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A SURVEY ON THE TOPOLOGY OF FRACTAL SQUARES

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Abstract. We consider a special type of self-similar sets, called fractal squares, and give a brief review on recent results and unsolved issues with an emphasis on their topological properties.

Contents

1. Introduction

A self-similar set in $\mathbb{R}^d$ means the attractor of an iterated function system (shortly, an IFS), which is a family $\mathcal{F}=\{f_j : 1\le j\le q\}$ consisting of $q\ge 2$ contractions of $\mathbb{R}^d$ into itself. Due to Hutchinson [1], there is a unique nonempty compact set K with $K=\bigcup_j f_j(K)$. In deed, if we set $\Phi(X)=\bigcup_j f_j(X)$ for nonempty compact sets $X\subset\mathbb{R}^d$, then $\lim\limits_{n\to\infty}\Phi^n(X)=K$ under the Hausdorff distance for any compact $X\ne\emptyset$. Here the Hausdorff distance between two nonempty sets X and Y is the infimum of all $\varepsilon>0$ such that $X_\varepsilon=\{z:\exists x\in X \text{ with } |x-z|<\varepsilon\}\supset Y$ and $Y_\varepsilon\supset X$ both hold. See for instance [1, §2.4].

Definition 1. *We call Φ the Hutchinson map of $\mathcal{F}$ and K the attractor of $\mathcal{F}$. If further every f_j is a similitude, we call K a* self-similar set.

Given an IFS $\mathcal{F}=\{f_j : 1\le j\le q\}$ and a sequence $w=i_1i_2\cdots i_n\in\{1,\ldots,q\}^n$, we denote by f_w the n-fold composite $f_{i_1}\circ\cdots\circ f_{i_n}$. For any infinite sequence $\omega=(i_n)_{n\ge1}\in\{1,\ldots,q\}^\infty$, we denote by ω_n the prefix of ω of length $n\ge1$. Then $\{f_{\omega_n}(K):n\ge1\}$ is a sequence of nonempty compact sets decreasing to a single point, to be denoted by x_ω. Moreover, it is also routine to check that

$$\lim_{n\to\infty} f_{\omega_n}(y)=x_\omega$$

holds for all $y\in\mathbb{R}^d$. Let $\mathcal{S}=\mathcal{S}_{\mathcal{F}}:\{1,\ldots,q\}^\infty$ be the map that sends every ω to x_ω.

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Definition 2. *We call $\mathcal{S} = \mathcal{S}_{\mathcal{F}}$ the symbolic projection of $\mathcal{F}$ (or of K).*

In the sequel, we often consider $d = (u, v) \in \mathbb{Z}^2$ as a vector in $\mathbb{R}^2$ so that every translation $x \mapsto x + d$ with $d \in \mathbb{Z}^2$ is well defined. Similarly, $f_{d,N}(x) = \frac{x+d}{N}$ is a map of $\mathbb{R}^2$ onto itself, for any $N \in \mathbb{Z}_+$ and $d \in \mathbb{Z}^2$.

Definition 3. *Given $N \geq 2$ and $\mathcal{D} \subset \{0, 1, \ldots, N-1\}^2$, we call the unique nonempty compact set satisfying $K = \bigcup_{d \in \mathcal{D}} f_{d,N}(K)$ a* fractal square *of order N.*

When the order $N \geq 2$ is known from the context, we often write $K_{\mathcal{D}}$ instead of K to emphasize the digit set $\mathcal{D}$. In the meanwhile, we also write $K(N, \mathcal{D})$, when we want to emphasize the order N and the digit set $\mathcal{D}$ both.

Topology of self-similar sets has been a focus of much attention in the study of fractal geometry. For fundamentals of the general theory of fractal geometry, one may refer to [2] and the references therein. For results that center around self-affine tiles or relate to numeration systems, one may refer to [3]. In this survey, we focus on fractal squares and quickly review some results and open questions from recent studies. In the special case $\mathcal{D} = \{0, 1, \ldots, N-1\}^2$, we have $K(N, \mathcal{D}) = [0, 1]^2$. So we usually consider $K(N, \mathcal{D})$ with $\#\mathcal{D} \in [2, N^2 - 1]$ and concentrate on their topological properties. We will restrict ourselves to the following issues.

- (A) To determine whether the number $\pi_0(K)$ of components of $K = K(N, \mathcal{D})$ is finite or (uncountably) infinite and further consider the following.
- (A.1) To determine whether K has local (or global) cut points, if $\pi_0(K) = 1$.
- (A.2) To describe the topology of K and of its components, if $\pi_0(K) > 1$.
- (A.3) To determine whether K is totally disconnected.

- (B) To find conditions for some (or all) components of K to be points, line segments, or non-degenerate continua that contain no line segment.
- (B.1) To determine whether a disconnected K has no point component.
- (B.2) To determine whether all the non-degenerate components of K are line segments and whether they form a *null sequence*, in the sense that for any constant $C > 0$ at most finitely many of them are of diameter $\geq C$.

- (C) To classify all fractal squares in terms of the lambda function λ_K, that is introduced in [18] for any compact set $K \subset \mathbb{R}^2$.

- (D) To classify certain families of fractal squares from a topological viewpoint.

2. To Count the Components of $K(N, \mathcal{D})$

Fractal squares $K = K(N, \mathcal{D})$ are self-similar sets contained in the unit square $[0, 1]^2$. The underlying IFS

$$\mathcal{F} = \left\{ f_j(x) = \frac{x + d_j}{N} : 1 \leq j \leq q \right\}$$

always possesses three basic properties. First, the interior $U = \mathrm{Int}\left([0,1]^2\right)$ is an open set such that $f_1(U) \cup \cdots \cup f_q(U)$ is a disjoint union contained in U. Thus the well-known *open set condition* [1, (5.2)] is satisfied. Second, the **double** intersection $E_{ij} = f_i(K) \cap f_j(K)$ for any $1 \le i < j \le q$ is nonempty **only if** $d_i - d_j$ belongs to

$$\left\{ \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Moreover, if $E_{ij} \neq \emptyset$ then it may consist of finitely many points, countably many points, a line segment, or a Cantor set together with finitely many (or none) isolated points. Third, the **triple** intersection $E_{ijk} = f_i(K) \cap f_j(K) \cap f_k(K)$ for any $1 \le i < j < k \le q$ is either empty or a singleton. In the latter case, it is easy to infer that for all $u \neq v \in \{i,j,k\}$ either $d_u - d_v$ or $d_v - d_u$ is a corner of $[0,1]^2$. Therefore,

$$\{\pm(d_i - d_j), \pm(d_i - d_k), \pm(d_j - d_k)\} \ \cap \ \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

has exactly three elements, each of which belongs to $\mathcal{D}$. On the one hand, a fractal square $K = K(N,\mathcal{D})$ and its components demonstrate very rich properties from a topological point of view. On the other, the connections between those properties and the digit set $\mathcal{D}$ are direct and illustrate a variety of subtleties. A large part of those connections have been well understood, due to recent as well as classical results from the literature.

Let us start from how the digit set $\mathcal{D}$ is related to the number of components in $K = K(N,\mathcal{D})$, to be denoted by $\pi_0(K)$. To do that, we need a graph that is defined for all self-similar sets.

Definition 4. *Let $\mathcal{F} = \{f_i, \ldots, f_q\}$ be an IFS and K the attractor. The first Hata graph $\mathcal{G}_1(\mathcal{F})$ is the one with vertex set $\mathcal{V}_1 = \{1, \ldots, q\}$ in which two vertices $i \neq j$ are incident if and only if $f_i(K) \cap f_j(K) \neq \emptyset$. Similarly, for all $k \ge 2$ the k-th Hata graph $\mathcal{G}_k(\mathcal{F})$ is the one with vertex set $\mathcal{V}_k = \{1, \ldots, q\}^k$ in which two distinct vertices $u = i_1 \ldots i_k$ and $v = j_1 \cdots j_k$ are incident if and only if $f_u(K) = f_{i_1} \circ \cdots \circ f_{i_k}(K)$ intersects $f_v(K) = f_{j_1} \circ \cdots \circ f_{j_k}(K)$.*

The graph $\mathcal{G}_1(\mathcal{F})$ is closely related to the connectedness of K. Indeed, we have.

Theorem 1 ([4, Theorem 4.6]). *$\mathcal{G}_1(\mathcal{F})$ is connected if and only if K is. In such a case, K is a locally connected cotinuum.*

If $K = K(N,\mathcal{D})$ is a fractal square, we also write $\mathcal{G}_1(\mathcal{D})$ instead of $\mathcal{G}_1(\mathcal{F})$. Set $\mathcal{D}_1 = \mathcal{D}$ and for $j \ge 1$ further set

$$\mathcal{D}_{j+1} = N\mathcal{D}_j + \mathcal{D} = \{Nd_1 + d_2 : d_1 \in \mathcal{D}_j, d_2 \in \mathcal{D}\} \tag{1}$$

$$K^{(j)} = \bigcup_{d \in \mathcal{D}_j} \frac{[0,1]^2 + d}{N^j}. \tag{2}$$

Call $K^{(j)}$ the *j-th approximation* of K. The sequence $\left\{K^{(j)} : j \geq 1\right\}$ decreases to K. Moreover, for any $d' \neq d'' \in \mathcal{D}$ the common part of $K + d'$ and $K + d''$ is nonempty if and only if $K^{(2)} + d'$ intersects $K^{(2)} + d''$. Therefore, one can induce a simple characterization of $\mathcal{G}_1(\mathcal{D})$.

Theorem 2 ([5, Theorem 2.2]). *$\mathcal{G}_1(\mathcal{D})$ has an edge between $d' \neq d''$ if and only if $K^{(2)} + d'$ intersects $K^{(2)} + d''$. Consequently, K is connected if and only if $K^{(3)}$ is.*

In order to determine whether a fractal square has finitely many components, one may use the approximations $K^{(n)} (n \geq 1)$ instead of $\mathcal{G}_n(\mathcal{D})$. We can rephrase the second part of Theorem 2 as follows.

Theorem 3. *$\pi_0(K) = 1$ if and only if $\pi_0\left(K^{(3)}\right) = 1$.*

Example 1. *Let $\mathcal{D}_0 = \{(0,0), (1,0), (1,1), (1,2), (2,1)\} \subset \{0,1,2\}^2$. The resulting fractal square $K(3, \mathcal{D}_0)$ is given in* [6, Example 2.3]. *See Figure 1 for a depiction of the approximations $K^{(1)}, K^{(2)}, K^{(3)}$ and check that $\pi_0\left(K^{(1)}\right) = \pi_0\left(K^{(2)}\right) = 1$ while $\pi_0\left(K^{(3)}\right) = 3$. In other words, $K^{(1)}$ and $K^{(2)}$ are connected (hence have one component) while $K^{(3)}$ hence $\mathcal{G}_1(\mathcal{D}_0)$ has three components.*

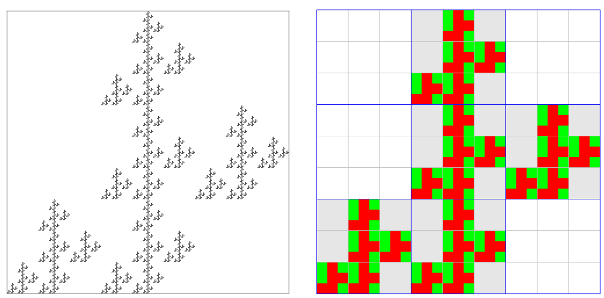

FIGURE 1. $K(3, \mathcal{D}_0)$ and the approximations $K^{(j)} (1 \leq j \leq 3)$.

Remark 1. *Given $\mathcal{D} \subset \{0, 1, \ldots, N-1\}^2$, $K(N, \mathcal{D}) = K(N^j, \mathcal{D}_j)$ for all $j \geq 2$.*

Let $q = \pi_0\left(K^{(1)}\right)$ and denote by $\mathcal{P}_1, \ldots, \mathcal{P}_q$ the components of $K^{(1)}$. Also denote by $\mathcal{V}_i^{(1)} (1 \leq i \leq q)$ the set of digits $d \in \mathcal{D}$ which are vertices of the subgraph $\mathcal{P}_i$. Further set $K_i^{(1)}$ to be the union of all the translates $\frac{K+d}{N}$ with $d \in \mathcal{V}_i^{(1)}$.

For the sake of convenience, we let $\alpha_i = \{i\} \times [0,1]$ and $\beta_i = [0,1] \times \{i\}$, where $i = 0, 1$. In the sequel, we give in several lemmas some observations that are either immediate or may be obtained by using self-similarity and standard arguments from plane topology.

Lemma 1. *Every homothety $NK_i^{(1)} = \left\{Nx : x \in K_i^{(1)}\right\} (1 \leq i \leq q)$ is a finite union of certain translates of the compact sets $K_j^{(1)}$. Thus $\left\{K_i^{(1)} : 1 \leq i \leq q\right\}$ becomes the system of attractors for a graph-directed IFS in the sense of* [7].

Lemma 2. *If no component of $K^{(1)}$ intersects each of the four sides $\alpha_0, \alpha_1, \beta_0, \beta_1$, then one can find either a Jordan arc γ in $[0,1]^2 \setminus K^{(1)}$ that intersects both $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ or one that intersects both $(0,1) \times \{0\}$ and $(0,1) \times \{1\}$.*

Lemma 3. *If $K^{(1)}$ has a component that is disjoint from one of the four broken lines $\alpha_i \cup \beta_j$ (with $i, j = 0, 1$) then K has a point component and $\pi_0(K^{(1)}) < \pi_0(K^{(2)})$.*

Lemma 4. *If $\pi_0(K^{(1)}) = \pi_0(K^{(2)}) > 1$ then there are two possibilities: (1) every component of $K^{(1)}$ intersects both $(0,1) \times \{0\}$ and $(0,1) \times \{0\}$, (2) every component of $K^{(1)}$ intersects both $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$. In such cases, we respectively call $\mathcal{D}$ a digit set with the* **north-south pattern** *and one with the* **east-west pattern***.*

The next result provides further insight, concerning point components.

Theorem 4 ([8, Theorem 1.1])**.** *If K has a point component then all its non-degenerate components form a subset K_c whose Hausdorff dimension $\dim_H K_c$ is strictly less than $\dim_H K$.*

The next result extends Theorem 3.

Theorem 5 ([9, Theorem 2.6])**.** *If $\pi_0\left(K^{(n)}\right) = \pi_0\left(K^{(n+1)}\right)$ for $n \geq 2$ then $\pi_0(K) = \pi_0\left(K^{(n)}\right)$. In particular, if $\pi_0\left(K^{(3)}\right) = 1$ (hence $\pi_0\left(K^{(2)}\right) = 1$) then $\pi_0(K) = 1$.*

The following dichotomy provides a fundamental starting point in exploring the topology of a fractal square.

Theorem 6 ([10])**.** *K has either finitely many or uncountably many components.*

Actually, by combining Lemmas 3 and 4 with Theorem 5 and [9, Theorem 3.3], one may strengthen Theorem 6 in the following way.

Theorem 7. *Every fractal square K falls into just one of two possibilities, either $\pi_0\left(K^{(n)}\right) = \pi_0\left(K^{(n+1)}\right) = \pi_0(K)$ for some $n \geq 2$ or $\pi_0\left(K^{(n)}\right) < \pi_0\left(K^{(n+1)}\right)$ for all $n \geq 2$. In the latter case there are just two sub-cases, either K has uncountably many point components or K is the product of a Cantor set and $[0,1]$.*

Example 2. *In Figure 2 we present $K^{(1)}$ and $K^{(2)}$ of a fractal square $K = K(5, \mathcal{D})$, whose digit set $\mathcal{D}$ is a subset of the one given in* [10, Figure 1]*. Notice that $\pi_0\left(K^{(1)}\right) = \pi_0\left(K^{(2)}\right) = 2$. From this, we can infer that $\pi_0\left(K^{(n)}\right) = 2$ for all $n \geq 3$ and that K has exactly two components. Such an inference is used in the proof for Theorem 5.*

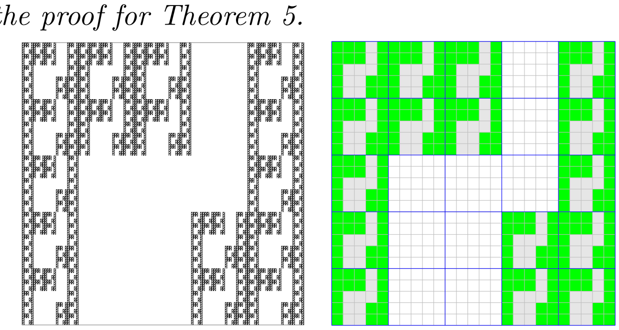

FIGURE 2. $K(5, \mathcal{D})$ and the approximations $K^{(1)}, K^{(2)}$.

To conclude this sections, we propose the following.

Question 1. *Given a fractal square $K = K(N, \mathcal{D})$, is it possible to determine $\pi_0(K)$ by checking the initial approximations $K^{(j)}$, say for $j \le 4$?*

3. Local Cut Points of $K(N, \mathcal{D})$

With Theorems 1 and 6, we know that a fractal square $K(N, \mathcal{D})$ has either one component (thus is connected), or more than one but finitely many components, or uncountably many components. In the current section we restrict to those that have at most finitely many components. The key issue is to determine whether a particular component of $K(N, \mathcal{D})$ is a universal plane curve, in the sense that it is homeomorphic with a classical fractal square $K(3, \mathcal{D}_S)$ known as the Sierpiński carpet, where $\mathcal{D}_S = \{0, 1, 2\}^2 \setminus \{(1, 1)\}$.

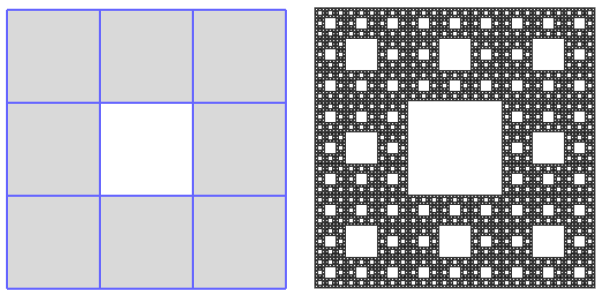

Figure 3. The digit set $\mathcal{D}_S$ such that $K(3, \mathcal{D}_S)$ is the classical Sierpiński carpet.

Recall that if a self-similar set K is connected, then the self-similarity from the underlying IFS $\mathcal{F} = \{f_1, \ldots, f_q\}$ ensures that for any $\varepsilon > 0$ we can find $n = n_\varepsilon > 0$ such that every $f_w(K)$ with $w = (w_1, w_2, \ldots, w_n) \in \{1, \ldots, q\}^n$ a word of length $|w| = n$ is of diameter less than ε. Since $\{f_w(K) : |w| = n\}$ is a finite cover of K and ε is arbitrary, K has *property S* and hence by [11, p.20, (15.1)] the following is immediate.

Theorem 8. *A connected self-similar set is locally connected thus is a Peano continuum.*

The local connectedness of general self-similar sets is characterized as follows.

Theorem 9 ([12, Theorem 1])**.** *A self-similar set is locally connected if and only if it has finitely many components.*

Here a Peano continuum just means the image of $[0, 1]$ under a continuous map. By Hahn-Mazurkiewicz Theorem, a continuum is locally connected if and only if it is a Peano continuum. The notion of Peano continuum is naturally generalized.

Definition 5. *A* Peano compactum *is a compact metric space with locally connected components such that for any constant $C > 0$ at most finitely many components may have a diameter greater than C.*

Remark 2. *We refer to* [13, Theorems 1-3] *concerning how the notion of Peano compactum is related to the study of polynomial Julia sets. In the next section, we will consider a special upper semi-continuous decomposition, having a Peano compactum as the quotient space, that is well-defined for any compact set K in the plane. With the help of this, we can define the so-called lambda function λ_K and apply it to the study on the topology of fractal squares.*

Notice that a connected Peano compactum is a Peano continuum. Moreover, by Theorem 6 a fractal square either is the union of finitely many Peano continua or has uncountably many components. In the former case, one may further wonder whether a component of K is an *S-curve* in the sense of Whyburn [14]. See Definition 6 below. In particular, when K is connected, we want to seek conditions under which K itself is an S-curve.

Definition 6. *An S-curve means a locally connected one-dimensional continuum S such that the boundary of each complementary domain of S is a simple closed curve and no two of these complementary domain boundaries intersect.*

The following is well known.

Theorem 10 ([14, Theorem 3])**.** *All S-curves are homeomorphic. In particular, every S-curve is homeomorphic with the Sierpinski carpet $K(3, \mathcal{D}_S)$.*

When it comes to the study of fractal squares, the characterization below is useful.

Theorem 11 ([14, Theorem 4])**.** *A one-dimensional Peano continuum in the plane is an S-curve if and only if it has no local cut point.*

Here a point x in a continuum X is said to be a *local cut point* provided that it has a connected neighborhood V_x with $V_x \setminus \{x\}$ a disconnected set. And a (global) *cut point* of X is just one whose complement in X is disconnected.

Theorem 12 ([6, Theorem 1.5])**.** *A fractal square $K = K(N, \mathcal{D})$ has a cut point if and only if every $\mathcal{G}_k(\mathcal{D})$ with $k \geq 2$ contains a vertex v_k such that at least two components of $\mathcal{G}_K(\mathcal{D}) \setminus \{v_k\}$ have more than $\#\mathcal{D}^{k-1}$ vertices.*

Assume that K is connected and $x_0 \in K$ is a local cut point. The point inverse $\mathcal{S}^{-1}(x_0)$, under the symbolic projection $\mathcal{S}$ given in Definition 2, falls into one of four possibilities. Indeed, its cardinality $\#\mathcal{S}^{-1}(x_0)$ equals either 1 or 2 or 3 or 4. If $\#\mathcal{S}^{-1}(x_0) = 1$, then K has a global cut point. Otherwise, one can find $n \geq 1$ and distinct digits $d_1, \ldots, d_j \in \mathcal{D}_n$ (with $2 \leq j \leq 4$) such that $\left(\bigcup_i \frac{K+d_i}{N^n}\right) \setminus \{x_0\}$ is disconnected and the following two requirements are both satisfied:

$$\{x_0\} = \bigcap_{i=1}^{j} \frac{K + d_i}{N} \tag{3}$$

$$x_0 \notin \bigcup_{d \in \mathcal{D}_n \setminus \{d_1, \ldots, d_j\}} \frac{K + d}{N^n} \tag{4}$$

For related discussions, we refer to [6, Theorem 1.6]. An issue of interest is to characterize fractal squares that are S-curves, and even those that have finitely many components each of which is an S-curve. Moreover, we wonder whether we can do that by checking the approximations $K^{(n)}$ for small n and taking distance from the Hata graphs $\mathcal{G}_j(\mathcal{D})$. More precisely, we propose the following.

Question 2. *Is it possible to determine whether a connected fractal square K is an S-curve by just checking the initial approximations $K^{(j)}$, say for $j \leq 4$?*

4. To Analyze the Components of $K(N, \mathcal{D})$

The previous two sections respectively ask about the number of components in a fractal square $K = K(N, \mathcal{D})$ and the existence of global or local cut points. In this section we investigate into the geometry of the components of K, such as the local connectedness of non-degenerate components and the existence of infinitely many components whose diameters are bounded from below by a constant $C > 0$.

The components P of K may be divided into three types: (1) a point, (2) a line segment, (3) a continuum that is not a line segment. To carefully analyze the components of K, we may need to employ the following sets and their complements: (1) the union $H = K + \mathbb{Z}^2$; (2) the sequence $H_j = K^{(j)} + \mathbb{Z}^2$ for all $j \geq 1$.

Let us recall some fundamental properties concerning H and H_j.

Lemma 5 ([15, Equations (2.1)])**.** *The containment $H_j \supset NH_{j+1} = \{Nx : x \in H_{j+1}\}$ holds for all $j \geq 1$ hence $H \supset NH$. Consequently, if H contains a line segment of irrational slope then $H = \mathbb{R}^2$, or equivalently, $K = [0, 1]^2$.*

Lemma 6 ([15, Lemma 2.1 and Theorem 2.2])**.** *Either all components of H^c are unbounded or none of them is. In the latter case, all components of H^c are of diameter $\leq \dfrac{\sqrt{2}(N^2+1)^2}{N}$.*

Lemma 7 ([15, Lemma 2.3, Corollary 2.4, and Theorem 2.5])**.** *If K contains a line segment of slope k_0 then H contains an infinite line of slope k_0. If K contains two line segments of distinct slope then K has a component which is not a line segment. Moreover, K has a component which is not a line segment if and only if every component of H^c is bounded.*

With the above results, Lau et al [15] culminate in the following dichotomy.

Theorem 13 ([15, Theorems 1.1 and 2.2])**.** *A fractal square $K = K(N, \mathcal{D})$ falls into one of two possibilities: (1) all components of $\mathbb{R}^2 \setminus H$ are unbounded and every component of K is either a point or a line segment; (2) every component of $\mathbb{R}^2 \setminus H$ is of diameter no greater than $\dfrac{\sqrt{2}(N^2+1)^2}{N}$. In the former case, all the line segments in K, if there are any, are of the same* **rational** *slope.*

In case one, there are three sub-cases:

(1.a) all components are points hence K is totally disconnected;

(1.b) no component is a point;

(1.c) there are both points and line segments.

In case two, there are two subcases:

(2.a) there are finitely many components, each of which is a Peano continuum;

(2.b) there are uncountably many point components and infinitely many non-degenerate ones.

The totally disconnected fractal squares, for sub-case (1.a), may be characterized by the existence of special sub-continua of $[0,1]^2$ that are disjoint from the interior of $K^{(n)}$, for some $n \geq 1$.

Definition 7 ([16, p.36, Definition 27]). *Given a fractal square $K = K(N,\mathcal{D})$ of order $N \geq 2$, a* complete path *P at level $n \geq 1$ means the union of squares $\frac{[0,1]^2+d}{N^n}$ for d belonging to a subset $\mathcal{D}_P$ of $\{0,1,\ldots,N^n-1\}^2 \setminus \mathcal{D}_n$ such that the next two requirements are both satisfied:*

(a) *There exist $0 \leq i,j \leq N-1$ with $\mathcal{D}_P \supset \{(i,0),(i,N-1),(0,j),(N-1,j)\}$*

(b) *The interior of P is connected and hence P itself is a continuum disjoint from the interior of $K^{(n)}$.*

The above notion is more general than the one given in [16]. Using Theorem 13 and standard arguments, one may easily infer the following.

Theorem 14 (see [16] or [17, Proposition 2.3]). *A fractal square $K(N,\mathcal{D})$ is totally disconnected if and only if there is a complete path, at level n for some $n \geq 1$.*

Remark 3. *An issue of interest is to estimate the level n from above. That is to say, one may wonder whether there is a universal bound, say n_0, such that if there is a complete path then there is one at level $n \leq n_0$. For fractal squares of order $N = 3$, the five $K(3,\mathcal{D})$ given in* [17, Fig. A.1] *are totally disconnected. Clearly, for none of them one can find a complete path at level one.*

Due to recent studies from the literature, we can further clarify all sub-cases in the following way. By [8, Theorem 1.1], see also Theorem 4, if K has a point component then the non-degenerate components form a subset whose Hausdorff dimension is strictly less than that of K. This points out a significant property shared by fractal squares that have point components.

We may characterize sub-case (1.b) as follows.

Theorem 15 ([9, Theorem 3]). *If $K = K(N,\mathcal{D})$ has uncountably many components and none of them is a point, then K is the product of a Cantor set with $[0,1]$. Moreover, this happens if and only if $\#\mathcal{D} = Nq$ for some $q \geq 2$ and $K^{(1)}$ consists of either q rectangles of the form $\left[\frac{i}{N},\frac{i+1}{N}\right] \times [0,1]$ or q ones of the form $[0,1] \times \left[\frac{i}{N},\frac{i+1}{N}\right]$.*

In sub-case (1.c) there are at least one hence infinitely many line segments each of which is a component of K. Moreover, we have the following.

Theorem 16 ([15, Corollary 2.6]). *If all components of $\mathbb{R}^2 \setminus H$ are unbounded and if K has both point components and non-degenerate ones then there is a constant rational k_0 such that every non-degenerate component is a line segment of slope k_0.*

In each of the sub-cases (1.a), (1.b), (1.c) and (2.a), all the components of K are locally connected. In sub-case (2.b), it is also known that every component of K is locally connected. Indeed, we have.

Theorem 17 ([12, Theorem 2])**.** *Every component of K is a Peano continuum.*

Based on the core decomposition of planar compact sets (obtained in [13]), Feng et al introduce in [18] the notion of lambda function for all compact sets in the plane. In particular, one may employ the lambda function in studying the topology of fractal squares. To illustrate that, let us prepare some terminology.

For any compact set $K \subset \mathbb{R}^2$, let $\mathfrak{M}_K$ consist of all the upper semi-continuous (shortly, usc) decompositions of K that satisfy two requirements: (1) every element is a sub-continuum of K, (2) the resulting quotient space is a Peano compactum. By [13, Theorem 7], there exists an usc decomposition $\mathcal{D}_K^{PC} \in \mathfrak{M}_K$ that refines every other $\mathcal{D} \in \mathfrak{M}_K$.

Definition 8. *Call $\mathcal{D}_K^{PC}$ the core decomposition of K. An element of $\mathcal{D}_K^{PC}$ is referred to as an (order-one) atom of K. If $\delta \in \mathcal{D}_K^{PC}$ then every atom of δ is called an order-two atom of K. Similarly, every atom of an order-n atom is called an order-$(n+1)$ atom of K.*

Now, from the nets of "atoms within atoms" one may define the lambda function as in [18]. Given a compact set $K \subset \mathbb{R}^2$, define $\lambda_K : \mathbb{R}^2 \to \mathbb{N} \cup \{\infty\}$ as follows, to be called the *lambda function* of K.

First, set $\lambda_K(x) = 0$ for all $x \notin K$ and all x such that $\{x\}$ is an order-one atom of K. Second, let $\lambda_K(x) = m - 1$ for all $x \in K$ if there is a minimal integer $m \geq 2$ such that $\{x\}$ is an order-m atom of K. Finally, set $\lambda_K(x) = \infty$ for $x \in K$ if such an integer m does not exist.

Definition 9. *Call $\lambda_K(K) = \{\lambda_K(x) : x \in K\}$ the* lambda range *of K.*

The lambda function is useful in the study of plane topology. For instance, a compact set $K \subset \mathbb{R}^2$ is a Peano compactum if and only if its lambda function vanishes everywhere. This is equivalent to $\lambda_K(K) = \{0\}$. In particular, a planar continuum K is a Peano continuum if and only if $\lambda_K(K) = \{0\}$. The classical Torhorst Theorem [11, p.106] states that, for any Peano continuum $M \subset \mathbb{R}^2$ and any component U of $\mathbb{R}^2 \setminus M$, the boundary ∂U is also a Peano continuum. Below we copy a quantified version of Torhorst Theorem from recent study.

Theorem 18 ([18, Theorem 2])**.** *For any compact $K \subset \mathbb{R}^2$, any component U of $\mathbb{R}^2 \setminus K$ and any compact $L \subset \partial U$, $\lambda_L(x) \leq \lambda_{\partial U}(x) \leq \lambda_K(x)$ holds for all $x \in \mathbb{R}^2$.*

Now we apply the lambda function to the study of fractal squares.

Theorem 19 ([9, Theorems 2 and 3])**.** *For any fractal square K, $\lambda_K(K) \subset \{0, 1\}$. Moreover, $\lambda_K(K) = \{1\}$ if and only if K is the product of a Cantor set and $[0, 1]$.*

It follows that a fractal square K falls into just three possibilities.

(i) $\lambda_K(K) = \{1\}$ and K is the product of a linear Cantor set and $[0, 1]$.

(ii) K is a Peano compactum hence $\lambda_K(K) = \{0\}$. In such a case, either K has finitely many components or for any constant $C > 0$, K has at most finitely many components that are line segments of diameter $\geq C$.

(iii) K is not a Peano compactum and $\lambda_K(K) = \{0, 1\}$. In such a case, all the non-degenerate components of K form an infinite family of line segments with the same slope; moreover, there is a constant $C > 0$ such that infinitely many of those line segments are of diameter $\geq C$.

It is easy to find simple fractal squares with $\lambda_K(K) = \{0\}$ or $\{1\}$. Let us give two fractal squares $K = K(5, \mathcal{D})$ with $\lambda_K(K) = \{0, 1\}$, where $\mathcal{D} = \mathcal{D}_A$ or $\mathcal{D}_B$. See Figure 4 for the depiction of the first approximation of $K(5, \mathcal{D}_A)$ and $K(5, \mathcal{D}_B)$. We can verify that for $K = K(5, \mathcal{D}_A)$ the level set $\lambda_K^{-1}(1)$ contains infinitely many line segments and its Hausdorff dimension $\dim_H \lambda_K^{-1}(1)$ is one. On the other hand, for $K = K(5, \mathcal{D}_B)$ the level set $\lambda_K^{-1}(1)$ contains uncountably many line segments and satisfies $1 < \dim_H \lambda_K^{-1}(1) < \dim_H K$.

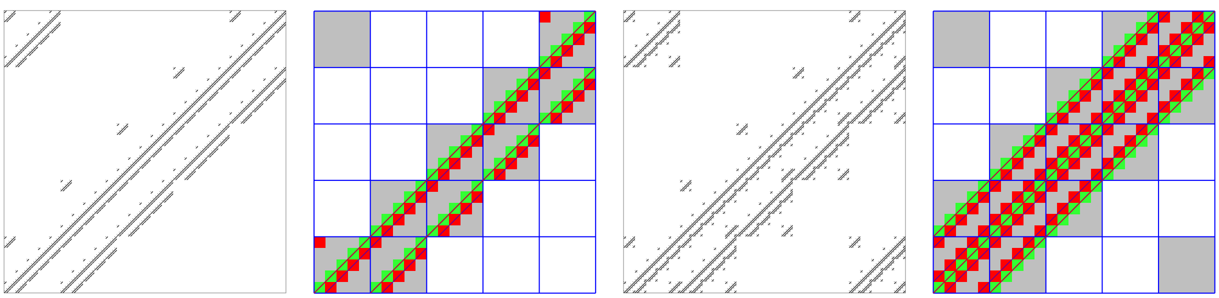

FIGURE 4. $K(5, \mathcal{D}_A)$ and $K(5, \mathcal{D}_B)$ together with the first approximations.

Fractal squares K with $\lambda_K(K) = \{1\}$ is characterized by [9, Theorem 3]. See also Theorem 19. The following Question 3 are of natural interest.

Question 3. *Under what conditions, on the approximations $K^{(j)}$ with small j, do we have $\lambda_K(K) = \{0, 1\}$? Moreover, when do we have $\dim_H \lambda_K^{-1}(1) = 1$ or > 1?*

5. Topological Classification of Fractal Squares $K(3, \mathcal{D})$

In this section we focus on the fractal squares $K(3, \mathcal{D})$ for all $\mathcal{D} \subset \{0, 1, 2\}^2$ with $2 \leq \#\mathcal{D} \leq 8$. Based on recent studies, we can tell apart most of those $K(3, \mathcal{D})$. Indeed, we have a nearly complete classification of those fractal squares in terms of topological equivalence.

For the sake of convenience, let $\chi(q)$ be the number of topological equivalence classes in the family of all $K(3, \mathcal{D})$ with $\#\mathcal{D} = q \in \{2, \ldots, 8\}$, to be denoted by $\mathcal{K}_{3,q}$. Generally, we may denote by $\mathcal{K}_{N,q}$ the family of all fractal squares $K(N, \mathcal{D})$ of order N such that $\#\mathcal{D} = a$. Knowledge of those numbers $\chi(q)$ may be summarized in the table below.

q	2	3	4	5	6	7	8
$\chi(q)$	1	2	2	≤ 8	13	8	2

The topology of $K \in \mathcal{K}_{3,q}$ is not complicated for small q. It is trivial that every $K \in \mathcal{K}_{3,2}$ is a Cantor set hence we have $\chi(2) = 1$. It is also transparent that every $K \in \mathcal{K}_{3,3}$ is either a Cantor set or a line segment, whose slope may be 0, ∞, or ± 1. Moreover, every $K \in \mathcal{K}_{3,4}$ is either a Cantor set or homeomorphic with $K(3, \mathcal{D})$ with $\mathcal{D} = \{(i,i) : 0 \le i \le 2\} \cup \{(1,0)\}$, which is a Peano compactum containing infinitely many line segments of slope 1. See Figure 5 for a depiction of the digit set $\mathcal{D}$ and the approximations $K^{(1)} \supset K^{(2)}$.

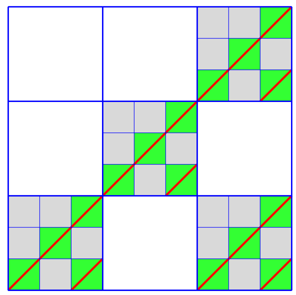

FIGURE 5. A digit set $\mathcal{D} \subset \{0,1,2\}^2$ and some line segments contained in $K(3, \mathcal{D})$.

Therefore, we have.

Theorem 20 ([17, Theorem 3.3] **and** [19, Theorem 4.5])**.** $\chi(3) = \chi(4) = 2$.

The fractal squares $K(3, \mathcal{D})$ with $\#\mathcal{D} = 5$ are divided into 21 congruent classes. See [17] for a complete list.

For five of them, as illustrated in [17, Fig. A.1], $K(3, \mathcal{D})$ is a Cantor set.

For six of them, as illustrated in [17, Fig. A.2], $K(3, \mathcal{D})$ is connected hence is a Peano continuum. Two of those continua are homeomorphic, with nontrivial fundamental group. The other four are dendrites, such as $K(3, \mathcal{D}_i)(i = 1, 2)$ with

$$\mathcal{D}_1 = \{(i,0) : 0 \le i \le 2\} \cup \{(1,1),(1,2)\} \tag{5}$$

$$\mathcal{D}_2 = \{(i,1) : 0 \le i \le 2\} \cup \{(1,0),(1,2)\} \tag{6}$$

It is known that $K(3, \mathcal{D}_1)$ and $K(3, \mathcal{D}_2)$ are not homeomorphic [17, Theorem 3.10]. On the other hand, the other two are homeomorphic with $K(3, \mathcal{D}_2)$. Here, by checking the digit sets one may figure out how to choose the underlying homeomorphisms to be even an affine map.

The rest ten, as illustrated in [17, Fig. A.3], have both point components and non-degenerate ones, which are parallel line segments. The one with digit set $\mathcal{D}_0$ (as given in Figure 1) has been addressed in Example 1. Further set

$$\mathcal{D}_3 = \{(i,0) : 0 \le i \le 2\} \cup \{(0,2),(1,2)\} \tag{7}$$

$$\mathcal{D}_4 = \{(i,0) : 0 \le i \le 2\} \cup \{(0,2),(2,2)\} \tag{8}$$

$$\mathcal{D}_5 = \{(i,i) : 0 \le i \le 2\} \cup \{(1,0),(0,2)\} \tag{9}$$

$$\mathcal{D}_6 = \{(i,0) : 0 \le i \le 2\} \cup \{(0,2),(2,1)\} \tag{10}$$

$$\mathcal{D}_7 = \{(i,0) : 0 \le i \le 2\} \cup \{(1,2),(2,1)\} \tag{11}$$

In Figure 6, we present for $3 \le j \le 6$ the first two approximations of $K(3, \mathcal{D}_j)$.

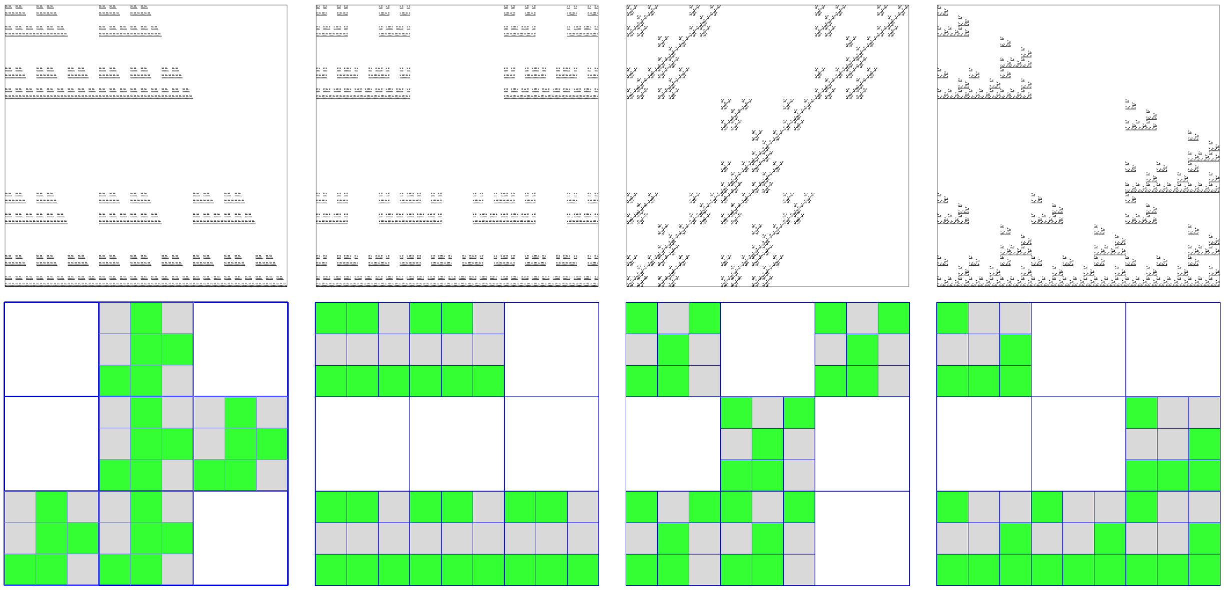

FIGURE 6. $K(3, \mathcal{D}_j)$ with $3 \le j \le 6$ and the first two approximations.

Thanks to [20, Theorems 2.1 and 5.1] and a family of self-similar sets constructed in [20, Example 2.1], we know that $K(3, \mathcal{D}_3)$ differs from $K(3, \mathcal{D}_0)$ by a bi-Lipschitz map. On the other hand, with the help of finite state automata a bi-Lipschitz map is constructed in [21] between $K(3, \mathcal{D}_0)$ and $K(3, \mathcal{D}_7)$. See Figure 7.

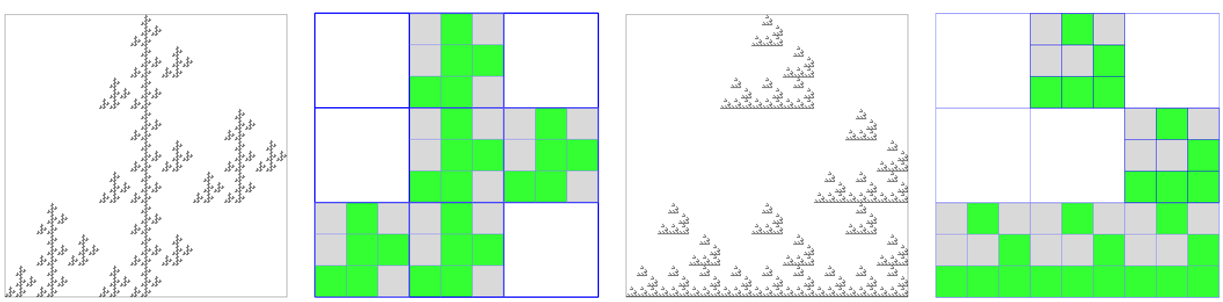

FIGURE 7. $K(3, \mathcal{D}_j)$ with $j = 0, 7$ and the first two approximations

In other words, we have.

Theorem 21 ([20, 21]). *Fractal squares $K(3, \mathcal{D}) \in \mathcal{K}_{3,5}$ with the next three choices of $\mathcal{D}$ are bi-Lipschitz equivalent to each other hence they are all homeomorphic.*

By summarizing those known results with basic observations, we eventually infer that each $K \in \mathcal{K}_{3,5}$ with both point components and line segment components is homeomorphic with one of $K(3, \mathcal{D}_i)$ with $3 \le i \le 6$. Thus there are at most $8 = 1 + 3 + 4$ topological equivalence classes among fractal squares $K \in \mathcal{K}_{3,5}$. In other words, we have $\chi(5) \le 8$. To determine the exact value of $\chi(5)$, it suffices to resolve the following.

Question 4. *Is $K(3, \mathcal{D}_i)$ homeomorphic with $K(3, \mathcal{D}_j)$ for some $3 \le i < j \le 6$?*

Fractal squares $K \in \mathcal{K}_{3,6}$ are carefully discussed in [23]. Every of those $K(3,\mathcal{D})$ is isometric with one of the sixteen given in [23, Fig.1]. In other words, there are sixteen isometric classes. By [23, Theorem 1], six of those $K(3,\mathcal{D})$ are special. Among them, two of them are the product of a Cantor set with $[0,1]$ hence are affine equivalent; two are affine equivalent but they only contain countably many line segment components; and the last two are Lipschitz equivalent. The rest ten satisfy two requirements. First, no two of them are topologically equivalent. Second, none of them is homeomorphic with any of the previous six $K(3,\mathcal{D})$. Therefore, we have the following.

Theorem 22 ([23, Theorem 1])**.** $\chi(6) = 13$.

Fractal squares $K \in \mathcal{K}_{3,7}$ are thoroughly analyzed in [5], which concentrates on bi-Lipschitz equivalence between those fractals squares. By basic observations, it is easy to check that every $K \in \mathcal{K}_{3,7}$ is isometric to one of the eight fractal squares given in [5, Fig.4 to 11]. The complements in $\{0,1,2\}^2$ of their digit sets are respectively given below:

$$\{(1,1),(1,2)\},\ \{(1,0),(1,2)\},\quad \{(1,2),(2,1)\},\ \{(0,2),(2,2)\}, \tag{12}$$

$$\{(0,1),(2,2)\},\ \{(1,2),(2,2)\},\quad \{(0,0),(2,2)\},\ \{(1,1),(2,2)\}. \tag{13}$$

For the sake of convenience, denote the resulting fractal squares by $K_j (1 \le j \le 8)$. The following five results are known, respectively from [5, Lemma 4.1 to 4.5].

(1) Each of K_1 and K_2 has a unique (global) cut point.
(2) Each of $K_j (3 \le j \le 5)$ has infinitely many (global) cut points.
(3) K_6 has cut sets that contain exactly two points.
(4) K_7 has cut sets that contain exactly three points.
(5) K_8 does not have finite cut sets.

Note that in t[5, Lemmas 4.7 and 4.8] the authors actually show that no two of $K_j (3 \le j \le 5)$ are homeomorphic. Therefore, $\chi(7) \ge 7$ and the only issue is to determine whether K_1 and K_2 are homeomorphic or not. To this question, a definite answer is given in the following.

Theorem 23 ([24, Theorem 2])**.** *Let $G_i (i = 1,2)$ be the group of homeomorphisms of K_i onto itself. Then G_1 has two elements while G_2 has eight.*

Therefore, we have.

Theorem 24. $\chi(7) = 8$.

Fractal squares $K(3,\mathcal{D})$ with $\#\mathcal{D} = 8$ allow three possibilities. First, it is isometric with the classical Sierpinski's carpet $K(3,\mathcal{D}_S)$ with $\mathcal{D}_S = \{0,1,2\}^2 \setminus \{(1,1)\}$. Second, it has no local cut point hence is homeomorphic with $K(3,\mathcal{D}_S)$. See for instance the proof for [5, Theorem 1.2]. Third, it is isometric to $K(3,\mathcal{D}_L)$ with $\mathcal{D}_L = \{0,1,2\}^2 \setminus \{(2,2)\}$. See Figure 8 for a depiction of $K(3,\mathcal{D}_L)$ and of its first two approximation.

It is routine to verify that $K(3,\mathcal{D}_L)$ has infinitely many local cut points, one of which is $x_0 = (2,2)$. Therefore, the following is immediate.

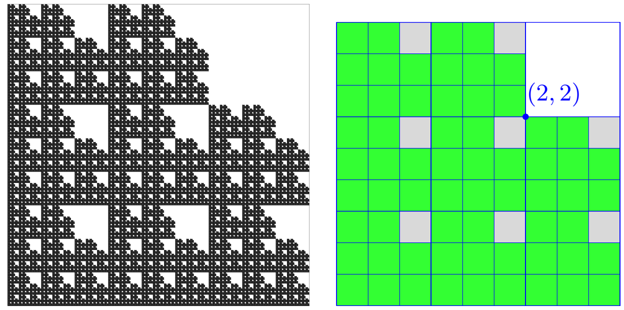

FIGURE 8. The fractal square $K(3, \mathcal{D}_L)$ and its first two approximation.

Theorem 25. $\chi(8) = 2$.

To conclude this section, we want to mention a basic question in response to Theorem 16. For related details, one may refer to [15, Corollary 2.6].

Given $N \geq 3$, let $\mathcal{L}_N$ consist of 0, 1 and all numbers $\tau \in (0, 1)$ such that there is a fractal square $K = K(N, \mathcal{D})$ whose non-degenerate components are line segments of the same slope τ. Such a slope τ is necessarily rational. Moreover, let $\mathcal{L}_N^{\#}$ consist of 0, 1, and all numbers $\tau \in (0, 1)$ such that there is a fractal square $K = K(N, \mathcal{D})$ with $\mathcal{D} \neq \{0, 1, \ldots, N-1\}^2$ that contains a line segment of slope τ. By the symmetry of K, the set $\mathcal{L}_N^{\#}$ essentially gives all the possible slopes of line segments contained in K, while $\mathcal{L}_N$ gives the same information for those K whose non-degenerate components are line segments.

Write every rational $\frac{r}{s} \in \mathcal{L}_N^{\#}$ in reduced form, so that the numerator r and the denominator s are coprime. Then the denominator s is strictly smaller than N. Indeed, if K contains a line segment with slope $\frac{r}{s} < 1$ for some $s \geq N$, then $H = K + \mathbb{Z}^2$ contains an infinite line of the same slope, which implies that $\mathcal{D} = \{0, 1, \ldots, N-1\}^2$.

We propose the following.

Question 5. *How to determine $\mathcal{L}_N$ and $\mathcal{L}_N^{\#}$ for any $N \geq 3$? Particularly, given $\mathcal{D} \subset \{0, 1, \ldots, N-1\}^2$, how to determine all $\tau \in \mathcal{L}_N^{\#}$ such that $K(N, \mathcal{D})$ contains a line segment of slope τ? Moreover, is it always true that $\mathcal{L}_N \subsetneq \mathcal{L}_N^{\#}$?*

6. Questions Related to Algebraic Topology of Peano Continua

Fractal squares $K = K(N, \mathcal{D})$, except for the case $\mathcal{D} = \{0, 1, \ldots, N-1\}^2$, have no interior point. By Theorem 17 (see also [12, Theorem 2]), every component P of K is a Peano continuum. Moreover, the topological dimension of P is just one. If K is actually connected then it is locally path-connected; if further it has a nontrivial fundamental group then it is semi-locally simply connected at none of its points. The following result sheds some light on how the topology of such a continuum is related to its fundamental group.

Theorem 26 ([22, Theorem 1.3])**.** *Let X, Y be one-dimensional Peano continua that are semi-locally connected nowhere. Then for X, Y to be homeomorphic it is necessary and sufficient that the fundamental groups $\pi_1(X)$, $\pi_1(Y)$ are isomorphic.*

More precisely, one may recover the topology of such a Peano continuum from its fundamental group. See for instance [25]. Thus we are motivated to propose the following.

Question 6. *Given a connected fractal square $K = K(N, \mathcal{D})$, find conditions on the initial approximations $K^{(j)}$, say for $j \leq 4$, for $\pi_1(K)$ to be nontrivial. The same issue for components of K, when it is not connected.*

There are many examples of fractal squares K, so that some of its components or K itself are dendrites (possibly line segments). So, one may further wonder whether such a dendrite is actually a Jordan arc. Therefore, we also propose the following.

Question 7. *Fix a fractal square $K = K(N, \mathcal{D})$ and one of its non-degenerate components, say P, with trivial fundamental group. Find conditions on the initial approximations $K^{(j)}$, say for $j \leq 4$, for P to be a Jordan arc.*

From the literature one can find quite a lot efforts of characterizing special or even general self-similar sets K that are homeomorphic with $[0, 1]$. See for instance [4, Theorem 6.7]. See also [26] for useful description of the topology of a self-similar set (resulted from an IFS $\mathcal{F} = \{f_1, \ldots, f_q\}$) based on the symbolic representation $\mathcal{S} : \{1, \ldots, q\}^\infty$ and [27] for special interests focusing on graph-directed IFS's that generate attractors each of which is a Jordan arc.

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