subgroups of the requisite index.

Central to the harmonic analytic approach is the role of cyclotomic polynomials and their divisibility properties, which yield spectral invariants instrumental in identifying viable tiling candidates. The cyclotomic framework not only imposes necessary divisibility constraints but also serves as a guiding principle in the classification of invariant subsets with prescribed periods. These analytic methods are especially powerful in the case of square-full periods, such as  $M = (pqr)^2$  for distinct odd primes  $p, q, r$ . In this regime, it has been established that every tiling must satisfy the Coven-Meyerowitz T2 property—an analytic criterion whose general confirmation has long been conjectured and only recently established for these complex periods. The depth and generality of this analytic approach are underscored by its success in settings characterized by the absence of significant subgroup structure.

Importantly, contemporary advances are not confined to existential results; they encompass constructive methodologies for reconstructing tiling sets from incomplete combinatorial or spectral information. Reduction techniques—employing saturation concepts and fiber decomposition—allow for the transformation of intricate tiling challenges into more tractable problems over groups with simpler algebraic or combinatorial configurations. Such methodological reductions are vital for resolving computational complexity bottlenecks and for mitigating the combinatorial intractability associated with the rapid growth of possible tiling configurations as periods become large and increasingly composite.

The principal advantage of these modern approaches is their adaptability regarding the structure of the underlying group. By dispensing with reliance on subgroup theorems—which restrict applicability to groups with straightforward prime power decompositions—the current framework extends naturally to a wider spectrum of finite abelian groups. Moreover, it accommodates tiling phenomena



in non-abelian and higher-rank groups, expanding the scope of structural decomposition techniques in group tiling theory.

### 3.2 Summary of Key Advances in Tiling Structures

To succinctly encapsulate the methodological advances and their domains of efficacy, the following overview is presented in Table 4.

### 3.3 Open Problems and Future Directions

This subsection outlines the main open challenges and forward-looking research paths in tiling theory for integers and groups, with emphasis on spectrality, group actions, computational aspects, and algebraic generalization. The objective is to orient the reader to current knowledge gaps and priorities for the field's continued advancement.

Despite the considerable progress made in tiling theory for integers and groups, several deep and challenging problems persist. Prominent among these is the intricate relationship between tiling and spectrality, particularly as articulated in the spectral set conjecture, commonly known as Fuglede's conjecture. The precise interplay between the tileability of a set and the existence of an orthogonal basis of group characters for its indicator function—a property defining spectral sets—remains incompletely understood, most notably in higher dimensions, for non-abelian groups, and in cases involving fractal or aperiodic tilings.

Emerging research trajectories aim to extend the combinatorial-harmonic analytic toolkit to higher-dimensional integer lattices and non-commutative group actions. A systematic study of fibered and self-similar tilings, especially in relation to substitution dynamics and symbolic dynamical systems, has become increasingly prominent. Computational complexity continues to pose persistent issues, motivating work on the efficient classification and enumeration of tilings in groups with many irreducible factors. There is also growing interest in investigating tilings in broader algebraic structures, such as non-abelian and finite simple groups, and in exploring the limits of current decomposition and reduction methods.

The confluence of these directions renders the classification of group tilings, their spectral properties, and their algorithmic construction a central and dynamic field for future exploration [33].

#### Key Takeaways:

- The relationship between tiling and spectral sets, especially as conjectured by Fuglede, remains largely unresolved outside the abelian, low-dimensional setting.
- Recent combinatorial and harmonic-analytic advances provide new tools for classifying and constructing tilings, particularly for complex periods and factorizations.
- Algorithmic and computational issues are central to advancing knowledge, notably for enumeration and reconstruction in highly composite or non-abelian groups.
- Systematic study of tilings in broader algebraic and dynamical frameworks is the next frontier for interdisciplinary research.

### 3.4 Approximation, Banach/Function Spaces, and Topological Invariants

This section aims to explore the interplay between approximation theory, Banach and more general function spaces, and the emergence of topological invariants within this landscape. The objectives here are: (1) to clarify the foundational roles played by approximation methods across various function spaces; (2) to examine how the structure of Banach and related spaces informs analytical and topological properties; and (3) to interpret the appearance of topological invariants as bridges between analytic approximation and geometric insight.

Approximation theory provides a crucial framework for studying how functions in complex spaces can be closely represented by simpler or more tractable elements, often leveraging the normed structure of Banach spaces. The selection of function spaces—such as  $L^p$ , Sobolev, or Hölder spaces—significantly impacts the precision of approximation and the types of error bounds achievable. Banach spaces, with their completeness and well-defined duality, underpin the rigorous analysis of operator behavior, compactness, and convergence properties in a host of applications.

Topological invariants, such as Betti numbers or cohomological characteristics, frequently surface when examining the deeper structure of function spaces, especially in contexts involving fractal geometry or operator algebras. These invariants serve as potent tools for distinguishing otherwise similar spaces and for revealing intrinsic geometric and dynamical properties hidden within analytical frameworks.

At the intersection of these themes lies a rich area of study: quantifying how approximation procedures reflect, respect, or sometimes obscure the underlying topological features of the function spaces in question.

Summary of key takeaways for this subsection: The objectives addressed include clarifying how approximation operates across diverse function spaces, with Banach space structure guiding analytic rigor and convergence. Topological invariants are interpreted as essential tools that bridge analytic and geometric understanding, revealing subtle properties fundamental to modern functional analysis and its applications. This section therefore lays the groundwork for understanding complex interactions between analytic approximation, space structure, and topological complexity.

**3.4.1 Stability and Counterexamples.** The landscape of Banach space approximation theory is structured around the persistence—or failure—of vital geometric and topological properties under various algebraic operations. Dominant among these features are  $M$ -ideals, the generalized center property ( $GC$ ), central subspaces, and a collection of best approximation frameworks (notably,  $\mathcal{F}$ -rcp,  $(P_1)$ , and SACP). Traditionally, the prevailing perspective assumed that such properties—particularly  $M$ -ideality—are robust under the formation of subspace sums: specifically, if  $Y$  and  $Z$  are  $M$ -ideals in a Banach space  $X$  with a closed sum  $Y + Z$ , then  $Y + Z$  would also be an  $M$ -ideal. Parallel expectations were extended to the ( $GC$ ) property and the notion of central subspaces.

However, recent research has challenged this paradigm by constructing explicit counterexamples and formulating constructive theorems that systematically reveal the precariousness of such inheritance assumptions. In particular, it has become clear that,

**Table 4: Major Approaches in Tiling Theory and Their Domains of Applicability**

Approach	Description	Applicable Settings
Combinatorial Period Bounds	Exponential bounds on minimal period	Cyclic groups, classical settings
Box Product & Multiscale Cuboids	Decomposition into product/cuboid structures	Cyclic groups with multiple prime divisors, squarefull periods
Cyclotomic Harmonic Analysis	Use of cyclotomic polynomials and spectral invariants	Any cyclic group, especially for squarefull periods
Saturating Set	Decomposition via saturation, fiber analysis	Finite abelian groups, complex period structure
Reduction Techniques	Reduction to simpler subgroup/configuration problems	Groups with few or no strong subgroups

although the sum of two semi  $M$ -ideals can retain  $M$ -ideality under certain technical restrictions, key properties such as the generalized center property (GC) and centrality may decisively fail to be preserved in the transition to  $Y + Z$ . This breakdown is not peripheral but is often present even when  $Y$  and  $Z$  themselves satisfy the relevant property and their sum is closed.

The mechanisms behind these instabilities are illuminated through carefully crafted counterexamples, which demonstrate that the criteria for inheritance depend on more refined notions than mere class membership—such as “almost constrainedness.” Theorem 2.2, for instance, establishes that if  $X$  satisfies (GC) and  $Y$  is almost constrained in  $X$ , then  $Y$  inherits (GC), an outcome that stands as the exception rather than the general rule for arbitrary subspace summations. These developments are unified through investigations into sums in contexts such as Köthe-Bochner spaces, polyhedral subspaces, and injective tensor products, all accentuating the nuanced interaction between algebraic formation and topological invariants.

Moreover, the research clarifies which best approximation properties—such as SACP and  $(P_1)$ —may be preserved through the sum of subspaces, while also specifying those that are susceptible to structural disruption. Notably, this line of inquiry resolves certain longstanding open problems regarding the characterization of  $M$ -ideals under secondary, often local, constraints [34].

The chief strength of this analytical methodology lies in its explicit demarcation of stability boundaries. By providing concrete countermodels, it does more than contest overly optimistic conjectures; it also sharpens the understanding of precisely where and how preservation can be expected. Nonetheless, a limitation of existing literature is its predominant focus on manageable or specifically constructed instances; thus, systematic exploration across broader Banach-lattice-like settings remains an open avenue for future work.

The findings summarized in Table 5 underscore the selective and situation-dependent character of property inheritance in Banach space approximation theory.

**3.4.2 Structural Criteria and Open Questions.** These results necessitate a critical re-examination of the structural criteria that govern the inheritance of approximation properties in infinite-dimensional settings. The demonstrable failure of overarching stability through the operation of subspace summation compels the formulation of finer-grained constraints, with properties such as almost constrainedness emerging as crucial—albeit not universally sufficient—conditions. The upshot is a recognition that inheritance phenomena in Banach spaces are dictated by subtle interrelations between algebraic composition and underlying topological complexity, thus invalidating any simplistic generalization from more elementary scenarios.

Several open questions persist at the conjunction of best approximation theory and topological invariants. Prominent among these is the unresolved existence of distinct subspaces  $Y$  and  $Z$ , each exhibiting the SACP property, yet whose closed sum fails to retain SACP. This issue highlights the intricate dependencies that bind algebraic operations, metric stabilities, and topological subtleties—challenges that current theories struggle to capture comprehensively [34]. Furthermore, the delineation of sharp boundaries for the stability of best approximation properties—especially concerning  $M$ -ideality and centrality—remains a focal research direction. Progress in this area is expected to not only expose the internal architecture of Banach and function space approximation but also to influence the investigation of tensor products, operator spaces, and the geometry of infinite-dimensional functional systems.

To summarize, the deployment of explicit counterexamples has substantially refined the understanding of property inheritance in Banach spaces, illuminating both stable and unstable regimes, and shaping new conceptual and technical pathways for ongoing research. Outstanding questions, distilled as follows, continue to define the frontier of the field:

What additional structural properties, beyond almost constrainedness, are necessary and sufficient for preservation of best approximation properties under subspace sums? Can general inheritance results be established for broad classes of function or operator spaces, or do instabilities dominate in most infinite-dimensional constructions? How do these phenomena manifest in more complex constructions, such as tensor products and spaces of vector-valued functions? Does there exist an explicit pair of SACP subspaces whose closed sum fails SACP, or can this exception be categorically ruled out?

In effect, the field is called to sharpen both its tools and its conceptual vocabulary, fostering a more sophisticated appreciation of the intricate interplay between algebraic operations and topological properties that lies at the core of Banach space approximation theory.

## 4 Fractal Dimension Theory and Incidence Structures

This section aims to introduce and analyze the intersection between fractal dimension theory and incidence structures. The primary objectives are to clarify foundational concepts, to examine key theoretical developments, and to bridge connections that illuminate the roles of fractals in the combinatorial and geometric study of incidence relations. In doing so, we emphasize both theoretical advances and practical relevance for readers from diverse disciplinary backgrounds.

**Table 5: Summary of Stability and Inheritance for Key Properties under Subspace Sum**

Property	Preservation under $Y + Z$ (Closed)	Critical Requirement	Known Counterexample
$M$ -ideality	Sometimes	Technical constraints (e.g., semi $M$ -ideals)	Yes
Generalized center ( $GC$ )	Rarely	Almost constrainedness	Yes
Central subspace	Rarely	Additional structure	Yes
$SACP / (P_1)$	Sometimes	Depends on precise subspace interplay	Yes

## 4.1 Fractal and Hausdorff Dimension

**Objectives and Context.** This subsection aims to present key advances in the quantification and characterization of geometric complexity via fractal and Hausdorff dimension, emphasizing foundational concepts, recent theorems, and their implications for projection and incidence geometry, as well as modern tangent and metric frameworks. The discussion is designed to be accessible to readers approaching from outside the immediate field by first delineating core definitions and goals.

The theory of fractal dimension lies at the nexus of geometry, analysis, and combinatorics, offering robust frameworks for quantifying the complexity of sets that transcend classical Euclidean constructs. Foundational notions—including the Hausdorff, packing, and box-counting dimensions—provide complementary perspectives on the size, irregularity, and scaling phenomena within sets that exhibit self-similarity, statistical self-affinity, or broader scaling behaviors. Among these, the *Hausdorff dimension* is particularly esteemed for its sensitivity to fine-scale structure, playing a central role in the interface between fractal geometry, incidence theory, and metric geometry [35].

A significant breakthrough in this domain concerns  $(s, t)$ -Furstenberg sets: subsets of  $\mathbb{R}^2$  that intersect every line within a direction family (whose parameter space possesses dimension at least  $t$ ) in a set of Hausdorff dimension at least  $s$ . Recent achievements have shown that for  $s \in (0, 1)$  and  $t \in (s, 2]$ , the dimension of such sets must exceed  $2s$  by a positive quantity  $\epsilon$  dependent only on  $s$  and  $t$ —a notable refinement of Wolff’s classical  $2s$  bound. This result, proved by Orponen and Shmerkin, employs techniques such as induction on scales and discrete incidence geometry of tubes, and has considerable implications for both projection and incidence geometry [35]. In particular, for regular sets  $K \subset \mathbb{R}^2$ , it yields improved upper bounds on the dimension of the set of directions for which orthogonal projections produce anomalously small images, thereby deepening connections between projection theory and fractal geometry [35].

These themes are echoed in the context of radial projections. The result  $\sup_{x \in X} \dim_H \pi_x(Y) \geq \min\{\dim_H X, \dim_H Y, 1\}$ , established for appropriate Borel sets  $X, Y \subset \mathbb{R}^2$ , addresses longstanding conjectures and elegantly bridges to continuum analogues of classical incidence combinatorics, such as Beck’s theorem [28]. The work of Orponen, Shmerkin, and Wang further strengthens lower bounds when  $\dim_H Y > 1$ , giving precise continuum analogues and generalizations. This illustrates a recurrent pattern: techniques originating in discrete incidence geometry, when skillfully combined with multi-scale decomposition methods, yield powerful tools to analyze the fractal geometry of projections and intersections.

The study of attractors for iterated function systems (IFS)—notably, self-affine constructions—is another vibrant area. The Ledrappier–Young formula extends to dimension results for measures invariant under affine IFS, encompassing instances where contraction is achieved “on average” and thus settling pivotal questions regarding the precise dimensionality of certain attractors [36]. For example, Cao and Zhu apply such dimension theory to singularity analysis in fluid models, establishing sharp Hausdorff dimension upper bounds on the temporal set where singularities may occur, which matches optimal estimates for related Navier–Stokes equations [36]. The treatment of random limsup sets, including those within the Heisenberg group, broadens this theory; here, Hausdorff dimensions can be related to singular value functions and affine invariants, spanning both commutative and non-commutative frameworks as shown by Myllyoja [37].

A key advancement in this stream involves the analysis of microsets and tangent structures. Fraser, Howroyd, Käenmäki, and Yu establish that the set of microset dimensions of any given set accurately characterizes both its Assouad (maximal) and lower (minimal) dimensions; furthermore, they show that every  $\mathcal{F}_\sigma$  set satisfying certain natural constraints can be realized as the spectrum of microset dimensions for a compact set, thus capturing the nuanced relationship between local geometric complexity and global fractal behavior [38]. Collectively, these lines of inquiry deepen the understanding of intrinsic dimension, tangent geometry, and the representation of set complexity at small scales.

Moving beyond classical settings, refinements such as the G-Hausdorff space and functional calculus for G-metric structures facilitate the extension of fractal interpolation and IFS techniques to non-affine, highly non-Euclidean regimes. The advent of G-IFS and their associated fractals epitomizes the trend toward abstraction and universality, enabling the treatment of heterogeneous metrics and transformations. This methodological progression is of particular importance for the burgeoning range of applications in analysis, modeling, and data science.

## 4.2 Projections, Slices, and Intersections

**Objectives and Context:** This subsection examines the fundamental roles of projections, slices, and intersections in fractal geometry, with particular emphasis on their impact on the measure-theoretic and dimensional properties of sets. The goal is to clarify the underlying principles governing these operations—both in classical Euclidean spaces and in more abstract or general metric settings—and to present recent advances that unify and extend classical results. This section aims to make the technical developments accessible by outlining their motivations, significance, and methodological innovations.

Projection theorems, along with the study of slices and intersections, are foundational within fractal geometry: they govern the measure-theoretic and combinatorial properties of sets in both high-dimensional and non-Euclidean contexts. Classical results, notably those of Marstrand and Mattila, prescribe almost sure behaviors for projections onto subspaces. However, recent research has markedly extended these principles to abstract spaces and highly structured families of projections [4, 5, 7, 17, 21, 28, 39–41].

A principal innovation in this area is the integration of energy estimates and Frostman's lemma, which facilitate quantitative lower bounds on intersection dimensions and merge incidence-theoretic with measure-theoretic approaches. More specifically, for analytic sets  $A \subset \mathbb{R}^n$  having positive and finite  $s$ -dimensional Hausdorff measure, and for correspondingly parameterized families of projections, the dimension of typical slices and intersections can be described with remarkable precision [40].

These techniques have yielded general criteria for the dimensions of unions and intersections among affine subspaces, thereby extending classical results pertaining to Furstenberg and Besicovitch sets and addressing longstanding conjectures regarding the minimal dimensions required for certain intersection phenomena. For example, it is established that the union of a nonempty  $s$ -dimensional family of  $k$ -dimensional affine subspaces must attain dimension at least  $k + s$ . Furthermore, sets intersecting every member of such a family in sizable subsets themselves possess necessarily large dimensions, in a quantifiable sense.

When classical projection theorems break down outside Euclidean settings, new frameworks have proved essential. In general complete metric spaces, the Besicovitch-Federer projection theorem may fail; yet, by analyzing generic perturbations within the space of 1-Lipschitz functions, one can recover analogues of measure-zero or measure-positivity results regarding images of  $n$ -unrectifiable and  $n$ -rectifiable sets, respectively [4]. These findings notably extend projection theory to more general contexts and create a synthesis between geometric measure theory and functional analysis.

The sophistication of approaches to Hausdorff measures on subspaces now enables sharp characterizations of “typical” versus “exceptional” behavior, encompassing directions, slice locations, and parameter sets of projections.

The following table succinctly organizes some notable advances in projection and slicing theorems, emphasizing their setting, main results, and methodological innovations:

As highlighted in Table 6, these developments represent significant steps toward a systematic description of projection and slicing phenomena in both classical and abstract metric settings [4, 40, 41].

Collectively, these innovations have illuminated sharp thresholds, regularity properties, and even optimality in the dimensional bounds governing intersections and slicing in fractal geometry. Applications pervade the study of Furstenberg-type sets, Besicovitch sets, and intricate families of fractal configurations, firmly anchoring these tools in the landscape of modern geometric analysis.

### 4.3 Fractal and Metric Structures in Advanced Spaces

This subsection aims to outline foundational objectives and recent breakthroughs in extending fractal dimension theory and metric

analytic frameworks to modern, non-classical spaces. Our focus is threefold: (i) characterizing the subtle fractal structure of measures arising from mathematical physics; (ii) synthesizing new structures and convergence paradigms in Lorentzian metric spaces; and (iii) evaluating operator-algebraic invariants as robust classifiers for large-scale geometry. Each thread emphasizes the interplay between abstract analytic techniques and the geometry of spaces with applications in relativity, quantum gravity, and topological physics.

Modern research has substantially expanded the reach of fractal dimension and metric analytic concepts into advanced, non-classical spaces, driven by challenges in mathematical physics, non-commutative geometry, and the pursuit of large-scale geometric invariants. Critical Liouville Quantum Gravity (cLQG) exemplifies such a context: here, the support of the random measure derived from the planar Gaussian Free Field exhibits extreme fractality. It has been rigorously demonstrated that the cLQG measure is supported on sets of vanishing Hausdorff dimension for broad classes of gauge functions, thereby confirming conjectures on the exceptional thinness of such random fractals [42]. These results, grounded in necessary and sufficient conditions for finite gauge-Hausdorff measures, are of central importance both for probability theory and for elucidating universality features within mathematical physics.

The development of Lorentzian metric spaces and synthetic approaches to spacetime geometry, motivated by both causet theory and general relativity, signals a further front of progress. Recent advances have generalized the concept of Lorentzian metric spaces by removing boundedness limitations, instead providing minimal and robust conditions for the distance function: the reverse triangle inequality, continuity, and a distinguishing property via the Lorentzian distance [11]. When augmented by a countably generating condition, these spaces retain desirable analytical features, such as the Polish property, and the generalization to (pre-)length spaces is shown to be stable under Gromov-Hausdorff convergence. The framework thus supports a synthetic analysis of convergence, stability, and precompactness in both random and non-smooth models of spacetime—fertile ground for integrating fractal geometry with open questions in mathematical relativity and quantum gravity.

Additionally, group-theoretic constructs and large-scale geometric invariants have further enriched the discourse. *Roe algebras*, arising from uniformly locally finite metric spaces, have been proved to encode coarse geometric properties, with isomorphic Roe algebras corresponding bijectively to coarse equivalence of the underlying spaces [18]. This phenomenon of “C\*-rigidity” demonstrates the potency of operator-algebraic invariants as classifiers for the large-scale geometry of spaces. Notably, the group of outer automorphisms of the Roe algebra,  $\text{Out}(C_{\text{Roe}}^*(X))$ , is explicitly identified with the group of coarse equivalences modulo closeness, clarifying the structure and uniqueness of these invariants. The correspondence is summarized in the following table:

Taken together, these rapidly advancing lines of research underscore the profound interplay between fractal dimension theory, incidence geometry, and advanced metric or group-theoretic structures. The accelerating trend toward abstraction and generality is not merely a technical achievement but also a testament to the universality and subtlety of fractal structures in both pure mathematics and its wide-ranging applications.

**Table 6: Recent Advances in Projection and Slicing Theorems**

Setting/Context	Main Result	Dimensional Thresholds	Key Techniques
Euclidean spaces ( $\mathbb{R}^n$ ), classical projections	Marstrand/Matila type theorems: almost all projections preserve the minimal dimension or measure	Dimension equal to or exceeding $\min(\dim_H E, m)$	Energy methods, measure theory, incidence geometry
Affine subspace unions	Union of $s$ -dimensional family of $k$ -dimensional subspaces has dimension $\geq k + s$	$s$ parameter family, $k$ -dimensional subspaces	Energy estimates, generalization of Furstenberg/Besicovitch arguments
General complete metric spaces	Generic 1-Lipschitz functions preserve measure-zero/positivity of certain sets	$n$ -rectifiable/unrectifiable sets	Baire category, functional analysis, 1-Lipschitz perturbations

**Table 7: Correspondence between Roe algebra and coarse geometric equivalence [18]**

Roe algebra property	Coarse geometry property
Isomorphism of Roe algebras	Coarse equivalence of spaces
Outer automorphism group $\text{Out}(C_{\text{Roe}}^*(X))$	Coarse equivalences modulo closeness

## 5 Function Spaces, Wavelets, and Fractal Analysis

This section aims to clarify the objectives and scope related to function spaces, wavelet theory, and fractal analysis, as they pertain to the broader themes of the survey. Readers should expect a guided synthesis of foundational results, their interconnections, and recent advances relevant to current and emerging applications.

### 5.1 Function Spaces

A principal objective of this subsection is to highlight the foundational role of function spaces in modern analysis, especially their importance in the representation and study of signals, images, and complex data arising in both theory and applications. We focus on  $L^p$  spaces, Sobolev spaces, and Besov spaces, outlining their essential properties and mapping out their relationships to subsequent topics in this survey.

**Key takeaway:** An in-depth understanding of function spaces creates the backbone for analyzing and advancing new representations, particularly in high-dimensional and irregular data settings. However, open challenges remain in identifying generalizable functional frameworks that can unify diverse application domains.

### 5.2 Wavelets

The goal of this subsection is to synthesize the motivations for and developments of wavelet frameworks, with an emphasis on their applicability to both classical and fractal data. This includes a review of multiresolution analysis, construction principles for wavelet bases, and the adaptation of wavelets to irregular domains.

**Key takeaway:** Wavelets enable localized analysis and efficient representation of functions with singularities or self-similar patterns. Future research must address the limitations of current wavelet methods when applied to highly non-uniform or data-driven domains, marking a prominent direction for further investigation.

### 5.3 Fractal Analysis

This subsection focuses on the objectives and techniques of fractal analysis within the context of function spaces and wavelets. We examine the quantification of regularity via fractal dimensions and the interplay between fractal geometry and functional representations.

**Key takeaway:** Integrating fractal analysis with function space and wavelet methods provides powerful insights into the complexity of analytic and real-world structures. Yet, robust frameworks

that systematically bridge traditional analysis and fractal processes remain an open and active area of research.

*Section synthesis:* Together, these subsections illustrate the deep connections between function spaces, wavelet theory, and fractal analysis. While significant progress has been made in each area, there remain unresolved issues regarding the unification and extension of their methods to tackle increasingly complex datasets. Future developments are likely to hinge on a more integrative approach that explicitly leverages their interplay.

### 5.4 Tight Wavelet Frames and Harmonic Analysis

The construction of tight wavelet frames (TWFs) within  $L^2(\mathbb{R}^n)$  is foundational in contemporary harmonic analysis, underpinning crucial developments in both pure and applied mathematics. TWFs are notable for permitting perfect reconstruction of signals while preserving redundancy, thereby providing a robustness and adaptability that surpass traditional orthonormal bases. Despite their theoretical appeal, the explicit and systematic construction of TWFs remains a prominent challenge. Conventional extension-based procedures frequently suffer from a lack of transparent strategies for generating mother wavelets, rendering such approaches theoretically comprehensive but practically inaccessible for explicit applications. Meanwhile, frameworks based on the sum-of-squares (SOS) criterion reduce the problem to an algebraic condition; however, even with an explicit mother wavelet, verifying or fulfilling the SOS requirement can be highly nontrivial for general refinable functions, posing an obstacle to broader applicability.

Recent progress has mitigated these constraints by introducing burden-sharing frameworks, in which the complexity of constructing TWFs is judiciously allocated between refinable functions and associated mother wavelets. By concurrently balancing the design criteria for refinable functions with those for mother wavelet selection, these approaches render the SOS conditions more tractable, thereby enabling concrete constructions that would otherwise be out of reach using traditional methods. This evolution has considerably expanded the class of accessible TWFs and equipped analysts with novel methodologies for tailoring wavelet systems to the nuanced demands of practical applications, particularly in contexts characterized by irregular, self-similar, or fractal geometries [43].

Of particular significance is the interplay between tight frames and multiresolution analysis on fractal domains. The recursive architecture of wavelet decompositions resonates with the inherent self-similarity of fractals, thereby enhancing both theoretical insight and computational capacity in the study of complex measures and datasets. In such settings, the synthesis of tight frame theory and multiresolution techniques affords powerful analytic and algorithmic tools, advancing the study of functions and distributions on highly non-regular domains.

## 5.5 Hardy–Rellich Inequalities and Non-Euclidean Settings

The generalization of classical  $L^p$ -Rellich and Hardy–Rellich inequalities to non-Euclidean and singular geometric environments has catalyzed significant advances in the analysis of spaces characterized by atypical symmetries and degeneracies. Notably, the Baouendi–Grushin vector fields provide a salient example of degenerate elliptic structures that necessitate innovative analytical techniques to address their pronounced anisotropy and degeneracy. In these contexts, identifying sharp constants and demarcating the precise boundary between subcritical and critical regimes for the relevant inequalities constitute central analytical challenges.

Recent investigations have made substantial contributions by deriving new identities that unify subcritical and critical Hardy inequalities, thereby establishing their equivalence and substantially broadening their applicability—including extensions to higher-order and radial differential operators. Through rigorous stability analyses applied to extremal functions, these studies have identified optimal constants and, particularly in the  $L^2$  case, have elucidated explicit remainder terms. Such precision—unattainable by classical approaches—has profound implications for the study of potential theory and partial differential equations in singular, fractal, and non-Euclidean settings. These innovations not only reinforce the synergy between harmonic analysis, functional inequalities, and the geometry of singular spaces but also lay the groundwork for new, rigorous analytic methodologies applicable to complex, degenerate domains [44].

As illustrated in Table 8, the evolution from classical extension-based and SOS methodologies toward burden-sharing frameworks reflects a shift towards greater explicitness and applicability, particularly in fractal and self-similar settings. The resulting analytic tools continue to enrich the study of nonclassical function spaces and provide new avenues for addressing the intricate geometry of singular and irregular domains.

## 6 Analytical Methods for PDEs, SPDEs, and Evolution Equations

This section surveys the principal analytical methods developed for the study of partial differential equations (PDEs), stochastic partial differential equations (SPDEs), and general evolution equations. Our objectives are to elucidate the foundational approaches that underlie analysis in these fields, delineate their scope and interconnectedness, and highlight the current challenges and emerging directions.

### 6.1 Classical Analytical Methods for PDEs

At the outset of studying PDEs, classical analytical methods—including separation of variables, transform techniques, and energy methods—form the core toolkit for both qualitative and quantitative analysis. The scope of this subsection is to summarize these classical tools and demonstrate how they serve as stepping stones for more advanced frameworks.

One key takeaway is that while classical techniques are widely effective for linear and certain nonlinear equations with sufficient regularity, their limitations in handling rough data, irregular domains, or pronounced nonlinearity motivate the development of more sophisticated approaches. Ongoing challenges include extending these methods to broader classes of PDEs and integrating them with computational tools.

### 6.2 Functional Analytic Approaches

Functional analysis provides a robust foundation for addressing existence, uniqueness, and regularity of solutions to PDEs and evolution equations. Methods based on Sobolev spaces, semigroup theory, and operator theory are particularly central. This subsection aims to clarify how these frameworks generalize the notion of solutions and enable the systematic treatment of both linear and nonlinear evolution problems.

It is important to note that while powerful, functional analytic approaches may encounter obstacles dealing with critical nonlinearities or nonstandard boundary conditions. Open problems remain in refining solution concepts and identifying optimal regularity results for complex systems.

### 6.3 Analytical Techniques for SPDEs

The development of SPDE theory introduces randomness into evolution equations, requiring new analytical techniques that blend probability and functional analysis. Here, we survey key methodologies such as martingale solutions, mild formulations via stochastic integrals, and regularity structures for rough paths. The objective of this section is to highlight how these approaches enable the treatment of noise and randomness in infinite-dimensional settings.

Despite significant progress, the rigorous analysis of SPDEs with irregular coefficients or singular noise remains a fundamental challenge. Future directions involve developing unified frameworks that can accommodate both spatial and temporal irregularities, as well as deepening the interplay with numerical analysis.

### 6.4 Synthesis and Open Directions

The analytical methods reviewed above collectively underpin the modern study of PDEs, SPDEs, and evolution equations. As research evolves, key open challenges persist—such as formulating robust solution concepts for highly irregular problems, bridging analytical and computational techniques, and extending stochastic analysis to novel application domains. Continued progress will likely depend on forging stronger connections and synthesizing ideas across these subfields.

Throughout this section, we have aimed to clarify the objectives and limitations of each approach, while emphasizing enduring questions for future investigation.

**Table 8: Comparison of Construction Methods for Tight Wavelet Frames (TWFs)**

Method	Advantages	Limitations
Extension-Based Techniques	Theoretically broad and capable of handling general frame extensions; flexible in abstract settings.	Lack explicit procedures for mother wavelet generation; limited practical accessibility for concrete constructions.
Sum-of-Squares (SOS) Paradigm	Reduces construction to algebraic verification; mathematically principled.	SOS condition may be difficult to verify or to satisfy for generic refinable functions; limited explicitness.
Burden-Sharing Frameworks	Balances conditions between refinable functions and mother wavelets; facilitates explicit constructions; enables application to fractal and irregular domains.	Frameworks are relatively recent; generality and effectiveness dependent on the refinement of dual criteria.

## 6.5 Variational and Metric Methods

This subsection surveys the foundational variational and metric geometric frameworks underlying modern analysis of partial differential equations (PDEs) such as the Fokker–Planck equation, with a focus on the role of gradient flows in Wasserstein space and advances in discretization and regularity theory.

The synergy between variational principles and metric geometric frameworks has profoundly advanced the analysis of partial differential equations, especially regarding the characterization and resolution of Fokker–Planck-type equations. Building on the pioneering approach of Jordan, Kinderlehrer, and Otto, subsequent research has reformulated the Fokker–Planck evolution as a gradient flow of the free energy functional in the space of probability measures equipped with the 2-Wasserstein distance. This viewpoint both establishes a robust variational framework and integrates optimal transport and metric measure theory, enabling the investigation of qualitative and quantitative solution properties.

Recent developments have notably focused on discretization strategies in Wasserstein space, which facilitate improved convergence for time-discrete approximations. Under appropriate regularity hypotheses on the domain and initial data, these approximations converge not merely in weak topologies but, crucially, in strong Sobolev norms such as  $L_t^2 H_x^2$ , attributed to refined transport inequalities that consistently retain higher-order regularity at each discretization step. For example, Santambrogio and Tosupulatov [30] establish that, provided the domain is smooth and convex and the initial data is bounded and sufficiently regular, the classical time-discrete JKO scheme achieves strong  $L_t^2 H_x^2$  convergence to solutions of the Fokker–Planck equation. Their results significantly improve upon previously known convergence properties, which were typically limited to weak or low-order norms, by exploiting discrete analogues of optimal transport inequalities to recover higher regularity at the level of approximations.

Despite these advances, several limitations remain. Most results require stringent regularity on the domain and data, such as convexity and uniform positivity, and the extension of strong convergence or higher-order regularity to more general settings is often open or unresolved. Furthermore, the technical complexity of discrete optimal transport inequalities presents challenges both to further generalizations and to the analysis of more nonlinear equations.

In summary, the variational and metric approach, especially via Wasserstein gradient flows, constitutes a powerful methodology for both qualitative and quantitative PDE analysis. Notable progress has been made in understanding strong convergence and regularity of time-discrete approximations; however, further work is needed to relax structural requirements and address broader classes of evolution equations.

## 6.6 Nonlocal and Stochastic PDEs

Analyzing nonlocal and stochastic partial differential equations (PDEs) demands both the modernization of classical potential theory and the adaptation of regularity tools to account for the effects of critical drift and stochastic perturbations. In the nonlocal regime, drift-diffusion equations with critical scaling—particularly those in which the drift belongs to the  $BMO^{-1}$  space—exhibit a complex interplay among dispersive, advective, and nonlocal phenomena. Utilizing advanced potential-theoretic arguments, solutions to such operators are now controlled via Riesz potentials, yielding novel a priori estimates that tightly relate pointwise solution bounds to fractional integrals of the source terms. This methodology underpins the derivation of parabolic regularity results, including Harnack inequalities and Hölder continuity, under minimal structural assumptions. Of particular note are sharp two-sided estimates for heat kernels associated with these nonlocal, critical-drift operators; these broaden well-known classical results and lay a rigorous foundation for applications ranging from geophysical models to abstract analysis. The analytical framework’s versatility extends to fractal domains and operators with intricate, non-Euclidean scaling, reflecting both the maturity and adaptability of nonlocal potential methods [45].

In the context of stochastic PDEs, recent progress in renormalization theory and the analysis of singular quasilinear equations—including complex KPZ-type models with nonlinear coefficients and multiplicative space-time white noise—has dramatically broadened the applicability of the regularity structures program. Through the deployment of multi-component modelled distributions and the enrichment of the combinatorial renormalization machinery (notably BPHZ-type counterterms), researchers have constructed robust function space architectures that support well-posedness for quasilinear and highly nonlinear non-polynomial systems. These analytic advancements, complemented by generalized versions of Taylor expansions, now permit rigorous convergence analysis of discrete schemes and make possible explicit links between renormalized and linearized models via probabilistic transforms, such as the Hopf–Cole transformation. Such developments resolve longstanding technical challenges in the theory of singular SPDEs and offer blueprints for addressing broader classes of nonlinear stochastic evolution equations [46].

The developments summarized in Table 9 epitomize the integration of analytic and probabilistic techniques, forming a foundation for tackling both nonlocal and stochastic effects in contemporary evolution equations.

## 6.7 Differentiability, Connectivity, and Poincaré Inequalities

A nuanced comprehension of the fine geometry in metric measure spaces—and the associated analytic function theory—fundamentally relies on recent progress in differentiability theory and generalized

**Table 9: Core Advances in Nonlocal and Stochastic PDE Analysis**

Thematic Focus	Analytical Advances	Consequences/Applications
Nonlocal PDEs	Control via Riesz potentials; sharp heat kernel bounds; minimal structural assumptions	Harnack inequalities; Hölder regularity; application to fractal and irregular domains
Stochastic PDEs	Extension of regularity structures; robust renormalization (BPHZ); generalized modelled distributions	Rigorous well-posedness for singular, quasilinear SPDEs; convergence analysis of numerical schemes

Poincaré inequalities. Groundbreaking results now characterize complete Radon–Nikodym property (RNP) differentiability spaces as those admitting countable coverings, up to sets of measure zero, by biLipschitz images of subsets from doubling metric measure spaces satisfying local  $(1, p)$ -Poincaré inequalities. Central to these achievements is a "thickening" construction, predicated on quantitative connectivity defined by  $(C, q, \delta)$ -connectedness, which underpins a robust structural theorem equating such connectivity properties with the existence of local Poincaré inequalities. These insights resolve longstanding open problems in the field, including Cheeger's queries concerning the necessity and sufficiency of differentiability structures, and Rajala's questions about the preservation of Poincaré inequalities under curvature-dimension constraints such as measure contraction properties and lower Ricci curvature bounds.

Beyond existential characterizations, these methods facilitate powerful techniques for the analysis and manipulation of Poincaré inequalities. In particular, they unveil remarkable "self-improvement" phenomena, wherein weak or non-homogeneous, including Orlicz-type, Poincaré inequalities are compelled to strengthen automatically to classical  $(1, q)$ -Poincaré inequalities for some exponent  $q > 1$ . This enhancement provides stability under a wide range of perturbations, including:

deformations induced by weights—especially Muckenhoupt  $A_p$  weights in geodesic metric spaces; transformation under various geometric flows; structural perturbations affecting connectivity and doubling properties.

Collectively, these advances markedly reinforce the metric measure theoretic approach, empowering the application of sophisticated methods from calculus of variations, harmonic analysis, and potential theory well beyond the confines of classical Euclidean analysis. This, in turn, deepens the foundational nexus between geometric and analytic regularity, broadening both methodological scope and theoretical depth [6].

## 7 Inverse Problems and Uniqueness in Conductivity

This section aims to survey the main objectives, scope, and recent progress regarding inverse problems and uniqueness in conductivity. We focus on fundamental theoretical advances, synthesis of recent methods, and key open problems that drive this research area. By clarifying the landscape and highlighting open challenges, we intend to guide both new entrants and experienced researchers toward deeper insights and future directions.

Inverse problems in conductivity seek to determine the internal conductivity distribution of a medium from indirect, often boundary-based, physical measurements. The central question is whether and how uniquely the conductivity can be recovered from

given measurement data. This inquiry has led to diverse mathematical models and analytic techniques, with significant applications in medical imaging, geophysics, and nondestructive testing.

Throughout this section, we highlight core concepts, synthesize major results, and explicitly indicate open problems and research directions within each sub-topic. At the end of each major subsection, we provide brief highlighting sentences to reinforce key takeaways and identify ongoing challenges.

To ensure clarity and navigability, we reiterate the main objectives at the outset, integrate transitions between subsections, and provide synthesis paragraphs summarizing critical insights. The interplay among the various aspects of inverse conductivity problems is emphasized, mapping recent advances to unresolved gaps in the literature.

In summary, this section provides a focused and integrative overview of inverse problems and uniqueness in conductivity, offering a foundation for further inquiry and innovation within the field.

### 7.1 Calderón's Problem

The investigation of inverse boundary value problems, with Calderón's problem as a prototypical example, forms a cornerstone of both theoretical and applied analysis in the context of electrical impedance tomography. The central question is whether the electrical conductivity of a domain can be uniquely reconstructed from knowledge of the boundary measurements, formalized through the Dirichlet-to-Neumann (DtN) map. Early groundwork, particularly the seminal contribution of Sylvester and Uhlmann, established the uniqueness of the solution for conductivities of class  $C^2$  in dimensions three and higher. However, extending this result to conductivities with lower regularity—most notably, Lipschitz continuous functions—posed significant analytical hurdles.

Progress toward resolving Calderón's problem for less regular conductivities highlighted the nuanced role of regularity in inverse problems:

- Initial extensions achieved uniqueness for  $C^1$  conductivities and demonstrated uniqueness for Lipschitz conductivities sufficiently close to the constant identity matrix.
- The technical bottleneck lay in constructing complex geometrical optics (CGO) solutions with minimal regularity assumptions, necessitating advanced Carleman estimates and sophisticated harmonic analysis techniques.

A major leap forward occurred with recent breakthroughs establishing the uniqueness of the inverse problem for arbitrary Lipschitz conductivities in dimensions three and higher, thereby resolving a central conjecture posed by Uhlmann and collaborators. These advances arise from an interweaving of methodologies originating in metric geometry, topology, and analytic PDE theory. The effectiveness of this interdisciplinary approach lies in the following:



- **Metric and analytic structure:** The metric properties of the underlying space critically influence the propagation of estimates required for the construction of CGO solutions.
- **Topological considerations:** These determine the possibility of asserting global uniqueness as opposed to merely local statements, thereby broadening the scope of the uniqueness results to non-Euclidean and general metric-topological environments [47].

As shown in Table 10, these methodological developments have progressively broadened the class of admissible conductivities and geometries for which uniqueness can be assured. Importantly, the successful resolution of the Lipschitz case provides a rigorous theoretical underpinning for realistic inverse problems encountered in applications such as medical imaging and geophysical exploration, where high smoothness cannot be assumed.

Nonetheless, several open challenges remain:

- The current techniques, while powerful, are analytically intricate and not easily generalized to domains with irregular boundary geometry or to anisotropic conductivity tensors.
- There remains a compelling need for a finer analysis of stability, as well as the development of robust and practical reconstruction algorithms in regimes of minimal regularity.
- The boundary between uniqueness and non-uniqueness has been further delineated, but questions of stability and quantitative reconstruction persist as fundamental areas of future research.

In summary, modern research on Calderón’s problem exemplifies a convergence of tools and ideas from partial differential equations, harmonic analysis, and geometric topology. These advances collectively mark a transformative milestone in the theory of inverse problems, furnishing both new avenues for mathematical exploration and enhanced frameworks for applied imaging [47].

## 8 Operator Algebras, Noncommutative Function Theory, and Topological Invariants

This section aims to elucidate the core interconnections between operator algebras, noncommutative function theory, and topological invariants, with an emphasis on recent developments and open challenges in the field. Our objectives are to clarify how advances in each topic inform the others, synthesize main findings, and highlight unresolved problems guiding future research.

We begin by surveying fundamental advances in operator algebras, focusing on the role of structural properties and classification results. This provides a foundation for understanding how algebraic frameworks facilitate new analytical approaches in noncommutative settings. In particular, the next subsection outlines the main constructs and challenges in noncommutative function theory, underscoring its dependence on operator algebra techniques. Here, we provide succinct summaries at the end of each discussion to encapsulate key ideas and identify persistent open questions.

Transitions between major subsections are emphasized via brief synthesis paragraphs, ensuring that the interplay between algebraic and topological perspectives remains clear to the reader. Specifically, when discussing topological invariants, we clarify how concepts from operator algebras and function theory coalesce to yield novel classification tools, as well as highlighting outstanding limitations.

Throughout this section, we explicitly state the main objectives at the start of each substantial part and employ concise take-home messages to reinforce learning and orient the reader as technical material develops. Open research directions and limitations are interspersed, rather than isolated to dedicated sections, to provide a continuous guide for ongoing exploration. The integrative perspective underscores both advances and the key gaps that remain, ultimately setting the stage for future developments at the interface of these fields.

### 8.1 Operator Algebras, Groupoids, and Quantum Invariants

The interplay between operator algebras and topological invariants constitutes a powerful framework for understanding both continuous and discrete systems within mathematical physics, especially concerning quantum matter, groupoid theory, and topological phases. Operator algebras encapsulate symmetries and large-scale features of noncommutative spaces, providing a robust apparatus for analyzing invariants associated with group actions and quantum symmetries. Notably, the classification of symmetry-protected topological (SPT) phases in two-dimensional quantum spin systems utilizes operator algebraic invariants—such as the  $H^3(G, \mathbb{T})$ -valued index—to capture nuanced equivalence classes of gapped Hamiltonians and their ground states. This approach surpasses traditional topological classification techniques by ensuring stability under automorphisms and equivalence under finite-depth quantum circuits, thus aligning with physical principles like locality and short-range entanglement [48].

Tensor networks operationalize these operator-algebraic notions by translating algebraic and categorical data—such as modular matrices and algebraic invariants—into physical observables characteristic of topological phases, including defect properties and the modular statistics of quantum systems endowed with symmetry or intrinsic topological order [4]. The tensor network formalism is particularly advantageous for addressing both bosonic and fermionic systems, capturing complex phenomena such as Majorana defects and supercohomology phases. Through this approach, the algebraic data of matrix product operator algebras maps directly onto measurable invariants in quantum materials.

Groupoid-theoretic methods, especially those developed for ample groupoids and topological full groups, furnish another vital pillar within this framework. The correspondences between groupoid homology, algebraic  $K$ -theory spectra, and infinite loop spaces have enabled explicit computations of rational and integral homological invariants across broad classes of topological full groups [49]. Frequently, these invariants display vanishing or acyclicity, reflecting deep structural features as seen in Brin-Higman-Thompson groups and yielding generalizations of the Matui AH-conjecture. Progress in the construction of permutative categories applicable to non-Hausdorff or minimal ample groupoids, and the establishment of new Morita invariance results, have significantly broadened the computational and conceptual reach of operator algebraic methods, enabling profound insights into groupoid-controlled dynamical systems and their topological types [5, 49].

The connection between operator algebras and coarse geometry exhibits a remarkable degree of rigidity exemplified by the theory

**Table 10: Summary of uniqueness results for Calderón’s problem under varying conductivity regularity and spatial dimension.**

Regularity of Conductivity	Dimension	Known Uniqueness Results
$C^2$	$n \geq 3$	Uniqueness established ( <i>Sylvester-Uhlmann</i> )
$C^1$	$n \geq 3$	Uniqueness established (via extensions of CGO and Carleman estimates)
Lipschitz (close to identity)	$n \geq 3$	Uniqueness established (perturbative regime)
Lipschitz (arbitrary)	$n \geq 3$	Uniqueness established (recent advances, resolves Uhlmann conjecture)
General Lipschitz	$n = 2$	Partial results; full uniqueness more subtle and context-dependent

of Roe algebras. A recent breakthrough asserts that isomorphic Roe algebras imply coarse equivalence for spaces of bounded geometry, reinforcing the comprehensiveness of operator-algebraic invariants as classifiers of large-scale spatial structure and creating a functorial bridge between geometric and operator-theoretic domains [27]. This correspondence enables bidirectional transfers of structural information, facilitating the derivation of spatial properties from algebraic data alone.

Operator-algebraic invariants also play a pivotal role in the study of quantum invariants for noncrystalline and patterned systems. Extensions of Chern number formulas to amorphous and quasicrystalline systems—lacking canonical site labeling—demonstrate the efficacy of operator-theoretic tools for establishing robust topological quantization across diverse physical models [26]. Index theorem formulations, in these settings, guarantee the robustness and quantization of topological invariants such as Hall conductance, even amidst pronounced disorder and synthetic configurations.

Intersections between operator algebras, tensor networks, and Morse or modular invariants have proven fruitful for rigorous determination of fractal dimensions and self-similarity within quantum and classical systems. Methods from complex and fractal geometry—including analyses of singularities in zeta functions and the calculation of parabolic Hausdorff dimensions—reveal new invariants of geometric and spectral significance for self-similar and aperiodic structures [13, 19, 50]. Taken collectively, these developments underscore the unifying capacity of operator-algebraic frameworks to encode and integrate topological, categorical, and analytical invariants across formerly distinct branches of mathematics and physics.

## 8.2 Analytical Methods and Spectral Theory

This section clarifies the survey’s objectives of bridging operator-algebraic, noncommutative, and analytical perspectives to illuminate spectral theory’s fundamental mechanisms and their broader applications within mathematical physics. Focusing on cocycle and Schrödinger operator theory, we highlight how recent analytical developments sharpen both our conceptual understanding and the available mathematical tools.

The integration of analytical methods, especially in the study of analytic  $SL(2, \mathbb{C})$  cocycles, has transformed spectral analysis by advancing the theory of almost reducibility as a cornerstone for probing spectral properties of analytic one-frequency Schrödinger operators. This approach forges an intricate dialogue between dynamical systems, analysis, and spectral theory, unifying operator-theoretic and dynamical invariants [51]. Importantly, novel analytical tools now confirm Avila’s Almost Reducibility Conjecture for

cocycles with non-exponentially approximated frequencies [51], expanding spectral characterizations and resolving longstanding obstacles in accessing non-trivial dynamical regimes.

A distinctive feature of these advances is the interplay between dynamical properties—such as reducibility and Lyapunov exponents—and the appearance of fractally structured spectral sets, which frequently manifest as Cantor-type or otherwise complex spectra. Functional-analytic methods complement operator algebraic invariants by facilitating explicit classifications of spectral types (absolutely continuous, singular, or pure point), with the classification dependent on the dynamical regime, such as subcriticality or criticality within the cocycle system. Yet, while these methodologies enable nuanced insights into spectral types and transitions, challenges remain in systematically describing the global versus local behavior of critical spectra and in classifying irregular or higher-rank dynamical scenarios.

Section-specific open problems persist. For example, a complete quantitative theory for critical and supercritical regimes of analytic cocycles remains unsettled, as does the effective characterization of spectra for multidimensional or higher-rank Schrödinger systems. Furthermore, debates are active regarding the universality and applicability of almost reducibility techniques beyond analyticity, and limitations of currently known methods when handling low-regularity or non-uniform hyperbolicity.

In line with survey-wide goals, future research is expected to focus on consolidating functional-analytic and operator-algebraic methods for a more unified framework, extending almost reducibility results, and developing sharper analytical invariants to capture the spectrum’s fine structure. These efforts are anticipated to yield applications in spectral theory, quantum dynamics, and the classification of topological phases, serving both mathematicians specializing in dynamical systems and analysts interested in spectral phenomena.

A dedicated subsection on open problems and research gaps follows, consolidating emerging directions, methodological limitations, and outstanding questions highlighted in the literature.

## 8.3 Noncommutative Function Theory

This subsection aims to survey recent developments in noncommutative (nc) function theory, emphasizing the operator-algebraic invariants that underpin rigidity phenomena and set the scope for current and emerging research directions. We particularly focus on how these invariants serve as unifying themes linking function theory, operator algebras, and broader geometric and physical

contexts. Our goal is to clarify the objectives of this area, highlight synthesized advances, and identify open questions central to guiding future investigations.

Progress in nc function theory highlights the depth and diversity of operator algebras as invariants in both function-theoretic and geometric settings. Recent research demonstrates that algebras of bounded noncommutative functions on operator balls, their homogeneous subvarieties, and the noncommutative unit polydisk naturally form operator algebras exhibiting pronounced rigidity properties; specifically, their isomorphism types are tightly controlled by geometric factors such as complete isometric isomorphism and nc biholomorphic equivalence [52]. These findings establish that the algebraic structure of uniformly continuous nc functions determines the corresponding noncommutative variety up to complete isometry, evidencing far greater rigidity than classical commutative function theory.

It is now clarified that, outside of particular contexts (such as the row ball), operator algebras of nc functions typically do not align with multiplier algebras of nc reproducing kernel Hilbert spaces, thereby exposing a fundamental non-universality in the noncommutative realm [52]. Extension and rigidity theorems for nc maps—especially those between subvarieties of injective balls—further highlight the centrality of operator-algebraic invariants in delineating isomorphism and extension boundaries, revealing subtle distinctions absent in the commutative context. One explicit open problem, as posed in [52], is to determine the extent to which the absence of universal multiplier structures in the general nc ball setting can be reconciled with deeper operator-algebraic or representation-theoretic frameworks.

In parallel, function theory on the symmetrized bidisc  $\Gamma$  and its spectral sets has achieved a compelling synthesis of operator theory, complex geometry, and algebraic curve theory. For any pair of matrices  $(S, P)$  with  $\Gamma$  as a spectral set, there exists a distinguished one-dimensional algebraic variety  $\Lambda \subset \Gamma$  such that for all matrix-valued polynomials  $f$ , the operator norm  $\|f(S, P)\|$  is controlled by  $\sup_{z \in \Lambda} |f(z)|$  [53]. The detailed classification of these varieties via determinantal representations linked to fundamental operators forges an explicit connection between operator algebras, determinantal geometry, and spectral theory. This framework suggests new directions for higher-dimensional generalizations and a need for exploring the full spectrum of spectral set phenomena in other noncommutative geometries. An open challenge is to systematically classify such varieties and their operator-theoretic invariants in substantially broader noncommutative domains.

The study of noncommutative function theory in self-similar and fractal contexts further reflects the presence of operator-algebraic invariants and rigidity phenomena, as seen in the classifications of algebras of bounded nc functions on “wild,” operator-theoretic analogues of fractal sets or varieties [19, 52]. These developments present several compelling but as yet unresolved questions regarding the limits of current classification techniques and the potential to realize new types of invariants in spaces with extreme topological or dynamical complexity.

The breadth and interplay of these operator-algebraic invariants, as summarized in Table 11, underscore their central role as unifying tools in the study of noncommutative geometry, topological phases, function theory, and quantum materials. Still, significant

open questions and limitations remain: the reconciliation of universal structures in nc function spaces, the explicit operator-algebraic classification of new geometric models, and a complete understanding of the boundaries between rigidity and flexibility in noncommutative domains. Strengthening the synthesis of operator theory with complex geometry and advancing the identification of actionable invariants in previously inaccessible settings represent prominent challenges for future research. These interconnections enable us to map current advances to both established and emerging research gaps, serving both experienced researchers and new entrants seeking to navigate the technical landscape and its applications.

## 9 Simplicial and Topological Structures; Persistent Homology

This section explores the foundational role of simplicial and topological constructs in contemporary data analysis, with an emphasis on persistent homology. Specifically, our objectives are threefold: to introduce the key mathematical frameworks underpinning these methods, to critically compare prominent approaches within the landscape, and to highlight open problems and future research directions in this domain. This focus aligns with the overarching goals of the paper, which seeks to provide both a comprehensive synthesis and a critical assessment of tools at the intersection of topology and data science.

### 9.1 Section Objectives and Relevance to Paper Goals

Simplicial and topological methods, including persistent homology, have emerged as essential for understanding high-dimensional and complex datasets. This section connects these algorithmic and theoretical developments directly to the paper-level aim of mapping the evolving interface between algebraic topology and data-driven inquiry. The explicit discussion of mathematical structures and computational frameworks provides readers with both the conceptual grounding and practical context needed to assess the comparative strengths and weaknesses of different approaches.

### 9.2 Comparative Assessment of Approaches

Throughout recent literature, various procedures for constructing simplicial complexes—notably the Vietoris–Rips, Čech, and witness complexes—have been proposed, each with specific computational trade-offs and domain-specific suitability. The Vietoris–Rips complex is lauded for its computational tractability, especially in scenarios involving large metric spaces, but can lead to high-dimensional simplices and, consequently, increased storage and processing demands. Čech complexes, though theoretically appealing due to their precise topological representation, often entail prohibitive computational cost as data size grows. The witness complex provides a compromise by leveraging a subset of “landmark” points to mitigate computational complexity, albeit potentially at the expense of topological fidelity. The choice between these complexes is thus an active subject of debate, tied to considerations of dataset size, structure, and the desired trade-off between accuracy and scalability.

Persistent homology, applied atop these complexes, provides a principled mechanism for capturing topological features across multiple spatial resolutions. However, current approaches to persistence

**Table 11: Summary of Key Operator-Algebraic Invariants across Topics**

Domain	Operator-Algebraic Invariant	Mathematical/Physical Impact
Quantum Spin Systems	$H^1(G, \mathbb{T})$ index	Classifies symmetry-protected topological phases; captures ground state equivalence beyond classical topology
Ample Groupoids and Topological Full Groups	Groupoid homology, $K$ -theory spectra	Enables explicit homological computations; generalizes Matui AH-conjecture; captures structure of dynamical systems
Coarse Geometry	Roe algebras	Establishes equivalence of large-scale geometry with algebraic isomorphism; functions as a complete invariant for bounded geometry spaces
Noncrystalline Quantum Models	Operator-index theorems, extended Chern numbers	Ensures robust topological quantization in disordered, amorphous, or synthetic materials
Noncommutative Function Theory	Isomorphism classes of nc function algebras, spectral set theory	Reveals rigidity phenomena; connects operator algebra structure to geometric, spectral, and function-theoretic properties

computation vary significantly. Matrix-reduction algorithms deliver rigorous guarantees but encounter scalability challenges, particularly in higher dimensions. On the other hand, techniques leveraging discrete Morse theory and cohomological methods promise improved efficiency but are not universally adopted, and the impact of these newer methods on result interpretability and stability remains a topic of ongoing inquiry.

Despite these advancements, certain limitations persist: the sensitivity of barcodes to noise in data, difficulties in vectorizing persistence diagrams for downstream learning tasks, and a lack of consensus regarding the most informative topological invariants for specific applications. Addressing these issues constitutes an active area of research. Moreover, debates continue around the interpretability of persistent features, particularly in domains outside of pure mathematics, where the mapping from algebraic invariants to domain-specific insights is often non-trivial.

### 9.3 Open Questions and Research Gaps

The rapid evolution of the field has surfaced a number of important open questions:

- How can persistent homology be further scaled to accommodate truly massive and streaming datasets, possibly in distributed environments?
- What are the theoretical limits of noise robustness for topological signatures derived from real-world data?
- Are there principled methods for selecting simplicial complex constructions, or must this remain heuristic and data-specific?
- How can the interpretability of persistent features be rigorously linked to domain semantics, especially in applied sciences?

Consolidating the above points, it is evident that while significant progress has been made in leveraging simplicial and topological structures for data analysis, a suite of critical methodological and theoretical challenges remains, shaping urgent directions for future research.

### 9.4 Simplicial Complexes and Computational Topology

The Vietoris–Rips and Čech complexes have profoundly influenced the field of topological data analysis (TDA) by providing a rigorous framework for extracting geometric and topological features from finite point sets and sampled manifolds. These complexes encode the structure of an underlying space via simplicial approximations, which are constructed according to proximity relationships controlled by a scale parameter. The scale parameter fulfills two crucial roles: it modulates the resolution of the topological summary and critically determines the stability and precision of computed invariants.

A central theoretical foundation in TDA is the Lipschitz-continuity of topological invariants—such as Betti numbers and Euler characteristic—with respect to the scale parameter in Vietoris–Rips

and Čech complexes [23]. This property ensures that minor perturbations in either the input data or the choice of scale produce only bounded changes in the computed invariants. As a result, the associated Betti curves, which track the evolution of homological features across scales, exhibit stability and, under appropriate sampling, converge on intervals to the true invariants of the underlying Riemannian manifold. These stability results are crucial for guaranteeing that topological signatures extracted from noisy or high-dimensional data reliably reflect intrinsic structures, thereby informing both the design and interpretation of filtrations in practical applications.

However, the computational cost associated with constructing Vietoris–Rips or Čech complexes escalates rapidly with increasing scale due to the combinatorial explosion of higher-dimensional simplices. This fundamental trade-off between representational fidelity and computational tractability remains a focal point in the development of scalable TDA, prompting active research into sparsified, localized, or otherwise optimized complex constructions.

Persistent homology operationalizes these theoretical insights by enabling the extraction of multiscale topological features, not only on sampled manifolds but also in more general spaces, including those with fractal or non-Euclidean metric structures. The persistence diagrams and associated Betti barcodes resulting from these analyses inherit the aforementioned stability properties, even in spaces exhibiting singularities or lacking a conventional manifold structure. Nonetheless, while current theoretical advances establish the methodological robustness of persistent homology, several open questions persist. Chief among these is the precise relationship between the persistence of topological features and the geometric or probabilistic properties of the underlying space, especially in the presence of non-manifold or singular geometric phenomena [23].

### 9.5 Bounded Cohomology, Aspherical Manifolds, and Differential Primitives

Bounded cohomology and its refinement, weakly bounded (or quasi-bounded) cohomology, play a pivotal role in geometric group theory and the study of aspherical manifolds, rigidity, and commutator lengths. A recent breakthrough has been the explicit construction of a finitely presented, locally CAT(0) group possessing a second cohomology class that is weakly bounded but not bounded, thus providing a sharp counterexample to a classic conjecture of Gromov regarding the existence of bounded primitives for differential forms on universal covers of closed manifolds [54].

This construction employs an explicit amalgamated free product of groups, carefully analyzed through the lens of piecewise Euclidean geometry. The resulting group yields a separation—previously unobserved for finitely presented groups—between bounded and

weakly bounded cohomology in low degrees. The key methodological innovation is the application of stable commutator length estimates and isoperimetric inequalities on the universal cover, which reveal a cohomology class lacking a globally bounded primitive yet still compliant with a linear isoperimetric bound, characterizing weak boundedness. The group admits a finite piecewise Euclidean model with local  $\text{CAT}(0)$  geometry, enabling the realization of aspherical manifolds that display these distinctive cohomological features.

The implications of this result extend broadly. By producing explicit aspherical manifolds in which the distinction between bounded and weakly bounded cohomology is realized, this work advances the understanding of the intricate structure of cohomological invariants in the realm of  $\text{CAT}(0)$  groups. It also informs ongoing investigations into the classification of aspherical spaces, properties of Kähler groups, and the invariants underlying quasi-isometric rigidity [54].

The integration of geometric, analytic, and combinatorial methods in addressing these problems underscores the nuanced interplay between group-theoretic properties, geometric models, and topological invariants. This approach not only disrupts previous conjectural boundaries but also suggests pathways for the systematic exploration of higher-degree analogues and the further refinement of isoperimetric and cohomological inequalities.

These insights highlight the ongoing evolution of both computational topology and the broader mathematical landscape, as new constructions and invariants continue to deepen our understanding of topological and geometric phenomena. Table 12 provides an overview of the stability results and computational obstacles characteristic of the principal simplicial complexes used in persistent homology.

## 10 Analytical, Algorithmic, and Data-driven Tools

This section systematically examines the core analytical, algorithmic, and data-driven tools employed in the field, with the objective of providing readers a clear and structured understanding of the main methods, their underlying principles, and their practical relevance. In particular, we highlight how these tools serve both foundational research and real-world applications, aiming to guide both newcomers and experienced researchers toward practical application domains and future research avenues.

Throughout this section, we draw particular attention to open problems and promising directions for future investigation, emphasizing unanswered theoretical questions, scalability challenges, and the adaptation of existing tools to emerging application domains. Transitional passages further reiterate the objectives of the survey, reinforcing how each tool or methodology aligns with overarching research goals and supports the broader progress in the field.

For the benefit of readers with varying backgrounds, we implement explicit signposting to clarify the intended audience and to identify which applications or problems are most relevant for each subtopic considered. The content herein provides a foundation for both a comprehensive academic understanding and an informed approach to practical implementation.

### 10.1 Analytical Methods in Fractal and Metric Analysis

This section aims to synthesize key analytical methodologies for the study of fractal and metric spaces, emphasizing both their comparative capabilities and the challenges they encounter in non-classical geometric contexts. The objectives are threefold: (1) to articulate how these tools contribute to dimensional, spectral, and variational analysis in spaces characterized by fractal geometry; (2) to examine their critical distinctions, practical limitations, and the current state of academic debate; and (3) to consolidate discussion of unresolved questions and research gaps that motivate ongoing inquiry in the field.

The study of fractal and irregular geometric structures has undergone significant advances, largely due to the synergy between thermodynamic formalism, Dirichlet forms, and heat kernel techniques. These frameworks collectively illuminate the interplay between geometry, spectral theory, and dynamics within non-smooth spaces. Initially rooted in statistical mechanics, thermodynamic formalism functions as a potent variational tool for determining the Hausdorff dimension of dynamically defined sets. It has proved especially valuable in number theory and multifractal analysis, where, for instance, continued fractions and sets classified by the growth properties of their coefficients have their Hausdorff dimensions precisely characterized via pressure equations [55]. Such approaches not only account for classical “dimension drop” phenomena but also generalize effectively to broader Diophantine scenarios [24]. By linking measure scaling and symbolic dynamics complexity to dimension theory, these methods underscore how statistical fluctuations translate into intricate multifractal spectra [56]. Nevertheless, core debates remain regarding the extent to which such pressure-based methods transfer to settings lacking clear dynamical invariance, and whether sharp multifractal formulas hold amid substantial disorder or irregular scaling.

Dirichlet forms, together with resistance metrics, have emerged as central analytical constructs on fractal spaces. They facilitate the extension of operators such as Laplacians to highly irregular sets, underpin the analysis of energy forms, and permit robust generalizations of partial differential equations (PDEs) beyond classical Euclidean contexts [11, 17, 57]. Noteworthy recent developments include the construction and study of Dirichlet forms on Laakso-type and IGS-fractals, leading to the precise definition of Sobolev spaces on fractals [24]. Revealingly, these studies have identified singularity phenomena—certain Sobolev exponents correspond to spaces that only intersect at constant functions—a marked departure from Euclidean intuition. The explicit demonstration of such intersection singularity for all exponents on self-similar models, as in [24], both highlights the analytic complexity of highly non-Euclidean spaces and poses open questions as to the generality of this behavior and its implications for potential theory and probability. Such singular analytical landscapes necessitate careful adaptation of potential-theoretic and variational techniques and underscore the entwined nature of energy measure geometry with self-similarity and capacity dimension [17, 57]. Active research is directed toward the non-symmetric theory and the stochastic processes naturally associated with these energy forms.

**Table 12: Summary of Key Properties and Computational Challenges of Vietoris–Rips and Čech Complexes**

Complex/Invariants	Key Stability Result	Computational Challenge
Vietoris–Rips Complex	Betti numbers/Euler characteristic are Lipschitz-continuous w.r.t. scale	Rapid growth of simplices with increasing scale
Čech Complex	Stability of persistent homology under input perturbation	High computational complexity for dense samples

Within this context, Harnack inequalities—traditionally cornerstones in the analysis of harmonic functions and PDEs on smooth manifolds—have demonstrated remarkable resilience. Recent findings establish the stability of the elliptic Harnack inequality under rough isometries and bounded perturbations for weighted graphs and more general metric measure spaces, even in the absence of volume doubling or classical heat kernel bounds [55, 57]. This versatility enables the application of analytic tools to spaces lacking regularity, signaling a broadening universality in the realm of metric potential theory. Nonetheless, precise boundaries remain unresolved: for which classes of irregular or singular structures (such as those with highly non-uniform local geometry) do classical inequalities break down or demand substantial refinement? The extension of parabolic or boundary Harnack inequalities in these settings is a prominent area of ongoing investigation.

The field of geometric function theory, and quasiconformal mapping in particular, continues to evolve, illuminating the subtleties of Ahlfors regularity, quasimetricity, and modulus in non-smooth contexts. Recent analytical results have extended to generalized Sobolev frameworks, where lower modulus bounds hold even in images of fractal nature [58]. Nevertheless, the anticipated equivalence between metric, analytic, and modulus-based definitions of quasiconformality can fail in the absence of strong geometric regularity conditions, such as upper Ahlfors regularity or projection finiteness. In fact, counterexamples demonstrate intricate failures in local linear connectivity and upper regularity, delineating intrinsic limits to generalizations previously considered robust [58]. These subtleties motivate ongoing work on characterizing fine-scale geometric obstructions and highlight open questions regarding the minimal conditions necessary for the full equivalence of quasiconformal definitions in fractal and metric settings.

Further significant progress is evident in harmonic and Fourier analysis on fractals, elucidating deep connections between geometry and analysis. Techniques such as spectral decimation and related Fourier-based methods permit explicit spectrum computations for operators defined on self-similar spaces and clarify a breadth of dynamical phenomena, from heat flow to quantum analogues within fractal domains. The spectral properties of Laplacians and Schrödinger operators form the foundation for understanding fractal measures and dynamical transport, as shown in studies of self-affine and projective fractal interpolation graphs [3, 25, 27, 59]. Explicit calculations of box-counting and Hausdorff dimensions—for instance, for graphs of fractal interpolation functions—often reduce to formulas involving the spectral radii of relevant scaling matrices, reflecting the profound structure underlying visually irregular sets [3, 15, 25, 27, 59]. These approaches explain the sharpness of dimension results in specific interpolation processes and help identify contexts where emergent regularity or randomness necessitates the development of even more refined analytical tools [3, 59].

However, open questions persist in extending explicit spectral and dimension results to less regular or non-affine scaling rules, to projective and higher-dimensional settings, and to settings lacking even basic separation conditions.

Despite these many strengths, analytic and geometric methodologies encounter notable obstacles when addressing spaces that lack separation conditions (such as the open set condition), exhibit high non-rectifiability, or fail to meet regularity benchmarks—for example, spaces lacking Ahlfors regularity or local linear connectivity. Fractals instantiated via irrational rotations, for instance, defy classical crystallographic analysis and necessitate new symmetry classifications that are often reliant on computational and combinatorial strategies [60]. Similarly, the analytical frameworks employed for PDEs and flow on fractal supports, though empowered by Dirichlet and energy-based approaches, frequently require sophisticated numerical discretization along with rigorous convergence analysis, as illustrated in recent studies of fractal Schrödinger equations [11]. This pressing need for new computational strategies and geometric invariants is central to current debates in the field.

As seen in Table 13, each analytical approach possesses distinctive strengths and corresponding constraints, necessitating careful selection and adaptation based on the properties of the underlying fractal or metric space.

*Open Problems and Future Directions.* Despite significant methodological progress, several open questions and research challenges shape the current landscape. Extending the reach of thermodynamic and multifractal formalism beyond symbolic or strongly invariant systems remains a major pursuit. For Dirichlet form methods, clarifying the prevalence and implications of Sobolev space singularity phenomena—notably the universality conditions identified on Laakso- and IGS-fractals [24]—raises deep analytic and probabilistic questions. In geometric function theory, identifying minimal geometric or analytic conditions that guarantee equivalence among various definitions of quasiconformality in fractal and wildly irregular settings is unresolved [58]. For spectral and Fourier analysis on fractals, unlocking explicit spectra and dimension results for non-affine, random, and higher-dimensional interpolation functions is an active frontier [3, 25, 27, 59]. On a computational and combinatorial front, developing robust invariants and analytic tools for spaces lacking separation or regularity—including those generated via irrational symmetries or exhibiting purely non-rectifiable behavior—remains a foundational challenge [17, 60].

These issues remain vibrant areas of ongoing theoretical and computational development, underscoring the need for further cross-pollination between analysis, geometry, probability, and computation in the exploration of fractal and metric spaces.

Table 13: Comparative Overview of Analytical Techniques for Fractal and Metric Spaces

Method	Key Features & Applications	Limitations
Thermodynamic Formalism	Variational characterization of Hausdorff dimension, multifractal analysis, symbolic dynamics [24, 50]	Relies on strong dynamical or scaling structure; less effective in non-invariant or irregular settings; unresolved in highly random/non-symbolic cases
Dierckx Form & Resistance Metrics	Extension of Laplacians, PDEs, energy forms to fractals; defines Sobolev spaces; potential theory; captures singularity phenomena [17, 24, 57]	Sensitive to singularities and irregularities; adaptations may be non-trivial on wild fractals; open questions on stochastic extensions and singular intersections
Harnack Inequalities	Robustness for elliptic PDEs; applications in varied metric measure spaces, including rough isometries [35, 37]	May lose efficacy without minimal geometric/analytic assumptions; boundary and parabolic cases still under study
Geometric Function Theory (Quasiconformal Maps)	Moduli and regularity in metric/quasi-metric settings; lower modulus bounds; analysis in fractal images [48]	Equivalences can break down in absence of strong regularity (e.g., Ahlfors irregularity, projection finiteness); delicate dependence on fine-scale geometry
Spectral & Fourier Analysis	Explicit spectra for fractal operators; links to transport and quantum systems; multifractal properties; projective and affine fractal interpolation functions [3, 25, 27, 59]	May require symmetry, self-similarity, or tractable scaling; complex on highly irregular spaces; extension to random, higher-dimensional, or non-separated settings remains open

10.2 Algorithmic, Machine Learning, and Data-driven Approaches

This section aims to survey recent, measurable advances and persistent challenges in algorithmic, machine learning, and data-driven methods for analyzing fractal and high-dimensional metric spaces, with clear objectives: to (a) summarize state-of-the-art strategies that complement traditional analytical techniques, (b) clarify their practical domains of success and failure, and (c) explicitly identify open problems that shape future research directions.

While analytical frameworks furnish essential theoretical insights, the rapid evolution of computational methods and data-driven paradigms is dramatically expanding the toolkit for probing, modeling, and quantifying fractal and high-dimensional metric spaces. Automated methodologies for extracting similarity variables—especially those grounded in optimization paradigms and symbolic regression—can discover latent self-similar or scaling structures directly from empirical data. This is particularly valuable in mathematical physics applications, where such approaches both rediscover established similarity variables (as in classical boundary layer or cavity collapse phenomena) and uncover new empirical scaling laws in less theoretically understood settings, such as multi-scale turbulent flows. Crucially, these strategies function without reliance on prior knowledge of the governing equations, instead inferring geometric regularities innate to the data [61]. Recent work has validated these methods’ ability to recover canonical similarity variables and to propose distinct scalings in novel settings, though their robustness in inherently multi-scale or noisy regimes remains an open research field <sup>1</sup>.

Machine learning techniques, especially when tested against detailed datasets like the MANTRA collection of triangulated manifolds and surfaces, introduce powerful quantitative metrics for scaling, complexity, and topological invariants of fractal and higher-order structures [16, 62]. Notably, neural network models based on simplicial complexes outperform conventional graph neural networks in tasks demanding robust topological invariance, such as Betti number prediction and orientability detection. However, as shown by comprehensive MANTRA benchmarks [62], even advanced higher-order architectures show substantial performance degradation when confronted with transformations like barycentric subdivision or with genuinely “wild” topological features, highlighting a gap between current model assumptions and geometric invariance observed in naturally occurring data. Moreover, there exists a dearth of standardized datasets that embody the full complexity of high-dimensional and topologically nontrivial objects, identified as a central bottleneck for empirical benchmarking and model development [16, 62]. Developing and diversifying such datasets is thus a crucial future direction.

<sup>1</sup>For instance, [61] demonstrates limitations in extending extraction methods to complex and multi-scale data.

Bridging these machine learning advancements with classical fractal insights, feature engineering driven by fractal geometry provides an interpretable and computationally efficient toolkit. The application of fractal dimensions and related analytic descriptors as features can yield classification performance comparable to, and in certain cases surpassing, that of deep neural networks—particular in domains requiring high geometric fidelity—while ensuring better interpretability and lower training cost [16, 62]. These approaches also benefit from robust statistical foundations [21], though ongoing research addresses the challenge of encoding higher-order or more abstract relations within these geometric features <sup>2</sup>.

Advances in algorithmic numerical analysis are instrumental in translating theory to practical computation. Finite element and other numerical schemes, specially adapted to self-similar and fractal measures, enable the reliable numerical solution of elliptic and parabolic PDEs on irregular or even “wild” domains. Results in this direction rigorously establish convergence rates and error bounds even in profoundly singular settings [3, 21, 59, 63–65]. These developments are further leveraged by visualization platforms for high-dimensional, projective, or Lorentzian geometries, making such complex domains accessible for exploratory analysis and hypothesis generation [63, 65, 66].

A key synthesis of topology and data science is achieved through persistent homology and combinatorial topology. Persistent homology robustly encodes topological features across scales, delivering a stable means to quantify structural invariants—such as loops, voids, and roughness—even amidst noise or discretization error [16]. These techniques have become indispensable for empirical validation, model selection, and for measuring topological “wildness” in real-world fractal datasets.

**Transitioning from these technical overviews to comparative evaluation,** Table 14 summarizes the landscape of contemporary approaches in terms of core applications and key limitations.

As detailed in Table 14, each data-driven and algorithmic approach comes paired with distinct strengths and constraints, necessitating their strategic pairing with analytical tools, depending on the complexity and features of the problem at hand.

**Looking forward,** several significant, unresolved challenges define the frontier for research in this domain. Most current deep learning architectures remain fundamentally limited in their capacity to enforce, or even leverage, intrinsic invariance to topological or geometric transformations, leading to dramatic performance drops on certain tasks and generally restricting model generalizability [16, 62]. Empirical progress is further hampered by the scarcity of datasets encompassing high-dimensional, “wild”, or topologically sophisticated geometries with ground truth labels, a limitation highlighted in both benchmarking studies and methodological reviews [16, 62]. Symbolic regression and similarity variable extraction, while showing flexibility and robustness in some classical

<sup>2</sup>See the limitations of fractal feature-based vs. deep models analyzed in [16].

**Table 14: Comparative Summary of Data-driven and Algorithmic Approaches**

Tool/Method	Strengths and Typical Applications	Key Limitations
Similarity Variable Extraction (Optimization & Symbolic Regression)	Identifies hidden self-similar/scaling laws from data; does not require prior equation knowledge	Sensitive to noise and multi-scale effects; less effective for non-classical or non-power-law scaling
Machine Learning on Geometric Data	Learns complex patterns and invariants; excels in tasks (e.g., Betti number prediction) when model bias fits data	Lacks intrinsic invariance to many geometric/topological transformations; performance highly data-dependent
Fractal-based Feature Engineering	Offers interpretability and computational efficiency; well-adapted to geometry-centric applications	Features may not capture higher-order relations; can be outperformed by deep models in more abstract contexts
Numerical PDE Solvers (FEM on Fractals)	Enables rigorous approximation of PDEs on irregular supports; establish convergence and error	May incur high computational cost; discretization and mesh quality (on fractals) pose major challenges
Persistent Homology & Topological Data Analysis	Quantifies topological features across scales; stable to noise/discretization; aids model validation	Computationally intensive for large/high-dimensional datasets; interpretation can be non-trivial

and empirical settings, are often challenged by noise, multi-scale structure, and the departure from power-law regimes [61]. Future research should thus prioritize: the design of models with built-in invariance to geometric and topological transformations, the expansion of comprehensive high-dimensional and topologically non-trivial datasets for benchmarking, and the synthesis of analytical and data-driven approaches to balance flexibility, interpretability, and computational practicality.

Overall, the evolving landscape of fractal and metric geometric analysis is increasingly shaped by the interplay of robust analytical techniques and fast-advancing computational and data-driven methodologies. The ongoing quest is for frameworks that balance flexibility, invariance, interpretability, and computational tractability. Meeting this challenge will require the continued integration of theoretical, algorithmic, and empirical advances, with each informing and refining the others to push the boundaries of understanding in fractal and metric analysis.

## 11 Self-Similarity, Scaling Laws, and Analytical/Stochastic Models

This section aims to systematically explore the measurable properties and analytical frameworks that underlie self-similarity and scaling phenomena observed in large-scale machine learning systems. We set out to (i) define and contextualize self-similarity and scaling laws in the context of deep learning, (ii) review analytical and stochastic models that have been proposed to explain these behaviors, and (iii) assess their explanatory and predictive power with respect to empirical and synthetic datasets. Additionally, we delineate unresolved challenges and research gaps to guide further inquiry.

Subsections below summarize key theoretical developments and juxtapose algorithmic narratives with analytical perspectives. Where applicable, we provide brief synthesis statements linking these facets before proceeding to summaries or tables.

At the conclusion of each substantive subsection, we highlight open questions relating to empirical validation, the generalization of models across architectures, and dataset-intrinsic sources of invariance.

### 11.1 Scaling and Self-Similar Structures

The study of scaling laws and self-similarity constitutes a cornerstone of the theoretical framework for understanding complexity across natural and engineered systems. Empirical scaling laws typically emerge from invariance under dilations or more general transformations, which can be precisely captured through concepts of universality, group-theoretic constructs, and quantitative measures such as fractal dimension and modulus [67, 68]. Self-similarity, in its essence, characterizes systems whose structures remain invariant under scaling transformations, producing recursive patterns

observable at multiple scales. This recursive nature underpins a wide spectrum of phenomena in physics, biology, and mathematics, including noise, amorphous geometries, and transcendental curves. Recent work has shown that transcendental scaling behaviors enable the construction of universal fitting functions for parametrizing complex data, even in cases where conventional models fail, thereby supporting robust quantitative descriptions of intricate systems [67, 68].

Both constructive and theoretical advances have clarified the multiplicity of self-similar structures. Notably, the distinction between combinatorial and analytic self-similarity has been elucidated by the analytic constructions and counterexamples presented via iterated graph systems (IGS). These constructions permit the rigorous formation of fractal spaces which display combinatorial self-similarity and satisfy Ahlfors regularity, yet do not exhibit analytic self-similarity in the sense of conformal dimension attainment or quasisymmetry to analytic Loewner spaces. Serving as concrete counterexamples to classical conjectures in geometric analysis, such as Kleiner’s conjecture, the IGS approach reveals the precise balances between regularity and symmetry that dictate fractal behavior [12]. The combinatorial Loewner property (CLP) is realized in these spaces, while explicit modulus and porosity computations verify that some constructed spaces remain outside the realm of quasisymmetric images of analytic Loewner spaces. The explicitness of the IGS framework further enables the direct calculation of energies and minimal potentials within these fractals, supporting fine-grained investigations into modulus, dimension, and analytic properties that were not previously accessible.

Recent advances in the classification of self-similar sets include the introduction of new symmetry types driven by rotations through irrational angles. Detailed computational searches have identified self-similar sets strongly connected and devoid of characteristic directions, organized around algebraic rather than integer parameters [64]. Explicit geometric examples, developed within this framework, mark a decisive move away from crystallographic restrictions and open new algebraic pathways for the synthesis and categorization of planar fractals.

Extensions to projective and amorphous self-similarity have expanded the scope of fractal theory beyond traditional Euclidean spaces. In particular, fractal interpolation, once limited to Euclidean settings, has recently been generalized to the real projective plane. Iterated function systems (IFS) composed of affine projective contractions generate attractors that are continuous topologically one-dimensional functions with fractal dimensions exceeding unity and, under suitable contraction ratios, approaching two [3]. This results in highly irregular or “wild” functions, highlighting both computational and visualization challenges unique to the projective setting. Importantly, these constructions reproduce classical



Euclidean results in the appropriate limit, providing a unified theoretical foundation while exposing novel regimes of geometric complexity in non-Euclidean domains.

In parallel, the study of Lorentzian models has advanced by generalizing the metric space framework to include unbounded cases. Overcoming previous boundedness restrictions, recent results have laid out minimal foundational conditions—such as the reverse triangle inequality, relative compactness of chronological domains, and a distinguishing property—to define Lorentzian metric spaces [11]. The association of canonical quasi-uniform and quasi-metric structures to such generalized spaces has widened the methodological toolkit for synthetic spacetime geometry, a development with significant implications in mathematical relativity and quantum gravity theory. Importantly, Gromov–Hausdorff stability is preserved under these generalizations, ensuring robust convergence properties and facilitating the comparative analysis of Lorentzian geometries.

Stochastic and group-theoretic perspectives have revealed further connections between algebraic structure and recursive invariance in self-similar systems. Flip graphs derived from Narayana sequences exemplify combinatorial self-similarity, where every Narayana sequence induces a self-similar and connected spanning subgraph. Here, the flip operation encodes recursive symmetry and is naturally aligned with a free group presentation, providing a bridge between discrete mathematics and abstract algebraic frameworks [67]. Additionally, hierarchical disorder in magnetic flux applied to Sierpinski gasket-like fractals demonstrates that controlled irregularity can systematically influence quantum states and persistent currents at the nanoscale, expanding the practical consequences of fractal geometry in materials science and nanoelectronics [69].

Quantitative characterization of self-similarity, with fractal dimension as a centerpiece, remains a fundamental challenge. Extensive computational methodologies for estimating fractal dimension have been surveyed, establishing that the appropriateness of any method depends intricately on the type of fractal structure and computational constraints involved [21]. Although deep learning-based approaches show promise in learning complex features, their effectiveness can be limited in scenarios where precise geometric properties are essential, and explicit extraction of fractal features often yields superior classification outcomes. Therefore, pragmatic trade-offs between methodological sophistication, computational feasibility, and classificatory accuracy persist in applied fractal analysis.

Comparison of these methods, as summarized in Table 15, highlights the pragmatic trade-offs between computational feasibility and classificatory efficacy in fractal analysis.

## 11.2 Stochastic Models in Physical Systems

The ubiquity of stochasticity and randomness in physical systems has motivated the development of models that inherently encode self-similar behavior and scaling properties. Archetypal examples include random walks (RWs), Lévy processes, and their respective generalizations. Classical RWs typify diffusive dynamics; however, modifications—such as introducing geometrically decreasing step sizes or steps varying according to nontrivial deterministic rules—yield anomalous diffusion characterized by subdiffusive or

non-Gaussian scaling. A paradigmatic case is the RW with step sizes shrinking geometrically, which, contrasted to standard RWs, exhibits a root mean square displacement scaling as  $t^{1/4}$ , diverging notably from the conventional  $t^{1/2}$ , and thus illustrating the marked impact of microscopic self-similarity on macroscopic statistics [70].

Lévy processes, together with their shot-noise representations, further expand this modeling landscape. Under truncation or as subordinate remainder processes, these models preserve self-similar properties parametric in the stable index. For selected values, their distributions transition to  $\alpha$ -stable forms, with the limiting case  $\alpha = 0$  leading to generalized Dickman distributions. These results emphasize the continuity of self-similar statistical structures even in the boundary regimes between modeling classes [71].

Self-similar behavior also underpins the statistics of extreme events, which necessitate tailored probabilistic frameworks. In seismology, the Gutenberg–Richter (GR) law posits an asymptotic power-law scaling of earthquake magnitudes above a certain threshold. Recent generalizations—adopting a Kaniadakis ( $\kappa$ -deformed) probability framework—demonstrate that the  $\kappa$ -GR law achieves excellent fits to observed magnitude distributions across all recorded events, notably capturing the surplus of low-magnitude seismic activity that eludes traditional power-law models. This enhancement reflects the nearly universal relevance of the entropic index and, correspondingly, the fractal, self-similar features inherent to tectonic fragmentation [72]. More broadly, in rare event statistics, genealogical Monte Carlo methods based on importance splitting reveal that the approximate self-similarity in “mean paths” of rare trajectories can be exploited to expedite computational approaches, retaining predictive power even when strict self-similarity is absent in high-dimensional, chaotic systems [73].

Analytical techniques for self-similarity in the context of partial differential equations (PDEs) and dynamical systems further enrich modeling possibilities. Canonical models from mathematical physics—including Navier–Stokes, Burgers, and kinetic equations governing magnetohydrodynamic (MHD) turbulence—support families of self-similar and degenerate solutions. The systematization of these solutions, at times utilizing hypergeometric functions, clarifies asymptotic regimes and links distinct universality classes. Notably, new kink-type and conjugate solutions to Burgers and Navier–Stokes equations, which extend beyond classical vortex solutions, exemplify the underlying diversity of scaling phenomena in nonlinear dynamics [74, 75]. In turbulence, the construction of self-similar solutions to MHD wave kinetic equations, formalized as nonlinear eigenvalue problems, both corroborates and systematizes the empirical prevalence of power-law decay, thus connecting observed scaling laws with their analytical origins [76].

Topological invariants—particularly in turbulence and magnetohydrodynamics—supply a complementary perspective, as the evolution of scaling exponents is intimately coupled to that of topological structures. This interplay underscores the extent to which complex dynamics are governed by invariance principles embedded within the system’s topology [76].

Contemporary data-driven and machine learning (ML) methodologies have inaugurated promising directions for the automated discovery of self-similar variables and scaling laws. Techniques

**Table 15: Selected Approaches to Fractal Dimension Estimation and Their Applicability**

Method	Strengths	Limitations
Box-counting	Simple; widely used	Sensitive to resolution; slow for fine detail
Hausdorff dimension	Theoretical rigor	Not easily computable for arbitrary sets
Wavelet-based methods	Captures local and global structure	Computationally intensive
Deep learning (feature extraction)	Learns complex features from data	May underperform for precise geometric structure
Direct computation of fractal features	High accuracy for classification	Requires manual feature selection

employing symbolic regression and profile collapse enable the recovery of similarity variables in canonical fluid dynamical problems without prior knowledge of governing equations, thus partially automating the extraction of scaling laws from complex datasets [61]. While such methods may be challenged by multi-scale or non-smooth data, they nonetheless establish a robust framework for law extraction across a spectrum of physical applications. Extensions to the analysis of spatio-temporal self-similarity, including video analysis and interpretability studies in neural network architectures (e.g., self-attention mechanisms), illustrate the expanding confluence of physical modeling and artificial intelligence [16].

In summary, the intricate interplay between self-similarity, fractal dimension, and stochastic modeling supplies not only powerful explanatory tools for tackling complex phenomena, but also paves the way for empirical and computational innovations across the sciences.

## 12 Topological, Quantum, and Game-Theoretic Invariants; Complexity

### 12.1 Quantum and Knot Invariants

This section aims to synthesize recent advances concerning the algebraic and topological behavior of quantum invariants for knots and three-manifolds, with an emphasis on the interplay between genus-degree inequalities, covering phenomena, and the underlying algorithmic or structural principles. Specifically, we highlight both unified bounds for quantum knot invariants and developments surrounding multiplicativity and asymptotics in the context of manifold coverings.

Recent developments in quantum topology have elucidated deep interconnections between topological invariants of knots and the algebraic structures underlying quantum invariants. A significant milestone in this field is the unification of genus-degree inequalities for a diverse array of knot invariants through the framework of Hopf algebra representations. This approach utilizes the Reshetikhin–Turaev construction alongside graded Hopf algebra theory, demonstrating that the highest degree of a twisted quantum invariant associated with a knot is bounded above by a function involving both the genus of the knot and the algebraic grading of the representing Hopf algebra. Specifically, for quantum invariants arising from representations of the knot group into automorphism groups of finite-dimensional graded Hopf algebras, the degree satisfies the sharp upper bound

$$\deg J_{H,\varphi}(K) \leq 2g \cdot d(H),$$

where  $g$  denotes the knot genus and  $d(H)$  represents the maximal grading in  $H$ 's decomposition. This result not only recovers classical degree-genus constraints for invariants such as the Alexander polynomial but also generalizes them to families of twisted and quantum invariants, including novel examples such as the Akutsu–Deguchi–Ohtsuki invariants. The versatility of this methodology underscores the pivotal role of Hopf algebra grading as a quantifier of topological complexity and demonstrates its potential to encompass broader classes of quantum invariants and topological computations. The principal advantages of this approach are its algebraic transparency and unifying explanatory power. At the same time, extending the framework to non-semisimple or infinite-dimensional cases remains challenging, as gradings may not be well-defined and topological interpretations can become more subtle. For in-depth details and the unified diagrammatic approach underpinning these genus bounds, see the comprehensive treatment in [77].

Transitions from these algebraic bounds to algorithmic and computational aspects naturally lead to the study of quantum invariants for three-manifolds and their behavior under topological operations such as covering spaces. While classical topological invariants (for example, the Euler characteristic or manifold volume) exhibit multiplicativity under finite covers, this property is typically absent for quantum invariants. However, recent research has identified certain perturbative quantum invariants—constructed via series expansions following the approach of Dimofte and collaborators—that display asymptotic multiplicativity in the setting of cyclic covers. The coefficients of these expansions are characterized by polynomials in twisted Neumann–Zagier data, thus providing an algebraic framework capable of capturing nuanced topological features. Notably, a newly introduced  $t$ -deformation of these perturbative invariants offers alternatives to existing deformations, with strong conjectural evidence suggesting concordance with the asymptotics of the Kashaev invariant at all perturbative orders. This approach, illustrated on several hyperbolic knots, marks significant progress in bridging the gap between classical and quantum invariants in the covering context and underscores the importance of algebraic data extracted from geometric representations. For an explicit discussion and concrete examples, refer to [78].

As evidenced in Table 16, the multiplicativity behavior sharply distinguishes classical invariants from most quantum invariants, with the latter exhibiting more nuanced responses to covering operations.

Despite these advances, several open challenges and future research directions persist. Chief among these are broadening the extension of degree-genus inequalities to quantum invariants based

Table 16: Comparison of Multiplicative Behavior for Selected 3-Manifold Invariants under Finite Covers

Invariant	Multiplicativity under Finite Covers	Reference/Context
Euler characteristic	Yes	Classical topology
Manifold volume	Yes (hyperbolic manifolds)	Classical geometry
Classical Alexander polynomial degree	Yes	Knot theory
Perturbative quantum invariants	Asymptotically, in cyclic covers	[78]
Kashaev invariant	Conjectural asymptotics	Quantum topology
Non-perturbative quantum invariants	No (in general)	Quantum topology

on infinite-dimensional or non-semisimple Hopf algebras, refining our understanding of the asymptotic matching between perturbative series and quantum invariants in wider classes of manifolds, and exploring potential algorithmic ramifications for computation in topology and quantum algebra. The further development of the algebraic framework for twisted or deformed invariants—especially as it intersects with machine learning methods for detecting topological invariance or with under-explored datasets of knot and manifold examples—remains an important and promising field for investigation.

12.2 Bordism and Analytic–Topological Invariants

**Objective:** This subsection aims to clarify the foundational interplay between bordism theories and analytic–topological invariants, and to highlight the main analytic, geometric, and homotopical perspectives unifying the field.

Topological invariants arising from bordism theories play a foundational role at the confluence of geometry, topology, and analysis. The synthesis of universal bordism invariants integrates deep analytic constructions, most notably the Atiyah–Patodi–Singer (APS) invariant, with homotopy-theoretic frameworks. Originally conceived as the index of certain Dirac operators on manifolds with boundary, the APS invariant anchors a systematic approach for linking analytic and topological invariants through secondary index theorems. This perspective naturally encompasses the Adams  $e$ -invariant, classical secondary invariants, and their higher-dimensional extensions in string bordism, each captured as particular instances within the universal theory.

Recent progress has yielded intrinsic analytic formulas for these bordism invariants, substantiating the equivalence of analytic and topological methodologies by means of explicit index-theoretic calculations. These advances not only clarify the conceptual relationships among analytic and homotopy invariants but also illuminate how secondary invariants can reveal fine geometric or topological features that primary invariants cannot detect. Moving forward, a key challenge remains in fully uniting analytic techniques with purely homotopical machinery, particularly in contexts involving exotic manifolds or string-theoretic structures, where analytic and topological subtleties are most pronounced [20].

**Key Points:** Universal bordism invariants include the APS index, Adams  $e$ -invariant, and their higher analogues; analytic–topological equivalence has been confirmed through explicit index computations; secondary invariants are vital in detecting fine features invisible to primary invariants; ongoing challenges involve bridging

analytic and homotopical perspectives, especially for exotic and string bordism.

**Summary Takeaway:** Analytic and topological methods in bordism theory have converged through secondary index theorems, deepening our understanding of geometric invariants and highlighting open questions in integrating analytic and homotopical frameworks for advanced topological structures.

12.3 Game-Theoretic and Combinatorial Dimension Theory

**Objective and Scope:** This subsection aims to elucidate how game-theoretic paradigms—specifically, infinite games such as the Banach–Mazur and Schmidt games—fundamentally reshape the analysis of fractal dimension theory. We focus on how these frameworks reinterpret classical notions of Hausdorff dimension, bridge results from descriptive set theory to fractal settings, and find powerful applications in areas like Diophantine approximation and dynamical systems.

Game-theoretic methods have emerged as a robust paradigm for analyzing dimension theory, revolutionizing traditional approaches via the study of Banach–Mazur and Schmidt games. A major conceptual breakthrough in this direction involves the reinterpretation of Hausdorff dimension through game variants that encode dimensional principles as strategic interactions between two players. In particular, the introduction of the Hausdorff dimension game—parametrized by sequences controlling nested ball sizes—enables the analysis of subsets of  $\mathbb{R}^d$  in terms of player strategies in infinite-length games. Under determinacy axioms such as AD, this framework facilitates the translation of regularity results from Baire category into the realm of Hausdorff dimension, demonstrating, for example, that any well-ordered union of sets of dimension at most  $\delta$  itself has dimension at most  $\delta$ .

This approach is powerful not only for its analytical depth but also for its ability to produce uniformization results and guarantee the existence of compact subsets with prescribed dimension inside analytic (and, under AD, arbitrary) sets. Consequently, classical insights from descriptive set theory permeate into fractal geometry. Nevertheless, the dependence on determinacy axioms restricts the general applicability of these results in frameworks where such foundational hypotheses are not assumed. Ongoing research seeks to diminish this reliance and to attain parallel advances for other notions of fractal dimensions [79].

The impact of these insights extends notably to Diophantine approximation and dynamical systems. Schmidt’s game, for example, is instrumental in examining the structure of badly approximable

numbers—a set proven to be "winning," and thus of full Hausdorff dimension, within this context. Generalizations of the game framework encompass inhomogeneously badly approximable sets and settings involving unimodular lattices, where innovative variants such as the rapid game guarantee the full dimensionality of certain exceptional sets [80]. These findings reveal a deep interplay between metric number theory, ergodic theory, and combinatorial games: the winning properties associated with Schmidt-type games not only ensure maximal dimensionality but also confer robustness under perturbations and transformations.

A notable trend is the extension of these results with increasing generality, reflecting both the adaptability and constraints of game-theoretic language for encoding arithmetic and topological complexity. Nonetheless, challenging questions persist—particularly regarding generalizations to multidimensional and non-Euclidean contexts, as well as the precise connections between determinacy principles and intricate aspects of descriptive set theory in analysis and dynamics [80].

**Key Points:** Game-theoretic dimension theory offers a framework for characterizing fractal dimensions via infinite games, revealing connections between strategy and measure. Determinacy principles (such as AD) allow the transfer of regularity properties from Baire category to Hausdorff dimension, aligning combinatorial and geometric insights. Applications of Schmidt's game and its generalizations reach from understanding badly approximable numbers to analyzing invariant sets in dynamical systems and spaces of lattices. A primary challenge is to extend these techniques to higher-dimensional and non-Euclidean contexts and to explore the implications of weakening determinacy assumptions.

**Practical Implications and Open Research Challenges:** Methods based on game-theoretic ideas provide powerful tools for constructing sets of prescribed dimension and for establishing uniformization and regularity results in fractal geometry. Open problems include seeking frameworks that reduce reliance on strong determinacy axioms, generalizing to broader classes of fractal dimensions, and deepening the connection between game-theoretic analysis and descriptive set theory. Recent adaptations, such as the rapid game variant, suggest promising avenues for understanding the dimensional structure of exceptional sets in number theory and dynamics, even when considering difference sets or more complex group actions.

**Takeaway:** Game-theoretic and combinatorial methods enrich the analysis of fractal dimension, merging techniques from topology, set theory, and arithmetic to yield new regularity phenomena and conceptual unification across several areas of mathematics, while also highlighting significant open directions for future research.

### 13 Diophantine Approximation and Fractal Geometry in Ultrametric and High Dimensions

**Objective and Scope:** This section aims to investigate the interplay between Diophantine approximation and fractal geometry, particularly as these topics emerge in ultrametric spaces and high-dimensional settings. We will clarify the main problems addressed, highlight thematic links, and point out open research directions.

Special attention is given to cross-paradigm synthesis and implications for modern mathematical analysis.

The section is structured as follows: We first motivate the study by outlining key questions and paradigms. Next, we explore foundational work in classical and ultrametric frameworks, followed by advances in high-dimensional generalizations. We conclude with a synthesis of results and emerging challenges.

**Summary of Key Takeaways:** This section: - Articulates the scope of Diophantine approximation in the context of fractal and ultrametric geometry. - Provides signposting between classical and modern results, organizing a complex landscape. - Identifies practical implications for high-dimensional geometry and outlines open challenges.

*Practical Implications and Open Research Challenges:* The synthesis presented here motivates further inquiry into algorithmic and computational aspects of fractal invariants in non-Euclidean settings, as well as the transferability of approximation theorems to novel topologies.

#### 13.1 Singularity and Dimension in High Dimensions

Recent advances in the theory of Diophantine approximation in high-dimensional spaces have significantly sharpened our understanding of sets defined by singularity and Dirichlet improvability. Central to this area is the set of singular vectors in  $\mathbb{R}^d$ , denoted  $\text{Sing}_d$ , whose Hausdorff dimension offers a precise quantification of the interplay between Diophantine conditions and the geometry of the ambient space. For  $d \geq 2$ , the dimension  $\dim_H \text{Sing}_d = \frac{d^2}{d+1}$  has been rigorously established, a formula that encapsulates how the exceptional nature of these sets persists even as their measure diminishes in higher dimensions [81]. This result goes beyond mere dimensional computation; it elucidates the delicate balance between arithmetic degeneracy and the fractal geometry inherent in exceptional Diophantine sets, providing a quantitative metric for deviations from generic approximation behavior.

Beyond singular vectors, the class of  $\varepsilon$ -Dirichlet improvable vectors, denoted  $\text{DI}_d(\varepsilon)$ , exhibits a more intricate dependency on the approximation parameter  $\varepsilon$ . Here, the Hausdorff dimension remains asymptotically close to  $\frac{d^2}{d+1}$ , yet it encodes a nuanced gradation shaped by the exponent of  $\varepsilon$ , interpolating between  $d^2$  and  $d$  as  $\varepsilon$  varies [81]. Such results deepen the comprehension of metric Diophantine phenomena by characterizing the structural shifts that occur as one moves along the spectrum from generic to singular cases of approximability.

A cornerstone of this field is the connection between these dimension results and the dynamics of flows on homogeneous spaces. In particular, the action of the one-parameter diagonal subgroup  $\text{diag}(e^t, \dots, e^t, e^{-dt})$  on  $SL_{d+1}(\mathbb{R})/SL_{d+1}(\mathbb{Z})$  provides an analytical framework for translating Diophantine properties into geometric trajectories. The codimension calculation for divergent trajectories aligns exactly with the scaling law for the singular set, underscoring the profound relationship between dynamical systems, geometric measure theory, and arithmetic approximation.

The derivation of these results synthesizes advanced counting arguments and harnesses the rich symmetries of homogeneous

Table 17: Hausdorff Dimension of Singular Vector Sets in Various Settings

Setting	Ambient Space	Hausdorff Dimension of $\text{Sing}_d$
Real ( $\mathbb{R}^d$ )	$d \geq 2$	$\frac{d^2}{d+1}$
Function field (Ultrametric)	$d \geq 2$	$\frac{d^2}{d+1}$

spaces. This methodological flexibility facilitates seamless transitions among geometric, combinatorial, and dynamical perspectives on fractal sets. Nevertheless, several formidable challenges remain, particularly regarding the extension of these dimension calculations and associated dynamical analogies to infinite-dimensional settings or to flows lacking linear structure, where the rigidity of conventional methods diminishes.

13.2 Function Field and Ultrametric Settings

The framework of Diophantine approximation admits a compelling generalization to function fields, whose ultrametric structure introduces unique phenomena absent from the real case. Although analogues of singular vectors and Dirichlet improvability remain central, the associated geometric and measure-theoretic properties diverge fundamentally due to the ultrametric norm and discrete valuation.

In this context, the Hausdorff dimension of the set of singular vectors has also been shown to be  $\frac{d^2}{d+1}$  for  $d \geq 2$ , paralleling the real case as summarized in Table 17 [82]. However, the proofs in the ultrametric setting rely on methodologies distinct from those employed in the Archimedean context, such as self-similar covering arguments, ultrametric inequalities, and specialized analogues of lattice theory adapted to finite fields and their completions. Structures arising from the inherent self-similarity and ultrametricity supplant the role of smooth geometry, enabling the identification of exact fractal characteristics as well as the existence of genuine "gaps" imposed by non-Archimedean constraints [82].

The strengths of ultrametric techniques are evident in their capacity to deliver precise dimension formulae and to highlight the rigidity and regularity of fractal sets over function fields. These approaches benefit from the discrete nature of finite base fields, facilitating combinatorial and self-similar constructions. However, notable limitations persist:

Many dimension results are contingent on the finiteness of the base field; it is presently unclear how these extend to function fields over infinite or more general bases. The self-similar and combinatorial tools critical in the ultrametric case may lose efficacy or applicability in broader settings, presenting significant technical and conceptual challenges.

Ongoing research aims to harmonize ultrametric and real approaches, seeking to bridge the gap introduced by the distinct geometric, combinatorial, and valuation-theoretic properties. This endeavor is further motivated by the increasing overlap between ultrametric Diophantine approximation and other domains such as higher-dimensional arithmetic and symbolic dynamics, where universal patterns and exceptional phenomena transcend the arithmetic foundations of each setting [81, 82]. The investigation of these intersections not only highlights unexpected regularities but

also exposes the distinct obstacles and prospects characteristic of ultrametric and high-dimensional geometric frameworks.

14 Synthesis and Thematic Interrelations

**Objective:** This section aims to synthesize key themes from earlier discussions, clarify their interrelations, and outline the broad practical implications and unresolved research challenges that arise at their intersection. By providing a cohesive overview, we seek to orient readers before delving into detailed paradigms or transitioning to subsequent analyses.

Throughout this section, we recap the principal mathematical paradigms and algorithmic approaches discussed previously, explicitly contrasting classical and modern developments where relevant. The synthesis also highlights conceptual bridges among different themes, helping to situate their individual contributions within the larger context of AI research on fractal and topological invariants.

**Summary of Key Takeaways:** - The mathematical structures underlying fractal and topological invariants serve as a unifying foundation for various AI algorithms, both classical (e.g., heuristic or rule-based methods) and modern (data-driven or deep learning approaches). - Recent advances demonstrate increasing convergence between theory-driven and data-driven perspectives, suggesting promising avenues for robust, interpretable, and scalable solutions. - While significant progress has been made in adapting classical invariants to computational paradigms, open questions remain about the transferability of these constructs to high-dimensional or non-traditional data domains.

**Practical Implications and Open Research Challenges:** - How can classical invariants be systematically integrated into modern learning architectures for improved generalization or interpretability? - What are the limitations of current AI methods in capturing complex geometric or topological structures present in real-world data? - There is a growing need for benchmark datasets and standardized evaluation protocols to facilitate comparison and reproducibility across approaches. - Interdisciplinary collaborations between mathematicians and AI practitioners are essential for translating theoretical results into scalable, real-world applications.

**Transition:** The next sections will examine in detail each major paradigm, providing both mathematical underpinnings and algorithmic case studies, and offering signposts to orient readers as distinct mathematical traditions are explored.

## 14.1 Thematic Synthesis

Contemporary mathematics and theoretical physics converge upon a persistent set of unifying themes, most prominently in the domains of self-similar structures, fractal geometry, topological invariants, operator theory, quantum methods, and algorithmic frameworks. Central to this convergence is the dynamic interplay between geometric regularity and irregularity, exemplified by self-similarity and the Hausdorff dimension. These concepts serve not merely as geometric curiosities, but as foundational principles interfacing multiple disciplines—from the microstructure of quantum spacetime to the combinatorial topology of high-dimensional data [2, 8, 11, 12, 16, 21, 22, 26, 49, 50, 55, 57, 60–62, 66–73, 75, 76, 83–88].

Traditional methodologies in fractal geometry and metric analysis have attained significant precision in the calculation and interpretation of fractal dimensions and scaling laws. Tools such as pressure equations and mass distribution techniques have linked the growth of continued fraction coefficients directly to Hausdorff dimension, elucidating transitions from full to vanishing dimension under varying constraints [2, 55]. Notably, the explicit construction of fractals with prescribed dimensional and topological properties—such as via iterated function systems, both in classical contexts and in new frameworks like those based on neutrosophic theory—have extended foundational tools, providing new routes for the analysis of product spaces and fuzzy  $\alpha$ -density concepts [85, 86]. The development of iterated graph systems and associated combinatorial Loewner properties has produced counterexamples that deepen the understanding of conformal dimension and quasymmetric equivalence, leading to tractable, richly structured fractal spaces with explicit modulus computations and non-classical geometric properties [12]. Recent advances further elucidate the stability and classification of analytic and geometric properties such as the elliptic Harnack inequality, even under rough isometries, demonstrating broad stability of invariants across varied geometric categories [57].

These rigorous frameworks have experienced substantial extension through operator theory, particularly via the spectral analysis of self-similar Laplacians and almost Mathieu operators. In these contexts, spectral decimation not only yields qualitative insights into underlying geometry but also facilitates explicit formulas for the density of states [26, 66]. Operator-algebraic approaches have furnished new invariants in the classification of symmetry-protected topological (SPT) phases, notably through  $H^3(G, \mathbb{T})$ -valued indices and the analysis of quantum invariants for knots and manifolds, merging algebraic, analytic, and topological data into a unified invariance principle [49, 50, 62, 67].

Parallel to these analytic developments, there is a decisive trend toward the integration of data-driven and machine learning methodologies. Contemporary research enables the automated discovery of scaling laws and the algorithmic extraction of self-similar variables directly from observations, obviating the need for prior knowledge of governing equations [21, 61]. These methodologies streamline the modeling of complex physical processes, facilitating the identification of nonclassical similarity variables, and have successfully recovered both classical and novel self-similar scalings in multiscale environments. Neural network architectures tailored for topological learning, especially when benchmarked against datasets like the

MANTRA suite, illustrate that simplicial complex-based models can better capture topological invariants compared to graph-based deep learning, although challenges persist—current methods often lack invariance under topological transformations and can underperform on tasks requiring explicit homeomorphism or orientability detection [62]. Thus, the development of topology-aware and genuinely invariant models remains an open problem.

Algorithmic and computational progress is further matched by innovative uses of fractal and spectral techniques for classification, prominently in the assessment of whether deep neural networks capture complex fractal features such as fractal dimension. Studies show that conventional deep models are often unable to internalize or represent such geometric complexity, whereas shallow classifiers leveraging explicit fractal features can outperform deep counterparts, particularly in structurally demanding domains, with gains in both efficiency and accuracy [16, 66]. Nonetheless, integrating these insights more deeply into learning architectures represents an ongoing challenge for the field.

Simultaneously, topological and quantum-inspired frameworks are reshaping classical invariants. Persistent homology has ushered in a comprehensive multi-scale approach to the analysis of spaces, bridging combinatorial and algebraic invariants with the geometry of data and physical fields [71]. Game-theoretic and non-Abelian frameworks have enriched the landscape further, uncovering new invariants through competitive optimization processes and self-similar reductions—such as matrix analogues of Painlevé equations—thus expanding the landscape of integrable models and invariants linked to geometric and combinatorial complexity [68, 88]. The resilience of invariants like the elliptic Harnack inequality across categories highlights the robust interplay between local analytic and global topological structures [57].

Quantum-theoretic and operator-algebraic methods continue to supply a complementary suite of techniques and invariants. In multifractional theories and quantum gravity, self-similar and scale-dependent structures underpin the concept of “dimensional flow”—the notion that effective spacetime dimension evolves as a function of scale, a principle sharpened by comprehensive comparative analyses across nontrivial geometric and quantum gravity models [60, 83, 84]. These multifractional frameworks allow for the empirical validation of theoretical predictions at the intersection of mathematics and physics. Within condensed matter, operator-algebraic perspectives facilitate deep connections between local symmetries, global invariants, and physically measurable quantities such as edge phenomena and conductance quantization [67, 70, 72]. Thus, both operator-theoretic and combinatorial invariants form an essential toolkit for addressing and linking frontier questions in geometry, physics, and information science.

## 14.2 Interrelations and Emerging Frontiers

Reiterating the core objectives of this survey, the aim is to illuminate how core themes—self-similarity, scaling laws, and invariance—bridge fractal geometry, metric space theory, partial differential equations, operator theory, and topological invariants, with particular attention paid to novel computational and operator-algebraic approaches. In synthesizing recent developments, the survey targets the following guiding questions: (1) How do self-similar and

scaling structures manifest in both classical (geometric, analytic) and computational (data-driven, machine learning) settings? (2) In what ways do operator-theoretic and topological invariants extend our understanding of geometric complexity and classification? (3) What are the state-of-the-art methodologies for extracting, quantifying, or leveraging these invariants across domains?

The preceding synthesis elucidates the extensive interconnections among these disciplines, which function not as isolated domains but as mutually reinforcing frameworks. Notably, self-similarity arises both as a geometric attribute and as an operator-theoretic principle, recurring across a spectrum of scales—from constructions of iterated graph systems yielding explicit counterexamples to conjectures interrelating self-similarity, modulus, and quasisymmetric mappings [22], to matrix generalizations in integrable systems where self-similar reductions yield new classes of multi-component Painlevé equations and non-commutative soliton dynamics [75]. Operator-algebraic and scaling-invariance languages unify these lines of inquiry.

This synthesis has been invigorated by recent expansions, including:

Operator algebras and their refined invariance notions, which encode subtle distinctions among SPT phases and surpass classical cohomological methods.

Advances in Lorentzian and quantum geometry, providing fresh tools for formulating and studying spatiotemporal invariants and connections to topological quantum field theory.

Ultrametric and multi-scale analysis, now better equipped to address the study of fine structural properties in projections and microsets.

A surge in the analysis of projections and microsets, offering new perspectives on geometric complexity across discrete and continuum models.

To clarify the unique contributions of this survey in light of prior reviews, we specifically foreground the integration of modern computational models and operator-algebraic invariants, and connect these to contemporary advances in data-driven frameworks and high-level topological machine learning benchmarks [16, 61, 62]. This sets the work apart from earlier surveys that primarily emphasized classical analytic or topological perspectives.

Recent state-of-the-art advancements directly connected to cited works include:

- The construction and benchmarking of intricate manifold triangulation datasets to probe topological invariance capabilities of deep learning models, with direct evidence highlighting the deficiency of current architectures in handling true topological invariance and subdivision [62].
- Empirical evidence that neural networks are currently ill-suited to extract or encode fractal dimensions, in contrast to shallow models leveraging explicit fractal features for significant performance and efficiency gains [16].
- General-purpose, data-driven methods for extracting self-similarity variables via symbolic regression and unsupervised profile collapse, capable of rediscovering analytic scaling laws from raw experimental and simulation data, and identifying previously unknown empirical scalings in turbulence [61].
- Identification of new asymptotic behaviors in quantum invariants of three-manifolds, including multiplicativity under finite covers and the establishment of genus-degree bounds

for twisted invariants, connecting quantum topology to geometric and algebraic complexity [77, 78, 89].

Key methodological frontiers, such as operator-algebraic invariants that distinguish SPT phases, the explicit computation of Euler characteristics and Chern classes for moduli spaces of Abelian differentials, and restrictions on symplectomorphism subgroups via flux homomorphisms, highlight both the technical sophistication and unresolved challenges still present in the theoretical landscape [89, 90].

Critical assessment of current limitations reveals several points of tension and controversy: For instance, deep learning models have not yet achieved invariance to topological transformations needed for robust topological data analysis, as exposed by benchmark datasets such as those in [62]. Similarly, data-driven discovery of self-similarity, while robust for certain classes of problems, struggles with truly multi-scale structures and lacks comprehensive solutions for processes with entangled dynamical regimes [21, 61]. Theoretical advancements in quantum invariants, while illuminating, raise open questions concerning the interplay between algebraic structure, topological complexity, and asymptotic behavior [77, 78, 89].

The utility of topological invariants and computational frameworks is further evident in persistent homology and multi-scale data analysis, where new models are evaluated against datasets designed to demand invariance under homeomorphisms and subdivision [87]. These developments point toward a prominent set of open problems:

- Computation of explicit invariants in group theory and operator algebras, particularly when classical invariants are insufficient for classification.
- Clarification of the precise role and scope of ultrametricity in number theory, dynamical systems, and analytic contexts.
- The development of robust topological invariants capable of distinguishing dynamical systems of nontrivial fractal architecture [77, 78, 89, 90].

At the methodological frontier, data-driven discovery is foundational: frameworks such as constrained symbolic regression and unsupervised profile-collapse are enabling the identification of similarity variables in the absence of explicit governing equations [61], yet challenges remain in capturing deep multi-scale structure and achieving true topological invariance in models [21, 62].

In summary, the evolving vision advanced in this survey is one of continued and deepening synthesis. Geometry, topology, operator theory, and analysis are increasingly recognized as interconnected pillars—with unification effected through pervasive motifs of self-similarity, scaling, and invariance. As these themes further intertwine, they are poised both to drive theoretical advances and to catalyze the creation of novel computational and experimental paradigms, impacting mathematics, physics, and data science alike.

## 15 Discussion and Outlook Synthesis, Open Problems, and Future Directions

The intersection of geometry, analysis, operator theory, and dynamical systems forms a fertile landscape for the development and unification of mathematical frameworks capable of addressing fractal and infinite-dimensional structures. Central to many recent advances is the synthesis of geometric intuition with analytic and

operator-theoretic rigor, notably in the study of fractal phenomena within both classical and abstract, infinite-dimensional contexts. The construction and analysis of invariants—quantities or structures preserved under transformations—has emerged as a unifying thread, connecting classical fractal theory to contemporary investigations and revealing both significant conceptual advances and persistent mathematical challenges.

A careful comparative analysis highlights both the synthesis already achieved and the open obstacles remaining within these domains. Classical fractal geometry relies on the concepts of dimension, measure, and self-similarity as its fundamental invariants, employing tools from real and harmonic analysis to probe rigidity phenomena—instances in which geometric or dynamic constraints enforce uniqueness or induce strong regularity on boundaries. These classical analysis methods have led to a deep understanding of how structure and restriction interplay in a variety of fractal objects. However, when these invariants are extended to more abstract settings—most notably operator algebras or infinite-dimensional vector spaces—significant obstacles arise. In such settings, commutative invariants often prove inadequate for fully capturing the noncommutative, or “quantum,” behaviors characteristic of operator-algebraic analogues of fractal sets. This inadequacy both underlines the necessity for novel theoretical constructs and exposes persistent gaps in the transfer of classical intuition to noncommutative geometries.

The development of invariant theory robust to the subtleties of infinite dimensionality and genuine fractal complexity remains a core open problem. While advances in noncommutative geometry and operator algebra have produced new classes of invariants—particularly through spectral triples and K-theoretic indices—the geometric interpretability of these invariants often lacks the granularity found in their classical counterparts. This limitation is especially evident in the analysis of boundary phenomena, where subtle interdependencies between regularity, rigidity, and analytic structure can defy straightforward generalization from the commutative case. An ongoing tension exists between operator-theoretic generalizations and the desire to retain the full geometric and dynamical richness of classical models.

As illustrated in Table 18, although certain operator-theoretic invariants can generalize classical concepts, they often fall short of reproducing the full richness—both geometric and dynamical—of their commutative analogues.

From the perspective of applications, these theoretical developments have significant implications beyond mathematics itself:

Operator-theoretic formulations foster finer classifications of partial differential equations on irregular domains, with techniques from fractal geometry suggesting innovative regularization strategies for equations with complex boundaries.

Fractal invariants, such as entropy and correlation dimensions, are being explored as analogues of measures of complexity and entanglement in quantum systems, spurring new investigations into the universality and limitations of invariant-based classification schemes.

The analysis of large, complex datasets—particularly those with network or point-cloud structures exhibiting self-similar or fractal characteristics—motivates the adaptation of both operator-theoretic

and geometric invariants for robust statistical and algorithmic methodologies.

Despite this progress, several open problems continue to shape the research agenda:

Extending the theory of rigidity and regularity to infinite-dimensional and noncommutative contexts remains underdeveloped, with fundamental questions concerning the existence and computability of operator-theoretic invariants still unsolved.

Bridging the divide between commutative and noncommutative models—specifically, constructing new invariants capable of interpolating and retaining both analytic tractability and geometric intuition—is a critical challenge.

The development of effective computational techniques, tailored to the complexity inherent to fractal and operator-algebraic structures, is increasingly urgent, driven by burgeoning applications in both physical sciences and large-scale data analysis.

Looking ahead, the continued synthesis of geometry, analysis, operator theory, and dynamical systems holds the promise not only of resolving some of the most persistent theoretical challenges but also of catalyzing transformative interdisciplinary advances. The anticipated progress will depend on the ingenious reinvention of invariant theories, undergirded by a profound understanding of rigidity and regularity within both traditional and abstract mathematical settings. Ultimately, such work stands poised to bridge classical theory with new frontiers in quantum science, data analysis, and the broader study of complex systems.

## 16 Conclusions

This survey set out to systematically synthesize and map recent advances across the interrelated domains of fractal geometry, operator theory, topological invariants, and computational approaches to geometric complexity. The primary objectives have been to: (1) clarify how foundational notions such as self-similarity, fractal dimensions, and metric/topological invariants have evolved in modern analysis; (2) highlight operator-theoretic and algebraic frameworks that enable deeper classification and rigidity results; (3) critically appraise new computational and data-driven paradigms; and (4) distill open problems at the interface of these perspectives. We aimed to address the following overarching questions: How do classical and novel invariants unify the understanding of fractal and self-similar structures? What are the limitations and advances in operator-theoretic and computational tools for analyzing such spaces? Which recent results best exemplify the confluence of geometry, analysis, algebra, and data science in this field?

The landscape of fractal geometry and its interconnected domains has matured from classical foundations into a sophisticated matrix of modern analytical, operator-theoretic, and data-driven paradigms. The confluence of these perspectives has cultivated a dynamic mathematical ecosystem wherein notions of self-similarity, metric invariants, topological complexity, and operator algebras collectively underpin theoretical breakthroughs and practical applications.

Early developments in fractal theory foregrounded self-similarity and dimensional analysis, culminating in rigorous geometric and



Table 18: Comparison of Classical and Noncommutative Fractal Invariants

Aspect	Classical Fractal Geometry	Noncommutative / Operator-Theoretic Approaches
Primary Invariants	Dimensions, measures, self-similarity	Spectral triples, K-theoretic indices
Main Analytical Tools	Real/harmonic analysis	Noncommutative geometry, operator algebras
Rigidity/Regularity	Well-characterized, geometric intuition	Partial, analytic; geometric interpretations limited
Extension Challenges	Intuition often transferrable	Commutative invariants insufficient for quantum aspects
Interpretability	High (geometry-driven)	Varies; depends on analytic construction

measure-theoretic invariants such as Hausdorff and box dimensions [13, 87]. These foundational notions afforded robust quantification of geometric irregularity and the structural oscillations observable in canonical examples, including the Cantor set and Sierpinski gasket [14]. Subsequent methodological innovations—most notably, the introduction of fractal tube formulas and the analytical exploitation of complex dimensions—expanded the available toolkit, enabling finer discrimination among fractal geometries and illuminating the profound interplay between dimensionality and oscillatory phenomena [14, 87].

In parallel, operator-theoretic approaches have forged a unifying framework, integrating fractal properties with spectral theory, harmonic analysis, and aspects of quantum mechanics. For example, the spectral decimation method has facilitated the precise analysis of Laplacians on fractals and supported extensions to self-similar quantum and almost Mathieu operators, anchoring the study of singular spectra and quantum phase classification in the arithmetic of fractal invariants [27, 36, 91]. Moreover, the emergence of operator algebras—exemplified by Roe algebras and their rigidity theorems—has reinforced the descriptive and classificatory power of algebraic structures for large-scale geometry, capturing metric properties up to coarse equivalence [18, 80].

A pivotal evolution in the field is the transition from static, highly idealized fractal models toward dynamically rich, often randomized, and data-informed settings. Recent research has employed stochastic processes on fractals—such as Lévy flights augmented with nontrivial drifts [92], studies of quantum Markov semigroups [77], and fractal analysis of KPZ-type stochastic PDEs [45]—to leverage probabilistic and operator-theoretic tools. These contributions elucidate subtle connections between analytic regularity, geometric measure theory, and the algebraic substrata of quantum systems, as highlighted in the rigorous classification of symmetry-protected topological phases and the construction of operator-invariant indices [35, 74, 93].

The interplay between metric and topological analysis with these advances remains particularly fruitful. Notable among recent results are explorations of universality and embedding in metric spaces, assessments of the stability of analytic inequalities (such as Harnack’s) under rough isometries, and detailed stratification of differentiability and connectivity properties within non-smooth spaces [10, 42, 66, 94]. The evolution of Gromov-Hausdorff type convergence concepts in both Lorentzian and Wasserstein settings further extends these themes beyond the Riemannian or finite-dimensional context, fostering a synthetic integration across geometric analysis, optimal transport, and theoretical physics [29, 32, 95].

Simultaneously, computational and data-driven advancements are reshaping the domain. Noteworthy innovations include the integration of topological deep learning architectures leveraging simplicial complexes, the creation of extensive datasets for geometric inference benchmarking, and the empirical discovery of self-similar variables in experimental data [33, 76]. These computational methods, while exposing current limitations in extracting fractal invariants through artificial intelligence [96], also reveal the transformative potential of combining geometric and fractal features to enhance classification, recognition, and scientific modeling—often surpassing purely data-driven models where structural priors are essential.

Compared to prior reviews, this survey not only traverses classical and modern developments but also uniquely synthesizes very recent (post-2023) advances in operator-algebraic rigidity [18], spectral self-similarity [66], and topological deep learning with high-order geometric data [62], as well as data-driven methodologies for extracting self-similarity directly from empirical observations [61]. It critically contrasts contemporary computational paradigms with structural mathematics, assessing limits and open problems (see below), and links analytic, topological, quantum, and data-oriented perspectives by explicit reference to their mutual and contrasting strengths.

As a brief enumeration, the following recent state-of-the-art advancements exemplify the field’s current trajectory (references point directly to the cited works for further detail): - Operator-algebraic rigidity: Roe algebra classification characterizes coarse metric equivalence at a new level of generality and functoriality [18]. - Spectral self-similarity: Spectral decimation and Laplacians on fractal spaces extend to almost Mathieu-type operators and quantum systems [66]. - Data-driven extraction of self-similarity: Symbolic regression and optimization enable the discovery of similarity variables from experimental and numerical data [61]. - High-order geometric datasets: The MANTRA dataset benchmarks topological deep learning models and reveals challenges in model invariance [62]. - Fractal structure in quantum gravity and KPZ/SPDEs: Quasilinear singular SPDEs and critical LQG measures are now rigorously linked to fractal support properties [42, 46]. - Recent computational limitations: Neural networks struggle to extract geometric fractal invariants, highlighting unresolved machine learning challenges [62, 96]. - Extended classification of dimensions: The exact dimensionality of projections and microsets enriches the landscape of metric and fractal dimension theory [38, 91].

Perhaps most significantly, the convergence of mathematics, physics, and computational science is forging genuinely interdisciplinary frameworks. Quantum invariants, spectral indices, and

algebraic structures are now not only objects of theoretical interest but also serve as bridges linking analysis, topology, mathematical physics, and information sciences [28, 61]. Integrative methodologies, such as multifractional theories for quantum gravity that generalize dimensional flow, constitute a synthesis of gravitational physics with fractal geometry, paving the way for both experimental confrontation and future theoretical integration [7].

At the same time, critical challenges and persistent limitations remain: - The classification of fractal sets and measures in higher dimensions remains incomplete, with new results revealing both rigidity and flexibility phenomena—such as the precise calculation of microset dimensions, spectra of invariant measures, and their projections [38, 91, 97, 98]. - The pursuit of explicit, computable invariants (and associated functional models) in noncommutative, sub-Riemannian, and quasilinear settings continues to drive innovation at the intersection of analysis, geometry, and dynamical systems [8, 45, 62]. - Ongoing limitations of contemporary computational models—including neural network invariance properties, the reach and granularity of operator-algebraic invariants, and the scaling of analytic versus heuristic methods—indicate urgent opportunities for the integration of structure-aware artificial intelligence with robust, scalable analytic algorithms [62, 76, 96]. - Unsettled controversies around the attainability, universality, and regularity of analytic inequalities (e.g., Poincaré, Harnack) under minimal regularity or in synthetic geometric settings [6, 57].

In summary, the evolutionary trajectory from foundational constructions in fractal geometry and dimension theory, through the synthesis of operator theory, quantum and algebraic invariants, and advanced computational paradigms, is inaugurating a new era of integrative mathematics. This progression advances not only the theoretical understanding of self-similar and fractal structures in mathematics and physics but also endows researchers with analytic and computational tools to approach the complexity of natural and engineered systems. The future of the field resides in the sustained dissolution of disciplinary boundaries, the identification of universal invariants and structures, and the invention of mathematical and computational languages that can faithfully articulate and manipulate the full richness of fractal, metric, operator, and quantum phenomena [1–109]. Future research will undoubtedly further unravel the interface among geometry, analysis, topology, operator algebras, quantum theory, and emergent computational paradigms, fostering new foundations and applications for fractal and self-similar modeling within mathematics and beyond.

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