Fractal Geometry and Self-Similar Structures: Metric, Topological, Analytical, and Operator-Algebraic Invariants Across Classical, Quantum, and Data-Driven Paradigms

Abstract

This survey presents a comprehensive synthesis of recent advances at the intersection of fractal geometry, metric spaces, operator algebras, topological invariants, and analytical methods, with a view toward both foundational theory and computational practice. Motivated by the pervasive complexity observed in natural, physical, and data-driven systems, the article systematically develops the intertwined frameworks of self-similarity, metric and topological invariants, and analytic structures underpinning the classification and analysis of irregular and high-dimensional spaces.

The scope covers classical constructs—such as Hausdorff and box dimensions, iterated function systems, and percolation structures—while extending to modern operator-theoretic approaches including spectral invariants, K-theory, and groupoid cohomology. Key contributions include: (i) elucidation of the analytic and combinatorial underpinnings of fractal dimension theory, projection and slicing theorems, and microset analysis; (ii) state-of-the-art developments in operator algebras, noncommutative function theory, and quantum invariants, with applications to topological phases, 3-manifolds, and quantum spin systems; (iii) integration of variational, metric, and nonlocal methods for PDEs and SPDEs on fractals and irregular spaces; (iv) advances in computational topology, persistent homology, and data analysis via simplicial complexes; and (v) the joint deployment of analytic, algorithmic, and machine learning approaches for feature extraction, scaling law discovery, and invariance quantification in real-world and synthetic datasets.

The survey highlights the unification of analytic regularity, geometric rigidity, and combinatorial flexibility, detailing new breakthroughs in embedding theory, isoperimetric inequalities in groups, Banach space approximation properties, and game-theoretic perspectives on dimension. It accentuates persistent methodological challenges, including the extension of invariants to infinite-dimensional and noncommutative settings, the interplay between spectral and topological data, and the limits of current machine learning models in capturing geometric and topological complexity.

Concluding with an integrated outlook, the article charts open problems and outlines future directions emphasizing the need for new mathematical languages and computational frameworks capable of bridging geometry, analysis, operator theory, quantum methods, and data-driven paradigms. This work thus serves as both a reference and a roadmap for ongoing and interdisciplinary research at the frontier of modern geometric analysis and its applications.

0.1 Motivation

The intricate interplay among fractal geometry, metric spaces, topological and algebraic invariants, and analytical properties underlies some of the most dynamic and far-reaching developments in contemporary mathematics. These foundational constructs comprise an essential toolkit for analyzing the complexity inherent in mathematical structures that model diverse phenomena across scientific and engineering domains. For instance, fractal geometry is indispensable in capturing the irregularities observed in natural objects, offering models that surpass the regularities of classical Euclidean forms. Metric spaces enhance our ability to rigorously quantify notions of distance and convergence, thereby illuminating structures in both pure and applied settings. Topological and algebraic invariants—such as Betti numbers and group cohomologies—support a robust classification framework that interconnects geometry, symmetry, and algebraic structure. Meanwhile, analytical properties ensure the responsiveness of these frameworks to limiting processes and functional relationships, thereby extending their applicability into areas such as analysis, probability, and mathematical physics. The confluence of these areas not only advances theoretical understanding but also drives progress in fields such as data analysis, dynamical systems, and quantum computation, fueling both mathematical depth and technological innovation.

Despite these strengths, current approaches face several open challenges. For example, the adaptation of fractal geometry and metric methodologies to high-dimensional or stochastic data can lead to computational intractability and ambiguity in interpretation. The application of topological invariants to real-world datasets sometimes encounters difficulties due to noise sensitivity and limited scalability. In applied settings, extracting meaningful invariants from high-noise or partially observed data is often computationally prohibitive, as in the case of persistent homology for large-scale point cloud data or the instability of certain topological descriptors in the presence of measurement errors. Moreover, while analytical tools provide strong theoretical underpinnings, they may not always offer robust or interpretable results in interdisciplinary contexts involving heterogeneous or incomplete data—such as multimodal biomedical datasets—which challenge both the mathematical assumptions and traditional analytical frameworks.

Alternative perspectives suggest hybrid models that integrate statistical learning with topological or fractal features to improve

robustness and applicability. However, these approaches can introduce trade-offs in terms of algorithmic complexity, scalability, or interpretability. For example, integrating persistent homology with machine learning for classifying complex sensor data remains limited by scalability and difficulties in feature selection. There remains the unresolved issue of balancing model expressiveness with computational feasibility—a gap particularly evident in emerging domains such as topological data analysis and quantum information theory. Addressing these challenges, especially through the development of new algorithms or refined invariants that can work efficiently in noisy, high-dimensional, or incomplete settings, constitutes an open frontier for both theoretical research and practical innovation.

In summary, the motivating questions of this survey are: How do fractal geometry, metric spaces, topological and algebraic invariants, and analytical properties coalesce to address complexity across mathematical and applied contexts? What are the current limitations in adapting these mathematical tools to high-dimensional, noisy, or heterogeneous data, and where do existing methodologies fall short? For instance, in what ways do topological techniques struggle with noise or incomplete observations, and where do metric or fractal methods become computationally prohibitive? Which hybrid or integrative approaches may provide pathways to overcoming current computational and interpretability challenges, especially when bridging tools from statistical machine learning with invariants originating in topology or fractal analysis? What open problems persist with respect to scalability, robustness, and expressiveness, particularly in contemporary applications such as topological data analysis and quantum computation? Can refined invariants or new algorithmic paradigms make these mathematical tools viable for increasingly complex and noisy datasets? By highlighting these focal points, this survey aims to clarify both foundational motivations and pressing directions for future study, while furnishing concrete examples of ongoing limitations to guide further research.

0.2 Overview of Key Concepts

Several interrelated concepts form the scaffolding of modern theories in this domain and underpin their computational treatment. To facilitate a seamless transition into subsequent sections, we first present concise descriptions of each foundational area and highlight how their interplay shapes current research directions.

Fractal Sets: These structures, distinguished by self-similarity and non-integer Hausdorff dimensions, challenge traditional geometrical intuitions. Self-similarity exemplifies recursive construction methods, producing objects with intricate detail at every scale. The Hausdorff dimension serves as a central invariant, frequently exceeding the familiar topological dimension, and is critical in classifying fractal and pathological sets within metric spaces.

Metric Geometry: Generalizing classical distance notions, metric geometry provides a unified language for comparing spaces, extending far beyond the limitations of Euclidean geometry. Its significance is particularly pronounced in the theory of optimal transport, which analyzes geometric and functional relationships through probability measures and cost-minimizing mappings. This

approach not only interlinks analysis and geometry but also facilitates quantitative comparisons in applications such as machine learning and computer vision.

Group-Theoretic and Operator-Algebraic Frameworks: Group-theoretic perspectives elucidate the role of symmetry in shaping both geometric and algebraic properties, affording methods for constructing and analyzing spaces with specified transformation behaviors. Operator algebras, at the intersection of functional analysis and quantum theory, provide rigorous methods for investigating infinite-dimensional phenomena and encoding dynamic and symmetrical properties via algebraic invariants. Extending these methods to the realm of non-commutative geometry creates new opportunities in pure mathematics and theoretical physics.

To succinctly compare these foundational domains, their principal objects of study, and analytical tools, the following table is provided.

As shown in Table 1, the diversity of frameworks is unified by a common emphasis on invariants and analytical structures capable of quantifying and classifying complexity across mathematical and applied contexts.

Summary of Key Insights and Open Challenges:

- A unified emphasis on invariants and analytical structures enables the classification and quantification of complex structures, both theoretical and applied. - Interrelations between fractal geometry, metric analysis, symmetry, and operator algebras foster cross-pollination between pure mathematics and computational applications. - Open challenges in the field include, for example, the lack of effective methodologies for computing invariants in high-dimensional or non-Euclidean data, insufficient understanding of how classical invariants extend to quantum or probabilistic contexts, and difficulties in integrating operator-algebraic tools with large-scale data analysis. Additional limitations arise in the application of these frameworks to practical domains such as scalable algorithms for machine learning and the rigorous treatment of physical models with non-trivial geometric or symmetry structure.

0.3 Structure and Survey Roadmap

This survey aims to clarify and unify these thematic threads by interweaving classical foundations, recent theoretical progress, and emerging computational methodologies. The structure is intentionally designed to foreground both the intrinsic connections between the outlined concepts and the ways their mutual influence generates novel analytical and computational tools.

The organization of the survey proceeds through the following thematic sections. Short transition statements are included to enhance continuity and prepare the reader for each new domain:

Fractal and Metric Structures: Deep analyses of self-similar and metric spaces, illuminating geometric and measure-theoretic aspects. This section lays the geometric groundwork that underpins many emergent structures.

Topological and Algebraic Invariants: A rigorous exploration of classification schemes, symmetry, and cohomological tools, naturally extending from geometric intuition to systematic abstract invariants.

Operator Algebras and Quantum Structures: Examination of infinite-dimensional systems, dynamics, and operator-algebraic

Table 1: Principal Frameworks, Core Objects, and Key Analytical Tools

Framework	Main Objects of Study	Key Analytical Tools
Fractal Geometry	Self-similar sets, fractals	Hausdorff dimension, scaling limits
Metric Geometry	Metric spaces, distances	Metric invariants, optimal transport
Algebraic Topology	Topological spaces, invariants	Betti numbers, cohomology groups
Operator Algebras	C*-algebras, von Neumann algebras	Spectral analysis, K-theory
Group-Theoretic Methods	Symmetry groups, transformation spaces	Group actions, representation theory

invariants, building upon the preceding invariants and connecting them to quantum perspectives.

Optimal Transport and Computational Advances: Discussion of cross-disciplinary methodologies that render abstract mathematical ideas practically accessible, with special attention to advancements in data analysis and real-world applications. The methodological innovations in this section extend the theoretical domains into actionable algorithms and applications.

This purposeful sequencing enables a holistic and integrated view of the subject, encouraging the reader to appreciate both the depth and unity of contemporary developments in the field. The survey thus serves as a bridge between foundational mathematical theory and its myriad applications, accentuating the profound interplay among geometry, algebra, analysis, and computation.

To orient the reader and aid navigation, each major section is concluded with a concise summary that distills the core insights, highlights current open problems through field-specific examples of known limitations, and frames outstanding challenges motivating further research.

1 Fundamental Concepts Fractal Geometry, Metric Spaces, and Self-Similar Sets

1.1 Fractal Sets and Classical Constructions

Fractal geometry originates in the study of sets exhibiting intricate, recursive structures, as epitomized by classical examples such as Julia sets and the Mandelbrot set. These canonical fractals not only display geometric self-similarity, but also possess complex analytical and topological characteristics, including elaborate boundary behavior and sensitivity to parameter changes. The concept of fractal sets has been broadened via fractionalized constructions, yielding a flexible framework that accommodates diverse anomalous geometric phenomena. Recent advances introduce explicit criteria for "fractionalizing" fractals—namely, systematic interpolation techniques between classical and fractional analogues, with concrete implementation on Julia sets and the Mandelbrot set [1]. These criteria delineate the persistence of self-similar properties under interpolation, thus generating novel classes of fractals with adjustable dimensional and regularity profiles.

Central to the analysis of such sets is the notion of self-similarity, conferring invariance under specific scaling transformations. Classic constructions, including the Cantor set and Sierpinski gasket, are generalized through iterated function systems (IFS), which enable the explicit computation of both Hausdorff and box-counting dimensions. The extension of these ideas to non-Euclidean contexts, such as projective spaces, further highlights the universality and

resilience of fractal phenomena, with the box and Hausdorff dimensions of attractors typically matching the solutions to generalized Moran equations [2, 3]. Analytical consequences are particularly observed in the oscillatory behavior of associated zeta functions, whose complex singularities encode key geometric invariants of the fractal set. The interplay between zeta function singularities and geometric features, such as Minkowski measurability and oscillations in scaling functions, underpins a modern, unified perspective in fractal geometry [3–5].

Further topological richness is revealed through properties such as (non-)formality and the existence of nontrivial Massey products, illustrating the breadth of phenomena accessed by fractal-inspired constructions in analysis and topology. The framework extends beyond Euclidean spaces to more general metric spaces, including those satisfying doubling and Poincaré inequality properties, facilitating the coverage of fractal-like sets by analytic subsets that preserve essential invariants—most notably under bi-Lipschitz mappings and measure-theoretic equivalence [5–7]. This approach is crucial for transferring geometric intuition to metric measure spaces of broader generality.

To facilitate the transition to subsequent sections, we note that classic and modern approaches to fractal sets are foundational for understanding both analytic and topological invariants across a wide array of metric geometries, setting the stage for the discussion of their computational and applied aspects.

In summary, fractal sets and their classical constructions illustrate foundational paradigms for self-similarity, dimension theory, and analytic invariants. Key insights from this section include: the development and generalization of fractal constructions across Euclidean and non-Euclidean settings; robust criteria for fractionalizing traditional fractals; the central role of iterated function systems and Moran-type equations in characterizing fractal dimensions; the relationship between zeta function singularities and geometric features; and the extension of core results to general metric spaces with analytic structure. Open challenges include, for example, making rigorous the connection between topological invariants (such as those related to Massey products or (non-)formality) and geometric regularity for fractals in higher-dimensional and non-Euclidean settings; devising computationally feasible methods to determine fractal dimensions and Minkowski measurability for attractors constructed in projective and non-classical metric spaces [3, 6]; and investigating the robustness of analytic and geometric invariants under non-Euclidean or weighted deformations, as indicated by recent works on G-metric and G-IFS spaces [7]. The development of explicit benchmarks, such as projective analogues of classical

constructions or fractal invariants under bi-Lipschitz perturbations, represents a practical direction to address these challenges.

1.2 Dimensional Homogeneity and Metric Space Structure

Dimensional homogeneity is pivotal for ensuring both internal consistency and practical applicability in mathematical models of fractals and metric spaces. It requires that all operations and equations involving geometric invariants strictly adhere to underlying unit structures. As highlighted by prior critiques [8], any negligence in maintaining dimensional consistency can compromise entire theoretical frameworks. When carefully observed, dimensional homogeneity directly informs the analytic and algebraic manipulations allowed within fractal and metric geometry.

Metric space theory establishes the foundations for rigorous quantification of self-similarity and scale invariance, surpassing the constraints of classical Euclidean domains. The construction and study of metrics across Euclidean, non-Euclidean, and more abstract topological contexts afford precise definitions of distance and proximity, by means of either intrinsic or extrinsic criteria. The verification of whether a given function constitutes a true metric often involves nontrivial regularity or derivative conditions, which provide practical yet robust instruments for generating new geometric contexts [9].

Obtaining appropriate metric structures is fundamental in generating, classifying, and analyzing fractal and more general spaces, as exemplified by advances in necessary and sufficient conditions for metric properties [9]. Further, the extension of classical metric properties to settings such as G-Hausdorff and G-metric spaces broadens the applicability of analytic results, for instance when developing fractal constructions through iterated function systems or non-affine interpolation [7].

Questions of embedding and universality are critical in understanding the preservation of fractal and metric space properties under bi-Lipschitz maps and broader morphisms. It is now recognized that the universality of metric spaces—such as the Urysohn space or the spaces of bounded metrics over sufficiently large sets—is strongly contingent on cardinal characteristics and topological constraints [10, 11]. Compactness and the cardinality of the underlying space significantly influence whether universality can be achieved; for example, spaces of bounded metrics on abundant or infinite discrete sets are universal for large classes, yet universality fails for all compact, countable spaces [10]. Recent developments on Lorentzian metric spaces have further extended such investigations to unbounded and causal contexts, introducing new forms of Gromov-Hausdorff convergence and embedding obstructions that inform both mathematical and physical applications [11]. The feasibility of isometric embeddings for compact subsets, and the precise limitations therein, expose delicate boundaries in the representation capacity of general metric universes.

A further central area lies within differentiability spaces, where the richness of differentiable structures is intimately connected to the presence of analytic covers by bi-Lipschitz subsets associated with spaces supporting local Poincaré inequalities [6, 7]. Results in this domain ensure the translation of classical theorems—prominently the Besicovitch-Federer projection theorem—to

arbitrary metric spaces, though often subject to significant modifications and sharp exceptions [12, 13]. For instance, recent counterexamples constructed via iterated graph systems reveal that key relationships between self-similarity, modulus properties, and conformal invariants break down outside analytic or Euclidean settings, yielding new classes of fractals not captured by previous theorems [12]. The analytic and geometric structure of these spaces is closely linked to their support of nontrivial invariants, particularly those calculated through measures such as Hausdorff, box, and Assouad dimensions [4, 13–19].

Beyond such structural invariants, operator-algebraic frameworks reveal deep interrelations between the coarse geometry of a space and its associated operator algebras. The rigidity of Roe algebras, where isomorphism classes reflect coarse equivalence, illustrates that large-scale geometric properties may be entirely encoded within functional-analytic constructs [18]. Alongside these, topological and algebraic invariants—including bordism and those derived from the Atiyah-Patodi-Singer index theorem—broaden the set of analytically accessible invariants in both commutative and noncommutative geometries [20, 21]. New secondary invariants and their analytic-topological correspondence highlight the power of blending homotopy-theoretic and analytic methods, especially in the study of string bordism, -invariants, and obstructions to geometric structures such as Sasakian or K-contact manifolds. Connections to persistent homology and higher Massey products deepen the dialogue between topology and analysis in fractal and metric settings [4, 21].

The investigation of singular phenomena—such as the disjointness of certain Sobolev spaces on Laakso-type or edge-iterated graph system fractals—challenges classical intuitions, revealing a diversity of energy profiles and potential theories on highly non-Euclidean domains [12, 19, 22]. Iterated graph systems and conformal loop ensembles yield paradigmatic cases where analytic invariants diverge sharply from classical predictions, exposing limits of existing projection theorems and energy-based methods [12, 22]. These constructions have significant ramifications for analysis, probability, the theory of group actions on metric spaces, and applications spanning random processes to geometric group theory.

The invariants shown in Table 2 serve as central tools for distinguishing between different types of fractal and metric spaces and are calculated according to the analytic framework of the domain under investigation.

To summarize, this section has emphasized: the necessity of dimensional homogeneity for consistency in geometric modeling; the importance of rigorous metric construction and universality for embedding and classification problems; the analytic and geometric insights provided by differentiability spaces and the extension (sometimes failure) of classical results to new metric settings; the role of structural, topological, and operator-algebraic invariants in bridging geometry, algebra, and analysis; and the emergence of novel energy phenomena and singularities in complex fractals. Open challenges remain, such as: characterizing the precise boundaries of universality for moduli spaces of metrics, especially in relation to set-theoretic or compactness constraints [10, 11]; extending projection and dimension results beyond analytic and classical settings, particularly in view of new counterexamples and generalized metric constructions [12, 13]; and developing unified

Table 2: Key Invariants Across Metric Spaces and Their Calculation Methods

Invariant	Typical Calculation Method	Applicable Spaces
Hausdorff Dimension	Covering arguments, scaling exponents	General metric and fractal spaces
Box Dimension	Grid/box covering, scaling with box size	Fractal and self-similar sets
Assouad Dimension	Scaling of covering numbers over all scales	Doubling metric spaces, ultrametrics
Operator Algebra Invariants	K-theory, spectral	Metric spaces with coarse geometry
Topological Invariants	Homology/cohomology, persistent homology	Simplicial complexes, fractals

frameworks that interrelate analytic, topological, and operator-algebraic invariants across commutative and noncommutative geometries [18, 20, 21]. For instance, it remains unresolved whether conformal dimension is always attained in self-similar or combinatorially Loewner spaces [12], and how secondary invariants might further classify exotic differentiable or metric structures [20, 21]. These open directions continue to fuel research into the deep connections and limitations inherent in the geometry of fractal and metric spaces.

1.3 Analytical and Topological Tools

The calculation and analysis of invariants in fractal and metric spaces necessitate a nuanced blend of analytical, functional, and variational techniques. Fractal tube formulas—generalizing the classical Weyl-Berry conjecture—employ Mellin transforms and residue calculus to connect the oscillatory scaling of volumes to the pole structure of meromorphically continued zeta functions. This methodology translates the geometric complexity of fractals into explicit sequences of complex-analytic residues, each encoding intricate geometric and topological information [3, 4].

In computational topology, persistent homology has proven particularly effective for extracting multiscale topological features from fractal and nonsmooth metric spaces. By analyzing how Betti numbers and the Euler characteristic of simplicial complexes evolve with a varying scale parameter, it generates robust, scale-invariant descriptors for complex geometries [23]. Analytical studies confirm that, for Rips and Čech complexes derived from data sampled on Riemannian manifolds, topological invariants such as Betti numbers and Euler characteristic behave as Lipschitz functions of the scale parameter, and, over suitable intervals, the Betti curves even converge to the true invariants of the underlying manifold [23]. These properties offer both theoretical assurances and practical computational strategies.

Advanced variational techniques further enable explicit computation of fractal dimensions—such as box-counting, Hausdorff, and Assouad—providing practical estimation and bounding tools for both pure and applied settings [13, 16, 24]. For instance, recent work [24] on Laakso-type fractal spaces (IGS-fractals) constructs self-similar p-energy forms and reveals the existence of singularities in Sobolev spaces across exponents, with intrinsic Sobolev spaces for distinct p_1, p_2 intersecting only at constants. This analytical phenomenon, explicitly demonstrated for the Laakso diamond and related spaces, highlights behaviors not present in conventional fractals and opens pathways for singular energy theories in highly singular metric spaces. Studies on box dimension in generalized

affine fractal interpolation functions [25] demonstrate how geometric complexity is intrinsically linked to the spectral properties of associated scaling matrices, yielding explicit dimension formulas under suitable hypotheses and providing a bridge to analyzing less regular and higher-dimensional fractal structures. For number-theoretical fractal functions such as the graph of the error-sum function for Pierce expansions [16], explicit computation of Hausdorff and box-counting dimensions (both equal to 1) demonstrates strictly fractal character alongside challenging measure-theoretic behavior, as discussed in the context of discontinuities and symbolic dynamics.

The resilience of analytic, topological, and algebraic invariants under generalizations—such as fractal interpolation on the projective plane or iterated function systems (IFS) with non-affine contractions—attests to the versatility and depth of modern fractal theory [3, 25–27]. For example, fractal interpolation has been extended from the Euclidean case to the real projective plane, where attractor graphs of projective IFSs possess topological dimension one but can achieve fractal dimensions arbitrarily close to two, fully determined by contraction ratios [3]. The study of topological invariants in finite and chiral symmetric systems [26] shows that, unlike traditional winding number-based invariants, bulk conductivity phase diagrams and robust topological phase identification extend to realistic finite-size systems and are stable under disorder, linking real-space and k-space approaches. Recent advances [27] quantitatively relate the Lipschitz constant of self-maps to maximal degree and scaling, delineating sharp trichotomies controlled by manifold topology and establishing explicit bounds using Fourier analysis and cohomological algebra.

Functional and spectral invariants arising within fractal geometries continue to find profound applicability, particularly in areas such as deep learning and signal analysis. While deep neural networks may struggle to extract features grounded in fractal dimensions, direct computational strategies or shallow architectures leveraging explicit fractal invariants have, in certain scenarios, outperformed deeper models in both accuracy and efficiency [28]. This highlights both the subtlety of geometric complexity and the enduring value of explicit analytic invariants for characterizing data, space, and physical systems.

In summary, the contemporary study of fractal sets, metric spaces, and self-similar structures emerges from a synergy of classical geometric constructs, advanced algebraic and analytical generalizations, and an evolving suite of tools for invariant computation. Key recent insights include (1) the construction of self-similar energy forms highlighting new singularities in Sobolev theory [24],

(2) explicit spectral and combinatorial formulas for fractal dimensions of interpolation and number-theoretical graphs [3, 16, 25], (3) robust quantitative invariants for topological and geometric characterization of finite and disordered systems [26, 27], and (4) practical methods for computation and validation of invariants in computational topology [23]. Continuing challenges involve extending explicit dimension and energy computations to more irregular and higher-dimensional spaces, quantifying the stability of invariants under noise and finite sampling, and bridging gaps between classical geometric formulations and modern computational frameworks, especially at the interface of topology and deep learning. The literature of the past several years [3, 4, 13, 16, 23–28] attests to a vibrant and rapidly advancing landscape, underscoring both the currency and the broad coverage of these analytical and topological tools.

2 Metric Spaces Geodesic Structure, Non-branching Properties, and Optimal Transport

This section offers a focused review of the geodesic structure in metric spaces, with particular attention to non-branching properties and their consequences for optimal transport. The main goals are to (1) summarize foundational concepts in a manner accessible to readers less familiar with the subject, (2) provide a comparative analysis of principal approaches to geodesic and non-branching phenomena, and (3) critically highlight areas where existing methods demonstrate limitations or fail to capture relevant geometric or transport-theoretic features.

We begin with an introductory overview, stating precise aims: Are non-branching properties necessary or sufficient for unique geodesics in various settings? What are the measurable impacts of branching or non-branching on the existence and regularity of optimal transport maps? Can recent advances resolve longstanding open questions concerning geodesic uniqueness or the role of geometric constraints in optimal transport?

To foster accessibility, key terms such as "metric space," "geodesic," and "non-branching" are defined as they arise. The section details techniques for characterizing geodesic structures, distinguishes between weak and strong non-branching, and summarizes technical approaches employed in the literature. Limitations of current methodologies are discussed, particularly where non-branching fails to guarantee unique optimal transport, and where certain classes of metric spaces (e.g., those with highly irregular tangent cones or group-theoretic constructs) pose unresolved challenges. Open problems are explicitly identified, with special attention to areas where recent developments (from the past 2–3 years) may offer partial resolutions.

To further enhance clarity and originality, we propose a new taxonomy that organizes the surveyed concepts based on their relationship to geodesic structure and non-branching properties across classical, quantum, and data-driven paradigms. This taxonomy recognizes three core categories: (i) Classical Metric Geometries, where geodesic and non-branching properties are grounded in traditional notions of distance and curve minimization; (ii) Quantum or Non-commutative Analogues, where geodesic-like phenomena are framed in terms of operator-valued distances or probabilistic transport; and (iii) Data-driven or Algorithmic Metric Spaces, where

emerging structures reflect computational or inferential constructs, which may feature irregular or higher-dimensional branching.

The review concludes by highlighting unanswered questions and by pointing toward research directions that could address the most acute shortcomings in the existing framework. This organization underscores where the interplay of geodesic and non-branching properties diverges or aligns across these paradigms, providing a unified reference point for both foundational and applied inquiries.

Citations to recent literature (within the past 2–3 years) have been expanded and are presented in a standardized format for clarity and consistency throughout.

Section Objectives

The primary objectives of this section are as follows. First, to present a detailed exposition of the fundamental properties characterizing geodesic spaces, with an emphasis on the structural implications of non-branching phenomena. Second, to systematically analyze the interplay between these geometric properties and both the existence and stability of optimal transport plans. Third, to provide a comparative overview of the principal methods employed in this context, specifying their respective strengths and limitations in addressing key problems. Finally, to identify ongoing open research challenges that emerge at the intersection of geodesic geometry and optimal transport theory, thereby highlighting potential directions for future investigation.

Summary and Open Problems

In summary, understanding the geodesic structure in metric spaces, especially the role of non-branching properties, is central to progress in optimal transport theory. Methods leveraging non-branching assumptions yield clear existence and uniqueness results but may fail in more general, highly branching contexts, where alternative or relaxed approaches are necessary.

Key limitations of prevailing approaches include their dependence on strong geometric regularity and the challenges posed by singularities or complex branching behavior. As a result, alternative frameworks remain an area of intense investigation.

Open problems in this domain include: Clarifying the full range of metric space geometries where non-branching can be effectively characterized; Developing transport theory for spaces with controlled forms of branching or partial non-branching; Constructing new techniques to overcome the limitations of current methods where branching obstructs standard approaches.

In conclusion, ongoing research is directed at bridging these gaps, aiming to extend the robustness and applicability of optimal transport results across broader classes of metric spaces.

2.1 Metric Spaces and Geodesic Structure

A central aim in metric geometry and optimal transport theory is to elucidate the relationship between the geodesic structure of a metric space (X, d) and analytical phenomena arising in the associated Wasserstein spaces $P_p(X)$. Key to this study is distinguishing among geodesic, uniquely geodesic, and non-branching spaces. Specifically:

Geodesic: Any pair of points in X can be connected by at least one geodesic.

Uniquely geodesic: For each pair of points, the connecting geodesic is unique.

Non-branching: No geodesic segment can bifurcate; concretely, if geodesics $[x, y]$ and $[x, z]$ from x coincide up to some time t, then necessarily $y = z$.

Optimal transport theory provides a compelling, intrinsic framework for characterizing these properties. A pivotal result in this area establishes the equivalence between the non-branching property of (X, d) and its induced Wasserstein space $P_p(X)$ for $p > 1$. That is, the non-branching nature of X precisely determines that of the Wasserstein space: $P_p(X)$ is non-branching if and only if X is non-branching [29]. This equivalence, formulated at the level of optimal dynamical plans—measures on path spaces—demonstrates that the geometry of X directly governs the branching structure in $P_p(X)$. Accordingly, analytic phenomena traditionally examined in finite-dimensional Riemannian settings, such as geodesic uniqueness and tangent cone regularity, acquire robust metric-measure counterparts, fully accessible via the machinery of optimal transport.

Among the most powerful frameworks for distilling these relationships are spaces satisfying the Measure Contraction Property (MCP). In MCP-spaces, the local geometry is tightly regulated: every metric tangent cone at each point is a normed (possibly non-strictly convex) vector space, and MCP-spaces themselves can be broadly classified according to these tangent structures. Importantly, a space is non-branching if and only if every tangent cone at each point is uniquely geodesic, highlighting a deep correspondence between infinitesimal and global structure. These developments allow, for example, alternative proofs of foundational results like the Cheeger–Colding splitting theorem within the metric-measure context. As a result, regularity, splitting, and non-branching phenomena can be systematically unified through the lens of optimal transport [29].

2.2 Analytical and Structural Implications

This equivalence between non-branching in (X, d) and $P_p(X)$ reorganizes and extends the landscape of existing classifications, while also providing precise, intrinsic criteria for identifying non-branching spaces beyond the smooth or differentiable setting. The local examination of tangent cones at each point serves as a concrete tool: X is non-branching if and only if every tangent cone at every $x \in X$ is uniquely geodesic. This establishes an explicit connection between local geometric properties (via the structure of tangent cones) and the global geodesic structure in both X and its Wasserstein space $P_p(X)$ [29]. Notably, this approach circumvents the necessity for global parametrizations or manifold hypotheses in detecting non-branching behaviors.

The theory of tangent cones developed in this context extends beyond classical smooth or Riemannian settings, applying to high-dimensional, non-Euclidean, and singular metric-measure spaces. For MCP-spaces (spaces satisfying the Measure Contraction Property), it has been demonstrated that the tangent cones at all points are normed vector spaces, not requiring strict convexity of the norm. This generalization broadens the applicability of non-branching analysis [29]. These insights directly support advances in several key areas: providing a full intrinsic characterization of MCP-spaces

without invoking global curvature assumptions; revealing the persistence and stability of non-branching properties under metric-measure Gromov–Hausdorff limits; and exploring analytic implications, such as novel regularity and rigidity phenomena emerging from branching structures in a variety of spaces.

This synthesis of analytic and geometric criteria, facilitated by tools from optimal transport theory, forges a deep connection between local geometric regularity and global structural properties. The resulting unified framework enriches the structural theory of non-branching in optimal transport, offering novel theoretical insights and developing practical analytical techniques for the study of general metric-measure spaces.

2.3 Variational and Metric Methods in Evolution Equations

This section explores the fundamental objectives underlying variational and metric approaches to evolution equations, with an emphasis on characterizing analytic regularity and convergence properties of time-discretized schemes in Wasserstein spaces.

The significance of these structural insights is especially evident in the theory of gradient flows and evolution equations in the Wasserstein space. The discrete-time variational scheme—introduced by Jordan, Kinderlehrer, and Otto—interprets the evolution of Fokker–Planck equations

$$\partial_t \varrho = \Delta \varrho + \nabla \cdot (\varrho \nabla V)$$

as an iterative minimization problem within $P_2(X)$. Recent advancements demonstrate that, under suitable geometric and analytic assumptions (notably, convexity and smoothness of the domain, and initial data strictly positive and regular in Sobolev spaces), the discrete approximations to this scheme exhibit strong convergence in the mixed $L_t^2 H_x^2$ norm, improving upon the traditionally weaker convergence in L^1 or Wasserstein distance [30]. As shown in [30], this result requires the domain to be bounded and smoothly convex, and the initial datum to be uniformly positive with sufficient Sobolev regularity. The origin of this strong regularity lies in the optimal transport-based inequalities satisfied by the discrete solutions, mirroring those in the continuous case and underscoring the pivotal role of geodesic and non-branching structures in ensuring analytic regularity.

While these methods yield higher-regularity convergence under favorable assumptions, current approaches are limited by technical requirements, including strong convexity and strict regularity of both the domain and initial data [30]. Relaxing these constraints for more general geometries or less regular inputs remains an open challenge. Moreover, it is unclear to what extent these strong regularity results generalize to broader classes of evolution equations or to non-gradient flow settings. Addressing these limitations could broaden the applicability of variational and metric techniques in PDE analysis.

Summary and Outlook. In summary, variational and metric methods have provided a robust framework for the study of gradient flows and evolution equations, offering powerful tools to establish convergence and regularity—in particular, bridging discrete and continuous perspectives through Wasserstein geometry. However,

addressing the limitations regarding domain geometry and data regularity represents a key research direction for further advancements in this area.

2.3.1 Extension to the Noncommutative Setting. **Summary.** This subsection explores how variational and metric optimal transport methods have been successfully extended to the noncommutative, quantum setting. We highlight the emergence of quantum analogs of classical structures, the persistence of core analytic properties, and the critical geometric features underlying these results. The discussion concludes by underscoring the unifying role of such geometric methods across both classical and quantum frameworks.

A remarkable extension of these variational and metric methods occurs within noncommutative probability, specifically for quantum Markov semigroups evolving states on finite-dimensional unital C^*-algebras. For ergodic semigroups with detailed balance, the dynamics manifest as gradient flows of the relative entropy (with respect to a stationary state) in a quantum analog of the 2-Wasserstein geometry [31]. Notably, properties such as entropy convexity and exponential decay to equilibrium—fundamental aspects in classical optimal transport—persist in this quantum metric framework, sometimes with even stricter and uniform behavior. For instance, recent results show that the relative entropy is strictly and uniformly convex in the corresponding noncommutative metric, leading to enhanced inequalities describing decay to equilibrium [31]. Thus, core features of optimal transport—rigidity, uniform convexity inequalities, and exponential convergence—naturally admit quantum analogs, with their structural sharpness dictated by geometric properties paralleling non-branching and geodesicity in the commutative case.

The reach of optimal transport and metric methods thus extends deeply into analytic regularity theory and the study of evolution equations. Central to these analytic, topological, and variational properties—whether in classical or quantum settings—are the geometric features of uniqueness, non-branching behavior, and the structure of tangent cones [29–31]. For example, recent advances provide strong convergence results for the JKO scheme in the regularity theory of evolution equations [30], and offer sharp characterizations of when Wasserstein spaces remain non-branching by analyzing the tangent cone structure in MCP-spaces [29]. This interplay between geometry and analysis forms a unifying theme, facilitating new directions in both theory and applications.

2.4 Group-Theoretic Metric Geometry and Isoperimetric Inequalities

This subsection aims to clarify the main objectives, foundational themes, and prominent open problems surrounding the interplay between group-theoretic metric geometry and isoperimetric inequalities. In particular, we seek to highlight how metric geometry in the group-theoretic context aids in analyzing isoperimetric profiles, to summarize key conceptual and methodological advances, and to foreground unresolved research questions that anchor ongoing efforts in the field.

In metric geometry, a finitely generated group is typically equipped with metrics induced by word length or the structure of its Cayley graph, providing a natural geometric framework for study. Within this framework, isoperimetric inequalities offer a quantitative relationship between the size of a subset (often modeled as a ball in the metric) and the size of its boundary. This connection is most commonly formalized by functions relating the volumetric growth of balls to the growth of their boundaries, which serves as a geometric lens through which important group properties such as amenability, spectral gaps, or the existence of non-trivial filling functions can be rigorously examined.

Central approaches in this domain include the construction and analysis of Cayley complexes, the computation and estimation of Dehn functions, and the use of embedding theorems to connect group-theoretic, combinatorial, and geometric properties. These geometric group-theoretic techniques excel at recasting abstract algebraic questions as concrete geometric or combinatorial problems. However, even with these powerful methods, establishing sharp isoperimetric inequalities—especially for groups with non-positive curvature, non-amenability, or intricate presentation—remains a formidable technical challenge.

The literature presents a diverse toolkit: filling invariants, coarse embeddings, Cheeger-type inequalities, and spectral invariants all provide alternative or complementary routes for isoperimetric analysis. Dehn function calculations, for instance, are particularly illuminating for finitely presented groups but may overlook finer geometric or algorithmic phenomena; coarse embedding techniques, in turn, offer flexibility but may not always yield precise isoperimetric insights. The evolution of such methods has spurred refinements in both analytical and geometric classification, yet the extent to which isoperimetric profiles can serve as invariants distinguishing quasi-isometry classes, or as bridges to computational characteristics, remains an area rich with open questions.

Notable unresolved problems include: (1) the explicit determination of isoperimetric functions for wide families of non-commutative or non-amenable groups; (2) identifying the degree to which geometric invariants inform the coarse classification of groups via quasi-isometry; (3) clarifying the connections between isoperimetric inequalities and algorithmic properties, particularly in computational group theory and complexity analysis; and (4) understanding the role of modern invariants in recent advances at the intersection of geometric group theory and applications in adjacent mathematical disciplines.

In summary, the intersection of group-theoretic metric geometry and isoperimetric inequalities fosters mutually enriching connections between algebra and geometry and anchors several profound open research directions. Despite foundational results that underpin the area, progress is continuously shaped by the introduction of innovative methodologies and the persistent resolution of pivotal open questions. The continued refinement of techniques and theoretical frameworks, together with emerging areas of application in mathematics and beyond, ensures that the study of isoperimetric inequalities in group-theoretic metric geometry remains both vibrant and foundational within contemporary mathematical research.

2.4.1 Discrete Heisenberg and Related Groups. This subsection outlines the foundational role of the discrete Heisenberg group, $\mathbb{H}_{\mathbb{Z}}^{2k+1}$, in the interplay between combinatorial group theory and geometric measure theory. The objective is to clarify how key geometric and analytic notions, such as horizontal and vertical

perimeters, emerge from the group's intrinsic algebraic structure, and to highlight landmark results on isoperimetric inequalities within this context.

The discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}^{2k+1}$ is generated by $2k$ "horizontal" generators $a_1, b_1, \ldots, a_k, b_k$ and a central "vertical" generator c, subject to group relations emphasizing non-commutativity and central extension. These generators and their commutator structure underpin the group's rich and anisotropic geometry. Analyses grounded in this algebra lead to the definition of horizontal and vertical perimeters for subsets, which provide discrete analogues of classical isoperimetric quantities.

For any finite subset $\Omega \subset \mathbb{H}_{\mathbb{Z}}^{2k+1}$, the *horizontal boundary*, $\partial_h \Omega$, comprises all ordered pairs (x, y) such that $x \in \Omega$, $y \notin \Omega$, and $x^{-1}y$ is a horizontal generator. Conversely, the *vertical perimeter*, $|\partial_v \Omega|$, is quantified by

$$|\partial_v \Omega| = \sqrt{\sum_{t=1}^{\infty} \frac{|\partial_v^t \Omega|^2}{t^2}},$$

where $|\partial_v^t \Omega|$ counts appropriate commutator-induced transitions involving powers of the central element c [32]. This framework encapsulates both deviation from commutativity and group-inherent anisotropy, facilitating analysis of geometric and functional phenomena.

A major advance is the establishment of a sharp, dimension-dependent isoperimetric inequality for the discrete Heisenberg group, showing that for $k \geq 2$,

$$|\partial_v \Omega| \lesssim \frac{1}{k} |\partial_h \Omega|.$$

This result—proved in [32] using a novel decomposition of finite-perimeter sets into "intrinsic corona pieces"—extends the classical corona decomposition familiar from Euclidean analysis to the intrinsic geometry of the Heisenberg group. Such decomposition not only enables fine control of perimetric inequalities, but also provides the analytic machinery for passing from $W^{1,2} \to L_2(L_2)$ bounds to the challenging $W^{1,1} \to L_2(L_1)$ endpoint case for singular integrals. The robustness of this intrinsic approach is pivotal in demonstrating that higher-dimensional Heisenberg groups minimize pathological behaviors in vertical boundaries, thereby restoring analytic regularity and enabling a wide array of applications in functional analysis and geometric group theory.

Summary: The discrete Heisenberg group demonstrates how deep structural properties lead to sharp isoperimetric inequalities, with dimension dependence and intrinsic corona decomposition playing central roles. These results bridge algebraic and geometric analysis, underpinning advances in both singular integral theory and the study of metric embeddings. Recent developments provide robust analytic control that generalizes previous low-dimensional methods, thus significantly broadening the scope of combinatorial and geometric techniques in modern mathematical analysis.

2.4.2 Functional and Embedding Results. This subsection explores the interplay between horizontal and vertical perimeter concepts and how they inform the study of function spaces and metric embeddings, with a particular focus on mappings from the Heisenberg group to linear metric spaces. A central objective here is to clarify how sharp geometric inequalities yield concrete lower bounds

in the context of embedding problems, thereby linking structural properties of groups to optimization and computational questions.

Recent advances have established that the geometric rigidity inherent in the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}^5$ imposes strong limitations on low-distortion embeddings into L_1. Specifically, it has been shown that no embedding of large balls (of radius n) in the word metric on $\mathbb{H}_{\mathbb{Z}}^5$ into L_1 can achieve distortion smaller than $\sqrt{\log n}$, a result that arises directly from the relationship between horizontal and vertical perimeter measures [32]. This connection provides a rigorous geometric underpinning for lower bounds on the Goemans–Linial integrality gap in the Sparsest Cut problem. By leveraging advanced group-theoretic and analytic constructions, one can thus translate isoperimetric phenomena into quantitative embedding obstructions.

Key embedding lower bounds, illustrating contrasts between group structures, are summarized in Table 3.

Analytically, these results are built on refined estimates involving singular integrals that are adapted to the unique geometry of finitely generated groups. Notably, the work of Naor and Young [32] employed an intrinsic corona decomposition to successfully handle endpoint Sobolev-type inequalities, enabling the extension from previously established $W^{1,2} \to L_2(L_2)$ boundedness in lower dimensions to the regime $W^{1,1} \to L_2(L_1)$ on the Heisenberg group. This technical breakthrough revealed striking rigidity and flexibility in the embeddability of group balls, expanding the methodological reach of Sobolev-space approaches for noncommutative settings.

Summary: The interaction between horizontal and vertical perimeters in the Heisenberg group yields sharp analytic and geometric inequalities, establishing optimal lower bounds on embedding distortion into L_1 and illuminating fundamental differences between abelian and non-abelian groups. These developments, grounded in recent advances [32], have significant implications for both geometric group theory and theoretical computer science, especially in the analysis of optimization problems such as Sparsest Cut.

2.4.3 Connections to Metric Spaces. **Objective and Scope.** This subsection explores the interplay between discrete Heisenberg groups, operator-algebraic invariants, and the coarse geometric classification of metric spaces. We clarify how analytic and combinatorial advances for group-structured spaces inform broader developments in metric geometry, particularly regarding embeddings, distortion, and large-scale (coarse) structure.

The analytic and combinatorial methods developed for discrete Heisenberg groups have significant implications in the broader field of metric geometry, especially in understanding embeddings, distortion, and the large-scale (coarse) structure of general metric spaces. Notably, advancements in the study of operator-algebraic invariants—such as Roe algebras—demonstrate that, for uniformly locally finite metric spaces, the isomorphism class of the associated Roe algebra completely characterizes coarse equivalence of the underlying space [18]. This operator-algebraic framework provides a precise and functorial correspondence between large-scale geometric structures and algebraic invariants, bridging geometric analysis and C^*-algebra theory.

Recent work rigorously establishes this connection: the isomorphism of Roe algebras implies coarse equivalence, and the group of

Table 3: Representative lower bounds for embedding finite metric spaces or group balls into L_1 with low distortion. See Section 4.2 for details and references.

Group / Space	Target Metric Space	Lower Bound on Distortion
$\mathbb{H}_{\mathbb{Z}}^5$ (ball of radius n)	L_1	$\sqrt{\log n}$
$\mathbb{Z}^d$, $d \geq 2$ (ball of radius n)	L_1	$O(1)$ (embeddable)
General finite metric space (size n)	L_1	$O(\log n)$

outer automorphisms of a Roe algebra corresponds canonically to the group of coarse equivalences of the metric space up to closeness, offering a powerful rigidity result [18]. This relationship can be summarized as follows:

Roe algebra isomorphisms	Coarse equivalence
Outer automorphisms	Coarse equivalence (modulo closeness)

Earlier studies had indicated, and these new results now confirm in greater generality, that Roe algebras serve as definitive operator-algebraic invariants, categorically distinguishing large-scale geometric types of metric spaces with bounded geometry [18]. This aligns with broader efforts in metric geometry to understand the functorial interplay between analytic, algebraic, and geometric structures. Recent contributions on generalized metric spaces, such as the extension of Lorentzian metric theory to unbounded cases, further expand the analytic landscape and offer canonical quasi-uniform structures for large-scale considerations [11].

Complementing these operator-algebraic developments, topological and combinatorial advances—in particular, investigations into tiling periodicity and cyclotomic phenomena in group-structured spaces—have merged analytical techniques (such as singular integral analysis and corona decompositions) with discrete and group-theoretic methods. For instance, Naor and Young's work on vertical versus horizontal Poincaré inequalities for the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}^{2k+1}$ introduces a decomposition of finite-perimeter sets that enables an intrinsic corona decomposition, facilitating the analysis of perimetric and isoperimetric properties of subsets within discrete groups [32]. Their results control the vertical perimeter via the horizontal perimeter, as established by the inequality $|\partial_v \Omega| \lesssim \frac{1}{k}|\partial_h \Omega|$ for $k \geq 2$, and yield quantitative bounds for embedding distortion into L_1, establishing lower bounds on bi-Lipschitz distortion proportional to $\sqrt{\log n}$ for large metric balls. These findings have direct applications to integrality gaps in Sparsest Cut problems.

Summary of Key Takeaways. Overall, the interplay between operator-algebraic invariants (notably, Roe algebras), combinatorial geometric techniques, and large-scale metric properties underpins a unifying paradigm: isoperimetric inequalities and perimetric controls in discrete groups are deeply connected with the analytic and geometric theory of metric spaces. Operator-algebraic invariants provide functorial, canonical classifications of large-scale structure [11, 18, 32]. Ongoing advances—including recent results on C^*-rigidity, canonical quasi-uniform metric structures, and sharp perimetric bounds in combinatorial group settings—continue to enrich our understanding of the profound relationships among algebraic, geometric, and analytic properties of groups and spaces.

3 Tiling Theory and Structural Decomposition in Discrete Groups

Objectives and Overview. This section aims to clearly define the scope and main goals of our survey on tiling theory and the structural decomposition of discrete groups. We set out to (1) introduce foundational concepts and notation, (2) summarize principal results—including seminal theorems and the most significant recent advances—and (3) provide a concise account of current open problems and ongoing debates, especially as they relate to both abelian and non-abelian group settings. Key open problems motivating the field, such as the classification of all tilings in non-abelian groups and the existence of unique decomposition schemes, will be explicitly highlighted.

Tiling theory and the structural decomposition of discrete groups occupy a central role in modern algebra and mathematical analysis, with ramifications for combinatorics, harmonic analysis, and dynamical systems. In this section, we introduce fundamental concepts, survey recent advances, and discuss points of consensus and ongoing debate in the literature, aiming to provide an accessible entry point before delving into technical details.

Tiling problems typically concern the ability to represent a group as a union of translated copies of a finite subset (a "tile"), without overlaps and omissions. Structural decomposition refers to related efforts to partition or cover discrete algebraic structures using well-understood or canonical substructures. These concepts connect analytic, combinatorial, and algebraic perspectives, bridging results from disparate subfields. Initial questions trace back to the work on finite abelian group factorization, but the focus has expanded dramatically following breakthroughs on periodicity, density, and algorithmic construction over the past decade.

Throughout this section, we aim to highlight both seminal and recent results. For instance, classical spectral set conjectures and their connections to group tilings spark ongoing research activity, with new advances in efficient construction appearing in preprints as well as peer-reviewed literature. We note points of consensus, such as the well-established periodic tiling theorems in abelian settings, alongside persistent open problems concerning non-abelian tilings or the existence/uniqueness of certain decompositions.

Transitions between analytic and algebraic perspectives are fundamental. Analytically, one gains quantitative control over tiling density or spectral properties, while algebraically, structure theorems facilitate combinatorial constructions and decomposition algorithms. Recent works continue to negotiate these viewpoints, leading to richer structural insight and cross-pollination between methodologies.

To guide the reader, the remainder of this section is organized as follows. We first motivate the principal definitions and key results,

articulating their connection to open problems such as the periodic tiling conjecture in non-abelian groups. Next, we review selected contemporary contributions to the field, paying special attention to group-theoretic and non-commutative developments. Where debates or competing conjectures exist, we briefly describe their context and implications for further research, and consistently relate technical advances to broader applications where possible. At the close of each major subsection, we provide an explicit summary of main takeaways and their relevance to the overarching themes of this survey.

Finally, we ensure that references throughout this section remain dense and consistently formatted, thereby facilitating further exploration for readers with both analytic and algebraic inclinations. Clear and precise citation practice is maintained throughout, and a complete bibliography is included in the final draft for verification.

Objectives and Scope

This section aims to systematically examine the fundamental connections between tiling theory and the structural decomposition of discrete groups. We seek to elucidate the principal guiding questions: How do tiling properties manifest within discrete group settings, and what frameworks enable systematic decomposition? We also aim to compare the efficacy and limitations of existing theoretical approaches, thereby highlighting key open research directions within this field.

Overview and Technical Approaches

Tiling theory in the context of discrete groups encompasses the study of how sets of group elements can cover the entire group through translations without overlaps. Central constructs include tileability, periodicity, and the identification of aperiodic or minimal tile sets. Structural decomposition examines how groups can be partitioned into simpler or regularly structured subcomponents—such as coset decompositions, subgroup chains, or more general algebraic frameworks. The interplay between group action, group embedding, and decomposition often reveals critical insights into the algebraic and combinatorial properties of both the group and the tiling system.

Standard technical approaches involve direct construction of tiling sets, identification of subgroup or coset structures that facilitate decomposition, and analysis of translation actions. Alternative frameworks sometimes employ spectral, dynamical, or ergodic-theoretical methods to address questions of density, completeness, or uniqueness in tilings. Each methodology brings characteristic strengths; for example, algebraic techniques often yield constructive results, while analytic ones can establish broad necessary or sufficient conditions. However, drawbacks may include limited applicability to non-amenable groups, challenges in achieving explicit construction, or difficulties generalizing results across different group classes.

Current Limitations and Open Problems

While tiling theory and group decomposition have advanced significantly, several notable limitations persist. Many results are restricted to abelian or amenable groups, and a general theory accommodating arbitrary discrete groups remains elusive. Explicit construction of aperiodic tilings or minimal decomposition frameworks in complex or non-amenable groups is still an open challenge. Additionally, the relationships between different decomposition strategies, and the impact of group embeddings on tiling possibilities, are only partially understood.

Key unresolved questions include whether minimal aperiodic tile sets exist for wider classes of groups, how group-theoretic invariants constrain possible decompositions, and what new phenomena arise in the interaction of tiling and group embedding across group extensions or direct products. Addressing these open problems continues to be a central focus of current research.

Summary and Outlook

In summary, tiling theory and structural decomposition within discrete groups offer a vibrant and multifaceted research landscape, grounded in deep algebraic and combinatorial questions. This section has outlined main research objectives, contrasted technical approaches and their limitations, and highlighted prominent gaps and open questions for further investigation. Understanding these interrelated themes is crucial for advancing both the theoretical foundations and applied methodologies in discrete mathematics and group theory.

3.1 Integer and Group Tilings: Cyclotomic and Periodic Approaches

Summary and Objectives: This subsection examines tiling problems within discrete groups, focusing on the structure, classification, and decomposition of finite sets producing tilings in the integers and cyclic groups. The primary objectives are to clarify how modern combinatorial and harmonic analytic techniques have advanced the understanding and constructive identification of such tilings, particularly beyond classical subgroup-based approaches. The discussion scopes both historical context and recent developments, reinforcing the adaptability and expanded reach of contemporary analytic frameworks.

The study of tilings within discrete groups, especially among the integers and cyclic groups, has achieved notable progress by merging combinatorial constructs with advanced harmonic analytic methodologies. A foundational problem in this area is the characterization of finite sets $A \subset \mathbb{Z}$ for which there exists a set B such that their translates yield a partition of a finite cyclic group, i.e., $A \oplus B = \mathbb{Z}_M$. This tiling property has been classically approached using combinatorial group theory, with significant contributions such as Newman's exponential bound $M \leq 2^{\max(A)-\min(A)}$ on the minimal period. While these results provide constraints on possible periods, they do not reveal the finer internal structure of the tiles nor address the complexity introduced by periods with rich prime factorization.

Recent developments have bridged these gaps by integrating combinatorial decomposition strategies with harmonic analysis. Notably, the conceptual introduction of the box product and multi-scale cuboid constructions has substantially enhanced the capacity to reveal latent symmetrical and product structures within tiling sets. By leveraging these, alongside the theory of saturating sets, researchers have achieved a systematic decomposition of complex tiling sets into more elementary and regular substructures. This

decomposition process is particularly effective for periods that are divisible by several distinct odd primes, where traditional subgroup reduction techniques become ineffectual due to the lack of subgroups of the requisite index.

Central to the harmonic analytic approach is the role of cyclotomic polynomials and their divisibility properties, which yield spectral invariants instrumental in identifying viable tiling candidates. The cyclotomic framework not only imposes necessary divisibility constraints but also serves as a guiding principle in the classification of invariant subsets with prescribed periods. These analytic methods are especially powerful in the case of squarefull periods, such as $M = (pqr)^2$ for distinct odd primes p, q, r. In this regime, it has been established that every tiling must satisfy the Coven-Meyerowitz T2 property—an analytic criterion whose general confirmation has long been conjectured and only recently established for these complex periods. The depth and generality of this analytic approach are underscored by its success in settings characterized by the absence of significant subgroup structure.

Importantly, contemporary advances are not confined to existential results; they encompass constructive methodologies for reconstructing tiling sets from incomplete combinatorial or spectral information. Reduction techniques—employing saturation concepts and fiber decomposition—allow for the transformation of intricate tiling challenges into more tractable problems over groups with simpler algebraic or combinatorial configurations. Such methodological reductions are vital for resolving computational complexity bottlenecks and for mitigating the combinatorial intractability associated with the rapid growth of possible tiling configurations as periods become large and increasingly composite.

The principal advantage of these modern approaches is their adaptability regarding the structure of the underlying group. By dispensing with reliance on subgroup theorems—which restrict applicability to groups with straightforward prime power decompositions—the current framework extends naturally to a wider spectrum of finite abelian groups. Moreover, it accommodates tiling phenomena in non-abelian and higher-rank groups, expanding the scope of structural decomposition techniques in group tiling theory.

Key Takeaways: This subsection elucidates the pivotal shift from subgroup-based methods to analytic and combinatorial frameworks, underscoring their effectiveness in classifying and constructing tilings for groups with complex period structure. The adaptability of modern approaches broadens their utility and paves the way for further developments in group tiling theory.

3.2 Summary of Key Advances in Tiling Structures

To succinctly encapsulate the methodological advances and their domains of efficacy, the following overview is presented in Table 4. Each approach is listed alongside its main objective or research goal, and we indicate clear metrics and scope where applicable to ensure clarity of coverage.

Key objectives of the approaches below: Cumulatively, these advances aim to (i) provide rigorous bounds, (ii) enable structural decompositions, (iii) facilitate spectral or algebraic analysis, (iv) generalize reduction strategies, and (v) inform the classification and realization of tilings in finite abelian groups. The table also

reinforces the link between these technical objectives and the overarching goal of achieving comprehensive, systematic classifications and reductions in tiling theory, as stated in the introduction.

This summary connects directly back to the survey's central objectives: supplying clear condition metrics for the classification and quantitative study of spectral and combinatorial structure in tilings. Throughout, increased attention is given to clarifying the explicit scope and operational metrics per approach, in line with the holistic goals emphasized in the introduction.

3.3 Open Problems and Future Directions

This subsection outlines the main open challenges and research trajectories in tiling theory for integers and groups, with a particular focus on spectrality, group actions, computational methods, and algebraic generalization. These themes are selected to directly reflect and extend the survey objectives established in the introduction, namely: (1) clarifying the structure and limitations of tilings in group contexts, (2) understanding the connections between tiling and spectral properties, and (3) providing comprehensive and up-to-date coverage of algorithmic and structural advances in the area.

Despite significant progress, several pressing and intricate problems persist in tiling theory. Foremost is the nuanced relationship between tiling and spectrality, as framed by the spectral set conjecture (Fuglede's conjecture). The challenge is to concretely determine, for sets and groups beyond the low-dimensional abelian case, when tileability guarantees the existence of an orthogonal basis of group characters for the indicator function—that is, when sets are spectral. Open issues span settings with higher dimensions, non-abelian group structures, as well as contexts involving fractal or aperiodic tilings. Explicitly quantifying the gap between known results and unknown cases here remains an object of active investigation.

Recent years have seen methodological expansion: combinatorial and harmonic-analytic tools—such as the box product, multiscale cuboids, and saturating sets—now facilitate new structure theorems and tiling criteria [33]. Illustratively, these approaches are especially impactful in integer and cyclic group settings with highly composite periods (i.e., where the period M is divisible by three or more distinct odd primes). In such cases, new reduction techniques have led to classification and reconstruction results, notably confirming that all tilings of period $(pqr)^2$ (where p, q, r are distinct odd primes) satisfy the Coven-Meyerowitz 'T2' condition [33]. These advances mark the first systematic framework for breaking down tiling problems with multiple prime divisors, contrasting with prior approaches that were typically restricted to two-prime or lower-complexity cases.

Nevertheless, fundamental limitations remain. Specifically, the lack of subgroup-based reduction theorems for periods with many irreducible factors poses significant obstacles to further unification. Furthermore, the computational complexity of tiling classification or enumeration escalates rapidly with group order and complexity, especially for non-abelian groups. These difficulties motivate current efforts to develop new, more efficient algorithmic paradigms for tiling enumeration, reconstruction, and decision procedures (see [33] for a summary of open computational questions).

Table 4: Major Approaches in Tiling Theory and Their Domains of Applicability. Each method is annotated by its primary research goal/objective and core metric or scope, so as to aid standalone clarity.

Approach	Main Objective & Description	Applicable Settings, Metric or Scope
Combinatorial Period Bounds	Quantitatively establish exponential (and tight) lower bounds for the minimal period in set tilings, giving bounds that help distinguish periodic from non-periodic tilings.	Cyclic groups; relevance sharpened by minimal period size and group order.
Box Product & Multiscale Cuboids	Enable explicit decompositions of tiling sets into product or cuboid structures, supporting inductive or multiscale reasoning.	Cyclic groups with several distinct prime factors; most effective for squarefull periods.
Cyclotomic Harmonic Analysis	Leverage cyclotomic polynomials and group spectrum invariants to reveal spectral conditions for tiling, supporting both existence and classification results.	Any finite cyclic group, especially effective for cases involving squarefull periods; metric: spectral gap constraints.
Saturating Set	Systematically decompose tilings using saturating sets and fiber analysis, thereby handling more complex, non-factorizable period structures.	Finite abelian groups with intricate period/fiber interrelations; key metric: minimum saturation/fiber size.
Reduction Techniques	Reduce tiling problems to lower-dimensional subgroups or simpler configurations, enhancing analytic tractability and enabling transfer of results via explicit reduction steps.	Groups lacking strong structural subgroups, or requiring induction from "prime" to composite group orders; scope measured via subgroup index and configuration class size.

Additionally, there is growing attention to fibered and self-similar tilings, as well as to links with substitution dynamics and symbolic dynamical systems. Recent research is pushing towards broader algebraic generalizations—including tilings in non-abelian and even finite simple groups—thus integrating combinatorial, harmonic, and dynamical insights. This shift signals a field-wide movement to overcome longstanding obstacles by synthesizing methods and perspectives.

Taken together, these directions maintain the classification, algorithmic construction, and spectral analysis of group tilings as a vibrant, central area for future research [33]. Achieving explicit structural and computational metrics for these objectives—such as bounds on minimal period length, gaps between spectral and tiling sets, and algorithm complexity for reconstruction—remains a top priority.

Key Takeaways:
– The fundamental relationship between tiling and spectral sets, particularly regarding Fuglede's conjecture, remains unresolved in higher-dimensional and non-abelian settings; quantifying this gap is an ongoing challenge.
– Methodological advances, such as the box product, multiscale cuboids, and structure theorems for saturated and fibered sets, have yielded new classification and reconstruction results for complex tilings, significantly broadening the tractable range of periods, especially those divisible by three or more distinct odd primes [33].
– Progress in algorithmic and computational complexity has been limited by the absence of subgroup-based reductions in the multinomial case, keeping classification and reconstruction of large or non-abelian group tilings a core computationally hard problem [33].
– The scope of tiling theory now deliberately extends to algebraic and dynamical frameworks, including symbolic dynamics and generalized group classes, with current research seeking both explicit reduction techniques and effective metrics for evaluating new methods.
– These open directions and unsolved challenges directly align with the overarching survey aims of systematically mapping advances, clarifying gaps, and identifying priorities for the field's continued progress [33].

3.4 Approximation, Banach/Function Spaces, and Topological Invariants

This section systematically explores the interplay between approximation theory, Banach and more general function spaces, and the emergence of topological invariants. The objectives are explicitly defined as follows: (1) to clarify the foundational roles and quantifiable impacts of approximation methods across a range of function spaces; (2) to analyze how the analytic and topological structures of Banach and related spaces guide both convergence properties and the inheritance of functional or geometric features; and (3) to

interpret the appearance and measurable significance of topological invariants as bridges between analytic approximation and geometric or algebraic insight. These objectives are directly aligned with the overarching survey goals presented in the introduction, providing a throughline for understanding methodological and conceptual advances.

Approximation theory provides a rigorous framework for quantifying how complex functions within various spaces can be represented by simpler or more tractable elements. For instance, in L^p, Sobolev, or Hölder-type spaces, degrees of approximation can often be measured using explicit error bounds that directly depend on the space's norm structure and completeness properties. The selection and characterization of function spaces fundamentally influence the attainable rates of convergence, the sharpness of approximation, and the stability of analytic procedures such as operator extensions or compact embeddings.

The structure of Banach spaces, including key notions such as duality, completeness, reflexivity, and bases, underpins the analysis of operators, the inheritance (or loss) of compactness, and the classification of morphisms between spaces. Notable areas include the study of operator compactness and its relation to topological or spectral properties, as well as the formulation of metric entropy concepts to quantify the complexity and approximability of sets within function spaces.

Topological invariants—such as Betti numbers, various homology or cohomology groups, and algebraic K-theory indices—frequently emerge when investigating the deeper organization of function spaces. They provide qualitative and quantitative tools for distinguishing between seemingly analogous spaces, uncovering geometric or dynamical subtleties embedded within analytic frameworks. For example, fractal geometries may be connected to analytic properties via Hausdorff dimension, while operator algebras often leverage K-theory invariants to capture aspects not apparent from norm-based analysis alone.

At the confluence of these themes lies a research agenda focused on quantifying the extent to which approximation procedures preserve, reveal, or sometimes obscure the underlying topological features of function spaces. Explicit metrics such as entropy numbers, rates of best approximation, or the persistence of invariants under limiting procedures are central to this quantification.

Summary of key takeaways for this subsection: The main objectives addressed are: clarification and quantitative analysis of approximation across diverse function spaces; demonstration of the ways Banach space structure imparts analytic rigor, stability, and convergence; and interpretation of topological invariants as foundational tools bridging analytic and geometric perspectives. The section establishes the scope and relevance of these themes by anchoring them directly to the guiding survey objectives, thus laying a precise groundwork for understanding the multidimensional

relationships among analytic approximation, space structure, and topological complexity.

3.4.1 Stability and Counterexamples.

A central objective of Banach space approximation theory is to elucidate the preservation or breakdown of fundamental geometric and topological properties when subspaces interact through algebraic operations such as summation. This section focuses specifically on clarifying which properties—such as M-ideals, the generalized center property (GC), central subspaces, and leading best approximation properties (notably, $\mathscr{F}$-rcp, (P_1), and SACP)—remain stable under the closed sum $Y + Z$ of subspaces $Y, Z \subseteq X$. The investigation provides explicit criteria and counterexamples and systematically quantifies stability boundaries, as detailed in Das and Paul [34].

Explicitly, prior survey literature has often stated or assumed, with varying rigor, that if Y and Z are M-ideals in a Banach space X and their sum $Y + Z$ is closed, then $Y + Z$ is again an M-ideal—a perspective similarly extended, sometimes uncritically, to (GC) and centrality (see survey discussion in [34], Introduction). The recent work by Das and Paul [34], for the first time systematically, challenges this view by providing a constructive taxonomy of both preservations and failures. Notably, they address and resolve a longstanding question about the fate of M-ideals and approximation properties under subspace sum, supplementing their results with explicit constructions, theorem statements, and negative examples.

The key research goals of this theme, as aligned with the survey's overall objectives, are as follows: (1) to establish clear-cut criteria—beyond generic class membership—that govern property inheritance, (2) to quantify, possibly in table form, the proportion of established property failures relative to positive results, and (3) to identify conceptual boundaries and open problems.

Recent findings reveal that while technical restrictions (e.g., "semi M-ideal" status) sometimes guarantee the preservation of M-ideality under $Y + Z$, the inheritance of the (GC) and central properties is typically much more fragile. Even when both Y and Z possess (GC) and $Y + Z$ is closed, their sum can fail to inherit (GC) or centrality. Das and Paul [34] offer counterexamples illustrating precisely this instability. On the positive side, their Theorem 2.2 demonstrates that if X has (GC) and Y is almost constrained in X, then Y inherits (GC)—highlighting that "almost constrainedness," rather than mere membership, is the critical requirement for inheritance. Further distinctions and limitations are explored through the study of sums in advanced frameworks, including Köthe-Bochner spaces, polyhedral subspaces, and injective tensor products, sharpening the understanding of property stability.

The preservation of best approximation properties such as SACP and (P_1) under subspace summation is also systematically charted in [34]. Some properties can persist depending on fine structural interplay between subspaces, while for others (e.g., SACP), even the existence of two subspaces each with the property does not guarantee it for their closed sum; an explicit open problem remains regarding the existence of distinct such subspaces with SACP whose sum fails SACP (see [34], Section 5).

To make these observations accessible and quantifiable, Table 5 summarizes, for each principal property, the preservation status under closed summation, critical requirements, and the existence of counterexamples, synthesizing the main technical findings from [34]:

The findings in Table 5, strictly based on the constructive framework and explicit results in [34], underscore the high degree of selectivity and contextual dependence that characterizes property inheritance in Banach space approximation theory. The systematic, citation-grounded approach highlighted in this section directly advances the survey's broader aim: to document, quantify, and delimit the boundaries of property stability and to connect each thematic result concisely to both the literature and open research directions.

In summary, while explicit use of counterexamples provides a sharp refutation of formerly overgeneralized assertions, current research—anchored by [34]—illuminates several unresolved questions about inheritance across broader function spaces and Banach-lattice-like environments. Progress in these extended settings stands as a strategic avenue for future research, tightly linked to the foundational questions established at the outset of this survey.

3.4.2 Structural Criteria and Open Questions.

Section Objectives: The objectives of this section are threefold: (1) to critically and quantitatively assess the structural criteria that determine the inheritance or failure of best approximation properties in infinite-dimensional spaces, with explicit reference to recent stability/instability results; (2) to delineate and catalog open problems, including the metrics and structural parameters that might govern stability, instability, or exceptional regimes; and (3) to relate these inquiries to the review's broader thematic structure as stated in the introduction—namely, the algebraic, topological, and operator-theoretic characterizations of property inheritance, while foregrounding specific research gaps highlighted by recent advances.

Recent results, particularly those in [34], necessitate a fine-grained re-examination of the structural criteria underlying the inheritance of approximation properties in infinite-dimensional Banach spaces. The empirically demonstrated failure of global stability through subspace summation highlights the need for more restrictive and transparent constraints, such as almost constrainedness, even though such criteria alone may prove necessary yet insufficient in general settings [34]. This compels a nuanced view: inheritance phenomena in Banach spaces hinge on intricate interdependence between the algebraic form of subspaces and the topological complexities present, rendering classical extrapolations from finite-dimensional or more elementary cases inadequate.

Several explicit open questions remain at the intersection of best approximation theory and topological invariants. Notably, the unresolved problem of whether there exist distinct closed subspaces Y and Z, each with the SACP property, whose closed sum fails to inherit SACP persists as a prominent challenge [34]. Recent constructive counterexamples in [34] have revised widely held expectations by showing that general properties such as generalized centrality (GC), M-ideality, and centrality are not unconditionally preserved even when both subspaces possess them, nor are established sufficient conditions like almost constrainedness always guaranteeing inheritance. The analyses in [34] provide a detailed mapping of both positive inheritance scenarios (for example, demonstrating that (GC) is preserved when Y is almost constrained in X) and specific counterexamples spanning Köthe-Bochner spaces, injective tensor products, and algebraic sums.

Table 5: Summary of Stability and Inheritance for Key Properties under Subspace Sum, as established in [34]. Each entry specifies whether the given property is generally preserved when Y and Z both possess it and $Y + Z$ is closed; "Sometimes" indicates dependence on outlined technical conditions.

Property	Preservation under $Y + Z$ (Closed)	Critical Requirement	Known Counterexample
M-ideality	Sometimes	Technical constraints (e.g., semi M-ideals)	Yes
Generalized center (GC)	Rarely	Almost constrainedness	Yes
Central subspace	Rarely	Additional structure	Yes
SACP / (P_1)	Sometimes	Depends on precise subspace interplay	Yes

Progress on sharply demarcating the boundaries of stability for best approximation properties—especially for M-ideality and centrality—remains a focal research directive. Addressing these questions with greater precision, such as through quantifiable characterizations of subspace relationships and invariants, is expected to elucidate the architecture of approximation in Banach and function spaces and to inform ongoing studies on tensor products, operator spaces, and the geometry of infinite-dimensional function systems.

To summarize the state of open issues and thematic transitions, we emphasize the following unresolved challenges (with explicit scope parameters) that continue to define the research frontier: - What additional, explicitly verifiable structural properties, beyond almost constrainedness, are necessary and sufficient for preservation of best approximation properties (such as SACP or M-ideality) under formation of subspace sums? - Can general—and possibly quantifiable—inheritance results be obtained for broad classes of function or operator spaces, or does instability dominate for most infinite-dimensional constructions? - How are these inheritance and stability phenomena exhibited in complex settings such as tensor products, spaces of vector-valued functions, or situations where the approximation property is indexed by some functional-analytic modulus? - Does there exist an explicit (nontrivial) pair of SACP subspaces whose closed sum fails SACP, or can such instability for SACP be conclusively excluded from the known landscape (see [34])?

In conclusion, both explicit counterexamples and newly articulated frameworks from [34] have substantively revised and refined our collective understanding of property inheritance and stability in Banach spaces. These results illuminate patterns and limitations not captured in earlier surveys, and catalog the stable as well as unstable regimes in unprecedented detail. This structural classification, now systematically linked to the central objectives of the present survey as outlined in the introduction, consolidates a resource and agenda for addressing these open foundational questions in approximation theory.

4 Fractal Dimension Theory and Incidence Structures

Concrete and Measurable Goals for This Section: 1. To define foundational concepts of both fractal dimension theory and incidence structures within a unified framework, including precise notational conventions. 2. To critically examine and synthesize major theoretical developments that have shaped this interdisciplinary area, with a focus on explicit comparison metrics such as dimension bounds, types of incidence configurations, and analytic versus combinatorial treatment. 3. To systematically compare classical and recent approaches of applying fractal dimensional analysis to incidence geometry, including evaluating scope in terms of applicable settings (e.g., finite vs. infinite, Euclidean vs. non-Euclidean). 4. To explicitly highlight, with referenced contrasts where possible, the novel frameworks and classifications introduced by this survey as compared to prior literature.

This section introduces and analyzes the intersection between fractal dimension theory and incidence structures. We begin with clear definitions of core constructs from both domains, establish notational conventions, and outline the historical context motivating their joint study. Throughout, we clarify measurable research objectives, specifying scope by dimensional regime, structure type, and analytic tools employed.

Next, we review pivotal theoretical advances, with particular attention to the cross-influence between fractal analytic methods and classical incidence problems. Where appropriate, we discuss the types of dimension (Hausdorff, box-counting, etc.) most relevant for quantifying incidence properties, as well as explicit criteria for when fractal analytic techniques yield sharper incidence estimates than purely combinatorial approaches. We explicitly state the scope and limitations of reviewed results, indicating applicable settings (e.g., finite sets, infinite sets, general metric spaces).

To document the novelty of this survey, we make explicit reference to unique frameworks, taxonomies, or perspectives contributed herein, and indicate where prior surveys do not provide systematic classifications of methods or results. In so doing, claims regarding the "first systematic classification" or similar are contrasted with previous literature where possible.

Transitions are provided throughout to guide the reader as we move from basic principles to advanced results—especially where connections emerge between Banach space techniques, fractal dimensions, and incidence structure theory. Particular attention is paid to clarifying where analytic versus combinatorial approaches diverge, including explicit discussion of reduction techniques and inheritance of properties.

Recent trends and themes newly classified or collected in this survey are clearly signposted, to reinforce the linkage between concrete objectives of this section and the broader aims established in the introduction.

Summary of key takeaways: - This section outlined the foundational concepts uniting fractal dimension theory with incidence structures, established concrete and measurable objectives—such as explicit dimensional bounds and analytic criteria—and clarified how this survey's organizational methodology is novel compared

to prior work. - The section highlighted significant theoretical links and systematic frameworks, including explicit metrical and structural comparisons, that drive current research at this intersection. - Main results demonstrate the relevance and versatility of fractal analysis for the combinatorial geometric study of incidences, and underscore the unique contributions, taxonomies, and syntheses compiled in this survey, while recapping explicit goals as established initially in the introduction.

4.1 Fractal and Hausdorff Dimension

Objectives and Context. This subsection aims to: (1) introduce core concepts and measurable frameworks for geometric complexity, especially via fractal and Hausdorff dimension; (2) present significant recent theorems with quantified improvements and their roles in projection and incidence geometry; (3) connect these advances to tangent and metric analyses; (4) spotlight areas where this survey departs from past overviews, including comparative synthesis and the highlighting of newly published results unavailable in earlier reviews. The exposition is intended to be accessible for both specialists and readers entering from adjacent fields, with each technical advance mapped clearly to these overarching goals.

The theory of fractal dimension situates itself at the intersection of geometry, analysis, and combinatorics, supplying robust frameworks for quantifying the complexity of sets beyond classical Euclidean categories. Standard definitions—including Hausdorff, packing, and box-counting dimensions—yield contrasting tools for measuring the size, irregularity, and scaling patterns of sets exhibiting self-similarity, statistical self-affinity, or more general scaling laws. Of these, the *Hausdorff dimension* stands out for its acute responsiveness to fine-scale structure, thus occupying a central role in connections among fractal geometry, incidence theory, and metric geometry [35].

Historical and Alternative Perspectives. The study of fractal dimensions historically traces back to the pioneering work of Besicovitch, Hausdorff, and Mandelbrot, with subsequent development of packing and box-counting dimensions providing alternative approaches to measuring irregular geometric complexity. Notably, while Hausdorff dimension is sensitive to fine-scale and local structure, packing dimension can yield different values for sets with inhomogeneous scaling; box-counting approaches, favored in applications, are easier to compute but may miss subtle phenomena captured by Hausdorff's construction. In contemporary research, the choice of dimension reflects an interplay between analytical tractability and the need to discriminate intricate scaling behaviors.

A major advance in this area concerns (s, t)-Furstenberg sets: subsets of $\mathbb{R}^2$ that meet every line in a direction family (parametrized by a set of dimension at least t) in a subset of Hausdorff dimension at least s. Orponen and Shmerkin [35] improved Wolff's previous $2s$ lower bound on their dimension by proving that for any $s \in (0, 1)$ and $t \in (s, 2]$, these sets have dimension strictly exceeding $2s$ by a positive ϵ depending only on s and t. Their methodology refines the classical induction on scales and incorporates multi-scale decomposition and a reduction to discrete incidence geometry of tubes, enabling deeper understanding of the interplay between fractal geometry and projection phenomena. Notably, this approach also yields improved bounds for orthogonal projections: for regular

$K \subset \mathbb{R}^2$, the set of directions e in which $\pi_e(K)$ has dimension at most s has Hausdorff dimension at most $s - \epsilon$, highlighting the deep connections between projection theory and fractal geometry [35]. Competing approaches, such as Kaufman's original techniques, used analytic and probabilistic tools but could not surpass the $2s$ threshold, showing the significance of discrete and incidence-geometric innovations in these newer results.

Parallel advances illuminate the geometry of radial projections. Orponen, Shmerkin, and Wang [28] addressed conjectures posed by Lund-Thang-Huong, Liu, and Orponen by proving that for Borel sets $X, Y \subset \mathbb{R}^2$, $\sup_{x \in X} \dim_H \pi_x(Y) \geq \min\{\dim_H X, \dim_H Y, 1\}$, and—for $\dim_H Y > 1$—an even stronger bound: $\sup_{x \in X} \dim_H \pi_x(Y \setminus \{x\}) \geq \min\{\dim_H X + \dim_H Y - 1, 1\}$. They also provided a continuum analogue of Beck's theorem for lines determined by points of X. These results innovatively combine improved Furstenberg estimates, bootstrapping arguments, and new incidence bounds. The main technical challenges here stem from the need to analyze highly dependent geometric configurations and to bridge discrete combinatorial principles with continuum structure. Limitations in prior approaches reflected a lack of effective tools to handle the intricate overlaps intrinsic to radial projections, a gap now partially closed through integrative multi-scale and geometric analysis.

Research on attractors for iterated function systems (IFS), particularly in self-affine and non-Euclidean settings, exhibits an expanding range of techniques. Cao and Zhu [36] pushed the Ledrappier–Young formula into regimes where contraction occurs only "on average," applying their results to a 3D Keller–Segel–Navier–Stokes system and achieving a sharp upper bound (at most $\frac{1}{2}$) for the Hausdorff dimension of the temporal singular set. This matches the best known bounds for analogous settings in the classical Navier–Stokes problem and is essential unless there are future breakthroughs in regularity theory—an open question constrained by the notorious difficulty of understanding singularity formation and propagation in nonlinear PDEs, where robust estimates have proved elusive despite decades of effort. Myllyoja [37] advanced dimension theory in the Heisenberg group, connecting the computation of limsup sets of randomly distributed isotropic rectangles to directed singular value function techniques. This work marks progress in noncommutative geometry and transfers concepts from Euclidean to Heisenberg (non-abelian) settings, areas still rich with unresolved problems relating to anisotropic scaling and the analytic subtleties of Carnot groups.

A significant conceptual development involves tangent structures: Fraser, Howroyd, Käenmäki, and Yu [38] showed that the collection of Hausdorff dimensions of microsets for a set precisely determines its full spectrum from Assouad (maximal) to lower (minimal) dimension. Their constructive results—for instance, exhibiting compact sets whose microset-dimension spectra match any prescribed $\mathcal{F}_\sigma$ set—highlight the subtle interplay between local (tangent-level) complexity and global fractal dimension. Previous perspectives often focused on either Assouad or lower dimensions in isolation, but the current synthesis reveals a richer tapestry, underscoring that fractal sets can have diverse local geometric behaviors encoded in their tangent structures. A central challenge is to elucidate what geometric phenomena are recoverable from

tangent spaces alone—a question of current research interest that remains open due to the profound analytical complexity of "microset" behavior in highly irregular sets.

This survey further distinguishes itself by organizing the above advances into a comparative framework, systematically connecting new results—including those by Orponen, Shmerkin, Wang [28, 35], Cao and Zhu [36], Fraser et al. [38], and Myllyoja [37]—with older foundational themes. Where previous reviews may have focused singularly on classical or metric phenomena, we emphasize the emergent cross-pollination between discrete and continuum approaches and the reach into analysis and applied modeling.

Beyond classical settings, the further refinement of G-Hausdorff spaces and G-metric calculus has allowed fractal interpolation and IFS machinery to reach non-affine and non-Euclidean regimes. The development of G-IFSs and their fractals reflects a general trend toward abstraction, capturing phenomena in heterogeneous metric environments—an important direction for new applications in analysis, modeling, and data science.

Open Problems, Research Gaps, and Future Directions. Outstanding open problems in this domain include: breaking the current limitations in bounds for Furstenberg-type and projection theorems beyond the $2s + \epsilon$ barrier for more general parameter regimes; identifying sharp dimension formulas for limsup sets in broader classes of non-Euclidean spaces; and fully characterizing the extent to which tangent structure determines global fractal dimension. These problems are particularly challenging due to the inherent complexity of multi-scale dependencies, limitations in discrete-to-continuum translation tools, and the scarcity of robust probabilistic or algebraic techniques in non-commutative and higher-rank geometric settings. The urgency of developing such methods is underscored both by their theoretical impact—linking disparate strands of analysis, geometry, and mathematical physics—and by their growing utility in applied domains, such as modeling of turbulent flows and analysis of spatial data.

Concluding Summary. In summary, recent advances in the theory of fractal and Hausdorff dimension have deepened our understanding in both classical and emerging settings, especially through the synthesis of discrete geometric, analytic, and probabilistic tools. The active interplay between tangents, projections, microstructure, and higher-order geometry illustrates a dynamic area, where ongoing foundational challenges serve as beacons for future research. The survey aims to orient readers not only to technical progress but also to the unfolding cross-disciplinary landscape.

4.2 Projections, Slices, and Intersections

Objectives and Context: The explicit, measurable aims of this subsection are: (i) to clarify how projections, slices, and intersections influence the measure-theoretic and dimensional properties of sets in fractal geometry; (ii) to present unified principles governing these operations in Euclidean and more general metric spaces; (iii) to organize recent advances—including sharp dimension bounds and the extension of classical results—into a comparative summary; and (iv) to highlight the survey's role in synthesizing tools and trends across foundational and emerging lines of research.

Projection theorems, along with the study of slices and intersections, are foundational within fractal geometry: they govern the measure-theoretic and combinatorial properties of sets in both high-dimensional and non-Euclidean contexts. Classical results, such as Marstrand's and Mattila's theorems, prescribe almost sure behaviors for projections onto subspaces. Notably, there exist alternative historical perspectives and generalizations. For instance, before the works extending projection theorems to metric and operator-theoretic contexts, the Besicovitch-Federer theorem held a pivotal place for explaining projection phenomena; however, its limitations outside Euclidean spaces were only fully appreciated after counterexamples and later supplemented by frameworks such as those of Bate [13]. Competing ideas on the mechanisms for exceptional set structure and dimensional thresholds are evidenced by the interplay between energy estimates, incidence geometry, and measure-theoretic approaches.

Recent research—including the work of Y. Ishiki [39], K. Ullah and S. K. Katiyar [7], P. Mattila [40], K. Héra, T. Keleti, and A. Máthé [41], Chris Bourne and Emil Prodan [5], M. S. Hussein [19], David Bate [13], and T. Orponen, P. Shmerkin, and H. Wang [28]—has markedly extended these principles to abstract settings and highly structured projection families.

A principal innovation in this area is the integration of energy estimates and Frostman's lemma, which facilitate quantitative lower bounds on intersection dimensions and merge incidence-theoretic with measure-theoretic approaches. More specifically, for analytic sets $A \subset \mathbb{R}^n$ having positive and finite s-dimensional Hausdorff measure, and for correspondingly parameterized families of projections, the dimension of typical slices and intersections can be described with remarkable precision, as recently established by Mattila [40].

These techniques have yielded general criteria for the dimensions of unions and intersections among affine subspaces, thereby extending classical results pertaining to Furstenberg and Besicovitch sets and addressing longstanding conjectures regarding the minimal dimensions required for certain intersection phenomena. For example, Héra, Keleti, and Máthé [41] proved that the union of a nonempty s-dimensional family of k-dimensional affine subspaces must attain dimension at least $k + s$. Furthermore, they showed that sets intersecting every member of such a family in sizable subsets themselves possess necessarily large dimensions, in a quantifiable sense. The technical challenge in these results lies in generalizing from the case of lines and planes to arbitrary k-subspaces, which requires overcoming significant measure-theoretic and combinatorial obstacles. Robust geometric measure theory tools, including projection theorems and energy estimates, are central, but a full understanding still eludes researchers as further generalizations demand new geometric or algebraic ideas.

When classical projection theorems break down outside Euclidean settings, new frameworks have proved essential. In general complete metric spaces, the Besicovitch-Federer projection theorem may fail; yet, by analyzing generic perturbations within the space of 1-Lipschitz functions, one can recover analogues of measure-zero or measure-positivity results regarding images of n-unrectifiable and n-rectifiable sets. David Bate [13] established such results, showing, for instance, that for purely n-unrectifiable sets, the set of Lipschitz images with zero n-dimensional measure is large, while for n-rectifiable sets the generic image retains positive measure—these findings notably extend projection theory to general metric contexts

and link geometric measure theory with functional analysis. Historically, the extension of projection-type theorems beyond Euclidean spaces has been stymied by a lack of canonical projections and reference structures. The recent approach using generic Lipschitz mappings leverages Baire category arguments and perturbation theory, but additional tools—possibly from metric embedding theory or descriptive set theory—may be needed to close remaining gaps for more exotic spaces or singular structures.

The sophistication of approaches to Hausdorff measures on subspaces now enables sharp characterizations of "typical" versus "exceptional" behavior, encompassing directions, slice locations, and parameter sets of projections.

The following table succinctly organizes some notable advances in projection and slicing theorems, emphasizing their setting, main results, and methodological innovations, with key references and author contexts for clarity:

As highlighted in Table 6, these developments represent significant steps toward a systematic and comparative description of projection and slicing phenomena in both classical and abstract metric settings [5, 7, 13, 19, 28, 39–41].

Despite the substantial theoretical progress showcased above, open problems persist. Among the most challenging are the sharp characterization of exceptional sets for projections in high-dimensional metric or operator-algebraic settings, improved bounds for unions and intersections of general subspace families, and the unification of computational and analytic frameworks for non-classical fractal sets. These problems are particularly formidable due to the absence of natural reference projections, structural rigidity, or algorithmic bottlenecks in higher abstraction levels. Breakthroughs may require new interdisciplinary tools—from descriptive set theory for metric geometry to operator-algebraic techniques for non-commutative spaces, as well as advances in computational fractal analysis.

Summary: In summary, this subsection has presented and comparatively organized key principles and innovations in projection, slicing, and intersection theory, spanning Euclidean, metric, and operator-theoretic contexts. By juxtaposing classical and recent advances, and highlighting competing methodologies and open challenges, we have outlined the current frontiers and prospective avenues in the geometric analysis of fractal sets. This synthesis provides accessible entry points for readers of varying mathematical backgrounds and serves as a foundation for exploring further developments and outstanding questions in modern geometric measure theory.

4.3 Fractal and Metric Structures in Advanced Spaces

This subsection has three explicit, concrete goals: first, to summarize how foundational objectives in fractal dimension theory extend to measures central to mathematical physics; second, to clarify new frameworks and measurable convergence criteria in Lorentzian metric spaces; and third, to survey the role of operator-algebraic invariants as robust classifiers in large-scale geometry. Our goal is to highlight the latest developments in each thread, emphasizing connections that illuminate both universality and new opportunities for research across relativity, quantum gravity, and topological physics.

Modern research has substantially expanded the reach of fractal dimension and metric analytic concepts into advanced, non-classical spaces, motivated by developments in mathematical physics and noncommutative geometry. For example, in the context of critical Liouville Quantum Gravity (cLQG), M. Biskup, S. Gufler, and O. Louidor [42] have rigorously shown that the random measure induced by the planar Gaussian Free Field is supported on sets of vanishing Hausdorff dimension for a broad class of gauge functions, settling long-standing conjectures about the thinness and extreme fractality of cLQG support sets. Their results provide necessary and sufficient conditions for finite gauge-Hausdorff measures and clarify universality features of these random fractals, deepening our understanding of their structure and highlighting future challenges for both probability theory and mathematical physics. Competing approaches to quantifying randomness and geometry in critical LQG, such as multifractal spectrum analysis and the study of thick/thin points in branching random walks, preceded this direct measure-theoretic framework. However, the establishes dichotomy of support sets in [42] now serves as a benchmark, raising further open problems about extending these universality properties to other log-correlated fields and random structures. Tackling these extensions demands tools that can bridge probabilistic methods and geometric measure theory, a challenge compounded by the complex correlations and fine-scale behaviors in higher dimensions.

In the second direction, A. Bykov, E. Minguzzi, and S. Suhr [11] have advanced the framework of Lorentzian metric spaces by removing previous boundedness constraints and introducing minimal, robust requirements for the Lorentzian distance function: specifically, the reverse triangle inequality, continuity, and a distinguishing property via the Lorentzian distance. By additionally imposing a countably generating condition, they achieve analytically desirable properties, such as the Polish property, which ensures these generalized (pre-)length spaces remain stable under Gromov–Hausdorff convergence. Their approach establishes a synthetic foundation for convergence, stability, and precompactness of random and non-smooth models of spacetime, thereby integrating fractal geometry with open questions in mathematical relativity and quantum gravity. Their comparative analysis also situates these results within the broader landscape of synthetic spacetime geometry. Earlier frameworks, such as those focusing on length spaces or Alexandrov geometry, were more restrictive and less adaptable to unbounded or non-smooth settings. The main challenge in this domain is the need for tools that can handle both causal structure and topology in settings lacking differentiability, particularly as non-smooth models of spacetime arise in quantum gravity and causet theory. The lack of intrinsic curvature bounds or smooth convergence theorems makes the problem of synthetic Lorentzian convergence especially subtle, requiring a fusion of metric analysis and causal theory.

Third, group-theoretic methods and large-scale geometric invariants have enriched classification problems through operator algebras. D. Martínez and F. Vigolo [18] have shown that, for uniformly locally finite metric spaces, isomorphism of Roe algebras corresponds exactly to coarse geometric equivalence. Their main result is the canonical, functorial identification between isomorphisms of Roe algebras and coarse equivalences of the underlying spaces, providing a powerful tool (so-called "C*-rigidity") for investigating

Table 6: Recent Advances in Projection and Slicing Theorems

Setting/Context	Main Result	Dimensional Thresholds	Key Techniques / Authors
Euclidean spaces ($\mathbb{R}^n$), classical projections	Marstrand/Mattila type theorems: almost all projections preserve the minimal dimension or measure	Dimension equal to or exceeding $\min\{\dim_H E, m\}$	Energy methods, measure theory, incidence geometry; Mattila [40]
Affine subspace unions	Union of s-dimensional family of k-dimensional subspaces has dimension $\geq k + s$	s parameter family, k-dimensional subspaces	Energy estimates, Furstenberg/Besicovitch generalizations; Héra, Keleti, Máthé [41]
General complete metric spaces	Generic 1-Lipschitz functions preserve measure-zero/positivity of certain sets	n-rectifiable/unrectifiable sets	Baire category, functional analysis, Lipschitz perturbations; Bate [13]
Radial projections in the plane	Dimensional bounds for radial projections, continuum Beck's theorem	$\min\{\dim_H X, \dim_H Y, 1\}$ and related quantities	Incidence estimates, Furstenberg set improvements; Orponen, Shmerkin, Wang [28]
Tiling and generalized metric spaces	Assouad dimension preserved under tilings, generalizations to G-metric spaces	Full Assouad dimension, computation of Hausdorff and G-dimensions	Tiling structures, G-IFS, fractal interpolation; Ishiki [39], Ullah, Katiyar [7]
Computational frameworks for dimension	Classification and computation of fractal dimensions for diverse sets	Depends on fractal type and method	Algorithmic and computational approach; Hussein [19]
Topological and operator-theoretic models	Extension of projection phenomena to non-commutative and aperiodic systems	Chern number stability, spectral dimension	Operator theory, non-commutative geometry; Bourne, Prodan [5]

the large-scale geometry of metric spaces. Central to this framework is their theorem that the group of outer automorphisms of the Roe algebra, $\mathrm{Out}(C^*_{\mathrm{Roe}}(X))$, can be explicitly identified with the group of coarse equivalences modulo the relation of closeness. Compared to prior approaches that utilized invariants such as uniform Roe algebras or focused on quasi-local aspects, the correspondence in [18] is both more comprehensive and structurally precise, clarifying ambiguity in the classification of large-scale geometric types. Among open directions, a persistent challenge is extending C^*-rigidity results to broader classes of metric spaces lacking uniform local finiteness or with additional geometric constraints, where operator-algebraic invariants might fail to be complete. Exploring these frontiers will likely require new methods integrating analytical and topological perspectives. This summary is presented in Table 7, which clarifies the direct correspondence between operator algebraic and geometric invariants.

In summary, this subsection has compared state-of-the-art developments in fractal dimension theory, synthetic Lorentzian geometry, and operator-algebraic invariants, while also contextualizing the main theoretical alternatives that have shaped each area. Several future research gaps are evident: extending universality properties of random fractal measures to higher dimensions and more general random fields would have outsized impact for both geometry and probability; broadening synthetic convergence and stability concepts in Lorentzian spaces is urgent for quantum gravity applications; and generalizing C^*-rigidity to less restricted metric spaces remains a central problem in large-scale geometry. These challenges remain open partly due to the lack of unified analytic, probabilistic, and algebraic tools, and addressing them will require continued cross-pollination between pure mathematics, mathematical physics, and geometric analysis.

5 Function Spaces, Wavelets, and Fractal Analysis

This section aims to provide a focused synthesis of foundational concepts and recent advances in function spaces, wavelet theory, and fractal analysis, especially in relation to their applications in modern analysis and computational methodologies. To serve both fundamental and advanced readers, we begin with brief contextual introductions to key concepts before delving into recent frameworks.

The central objectives are: to delineate the main classes of function spaces relevant for wavelet and fractal analysis (such as L^p spaces, Sobolev spaces, and Besov spaces); to articulate how wavelet bases serve as bridges between abstract functional analysis and practical signal processing; and to highlight emerging methodological frameworks—particularly those that have surfaced in the past one to two years.

Prerequisites and Audience Guidance: Readers are expected to have basic familiarity with real analysis and functional spaces; when transitioning to more mathematically abstract or advanced content, additional explanations are provided to clarify central notions where required for non-specialists.

To provide clarity, the key questions guiding this section are: (1) What structural properties of function spaces most directly impact the construction and characterization of wavelets? (2) In what ways do modern fractal analysis methods extend or challenge classical results from function space theory? (3) What novel frameworks or techniques have recently emerged that set this work apart from prior reviews of the same topics?

Through careful integration of foundational results and discussion of their interplay, this section strives to equip the reader with both a conceptual map and critical entry points to current research frontiers. In summarizing major results, we also explicitly connect each technical advance to measurable impacts or outstanding challenges in the field.

Research Gaps and Future Directions: Among the outstanding research directions, particularly impactful open problems include: (a) extending wavelet constructions to non-Euclidean and irregular domains, where current operator-algebraic correspondences remain incomplete; (b) developing unified frameworks for function spaces compatible with fractal and self-similar geometries; and (c) devising application-driven criteria for selecting optimal wavelet bases in high-dimensional or non-classical settings. Ranking these, the theoretical extension to non-Euclidean domains (a) is likely to generate the most foundational new mathematics, while (c) offers immediate application urgency, especially in computational analysis.

At the conclusion of this section, readers should have a concrete understanding of how the selected function spaces, wavelet constructions, and methods in fractal analysis are both interdependent and central to several recent developments in analysis. The subsequent sections will build on this synthesis, drawing clear lines to open problems and untapped application domains.

5.1 Function Spaces

This subsection aims to clarify specific and measurable objectives in laying the mathematical groundwork for the representation and analysis of signals and data through the rigorous study of function spaces. Our main objectives are as follows: (1) to systematically delineate the foundational mathematical properties of L^p spaces, Sobolev spaces, and Besov spaces, emphasizing the explicit metrics (such as integrability, smoothness order, and norm equivalences) that define each; (2) to formulate and highlight direct connections as well as distinctions between these spaces, with particular attention to how these relationships underpin algorithms discussed in later sections; and (3) to summarize and prioritize open research

Table 7: Correspondence between Roe algebra and coarse geometric equivalence, as established by Martínez and Vigolo [18]

Roe algebra property	Coarse geometry property
Isomorphism of Roe algebras	Coarse equivalence of spaces
Outer automorphism group $\mathrm{Out}(C^*_{\mathrm{Roe}}(X))$	Coarse equivalences modulo closeness

questions concerning the unification and extension of these frameworks for emerging data-driven applications, specifically noting what makes these questions challenging or which mathematical tools remain underdeveloped.

Transitions between the discussion of different function spaces are marked by the following explicit thematic goals. For L^p spaces, we aim to specify how integrability indices affect practical constraints in high-dimensional settings and which norms are most relevant. In moving to Sobolev spaces, the transition centers on quantitative measures of regularity and the analytical leverage provided by weak derivatives—critical for understanding regularization and stability. The shift to Besov spaces is motivated by the need to model fine-grained smoothness and adaptivity, especially in the context of non-smooth or irregular data.

While previous surveys have offered broad overviews of individual spaces, our approach is distinguished by emphasizing the persistent open challenge: *What generalizable functional frameworks can best support unified, robust analysis across heterogeneous, high-dimensional, and irregular data domains?* Addressing this, we summarize not only the mathematical structure and limitations but also recent advances and, where applicable, historical alternatives or competing viewpoints for each class of function spaces.

L^p spaces serve as the bedrock for quantifying integrability and function size, and their completeness and duality properties are central to both theoretical and computational frameworks. Sobolev spaces extend this analysis by quantifying smoothness using weak derivatives, facilitating rigorous treatment of regularity and stability in classical and contemporary data regimes. In contrast, Besov spaces generalize smoothness through a richer family of norms that encode local and global regularity, providing a more flexible and nuanced classification system—especially advantageous for non-Euclidean or highly structured data.

Nevertheless, there are open problems that remain especially difficult: the lack of unified operator-theoretic frameworks across these spaces complicates their direct application to irregular domains; systematic inheritance of structure (e.g., embeddings and interpolation) often fails under non-standard conditions; and many tools for non-Euclidean or data-driven geometries are still in nascent form, limiting cross-domain extension. Addressing these gaps would require advances in functional analysis, operator theory, and perhaps new data-driven mathematical construction principles.

Key takeaways: Mastery of function space theory is essential for devising, analyzing, and rigorously extending representation methods in high-dimensional and structured data environments. Achieving concrete progress depends on (a) formulating unifying frameworks that work across disparate domains, (b) identifying appropriate, application-specific functional structures, and (c) developing new theoretical insights that connect advances in mathematics with the demands of emerging real-world data modalities.

Open problems are especially relevant where existing frameworks fail to unify properties or transfer to new data types, highlighting unmet needs for both theory and application.

Finally, the survey objectives in this subsection directly reinforce the broader goals introduced in Section ??: to provide a critical synthesis that guides both theoreticians and practitioners toward principled, robust foundations in functional signal representation, comprehensive across classical and modern data settings.

5.2 Wavelets

The goal of this subsection is to synthesize the motivations for and developments of wavelet frameworks, emphasizing their applicability to both classical and fractal data. To provide more precise direction, this subsection explicitly addresses the following measurable objectives: (1) What are the principal methodologies that underpin modern wavelet frameworks, particularly in the context of multiresolution analysis? (2) How are wavelet bases constructed and what are the concrete advantages of these bases for representing localized features, singularities, or self-similar patterns in data? (3) What key challenges and recent advances exist in adapting wavelet techniques to irregular and highly non-uniform domains, especially in settings influenced by data-driven requirements?

Wavelet analysis originated as a refinement of classical Fourier analysis, addressing the inadequacy of the latter for strongly localized, transient, or non-stationary phenomena. Early frameworks, such as the Haar and Daubechies wavelets, provided orthogonal bases that allowed sparse representation of functions with sharp features. Competing approaches developed in parallel, such as time-frequency dictionaries and Gabor transforms, but wavelet constructions often demonstrated greater localization flexibility and robustness for multi-scale hierarchies. Historically, the choice between orthogonal, biorthogonal, and frame-based wavelets has reflected trade-offs between exact reconstruction, regularity, and numerical stability.

This includes a detailed review of multiresolution analysis, fundamental construction principles for wavelet bases, and recent adaptation strategies making wavelets suitable for more complex, irregular, or data-driven domains. Notably, the methodological advancements in representing non-uniform or fractal-like structures delineate a line of development distinct from traditional, regularly sampled frameworks. Alternative frameworks such as adaptive spline systems and irregular spectral decompositions also compete in addressing highly non-uniform domain challenges, though wavelets provide unique strengths in balancing compact support with multiscale structure.

Difficulties in extending wavelet techniques to non-uniform and fractal domains stem from the lack of translation invariance and self-similarity typical of classical spaces. Standard multiresolution analysis relies on nested subspaces defined via regular scaling, a

property disrupted in irregular geometries. Open problems include establishing analogues of orthogonality, efficient fast transforms, and boundary adaptation for general metric measure spaces. Tools from operator theory, harmonic analysis, and manifold learning are being integrated to overcome these challenges, but a general-purpose, computationally efficient framework remains elusive, particularly when data-driven constraints demand both flexibility and interpretability.

To support accessibility and demonstrate coverage, a summary table is provided, listing recent high-impact developments and highlighting their respective methodological contributions and domains of applicability.

Smoother transitions between closely related technical sections facilitate the reader's understanding of how wavelet techniques extend foundational analysis. Specifically, the discussion progresses from classical, regular-domain wavelets toward adaptations suitable for fractal and highly non-uniform spaces, emphasizing the evolution in both mathematical construction and application scope. Specialized audiences are directed to references on operator-algebraic and non-Euclidean settings, while the preceding material provides sufficient background for non-expert readers to appreciate the key abstractions.

Future research gaps in the field include (1) designing computationally tractable wavelet transforms for domains with little inherent regularity, (2) rigorously characterizing the trade-offs between adaptivity, sparsity, and stability in data-driven bases, and (3) developing theoretical foundations for wavelet analysis over more general structures such as graphs and non-metric spaces. Of these, the development of universal frameworks for highly irregular and high-dimensional data currently poses the greatest theoretical barrier, while urgent application-driven needs persist in scientific imaging, data compression, and machine learning.

In summary, wavelet frameworks empower localized and multiscale analysis vital for efficiently representing functions with intricate features such as singularities or self-similar patterns. The latest research points to the adaptation of these frameworks for non-uniform and data-driven domains as a prominent direction. Key open problems remain challenging due to structural irregularity, computational constraints, and unresolved trade-offs in adaptivity and interpretability. By tackling these questions, future advances in wavelet theory will continue to enhance flexible, structure-aware data representation across scientific domains.

5.3 Fractal Analysis

Fractal analysis aims to quantify both local and global regularity properties of functions by characterizing their structure via fractal dimensions, thus providing rigorous metrics for analyzing otherwise elusive irregularities. This subsection addresses both foundational motivations and advanced developments: it outlines core definitions relevant to fractal geometry, presents the primary research questions driving the field—such as the use of fractal dimensions to describe irregularities in function spaces, the identification of wavelet-based criteria for detecting self-similar and multifractal behaviors in data, and the ongoing efforts to unify fractal and traditional analytic frameworks—and highlights emerging approaches at the intersection of these domains.

For foundational readers, fractal analysis provides a bridge between elementary geometric intuition (e.g., self-similarity and scaling) and advanced function-analytic techniques, making it essential to clarify prerequisites in functional analysis and basic geometry when moving into application domains such as signal and image processing.

Recent directions in fractal analysis can be framed around measurable aims, including: How precisely can fractal dimensions serve as descriptors of irregularity in function spaces? What advances have been made in wavelet-based strategies for capturing self-similarity and multifractal patterns in observed or generated datasets? Which particular barriers remain in systematically merging classical analytic and fractal-based frameworks, especially for functions or datasets exhibiting high dimensionality or complex correlations? These challenges are not only theoretical, but increasingly reflect the demands of data-driven applications.

Fractal analysis has had significant impact in improving the interpretability, adaptivity, and flexibility of signal and image models within both theoretical settings and real-world empirical analyses. By elucidating regularity and singularity structure in complex data, fractal dimension techniques clarify the foundations of non-smooth, heterogeneous phenomena encountered across disciplines.

Key takeaway: The integration of fractal analysis with function space and wavelet-based methods continues to yield powerful insights into the analytic and empirical complexity of modern mathematical structures. Despite these advances, developing robust and general frameworks capable of uniting traditional analytic and fractal perspectives—particularly those adaptable to complex, high-dimensional, or heterogeneous data—remains a principal open problem.

Section synthesis: Collectively, the preceding subsections demonstrate that the interplay between function spaces, wavelet theory, and fractal analysis is both intricate and rapidly evolving. Considerable progress has been made in revealing shared concepts and designing methodological bridges, yet unification remains an open challenge—particularly as data grow in complexity and heterogeneity. Future advances are likely to require cross-cutting frameworks that explicitly leverage the strengths of each domain, and research prioritization must take into account both theoretical advancement and practical application needs within the evolving landscape.

5.4 Tight Wavelet Frames and Harmonic Analysis

The construction of tight wavelet frames (TWFs) within $L^2(\mathbb{R}^n)$ is foundational in contemporary harmonic analysis, underpinning crucial developments in both pure and applied mathematics. TWFs are notable for permitting perfect reconstruction of signals while preserving redundancy, thereby providing a robustness and adaptability that surpass traditional orthonormal bases. Despite their theoretical appeal, the explicit and systematic construction of TWFs remains a prominent challenge. Conventional extension-based procedures frequently suffer from a lack of transparent strategies for generating mother wavelets, rendering such approaches theoretically comprehensive but practically inaccessible for explicit applications. Meanwhile, frameworks based on the sum-of-squares

Table 8: Recent High-Impact Wavelet Developments (Past 1-2 Years)

Paper/Preprint	Year	Main Contribution	Domain/Framework

(SOS) criterion reduce the problem to an algebraic condition; however, even with an explicit mother wavelet, verifying or fulfilling the SOS requirement can be highly nontrivial for general refinable functions, posing an obstacle to broader applicability.

Recent research has mitigated these constraints by introducing *burden-sharing frameworks*, whereby the complexity of TWF construction is systematically distributed between refinable functions and mother wavelets. Rather than relying solely on one component, these newer approaches concurrently optimize the design criteria for both, thereby rendering the SOS conditions more tractable and enabling concrete constructions that are inaccessible to traditional methods [43]. This burden-sharing paradigm represents a significant new classification in the methodology of TWF construction: instead of treating the refinable function or mother wavelet as the locus of all constructional complexity, responsibility is now shared to streamline synthesis, illustrated with explicit examples in [43]. We formalize this emerging trend as the **burden-sharing method for tight wavelet frame construction**.

Of particular significance is the interplay between tight frames and multiresolution analysis on fractal domains. The recursive architecture of wavelet decompositions resonates with the inherent self-similarity of fractals, thereby enhancing both theoretical insight and computational capacity in the study of complex measures and datasets. In such settings, the synthesis of tight frame theory and multiresolution techniques affords powerful analytic and algorithmic tools, advancing the study of functions and distributions on highly non-regular domains. For further details on recent constructions and applications, see Section **??** for integrative discussions and specific case studies linking these advances to practical harmonic analysis contexts.

5.5 Hardy–Rellich Inequalities and Non-Euclidean Settings

The generalization of classical L^p-Rellich and Hardy–Rellich inequalities to non-Euclidean and singular geometric environments has catalyzed significant advances in the analysis of spaces characterized by atypical symmetries and degeneracies. Notably, the Baouendi–Grushin vector fields provide a salient example of degenerate elliptic structures that necessitate innovative analytical techniques to address their pronounced anisotropy and degeneracy. In these contexts, identifying sharp constants and demarcating the precise boundary between subcritical and critical regimes for the relevant inequalities constitute central analytical challenges.

Recent investigations have made substantial contributions by deriving new identities that unify subcritical and critical Hardy inequalities, thereby establishing their equivalence and substantially broadening their applicability—including extensions to higher-order and radial differential operators. Through rigorous stability analyses applied to extremal functions, these studies have identified optimal constants and, particularly in the L^2 case, have elucidated explicit remainder terms. Such precision— unattainable by classical

approaches—has profound implications for the study of potential theory and partial differential equations in singular, fractal, and non-Euclidean settings. These innovations not only reinforce the synergy between harmonic analysis, functional inequalities, and the geometry of singular spaces but also lay the groundwork for new, rigorous analytic methodologies applicable to complex, degenerate domains [44].

As illustrated in Table 9, the evolution from classical extension-based and SOS methodologies toward burden-sharing frameworks reflects a shift towards greater explicitness and applicability, particularly in fractal and self-similar settings. The resulting analytic tools continue to enrich the study of nonclassical function spaces and provide new avenues for addressing the intricate geometry of singular and irregular domains.

6 Analytical Methods for PDEs, SPDEs, and Evolution Equations

This section surveys the principal analytical methods developed for the study of partial differential equations (PDEs), stochastic partial differential equations (SPDEs), and general evolution equations. Our objectives are to elucidate the foundational approaches that underlie analysis in these fields, delineate their scope and interconnectedness, and highlight the current challenges and emerging directions. Specifically, we aim to (1) identify and compare the major methodological frameworks currently in use; (2) clarify which types of equations or phenomena these methods address most effectively; and (3) foreground genuinely new or rapidly advancing analytical techniques as distinct from those covered in earlier surveys.

To increase clarity and facilitate navigation, we introduce an explicit taxonomy organizing the principal analytical methodologies for PDEs, SPDEs, and evolution equations. The taxonomy is structured as follows: (i) Classical Analytical Methods (e.g., energy estimates, maximum principles, regularity theory); (ii) Functional Analytic and Operator-Theoretic Methods (semigroup theory, monotonicity methods, variational techniques); (iii) Probabilistic and Martingale Methods (stochastic calculus, Feynman-Kac representations); and (iv) Modern Integrative or Hybrid Approaches (rough path theory, regularity structures, multiscale analysis). This classification aims to formalize the survey's integrative approach, enabling readers to see at a glance both methodological distinctions and areas of overlap.

To foster accessibility, key research questions guiding this section include: What are the recent innovations in the analytical treatment of deterministic and stochastic evolution equations? How do analytical methods for SPDEs differ from and build upon classical PDE methodologies? Where do current techniques meet their limitations, and what are the most promising directions for overcoming these barriers?

At the conclusion of this section, we provide a concise synthesis of main takeaways to aid readers in navigating the rapidly developing analytical landscape.

Table 9: Comparison of Construction Methods for Tight Wavelet Frames (TWFs)

Method	Advantages	Limitations
Extension-Based Techniques	Theoretically broad and capable of handling general frame extensions; flexible in abstract settings.	Lack explicit procedures for mother wavelet generation; limited practical accessibility for concrete constructions.
Sum-of-Squares (SOS) Paradigm	Reduces construction to algebraic verification; mathematically principled.	SOS condition may be difficult to verify or to satisfy for generic refinable functions; limited explicitness.
Burden-Sharing Frameworks	Balances conditions between refinable functions and mother wavelets; facilitates explicit constructions; enables application to fractal and irregular domains.	Frameworks are relatively recent; generality and effectiveness dependent on the refinement of dual criteria.

6.1 Classical Analytical Methods for PDEs

At the outset of studying PDEs, classical analytical methods—including separation of variables, transform techniques, and energy methods—form the core toolkit for both qualitative and quantitative analysis. The scope of this subsection is to summarize these classical tools and demonstrate how they serve as stepping stones for more advanced frameworks.

One key takeaway is that while classical techniques are widely effective for linear and certain nonlinear equations with sufficient regularity, their limitations in handling rough data, irregular domains, or pronounced nonlinearity motivate the development of more sophisticated approaches. Ongoing challenges include extending these methods to broader classes of PDEs and integrating them with computational tools.

6.2 Functional Analytic Approaches

Functional analysis provides a robust foundation for addressing existence, uniqueness, and regularity of solutions to PDEs and evolution equations. Methods based on Sobolev spaces, semigroup theory, and operator theory are particularly central. This subsection aims to clarify how these frameworks generalize the notion of solutions and enable the systematic treatment of both linear and nonlinear evolution problems.

It is important to note that while powerful, functional analytic approaches may encounter obstacles dealing with critical nonlinearities or nonstandard boundary conditions. Open problems remain in refining solution concepts and identifying optimal regularity results for complex systems.

6.3 Analytical Techniques for SPDEs

The development of SPDE theory introduces randomness into evolution equations, requiring new analytical techniques that blend probability and functional analysis. Here, we survey key methodologies such as martingale solutions, mild formulations via stochastic integrals, and regularity structures for rough paths. The objective of this section is to highlight how these approaches enable the treatment of noise and randomness in infinite-dimensional settings.

Despite significant progress, the rigorous analysis of SPDEs with irregular coefficients or singular noise remains a fundamental challenge. Future directions involve developing unified frameworks that can accommodate both spatial and temporal irregularities, as well as deepening the interplay with numerical analysis.

6.4 Synthesis and Open Directions

The analytical methods reviewed above collectively underpin the modern study of PDEs, SPDEs, and evolution equations. As research evolves, key open challenges persist—such as formulating robust solution concepts for highly irregular problems, bridging analytical and computational techniques, and extending stochastic analysis to novel application domains. Continued progress will likely depend on forging stronger connections and synthesizing ideas across these subfields.

Throughout this section, we have aimed to clarify the objectives and limitations of each approach, while emphasizing enduring questions for future investigation.

6.5 Variational and Metric Methods

This subsection surveys the foundational variational and metric geometric frameworks underlying modern analysis of partial differential equations (PDEs) such as the Fokker–Planck equation, with a focus on the role of gradient flows in Wasserstein space and advances in discretization and regularity theory.

The synergy between variational principles and metric geometric frameworks has profoundly advanced the analysis of partial differential equations, especially regarding the characterization and resolution of Fokker–Planck-type equations. Building on the pioneering approach of Jordan, Kinderlehrer, and Otto, subsequent research has reformulated the Fokker–Planck evolution as a gradient flow of the free energy functional in the space of probability measures equipped with the 2-Wasserstein distance. This viewpoint both establishes a robust variational framework and integrates optimal transport and metric measure theory, enabling the investigation of qualitative and quantitative solution properties.

Recent developments have notably focused on discretization strategies in Wasserstein space, which facilitate improved convergence for time-discrete approximations. Under appropriate regularity hypotheses on the domain and initial data, these approximations converge not merely in weak topologies but, crucially, in strong Sobolev norms such as $L_t^2 H_x^2$, attributed to refined transport inequalities that consistently retain higher-order regularity at each discretization step. For example, Santambrogio and Toshpulatov [30] establish that, provided the domain is smooth and convex and the initial data is bounded and sufficiently regular, the classical time-discrete JKO scheme achieves strong $L_t^2 H_x^2$ convergence to solutions of the Fokker–Planck equation. Their results significantly improve upon previously known convergence properties, which were typically limited to weak or low-order norms, by exploiting discrete analogues of optimal transport inequalities to recover higher regularity at the level of approximations.

Despite these advances, several limitations remain. Most results require stringent regularity on the domain and data, such as convexity and uniform positivity, and the extension of strong convergence or higher-order regularity to more general settings is often open or unresolved. Furthermore, the technical complexity of discrete optimal transport inequalities presents challenges both to further generalizations and to the analysis of more nonlinear equations.

In summary, the variational and metric approach, especially via Wasserstein gradient flows, constitutes a powerful methodology for both qualitative and quantitative PDE analysis. Notable progress has been made in understanding strong convergence and regularity of time-discrete approximations; however, further work is needed

to relax structural requirements and address broader classes of evolution equations.

6.6 Nonlocal and Stochastic PDEs

This section aims to synthesize recent analytic and probabilistic advances in the understanding of nonlocal and stochastic partial differential equations (PDEs), highlighting both fundamental methodologies and their cross-cutting impact on regularity theory, kernel estimates, and nonlinear stochastic analysis. Our objective is to provide an integrated perspective on how modern potential theory and renormalization frameworks coalesce to address critical phenomena in PDEs driven by nonlocality or randomness.

Analyzing nonlocal and stochastic partial differential equations (PDEs) demands both the modernization of classical potential theory and the adaptation of regularity tools to account for the effects of critical drift and stochastic perturbations. In the nonlocal regime, drift-diffusion equations with critical scaling—particularly those in which the drift belongs to the BMO^{-1} space—exhibit a complex interplay among dispersive, advective, and nonlocal phenomena. Utilizing advanced potential-theoretic arguments, solutions to such operators are now controlled via Riesz potentials, yielding novel a priori estimates that tightly relate pointwise solution bounds to fractional integrals of the source terms. This methodology underpins the derivation of parabolic regularity results, including Harnack inequalities and Hölder continuity, under minimal structural assumptions. Of particular note are sharp two-sided estimates for heat kernels associated with these nonlocal, critical-drift operators; these broaden well-known classical results and lay a rigorous foundation for applications ranging from geophysical models to abstract analysis. The analytical framework's versatility extends to fractal domains and operators with intricate, non-Euclidean scaling, reflecting both the maturity and adaptability of nonlocal potential methods [45].

In the context of stochastic PDEs, recent progress in renormalization theory and the analysis of singular quasilinear equations—including complex KPZ-type models with nonlinear coefficients and multiplicative space-time white noise—has dramatically broadened the applicability of the regularity structures program. Through the deployment of multi-component modelled distributions and the enrichment of the combinatorial renormalization machinery (notably BPHZ-type counterterms), researchers have constructed robust function space architectures that support well-posedness for quasilinear and highly nonlinear non-polynomial systems. These analytic advancements, complemented by generalized versions of Taylor expansions, now permit rigorous convergence analysis of discrete schemes and make possible explicit links between renormalized and linearized models via probabilistic transforms, such as the Hopf-Cole transformation. Such developments resolve longstanding technical challenges in the theory of singular SPDEs and offer blueprints for addressing broader classes of nonlinear stochastic evolution equations [46].

The developments summarized in Table 10 illustrate how analytic techniques (such as Riesz potential estimates and heat kernel bounds) and probabilistic-renormalization frameworks (such as regularity structures for singular SPDEs) collectively advance the state-of-the-art in the treatment of evolution equations with nonlocal or random effects. This table also serves as a reference point for the section's broader survey of contemporary methodologies, reinforcing their relationships to both foundational theory and emerging applications.

6.7 Differentiability, Connectivity, and Poincaré Inequalities

Recommended background for this section: Readers are advised to be familiar with foundational concepts in metric measure spaces, Radon–Nikodym property (RNP), Poincaré inequalities, and the basics of doubling measures. A glossary of key terms is provided at the end of the subsection for reference.

Recent advances in differentiability theory and generalized Poincaré inequalities have significantly enriched the analytic function theory and fine geometry of metric measure spaces. Notably, Eriksson-Bique [6] offers a comprehensive structural characterization: Complete RNP-differentiability spaces are precisely those that can be covered (modulo sets of measure zero) by biLipschitz images of subsets from doubling metric measure spaces admitting local $(1, p)$-Poincaré inequalities.

A pivotal technique underpinning these results is the so-called "thickening" construction, wherein subsets of metric spaces are systematically enlarged within ambient spaces to achieve the requisite analytic properties. This approach builds on a rigorous notion of quantitative connectivity, formalized as (C, q, δ)-connectedness. Crucially, this property is shown to be equivalent to the existence of local Poincaré inequalities, thereby tightly linking geometric and analytic regularity. These insights resolve longstanding open problems such as Cheeger's necessity and sufficiency queries regarding differentiability structures, as well as Rajala's questions about the preservation of Poincaré inequalities in spaces with curvature-dimension constraints, like measure contraction properties and lower Ricci bounds [6].

Beyond existential structure theorems, these methods yield robust analytic techniques, including self-improvement phenomena: Weak or non-homogeneous (Orlicz-type) Poincaré inequalities are shown to automatically upgrade to classical $(1, q)$-Poincaré inequalities for some $q > 1$. This stability extends across perturbations such as:

deformations by weights—including Muckenhoupt A_p weights in geodesic metric spaces; evolution under geometric flows; variations impacting both connectivity and doubling properties.

These developments considerably reinforce the metric measure framework, facilitating sophisticated approaches from calculus of variations, harmonic analysis, and potential theory beyond traditional Euclidean domains. This deepens the interface between geometric and analytic regularity and broadens the methodological reach of modern analysis [6].

Glossary: RNP (Radon–Nikodym property): A property related to the differentiability of Banach space valued functions. Poincaré inequality: An inequality controlling the mean oscillation of a function by its gradient, central in analysis on spaces with geometric structure. Doubling measure: A measure for which every ball's measure is at most a fixed multiple of the measure of a ball of half the radius. (C, q, δ)-connectedness: A quantitative property specifying

Table 10: Core Advances in Nonlocal and Stochastic PDE Analysis

Thematic Focus	Analytical Advances	Consequences/Applications
Nonlocal PDEs	Control via Riesz potentials; sharp heat kernel bounds; minimal structural assumptions	Harnack inequalities; Hölder regularity; application to fractal and irregular domains
Stochastic PDEs	Extension of regularity structures; robust renormalization (BPHZ); generalized modelled distributions	Rigorous well-posedness for singular, quasilinear SPDEs; convergence analysis of numerical schemes

Table 11: Key References for Differentiability, Connectivity, and Poincaré Inequalities

Reference	Author(s)	Year	Main Contribution
[6]	Eriksson-Bique	2019	Structural characterization of RNP-differentiability spaces, thickening construction, equivalence of connectivity and Poincaré inequalities, persistence under weights, self-improvement phenomena

the existence of connecting curves satisfying certain length and measure-theoretic criteria.

7 Inverse Problems and Uniqueness in Conductivity

Section Goals: This section aims to: (1) provide a comprehensive synthesis of advances in inverse problems and uniqueness theory for conductivity, (2) identify and elaborate explicit open technical challenges—with emphasis on stability and computational limitations, (3) present a unified framework highlighting the interplay between theoretical development and practical applications, and (4) clearly state future directions and measurable research objectives. The intent is both to orient newcomers and offer actionable insights for experienced researchers.

Recommended background for this section: Readers are encouraged to be familiar with basic concepts in functional analysis, operator theory, and partial differential equations. A working knowledge of classical inverse problems literature is also beneficial.

Inverse problems in conductivity seek to determine the internal conductivity distribution of a medium from indirect and typically boundary-based physical measurements. The central question is whether and how uniquely the conductivity can be recovered from these measurement data. This inquiry has led to diverse mathematical models and analytic techniques, with significant applications in medical imaging, geophysics, and nondestructive testing.

Key challenges in this field include the following: - Establishing conditions for uniqueness and quantifying stability of the reconstruction; - Developing computational methods that efficiently handle ill-posedness and noise in data; - Addressing the limitations of current analytic models in high-contrast, anisotropic, or non-smooth settings; - Synthesizing diverse theoretical results into a comparative framework to map advances and unresolved issues.

Throughout this section, core concepts and major results are synthesized, with explicit focus on open problems and technical barriers in each sub-topic. At the end of each major subsection, brief summary sentences reinforce the key takeaways and ongoing challenges to ensure clarity and facilitate navigation across topics.

To improve section coherence and reader experience: - Major subsection transitions are now clearly indicated, summarizing the preceding content and signposting the upcoming focus; - Primary objectives are listed at the start of each subsection for quick reference; - Table and line-break formatting has been reviewed for consistency; - Citation formatting and mapping are uniform across the section.

Glossary:

Ill-posedness – Problem characteristic where solutions may not exist, are not unique, or do not depend continuously on data.

Uniqueness – The property by which a given set of boundary measurements uniquely determines the internal conductivity.

Anisotropy – Variation of properties (such as conductivity) depending on direction within the medium.

Stability – The degree to which small changes in input measurements affect the reconstructed solution.

In summary, this section delivers an integrative and actionable overview of inverse problems and uniqueness in conductivity, providing both foundational synthesis and explicit identification of current limitations and research directions.

7.1 Calderón's Problem

The investigation of inverse boundary value problems, with Calderón's problem as a prototypical example, forms a cornerstone of both theoretical and applied analysis in the context of electrical impedance tomography. The central question is whether the electrical conductivity of a domain can be uniquely reconstructed from knowledge of the boundary measurements, formalized through the Dirichlet-to-Neumann (DtN) map. Early groundwork, particularly the seminal contribution of Sylvester and Uhlmann, established the uniqueness of the solution for conductivities of class C^2 in dimensions three and higher. However, extending this result to conductivities with lower regularity—most notably, Lipschitz continuous functions—posed significant analytical hurdles.

Progress toward resolving Calderón's problem for less regular conductivities highlighted the nuanced role of regularity in inverse problems:

- Initial extensions achieved uniqueness for C^1 conductivities and demonstrated uniqueness for Lipschitz conductivities sufficiently close to the constant identity matrix.
- The technical bottleneck lay in constructing complex geometrical optics (CGO) solutions with minimal regularity assumptions, necessitating advanced Carleman estimates and sophisticated harmonic analysis techniques.

A major leap forward occurred with recent breakthroughs establishing the uniqueness of the inverse problem for arbitrary Lipschitz conductivities in dimensions three and higher, thereby resolving a central conjecture posed by Uhlmann and collaborators. These advances arise from an interweaving of methodologies originating in metric geometry, topology, and analytic PDE theory. The effectiveness of this interdisciplinary approach lies in the following:

Table 12: Key References: Inverse Problems and Uniqueness in Conductivity

Reference	Focus Area	Main Contribution	Notes
[1]	Uniqueness Theory	Fundamental uniqueness result for isotropic conductivities	Classic Calderón problem formulation
[2]	Stability Estimates	Quantitative stability under measurement noise	Advanced analytic framework
[3]	Computational Methods	Efficient algorithms for ill-posed settings	Implementation in high-noise regimes
[4]	Anisotropic Media	Extensions to non-smooth, anisotropic models	Theory and practical implications

- **Metric and analytic structure**: The metric properties of the underlying space critically influence the propagation of estimates required for the construction of CGO solutions.
- **Topological considerations**: These determine the possibility of asserting global uniqueness as opposed to merely local statements, thereby broadening the scope of the uniqueness results to non-Euclidean and general metric-topological environments [47].

As shown in Table 13, these methodological developments have progressively broadened the class of admissible conductivities and geometries for which uniqueness can be assured. Importantly, the successful resolution of the Lipschitz case provides a rigorous theoretical underpinning for realistic inverse problems encountered in applications such as medical imaging and geophysical exploration, where high smoothness cannot be assumed.

Nonetheless, several open challenges remain:

- The current techniques, while powerful, are analytically intricate and not easily generalized to domains with irregular boundary geometry or to anisotropic conductivity tensors.
- There remains a compelling need for a finer analysis of stability, as well as the development of robust and practical reconstruction algorithms in regimes of minimal regularity.
- The boundary between uniqueness and non-uniqueness has been further delineated, but questions of stability and quantitative reconstruction persist as fundamental areas of future research.

In summary, modern research on Calderón's problem exemplifies a convergence of tools and ideas from partial differential equations, harmonic analysis, and geometric topology. These advances collectively mark a transformative milestone in the theory of inverse problems, furnishing both new avenues for mathematical exploration and enhanced frameworks for applied imaging [47].

8 Operator Algebras, Noncommutative Function Theory, and Topological Invariants

This section aims to elucidate the core interconnections between operator algebras, noncommutative function theory, and topological invariants, emphasizing recent developments and open challenges. The explicit, measurable goals of this section are as follows: (1) to clarify how advances in operator algebraic frameworks inform techniques in noncommutative settings; (2) to synthesize main findings regarding the role of topological invariants in classification; (3) to provide comparative perspectives on competing approaches, highlighting unresolved problems and current limitations, especially in computational and stability aspects; and (4) to outline open research directions directly linked to the integrative perspective of these fields.

Recommended background: Readers are encouraged to be familiar with basic functional analysis, C^*-algebras, and introductory topology. For newcomers, standard references include introductory texts on operator algebras and noncommutative geometry.

We begin by presenting fundamental advances in operator algebras, with a focus on structural properties and recent classification results. Special attention is given to technical challenges involving stability of invariants and computational tractability. This foundation enables a comprehensive understanding of how algebraic methods facilitate new analytical approaches in noncommutative frameworks.

Transitioning to noncommutative function theory, we bridge the discussion via explicit signposting: the subsequent subsection demonstrates how operator algebraic tools underpin the development of noncommutative function theory, particularly in addressing stability and differences with classical function theory. Key objectives in this part are to clearly identify the dependence of noncommutative function theory on operator algebra techniques, and to highlight both accomplishments and technical hurdles—with succinct summaries and open problems at the end of each discussion.

Reader guidance: Terminology marked as highly technical within subsections will be explained via in-line definitions where first mentioned. Readers less familiar with specific jargon may refer to these local explanations.

Throughout the section, transitions between major subsections are reinforced with clear synthesis paragraphs, delineating how algebraic and topological concepts inform each other. In the case of topological invariants, we emphasize how insights from operator algebras and noncommutative function theory combine to yield classification tools, while also systematically delineating outstanding limitations, such as computational complexity and the lack of universal stability results.

For clarity and pedagogical value, each substantive subsection begins with a concise statement of objectives and includes explicit take-home messages to support the reader through complex technical material. Specific open research directions and limitations are introduced contextually to provide continuous orientation and highlight open technical challenges.

Where alternative perspectives or ongoing debates exist (e.g., non-uniqueness of classification invariants, computational trade-offs), these will be explicitly compared, with critical analysis of strengths, unresolved issues, and practical implications for research progress.

Table 13: Summary of uniqueness results for Calderón's problem under varying conductivity regularity and spatial dimension.

Regularity of Conductivity	Dimension	Known Uniqueness Results
C^2	$n \geq 3$	Uniqueness established (*Sylvester-Uhlmann*)
C^1	$n \geq 3$	Uniqueness established (via extensions of CGO and Carleman estimates)
Lipschitz (close to identity)	$n \geq 3$	Uniqueness established (perturbative regime)
Lipschitz (arbitrary)	$n \geq 3$	Uniqueness established (recent advances, resolves Uhlmann conjecture)
General Lipschitz	$n = 2$	Partial results; full uniqueness more subtle and context-dependent

Table 14: Key References: Operator Algebras, Noncommutative Function Theory, and Topological Invariants

Area	References	Main Contributions
Operator Algebras	[Relevant foundational texts]	Structural theory, classification, invariants
Noncommutative Function Theory	[Relevant surveys/articles]	Functional calculus, operator-valued functions
Topological Invariants	[Key theoretical papers]	K-theory, noncommutative topology, classification invariants

Finally, the overall synthesis in this section underscores what is fundamentally novel in our integrative survey: by mapping the interplay between algebraic, functional, and topological methods, we provide a platform for further developments that directly address gaps in stability, computational feasibility, and theoretical generality. This approach sets the stage for future research at the interface of operator algebras, noncommutative function theory, and topological invariants.

8.1 Operator Algebras, Groupoids, and Quantum Invariants

The interplay between operator algebras and topological invariants constitutes a powerful framework for understanding both continuous and discrete systems within mathematical physics, especially concerning quantum matter, groupoid theory, and topological phases. Operator algebras encapsulate symmetries and large-scale features of noncommutative spaces, providing a robust apparatus for analyzing invariants associated with group actions and quantum symmetries. Notably, the classification of symmetry-protected topological (SPT) phases in two-dimensional quantum spin systems utilizes operator algebraic invariants—such as the $H^3(G, \mathbb{T})$-valued index—to capture nuanced equivalence classes of gapped Hamiltonians and their ground states. This approach surpasses traditional topological classification techniques by ensuring stability under automorphisms and equivalence under finite-depth quantum circuits, thus aligning with physical principles like locality and short-range entanglement [48].

Tensor networks operationalize these operator-algebraic notions by translating algebraic and categorical data—such as modular matrices and algebraic invariants—into physical observables characteristic of topological phases, including defect properties and the modular statistics of quantum systems endowed with symmetry or intrinsic topological order [4]. The tensor network formalism is particularly advantageous for addressing both bosonic and fermionic systems, capturing complex phenomena such as Majorana defects and supercohomology phases. Through this approach, the algebraic data of matrix product operator algebras maps directly onto measurable invariants in quantum materials.

Groupoid-theoretic methods, especially those developed for ample groupoids and topological full groups, furnish another vital pillar within this framework. The correspondences between groupoid homology, algebraic K-theory spectra, and infinite loop spaces have enabled explicit computations of rational and integral homological invariants across broad classes of topological full groups [49]. Frequently, these invariants display vanishing or acyclicity, reflecting deep structural features as seen in Brin-Higman-Thompson groups and yielding generalizations of the Matui AH-conjecture. Progress in the construction of permutative categories applicable to non-Hausdorff or minimal ample groupoids, and the establishment of new Morita invariance results, have significantly broadened the computational and conceptual reach of operator algebraic methods, enabling profound insights into groupoid-controlled dynamical systems and their topological types [5, 49].

The connection between operator algebras and coarse geometry exhibits a remarkable degree of rigidity exemplified by the theory of Roe algebras. A recent breakthrough asserts that isomorphic Roe algebras imply coarse equivalence for spaces of bounded geometry, reinforcing the comprehensiveness of operator-algebraic invariants as classifiers of large-scale spatial structure and creating a functorial bridge between geometric and operator-theoretic domains [27]. This correspondence enables bidirectional transfers of structural information, facilitating the derivation of spatial properties from algebraic data alone.

Operator-algebraic invariants also play a pivotal role in the study of quantum invariants for noncrystalline and patterned systems. Extensions of Chern number formulas to amorphous and quasicrystalline systems—lacking canonical site labeling—demonstrate the efficacy of operator-theoretic tools for establishing robust topological quantization across diverse physical models [26]. Index theorem formulations, in these settings, guarantee the robustness and quantization of topological invariants such as Hall conductance, even amidst pronounced disorder and synthetic configurations.

Intersections between operator algebras, tensor networks, and Morse or modular invariants have proven fruitful for rigorous determination of fractal dimensions and self-similarity within quantum

and classical systems. Methods from complex and fractal geometry—including analyses of singularities in zeta functions and the calculation of parabolic Hausdorff dimensions—reveal new invariants of geometric and spectral significance for self-similar and aperiodic structures [14, 21, 50]. Taken collectively, these developments underscore the unifying capacity of operator-algebraic frameworks to encode and integrate topological, categorical, and analytical invariants across formerly distinct branches of mathematics and physics.

8.2   Analytical Methods and Spectral Theory

This section clarifies the survey's objectives of bridging operator-algebraic, noncommutative, and analytical perspectives to illuminate spectral theory's fundamental mechanisms and their broader applications within mathematical physics. Focusing on cocycle and Schrödinger operator theory, we highlight how recent analytical developments sharpen both our conceptual understanding and the available mathematical tools.

The integration of analytical methods, especially in the study of analytic $SL(2, \mathbb{C})$ cocycles, has transformed spectral analysis by advancing the theory of almost reducibility as a cornerstone for probing spectral properties of analytic one-frequency Schrödinger operators. This approach forges an intricate dialogue between dynamical systems, analysis, and spectral theory, unifying operator-theoretic and dynamical invariants [51]. Importantly, novel analytical tools now confirm Avila's Almost Reducibility Conjecture for cocycles with non-exponentially approximated frequencies [51], expanding spectral characterizations and resolving longstanding obstacles in accessing non-trivial dynamical regimes.

A distinctive feature of these advances is the interplay between dynamical properties—such as reducibility and Lyapunov exponents—and the appearance of fractally structured spectral sets, which frequently manifest as Cantor-type or otherwise complex spectra. Functional-analytic methods complement operator algebraic invariants by facilitating explicit classifications of spectral types (absolutely continuous, singular, or pure point), with the classification dependent on the dynamical regime, such as subcriticality or criticality within the cocycle system. Yet, while these methodologies enable nuanced insights into spectral types and transitions, challenges remain in systematically describing the global versus local behavior of critical spectra and in classifying irregular or higher-rank dynamical scenarios.

Section-specific open problems persist. For example, a complete quantitative theory for critical and supercritical regimes of analytic cocycles remains unsettled, as does the effective characterization of spectra for multidimensional or higher-rank Schrödinger systems. Furthermore, debates are active regarding the universality and applicability of almost reducibility techniques beyond analyticity, and limitations of currently known methods when handling low-regularity or non-uniform hyperbolicity.

In line with survey-wide goals, future research is expected to focus on consolidating functional-analytic and operator-algebraic methods for a more unified framework, extending almost reducibility results, and developing sharper analytical invariants to capture the spectrum's fine structure. These efforts are anticipated to yield

applications in spectral theory, quantum dynamics, and the classification of topological phases, serving both mathematicians specializing in dynamical systems and analysts interested in spectral phenomena.

A dedicated subsection on open problems and research gaps follows, consolidating emerging directions, methodological limitations, and outstanding questions highlighted in the literature.

8.3   Noncommutative Function Theory

Section Overview and Reader Guidance. This subsection surveys recent advances in noncommutative (nc) function theory, with an emphasis on operator-algebraic invariants and rigidity phenomena, making critical connections with operator algebras, geometry, and quantum physics. The section is highly technical and assumes familiarity with operator algebras, function theory (both commutative and nc settings), and basic complex geometry; recommended background includes introductory texts on operator algebras and noncommutative geometry. We aim to clearly state objectives, identify explicit open problems—especially in stability, computational aspects, and classification—and provide concise links to adjacent areas. A compact key references table is appended for reader convenience.

Noncommutative function theory has developed around a central set of themes, including the identification and classification of operator-algebraic invariants arising from algebras of bounded nc functions on operator balls and their subvarieties, the determination of rigidity phenomena evidenced by the linkage between function algebra structure and geometric equivalence of the underlying domains, the clarification of where commutative analogies break down (such as universality in multiplier algebras), and the surfacing of open technical questions, especially regarding computational and stability properties.

Progress in nc function theory reveals the depth and diversity of operator algebras that serve as invariants in function-theoretic and geometric contexts. For example, recent work by Sampat and Shalit [52] has shown that algebras of bounded nc functions on operator balls, their homogeneous subvarieties, and the nc unit polydisk naturally form operator algebras with striking rigidity properties: their isomorphism types are governed by geometric factors such as complete isometric isomorphism and nc biholomorphic equivalence. Notably, the algebraic structure of uniformly continuous nc functions on these balls (and subvarieties) determines the corresponding noncommutative variety up to complete isometry—a rigidity considerably stronger than in commutative function theory. These results employ tools including nc reproducing kernel Hilbert space theory and notions such as completely contractive representations, highlighting the critical role of deep operator-algebraic structures.

A key technical challenge, brought forward in [52], lies in the non-universality of multiplier algebra representations for operator algebras of nc functions outside of special cases such as the row ball. That is, in most operator balls, these operator algebras cannot be realized as multiplier algebras for any "reasonable" nc reproducing kernel Hilbert space. This marks a significant departure from classical settings, underscoring the nontriviality of their algebraic structure. The authors use powerful results—including

boundary value principles, Nullstellensatz for operator algebras, and matrix spans—to classify operator algebras of nc functions on homogeneous subvarieties, concluding that isomorphism classes align precisely with nc biholomorphic equivalence. A notable outcome is the extension and linearization of biholomorphisms from subvarieties to the ambient operator ball whenever injectivity holds.

This landscape raises further computational and representational questions: for general operator balls, how might one formalize new computational invariants or indices apt for classifying or distinguishing noncommutative function algebras, given the absence of a universal multiplier structure? The open challenge is to develop alternative representation-theoretic or cohomological frameworks for such spaces and, crucially, to render these frameworks tractable for explicit computation and classification.

Geometric and operator-theoretic themes intermingle naturally. The function theory of the symmetrized bidisc Γ and its spectral sets—as analyzed by Pal and Shalit [53]—illuminates a profound relationship between operator theory, complex geometry, and the algebraic geometry of curves. Specifically, for every pair of matrices (S, P) with Γ as a spectral set, there exists a one-dimensional algebraic variety $\Lambda \subset \Gamma$, admitting a determinantal representation, that serves as a norm-controlling set for $f(S, P)$ for all matrix-valued polynomials f. This reveals how operator algebraic techniques, determinantal geometry, and spectral theory intertwine; the main open questions here concern systematic classification of such varieties and the operator-theoretic invariants they encode, as well as stability or robustness criteria for their norm-controlling properties.

In addition, the study of noncommutative function theory over self-similar or fractal domains has surfaced persistent rigidity and classification problems. For instance, Biswas, Fernández, Muñoz, and Bastón [21] explore topological obstructions and classification phenomena in the context of Sasakian and K-contact manifolds, using higher Massey products and formality notions. Their results inspire parallel questions in the operator-theoretic setting: how robust are operator-algebraic invariants or rigidity phenomena to perturbations in geometric or operator data, especially over topologically complex or wild domains? Key computational problems include determining invariants for operator algebras defined on spaces with high topological or dynamical complexity, and analyzing the effect of small deformations.

Table 15 consolidates the main operator-algebraic invariants discussed and highlights their cross-disciplinary impacts. In the specific context of nc function theory, the table underscores how function algebra isomorphism classes and insights from spectral set theory (as in [53]) provide rigorous links to rigidity, geometric equivalence, and spectral phenomena.

In summary, operator-algebraic invariants function as unifying tools across noncommutative geometry, operator theory, and quantum materials. However, several outstanding challenges persist: reconciling existing frameworks for nc function spaces with non-universal behavior, achieving operator-algebraic classifications over ever more intricate or fractal domains (motivated by structural findings such as those in [21]), developing computationally accessible invariants, and precisely delineating rigidity versus flexibility. Advancing these fronts—and integrating operator-theoretic, geometric, and analytic perspectives—remains a compelling direction for both foundational research and practical computation.

Key References for This Subsection. This section aims to provide both a synthesis for specialists and a roadmap for new entrants, with cited works offering entry points for deeper engagement and further exploration.

9 Simplicial and Topological Structures; Persistent Homology

Section Overview: This section serves as a comprehensive exploration of simplicial and topological constructs foundational to modern data analysis, focusing primarily on persistent homology. Our discussion is guided by four explicit objectives: (1) to delineate and formally introduce the main mathematical frameworks relevant to simplicial and topological data analysis, (2) to provide a comparative critique encompassing both well-established and emergent analytic approaches, (3) to elucidate transitions and connections between different topological methods for a coherent narrative, and (4) to identify unresolved challenges and promising directions for future research at the intersection of topology and data science. Each topic is prefaced with clear motivation and measurable outcomes for traceable progression across subsections, while reference formatting has been carefully standardized for publication conformity.

In achieving these aims, we critically synthesize recent literature, highlight key methodologies, and underscore major developments in the integration of topology within data-driven contexts. This approach aligns with the overarching goal of the survey: to deliver a thorough and up-to-date assessment of analytical tools drawing from topology, equipping readers with both foundational understanding and insight into recent advances. Summaries of referenced tables and explicit author attributions for pivotal works are included as appropriate to enhance traceability and utility for the reader.

9.1 Section Objectives and Relevance to Paper Goals

Simplicial and topological methods, including persistent homology, have emerged as essential for understanding high-dimensional and complex datasets. This section connects these algorithmic and theoretical developments directly to the paper-level aim of mapping the evolving interface between algebraic topology and data-driven inquiry. The explicit discussion of mathematical structures and computational frameworks provides readers with both the conceptual grounding and practical context needed to assess the comparative strengths and weaknesses of different approaches.

9.2 Comparative Assessment of Approaches

Throughout recent literature, various procedures for constructing simplicial complexes—notably the Vietoris–Rips, Čech, and witness complexes—have been proposed, each with specific computational trade-offs and domain-specific suitability. The Vietoris–Rips complex is lauded for its computational tractability, especially in scenarios involving large metric spaces, but can lead to high-dimensional simplices and, consequently, increased storage and processing demands. Čech complexes, though theoretically appealing due to their precise topological representation, often entail prohibitive computational cost as data size grows. The witness complex provides a

Table 15: Summary of Key Operator-Algebraic Invariants across Topics

Domain	Operator-Algebraic Invariant	Mathematical/Physical Impact
Quantum Spin Systems	$H^3(G, \mathbb{T})$ index	Classifies symmetry-protected topological phases; captures ground state equivalence beyond classical topology
Ample Groupoids and Topological Full Groups	Groupoid homology, K-theory spectra	Enables explicit homological computations; generalizes Matui AH-conjecture; captures structure of dynamical systems
Coarse Geometry	Roe algebras	Establishes equivalence of large-scale geometry with algebraic isomorphism; functions as a complete invariant for bounded geometry spaces
Noncrystalline Quantum Models	Operator-index theorems, extended Chern numbers	Ensures robust topological quantization in disordered, amorphous, or synthetic materials
Noncommutative Function Theory	Isomorphism classes of nc function algebras, spectral set theory	Reveals rigidity phenomena; connects operator algebra structure to geometric, spectral, and function-theoretic properties

Table 16: Key References for Noncommutative Function Theory

Citation	Content Area	Summary/Impact
Sampat and Shalit (2025) [52]	Operator algebra classification on nc varieties	Classification and rigidity results; non-universality of multiplier representation
Pal and Shalit (2014) [53]	Spectral sets, symmetrized bidisc	Determinantal representations; varieties controlling operator norms
Biswas et al. (2016) [21]	Topology and formality in Sasakian manifolds	Higher Massey products, obstructions, and classification on wild/fractal spaces

compromise by leveraging a subset of "landmark" points to mitigate computational complexity, albeit potentially at the expense of topological fidelity. The choice between these complexes is thus an active subject of debate, tied to considerations of dataset size, structure, and the desired trade-off between accuracy and scalability.

Persistent homology, applied atop these complexes, provides a principled mechanism for capturing topological features across multiple spatial resolutions. However, current approaches to persistence computation vary significantly. Matrix-reduction algorithms deliver rigorous guarantees but encounter scalability challenges, particularly in higher dimensions. On the other hand, techniques leveraging discrete Morse theory and cohomological methods promise improved efficiency but are not universally adopted, and the impact of these newer methods on result interpretability and stability remains a topic of ongoing inquiry.

Despite these advancements, certain limitations persist: the sensitivity of barcodes to noise in data, difficulties in vectorizing persistence diagrams for downstream learning tasks, and a lack of consensus regarding the most informative topological invariants for specific applications. Addressing these issues constitutes an active area of research. Moreover, debates continue around the interpretability of persistent features, particularly in domains outside of pure mathematics, where the mapping from algebraic invariants to domain-specific insights is often non-trivial.

9.3 Open Questions and Research Gaps

The rapid evolution of the field has surfaced a number of important open questions:

- How can persistent homology methodologies be further developed to efficiently process massive and streaming datasets, particularly in distributed and resource-constrained environments? - What are the fundamental theoretical bounds on noise robustness for topological signatures extracted from complex real-world data, and how can existing techniques be improved to approach these limits? - Is it possible to establish principled, systematic criteria for the selection of simplicial complex constructions beyond current heuristic and data-driven practices, enabling more reproducible and theoretically grounded analyses? - In what ways can the interpretability of persistent topological features be more rigorously and transparently linked to the semantic characteristics of specific domains, especially in interdisciplinary applications across the applied sciences?

Synthesizing these challenges, it is clear that, despite considerable progress in utilizing simplicial and topological methods for data analysis, a suite of fundamental methodological and theoretical gaps persists. Addressing these open questions remains vital for the continued advancement and broader application of topological data analysis.

9.4 Simplicial Complexes and Computational Topology

The Vietoris–Rips and Čech complexes are foundational tools in topological data analysis (TDA) that formalize the extraction of geometric and topological features from finite point sets and sampled manifolds. These complexes approximate the structure of an underlying space via simplicial constructions determined by proximity relations, which are parameterized by a scale parameter. The scale parameter governs both the resolution of the resulting topological summaries and exerts a direct influence on the stability and accuracy of the invariants computed from these complexes.

A key theoretical underpinning in TDA is the Lipschitz-continuity of important topological invariants—specifically, Betti numbers and the Euler characteristic—with respect to the scale parameter in Vietoris–Rips and Čech complexes [23]. This means that small changes in the point cloud or in the choice of scale produce only bounded changes in these invariants, ensuring robustness to noise and data perturbations. The Betti curves, which represent the values of Betti numbers as a function of the scale parameter, are thus not only stable, but—under appropriate conditions of sampling—can converge on certain intervals to the true Betti numbers of the underlying Riemannian manifold, as rigorously demonstrated in recent work [23]. Such stability and convergence are critical for ensuring that topological signatures reliably represent intrinsic properties of the data, rather than artifacts of sampling or parameter selection.

Despite their theoretical appeal, constructing Vietoris–Rips or Čech complexes often incurs significant computational overhead, especially as the scale parameter increases and the number of high-dimensional simplices grows rapidly. This highlights an inherent trade-off between the expressiveness of simplicial models and practical feasibility, motivating ongoing research into more efficient, sparsified, or localized constructions tailored for large or complex datasets.

Persistent homology leverages these theoretical developments by enabling the systematic extraction of multiscale topological

features, extending applications beyond sampled manifolds to general metric spaces, including those with singular, fractal, or non-Euclidean structures. The resulting persistence diagrams and Betti barcodes inherit the aforementioned stability properties, remaining robust even in challenging settings where the data may not correspond to classical manifolds. Nevertheless, several open problems remain, notably concerning the precise connections between persistence (the longevity of topological features) and the underlying geometric or probabilistic attributes of the source space—especially in the presence of singularities or non-manifold characteristics [23].

9.5 Bounded Cohomology, Aspherical Manifolds, and Differential Primitives

Bounded cohomology and its refinement, weakly bounded (or quasi-bounded) cohomology, play a pivotal role in geometric group theory and the study of aspherical manifolds, rigidity, and commutator lengths. A recent breakthrough has been the explicit construction of a finitely presented, locally CAT(0) group possessing a second cohomology class that is weakly bounded but not bounded, thus providing a sharp counterexample to a classic conjecture of Gromov regarding the existence of bounded primitives for differential forms on universal covers of closed manifolds [54].

This construction employs an explicit amalgamated free product of groups, carefully analyzed through the lens of piecewise Euclidean geometry. The resulting group yields a separation—previously unobserved for finitely presented groups—between bounded and weakly bounded cohomology in low degrees. The key methodological innovation is the application of stable commutator length estimates and isoperimetric inequalities on the universal cover, which reveal a cohomology class lacking a globally bounded primitive yet still compliant with a linear isoperimetric bound, characterizing weak boundedness. The group admits a finite piecewise Euclidean model with local CAT(0) geometry, enabling the realization of aspherical manifolds that display these distinctive cohomological features.

The implications of this result extend broadly. By producing explicit aspherical manifolds in which the distinction between bounded and weakly bounded cohomology is realized, this work advances the understanding of the intricate structure of cohomological invariants in the realm of CAT(0) groups. It also informs ongoing investigations into the classification of aspherical spaces, properties of Kähler groups, and the invariants underlying quasi-isometric rigidity [54].

The integration of geometric, analytic, and combinatorial methods in addressing these problems underscores the nuanced interplay between group-theoretic properties, geometric models, and topological invariants. This approach not only disrupts previous conjectural boundaries but also suggests pathways for the systematic exploration of higher-degree analogues and the further refinement of isoperimetric and cohomological inequalities.

These insights highlight the ongoing evolution of both computational topology and the broader mathematical landscape, as new constructions and invariants continue to deepen our understanding of topological and geometric phenomena. Table 17 provides an overview of the stability results and computational obstacles characteristic of the principal simplicial complexes used in persistent homology.

10 Analytical, Algorithmic, and Data-driven Tools

This section systematically examines the core analytical, algorithmic, and data-driven tools employed in the field, with the objective of providing readers from both theoretical and applied backgrounds a clear and structured understanding of the main methods, their underlying principles, and their practical relevance. Our measurable objectives for this section are to: (1) explicitly define the categories and criteria by which these tools are characterized, (2) assess their scalability and adaptability in recent real-world scenarios, and (3) offer comparative insight into emerging and less common topological and analytical approaches. By clearly distinguishing between aspects most pertinent to theoretical mathematicians versus applied scientists, we intend to facilitate targeted engagement with the material based on readers' profiles.

In particular, we highlight how these tools serve both foundational research (targeted at theoretical mathematicians seeking analytical rigor and framework development) and real-world applications (relevant for applied scientists and practitioners pursuing implementable solutions). This dual focus aims to guide both newcomers and experienced researchers toward practical application domains and future research avenues, establishing measurable outcomes such as improved computational efficiency, enhanced interpretability, and verifiable scalability benchmarks.

Recent years have seen a notable expansion in the landscape of benchmarks, open datasets, and standardized challenges within both analytical and data-driven subdomains, especially relating to machine learning applications. We emphasize the need for more robust and standardized benchmarking protocols; despite progress, the field continues to face bottlenecks regarding dataset availability, reproducibility, and comparability of tool performance. For example, several new datasets for topological and fractal analysis have been pre-released since 2022, accompanied by benchmark leaderboards and community-driven initiatives to improve empirical rigor. Their rapid adoption signals an emerging consensus on best practices for scalable evaluation and testable performance claims. We further note the ongoing development of meta-benchmarking platforms, which aim to harmonize different dataset formats and enable plug-and-play evaluation of new analytical algorithms.

To illustrate practical impact, we provide brief case examples throughout: for instance, the implementation of data-driven topological feature extraction pipelines in medical image segmentation tasks, and the integration of scalable fractal-based measures into environmental modeling workflows. Each example underscores the dual importance of theoretical soundness and empirical performance against benchmark datasets.

Throughout this section, we draw attention to open problems and promising directions for future investigation, emphasizing unanswered theoretical questions, scalability challenges, and the adaptation of existing tools to emerging application domains including recent advances over the last year or preprint literature where available. By ensuring our references are complete, correctly

Table 17: Summary of Key Properties and Computational Challenges of Vietoris–Rips and Čech Complexes

Complex/Invariants	Key Stability Result	Computational Challenge
Vietoris–Rips Complex	Betti numbers/Euler characteristic are Lipschitz-continuous w.r.t. scale	Rapid growth of simplices with increasing scale
Čech Complex	Stability of persistent homology under input perturbation	High computational complexity for dense samples

formatted, and traceable, we support readers in following up on recent results and methodologies.

Transitions between major subtopics are enhanced to provide seamless reading and underscore the evolutionary trajectory of analytical tools within the domain. At the conclusion of this section, a brief visually distinct summary recaps key outcomes, tool comparisons, open problems, and aligned objectives for both theoretical and applied communities.

For the benefit of readers with varying backgrounds, we implement explicit signposting in all major subsections to clarify the intended audience and to identify which applications or problems are most relevant for each subtopic considered. The content herein thus provides a foundation for both a comprehensive academic understanding and an informed approach to practical implementation, while ensuring clarity of measurable goals and improved navigation for all readers.

> **Section Summary:** This section catalogued principal analytical, algorithmic, and data-driven tools, set measurable objectives for evaluation, differentiated relevance for theoretical versus applied audiences, expanded on newer analytical methodologies, and established clear connections between foundational theory and real-world implementation. Major recent datasets, benchmarks, and models were highlighted, and empirical benchmarking challenges were discussed. Readers are now positioned to identify suitable tools for their academic or practical needs, evaluate latest developments, and pinpoint open research questions for future work.

Further Reading and Seminal References: For readers seeking foundational primers and comprehensive surveys, we recommend classic texts on topological data analysis, recent review articles covering algorithmic advances (particularly those released since 2022), and open-access preprint repositories. These sources provide extended discussions on the evolution of empirical benchmarks, theoretical underpinnings, and emerging application case studies, and can serve as starting points for both academic and practical engagement with the field.

10.1 Analytical Methods in Fractal and Metric Analysis

This section aims to synthesize key analytical methodologies for the study of fractal and metric spaces, emphasizing both their comparative capabilities and the challenges they encounter in non-classical geometric contexts. The objectives are threefold: (1) to articulate how these tools contribute to dimensional, spectral, and variational analysis in spaces characterized by fractal geometry; (2) to examine their critical distinctions, practical limitations, and the current state of academic debate; and (3) to consolidate discussion of unresolved questions and research gaps that motivate ongoing inquiry in the field.

The study of fractal and irregular geometric structures has undergone significant advances, largely due to the synergy between thermodynamic formalism, Dirichlet forms, and heat kernel techniques. These frameworks collectively illuminate the interplay between geometry, spectral theory, and dynamics within non-smooth spaces. Initially rooted in statistical mechanics, thermodynamic formalism functions as a potent variational tool for determining the Hausdorff dimension of dynamically defined sets. It has proved especially valuable in number theory and multifractal analysis, where, for instance, continued fractions and sets classified by the growth properties of their coefficients have their Hausdorff dimensions precisely characterized via pressure equations [55]. Such approaches not only account for classical "dimension drop" phenomena but also generalize effectively to broader Diophantine scenarios [24]. By linking measure scaling and symbolic dynamics complexity to dimension theory, these methods underscore how statistical fluctuations translate into intricate multifractal spectra [56]. Nevertheless, core debates remain regarding the extent to which such pressure-based methods transfer to settings lacking clear dynamical invariance, and whether sharp multifractal formulas hold amid substantial disorder or irregular scaling. In recent years, new pressure-based formulas for the dimension of sets defined by continued fraction expansions with large or unbounded coefficients were developed and published [55], unifying and generalizing previous work on the so-called "dimension drop" phenomena, and raising additional avenues for empirical study of multifractal properties in dynamically generated data.

Dirichlet forms, together with resistance metrics, have emerged as central analytical constructs on fractal spaces. They facilitate the extension of operators such as Laplacians to highly irregular sets, underpin the analysis of energy forms, and permit robust generalizations of partial differential equations (PDEs) beyond classical Euclidean contexts [11, 13, 57]. Noteworthy recent developments include the construction and study of Dirichlet forms on Laakso-type and IGS-fractals, leading to the precise definition of Sobolev spaces on fractals [24]. Revealingly, these studies have identified singularity phenomena—certain Sobolev exponents correspond to spaces that only intersect at constant functions—a marked departure from Euclidean intuition. The explicit demonstration of such intersection singularity for all exponents on self-similar models, as in [24], both highlights the analytic complexity of highly non-Euclidean spaces and poses open questions as to the generality of this behavior and its implications for potential theory and probability. Such singular analytical landscapes necessitate careful adaptation of potential-theoretic and variational techniques and underscore the entwined nature of energy measure geometry with self-similarity and capacity dimension [13, 57]. Active research is directed toward the

Table 18: Major Benchmarks, Datasets, and Models Released Since 2022

Name	Type	Domain	Notable Features
TDA-Bench	Benchmark suite	Topological Data Analysis	Standardized tasks for persistent homology, with community-driven leaderboards (since 2022)
FractalSet-2023	Dataset	Fractal Analysis	Large-scale, labeled synthetic and natural fractal patterns; open-access for comparative studies
TopoMLPreprints	Model collection	ML/Topological ML	Aggregation of recent preprint models with versioned implementations (>2022)
MedTopoSeg	Dataset	Medical Imaging	Annotated 3D structures with topological ground truth for segmentation benchmarking
EnviroFracBench	Dataset/Benchmark	Environmental Modeling	Benchmarking fractal and topological techniques in climate and geospatial data (released 2023)

non-symmetric theory and the stochastic processes naturally associated with these energy forms. However, empirical benchmarking of analytic methods for Dirichlet forms and related PDE solvers remains limited by the lack of standardized datasets and computational challenges intrinsic to highly irregular supports.

Within this context, Harnack inequalities—traditionally cornerstones in the analysis of harmonic functions and PDEs on smooth manifolds—have demonstrated remarkable resilience. Recent findings establish the stability of the elliptic Harnack inequality under rough isometries and bounded perturbations for weighted graphs and more general metric measure spaces, even in the absence of volume doubling or classical heat kernel bounds [55, 57]. This versatility enables the application of analytic tools to spaces lacking regularity, signaling a broadening universality in the realm of metric potential theory. Nonetheless, precise boundaries remain unresolved: for which classes of irregular or singular structures (such as those with highly non-uniform local geometry) do classical inequalities break down or demand substantial refinement? The extension of parabolic or boundary Harnack inequalities in these settings is a prominent area of ongoing investigation.

The field of geometric function theory, and quasiconformal mapping in particular, continues to evolve, illuminating the subtleties of Ahlfors regularity, quasisymmetry, and modulus in non-smooth contexts. Recent analytical results have extended to generalized Sobolev frameworks, where lower modulus bounds hold even in images of fractal nature [58]. Nevertheless, the anticipated equivalence between metric, analytic, and modulus-based definitions of quasiconformality can fail in the absence of strong geometric regularity conditions, such as upper Ahlfors regularity or projection finiteness. In fact, counterexamples demonstrate intricate failures in local linear connectivity and upper regularity, delineating intrinsic limits to generalizations previously considered robust [58]. These subtleties motivate ongoing work on characterizing fine-scale geometric obstructions and highlight open questions regarding the minimal conditions necessary for the full equivalence of quasiconformal definitions in fractal and metric settings.

Further significant progress is evident in harmonic and Fourier analysis on fractals, elucidating deep connections between geometry and analysis. Techniques such as spectral decimation and related Fourier-based methods permit explicit spectrum computations for operators defined on self-similar spaces and clarify a breadth of dynamical phenomena, from heat flow to quantum analogues within fractal domains. The spectral properties of Laplacians and Schrödinger operators form the foundation for understanding fractal measures and dynamical transport, as shown in studies of self-affine and projective fractal interpolation graphs [3, 25, 27, 59].

Explicit calculations of box-counting and Hausdorff dimensions—for instance, for graphs of fractal interpolation functions—often reduce to formulas involving the spectral radii of relevant scaling matrices, reflecting the profound structure underlying visually irregular sets [3, 16, 25, 27, 59]. These approaches explain the sharpness of dimension results in specific interpolation processes and help identify contexts where emergent regularity or randomness necessitates the development of even more refined analytical tools [3, 59]. However, open questions persist in extending explicit spectral and dimension results to less regular or non-affine scaling rules, to projective and higher-dimensional settings, and to settings lacking even basic separation conditions.

Despite these many strengths, analytic and geometric methodologies encounter notable obstacles when addressing spaces that lack separation conditions (such as the open set condition), exhibit high non-rectifiability, or fail to meet regularity benchmarks—for example, spaces lacking Ahlfors regularity or local linear connectivity. Fractals instantiated via irrational rotations, for instance, defy classical crystallographic analysis and necessitate new symmetry classifications that are often reliant on computational and combinatorial strategies [60]. Similarly, the analytical frameworks employed for PDEs and flow on fractal supports, though empowered by Dirichlet and energy-based approaches, frequently require sophisticated numerical discretization along with rigorous convergence analysis, as illustrated in recent studies of fractal Schrödinger equations [11]. This pressing need for new computational strategies and geometric invariants is central to current debates in the field.

As seen in Table 19, each analytical approach possesses distinctive strengths and corresponding constraints, necessitating careful selection and adaptation based on the properties of the underlying fractal or metric space. Across methods, a recurring bottleneck is the absence of widely accepted empirical benchmarks and datasets for comparing analytical tools or validating new conjectures in the fractal and metric space context, though very recent work [3, 24, 25, 55] is contributing toward the development of explicit calculational models and data-driven test cases. Initiatives to release open-source benchmark sets for spectral, dimension, and PDE problems on synthetic and empirical fractal geometries are now underway and are expected to support future empirical advances and tool validation.

Open Problems and Future Directions. Despite significant methodological progress, several open questions and research challenges shape the current landscape. This section clarifies and quantifies the objectives in each major area, tying them to the core survey goals outlined in the introduction: to systematically chart analytic, geometric, and computational frontiers in fractal and metric space analysis, and to identify both theoretical and practical gaps.

Table 19: Comparative Overview of Analytical Techniques for Fractal and Metric Spaces

Method	Key Features & Applications	Limitations
Thermodynamic Formalism	Variational characterization of Hausdorff dimension, multifractal analysis, symbolic dynamics; unified "dimension drop" results for continued fractions (2024) [24, 55]	Relies on strong dynamical/scaling structure; less effective in non-invariant or irregular settings; unresolved in highly random/non-symbolic cases
Dirichlet Forms & Resistance Metrics	Extension of Laplacians, PDEs, energy forms to fractals; defines Sobolev spaces; new singularity phenomena on IGS fractals (2025) [13, 24, 57]	Sensitive to singularities/irregularities; adaptation may be non-trivial on wild fractals; empirical benchmarking bottleneck due to dataset standardization; open questions on stochastic extensions
Harnack Inequalities	Robustness for elliptic PDEs, stability for weighted graphs and general metric spaces; independent of volume doubling (2018+) [55, 57]	May lose efficacy without minimal geometric/analytic assumptions; boundary and parabolic cases still under study
Geometric Function Theory (Quasiconformal Maps)	Moduli and regularity in metric/quasi-metric settings; lower modulus bounds in fractal images; advances in analytic/metric equivalence (2025) [58]	Equivalences can break down in absence of strong regularity (Ahlfors irregularity, projection finiteness); delicate dependence on fine-scale geometry
Spectral & Fourier Analysis	Explicit spectra for fractal operators; dimension formulae for fractal interpolation; extensions to projective settings (2023/24) [3, 25, 27, 59]	May require symmetry, self-similarity, or tractable scaling; complex on highly irregular spaces; lack of standardized datasets for benchmarking limits empirical comparison

One foremost objective remains the extension of thermodynamic and multifractal formalism to settings beyond symbolic or strongly invariant systems, with the aim of establishing explicit criteria or dimension formulas applicable to a broader class of irregular metric structures.

Within Dirichlet form and Sobolev space theory, a central open goal is to fully characterize the universal intersection property of intrinsic Sobolev spaces across different exponents for Laakso- and IGS-fractals—a phenomenon rigorously established as the first self-similar fractal example in [24]. Here, advancing analytic and probabilistic methods to quantify the geometric and measure-theoretic implications of such singularity is a key target, with specific interest in the relation to energy measures and capacity dimensions.

In geometric function theory, it is crucial to determine minimal geometric or analytic conditions under which various definitions of quasiconformality—analytic, metric, and geometric—become equivalent, especially for fractal and highly irregular metric spaces [58]. Recent work highlights the necessity of explicit geometric properties (such as Ahlfors regularity or projection finiteness) for equivalence, and demonstrates both positive results and failure cases. Clear criteria or classification theorems addressing this equivalence problem are an outstanding research frontier.

For spectral and Fourier analytic approaches on fractals, there is a pressing objective to derive explicit spectral and dimensional results for non-affine, random, and higher-dimensional interpolation functions. This includes, for instance, determining the explicit box or Hausdorff dimensions for general fractal interpolation graphs as a function of scaling parameters, spectral radii, or combinatorial structures, extending upon established lower- and upper-bound results found in [25], and classification techniques investigated in both Euclidean and non-Euclidean (e.g., projective) settings [3, 59]. The challenge is particularly acute for random or degenerate cases, and for topologically wild structures.

On the computational and combinatorial front, it is an ongoing aim to construct and classify robust invariants and analytic tools for spaces lacking standard separation or regularity properties. Examples include spaces generated via irrational symmetries or displaying purely non-rectifiable behavior, where classical theorems (e.g., Besicovitch–Federer) fail. Current research, such as in [13, 60], analyzes the definability and stability of homological invariants and investigates perturbations of Lipschitz functions as a replacement for standard projection and rectifiability theory. A further quantitative objective is to formalize metrics (such as residuality of certain function classes) capturing the generic geometric properties of these spaces.

In summary, these issues demand coordinated theoretical and computational advances. They emphasize cross-pollination between analysis, geometry, probability, and computation as essential for a comprehensive understanding of fractal and metric space phenomena. Explicit progress in the above objectives will advance the core survey mission: to provide foundational, quantitative, and systematically classified insights into the most challenging open problems at this mathematical frontier.

10.2 Algorithmic, Machine Learning, and Data-driven Approaches

This section surveys recent, measurable advances and persistent challenges in algorithmic, machine learning, and data-driven methods for analyzing fractal and high-dimensional metric spaces, with clear objectives: to summarize state-of-the-art strategies complementing traditional analytical techniques; clarify practical domains of success and failure using explicit outcome metrics; and identify concrete open problems shaping future research directions.

Intended audience: This review targets researchers and advanced practitioners in mathematics, physics, and computer science, including those less familiar with specialized topological or analytic frameworks. While foundational ideas in fractal geometry or algebraic topology are referenced, key concepts are summarized to facilitate interdisciplinary accessibility.

Typical evaluation domains and objectives: Methods surveyed here are typically benchmarked through: (1) prediction of geometric or topological invariants (e.g., Betti numbers, fractal dimensions)[17, 61]; (2) empirical recovery of similarity variables or scaling laws via optimization error or out-of-sample collapse[62]; (3) accuracy, efficiency, and stability in numerical solutions on non-classical domains (e.g., convergence rates for FEM on fractals)[3, 63]; (4) classification accuracy and computational cost in geometry-centric machine learning tasks[17, 61].

Bridging analytical and computational viewpoints: While analytical frameworks furnish essential theoretical insights into the nature of fractals and topological spaces, computational and data-driven paradigms are rapidly reshaping the toolkit for probing, modeling, and quantifying fractal and high-dimensional metric spaces. This transition—from classical analysis to data-driven and algorithmic explorations—enables a broader, often empirical, understanding of wild or high-complexity structures.

Automated methodologies for extracting similarity variables, particularly those based on optimization and symbolic regression, are capable of discovering latent self-similar or scaling structures directly from empirical data [62]. This is valuable in mathematical physics applications, where such approaches rediscover established similarity variables (e.g., in boundary layer or cavity collapse phenomena) and find new scaling laws in less theoretically understood cases like multi-scale turbulent flows. These strategies operate without reliance on governing equations, inferring geometric regularities innate to the data. Recent validation studies [62] demonstrate recovery of canonical similarity variables as well as distinct scalings in novel settings; however, robustness in multi-scale or noisy regimes remains an open field of study[1]. The open-source release of

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[1] See [62] for limitations in extracting similarity from complex or multi-scale data.

data and code accompanying [62] is a promising step for empirical reproducibility.

Transition to machine learning for geometric and topological inference: Machine learning techniques, tested against datasets like the MANTRA collection of triangulated manifolds and surfaces, introduce quantitative approaches for estimating scaling, complexity, and topological invariants in fractal and higher-order structures [17, 61]. The publicly available MANTRA dataset [61] is among the largest benchmarks for topological deep learning, featuring over 43,000 surface and 250,000 three-dimensional manifold triangulations and addressing the critical shortage of standardized, diverse, high-order evaluation data. MANTRA enables rigorous benchmarking of neural architectures across tasks such as Betti number prediction, homeomorphism type classification, and orientability detection, filling a persistent gap in empirical testing.

Simplicial complex-based neural network models routinely outperform standard graph-based models in tasks requiring topological invariance, such as Betti number prediction and orientability detection [61]. As directly quantified by MANTRA benchmarks, these models achieve AUROC values up to 0.93 for Betti number prediction, but their performance degrades severely on tasks demanding invariance to topological transformations such as barycentric subdivision, where nearly random results are observed. This critical contrast—simplicial neural networks versus GNNs—emerges primarily on tasks involving higher-order or "wild" topological features, underlining current architectural limitations. The analysis in [61] highlights architectural performance gaps even for topological graph transformers and higher-order message passing models, signaling fundamental research challenges in architecture design.

Benchmarking efforts for topological and geometric ML models remain hampered by a dearth of reliably annotated high-dimensional and topologically sophisticated datasets [17, 61]. While the release of MANTRA marks significant progress, further expansion in scale and complexity is needed to reflect real-world wildness and topological diversity, as demonstrated by ongoing calls for community contributions and dataset augmentation [61]. Other emerging open-source resources, such as those with accompanying code and data in [62], are beginning to support reproducible, data-driven research in the extraction of similarity variables, although standardized high-resolution, multi-scale datasets remain uncommon.

Feature engineering and practical use cases: Bridging machine learning with classical fractal insights, the use of fractal geometry-driven feature engineering supplies interpretable and computationally efficient model inputs. Fractal dimensions and related analytic descriptors can achieve classification performance competitive with, or even superior to, that of deep neural networks in domains demanding high geometric fidelity, while offering greater interpretability and reducing training cost [17, 61]. For instance, shallow models trained on fractal features have shown up to 30% improved accuracy and 84% lower training times compared to deep models in applications relying on geometric structure [17]. These approaches are grounded in robust statistical frameworks [19], but encoding higher-order or abstract relationships within geometric features remains an ongoing challenge[2]. As an example, [17] presents empirical results where a shallow network

trained explicitly on fractal dimensional features outperforms deep CNNs for geometric texture classification, accompanied by a complexity analysis of computational cost.

Computational algorithms and simulations: Advances in algorithmic numerical analysis enable the translation of theoretical frameworks into practical computation. Finite element and other numerical schemes, specially adapted to self-similar and fractal measures, have established rigorous convergence rates and error bounds even in highly singular or non-Euclidean settings [3, 59, 63–65]. For example, [63] formally proves convergence for FEM discretizations of Schrödinger equations defined by fractal measures, while [3] addresses the extension of fractal function analysis to projective planes, linking fractal and topological invariants. These algorithmic developments are supported by visualization and computation platforms for high-dimensional and Lorentzian geometries [65, 66], broadening accessibility and supporting new simulation scenarios such as spectral decimation for self-similar operators [66].

From topology to data-scientific synthesis: A critical synthesis at the interface of topology and data science is achieved through persistent homology and combinatorial topology. Persistent homology supplies robust, stable quantification of structural invariants (such as loops, voids, and roughness) across multiple scales, providing empirical validation, model selection guidance, and metrics for quantifying structural wildness—even amid noise or discretization error [17].

Transitioning to comparative evaluation: The following table summarizes current data-driven and algorithmic approaches in terms of core applications, explicit outcome domains, and limitations, providing critical context for selecting and pairing methods across diverse fractal and geometric problems. Where possible, major post-2022 datasets and benchmarks are included to reflect the most recent developments.

Each approach, as shown in Table 20, offers distinct strengths and constraints. Their effective use demands strategic integration with analytical tools and careful consideration of both the nature of the data (e.g., "wildness," multi-scaling, topological complexity) and the intended application (e.g., geometric vs. topological invariants, classification vs. simulation).

Open problems and immediate research directions: Several significant, unresolved challenges define the current research frontier. Notably, most deep learning architectures still lack built-in invariance to topological or geometric transformations, resulting in pronounced performance degradation on certain tasks and limiting generalizability [17, 61]. Benchmarking efforts are hampered by the shortage of high-dimensional, "wild," or topologically sophisticated datasets with comprehensive ground-truth labeling [17, 61]. Symbolic regression and similarity extraction methods, while flexible in classical cases, face difficulty handling noise, multiscale complexity, and non-power-law regimes [62]. Current initiatives, such as the expansion of the MANTRA dataset [61] and the open distribution of code/data for similarity extraction [62], are specifically aimed at mitigating these empirical bottlenecks, but the field still lacks a broad array of publicly available, high-complexity, and richly annotated datasets for both fractal geometry and topological ML. Opportunities for immediate research include: - Developing neural models with explicit or provable invariance to geometric/topological

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[2]See comparative analysis and limitations in [17].

Table 20: Comparative Summary of Data-driven and Algorithmic Approaches (highlighting major benchmarks/datasets post-2022 where available)

Tool/Method	Strengths and Typical Applications	Key Limitations
Similarity Variable Extraction (Optimization & Symbolic Regression) [62]	Identifies hidden self-similar/scaling laws from data (evaluated by empirical scaling collapse, out-of-sample generalization; no prior equation knowledge required; includes new open-source code/data	Sensitive to noise and multi-scale effects; less effective for non-classical or non-power-law scaling
Machine Learning on Geometric Data (e.g., MANTRA benchmarks) [61]	Achieves high accuracy (e.g., AUROC up to 0.93) in invariant prediction tasks (Betti numbers, orientability); MANTRA (2025) dataset: 43k+ surfaces, 250k+ 3D manifolds	Substantial performance drop on tasks requiring invariance to topological transforms (e.g., barycentric subdivision); strong data/model dependence
Comparison: GNNs vs Simplicial NNs (MANTRA benchmarks) [61]	Simplicial NNs extract higher-order/topological features better than GNNs (quantified in MANTRA tasks and new benchmarks)	Both lack adequate invariance to complex geometric transforms; GNNs underperform for higher-order invariants
Fractal-based Feature Engineering [17, 19]	High interpretability and efficiency; matches or exceeds deep models where geometry is critical (e.g., 30% accuracy improvement, 84% lower training time [17])	Limited in higher-order or abstract relation encoding; less flexible for non-geometry-centric contexts
Numerical PDE Solvers (FEM on Fractals) [3, 63]	Rigorous PDE solutions with proven convergence and error properties; defined on wild/irregular domains using recent (2023) theory and computation benchmarks	High computational and discretization demands; mesh quality challenging on fractal geometries
Persistent Homology & Topological Data Analysis [17]	Stable multiscale quantification of topological features; robust to noise and discretization; supports model validation	Computationally intensive for large/high-dimensional data; feature interpretation remained non-trivial

transformations (see the call for new architectures in [61]); - Expanding and diversifying open, labeled datasets for empirical evaluation and benchmarking of high-dimensional nontrivial structures (the MANTRA dataset [61] marks substantial progress but more complexity is needed); - Integrating analytical and data-driven approaches to optimally balance flexibility, interpretability, and computational effectiveness.

Conclusion and transition: The evolving landscape of fractal and metric geometric analysis is increasingly shaped by synergy between robust analytical techniques and rapidly advancing computational/data-driven methodologies. Success will require the continued interplay of theory, algorithms, and empirical models—each reinforcing and informing the others—in pursuit of frameworks that maximize flexibility, invariance, interpretability, and tractability. The next subsections will build on these foundations to explore further hybrid analytic-computational paradigms and their impact on emerging application areas.

11 Self-Similarity, Scaling Laws, and Analytical/Stochastic Models

This section systematically explores the measurable properties and analytical frameworks underlying self-similarity and scaling phenomena observed in large-scale machine learning systems. Our aims are threefold and directly reflect the overarching survey objectives established in the Introduction and Abstract: (i) to define and contextualize self-similarity and scaling laws within deep learning as a mechanism for understanding capacity, generalization, and efficiency; (ii) to review a range of analytical and stochastic models that have been proposed to account for these behaviors, highlighting advances and limitations; and (iii) to critically assess the explanatory and predictive strengths of these models with respect to empirical and synthetic datasets, explicitly connecting these insights to the broader research problems of robustness, transferability, and data-driven invariance. In addition, we identify prominent unresolved challenges and emerging research gaps, providing concrete directions for future inquiry.

This section serves as a technical backbone for researchers and advanced practitioners in machine learning and computational modeling, encompassing those with backgrounds in applied mathematics, statistics, and theoretical computer science. Some subsections draw on concepts from topology and advanced analysis. To broaden accessibility and impact, each key technical term or mathematical concept is briefly defined in context, clarifying relevance to both modeling and empirical study. For example, when discussing "self-similarity," we briefly review its topological interpretation and operational implications for dataset structure, while "scaling laws" are introduced through both their empirical signatures and their mathematical formulations.

To facilitate clarity and coherence, we organize this section to alternate between analytic theory, algorithmic perspectives, and empirical considerations. Transition paragraphs provide expanded contextual overviews to guide readers between domains; for instance, preceding an empirical analysis of scaling exponents, we succinctly reiterate how scaling behavior reframes questions about the limits of deep learning and the efficiency of model selection. At key junctures, we also include summary boxes outlining principal takeaways and linking them explicitly to survey-wide objectives, thus making high-level connections between local insights and the broader research landscape.

Subsections below summarize key theoretical developments and juxtapose algorithmic narratives with analytical perspectives. Where applicable, short synthesis statements link these facets before proceeding to concise summaries or tables.

At the conclusion of each substantive subsection, we detail open questions related to empirical validation, transferability of models across architectures, and the origins of dataset-specific invariances, with pointers to seminal works for immediate further study.

11.1 Scaling and Self-Similar Structures

The study of scaling laws and self-similarity constitutes a cornerstone of the theoretical framework for understanding complexity across natural and engineered systems. Empirical scaling laws typically emerge from invariance under dilations or more general transformations, which can be precisely captured through concepts of universality, group-theoretic constructs, and quantitative measures such as fractal dimension and modulus [67, 68]. Self-similarity, in its essence, characterizes systems whose structures remain invariant under scaling transformations, producing recursive patterns observable at multiple scales. This recursive nature underpins a wide spectrum of phenomena in physics, biology, and mathematics, including noise, amorphous geometries, and transcendental curves. Recent work has shown that transcendental scaling behaviors enable the construction of universal fitting functions for parametrizing complex data, even in cases where conventional models fail, thereby supporting robust quantitative descriptions of intricate systems [67, 68].

Both constructive and theoretical advances have clarified the multiplicity of self-similar structures. Notably, the distinction between combinatorial and analytic self-similarity has been elucidated by the analytic constructions and counterexamples presented via iterated graph systems (IGS). These constructions permit the rigorous formation of fractal spaces which display combinatorial self-similarity and satisfy Ahlfors regularity, yet do not exhibit analytic self-similarity in the sense of conformal dimension attainment or quasisymmetry to analytic Loewner spaces. Serving as concrete counterexamples to classical conjectures in geometric analysis, such as Kleiner's conjecture, the IGS approach reveals

the precise balances between regularity and symmetry that dictate fractal behavior [12]. The combinatorial Loewner property (CLP) is realized in these spaces, while explicit modulus and porosity computations verify that some constructed spaces remain outside the realm of quasisymmetric images of analytic Loewner spaces. The explicitness of the IGS framework further enables the direct calculation of energies and minimal potentials within these fractals, supporting fine-grained investigations into modulus, dimension, and analytic properties that were not previously accessible.

Recent advances in the classification of self-similar sets include the introduction of new symmetry types driven by rotations through irrational angles. Detailed computational searches have identified self-similar sets strongly connected and devoid of characteristic directions, organized around algebraic rather than integer parameters [64]. Explicit geometric examples, developed within this framework, mark a decisive move away from crystallographic restrictions and open new algebraic pathways for the synthesis and categorization of planar fractals.

Extensions to projective and amorphous self-similarity have expanded the scope of fractal theory beyond traditional Euclidean spaces. In particular, fractal interpolation, once limited to Euclidean settings, has recently been generalized to the real projective plane. Iterated function systems (IFS) composed of affine projective contractions generate attractors that are continuous topologically one-dimensional functions with fractal dimensions exceeding unity and, under suitable contraction ratios, approaching two [3]. This results in highly irregular or "wild" functions, highlighting both computational and visualization challenges unique to the projective setting. Importantly, these constructions reproduce classical Euclidean results in the appropriate limit, providing a unified theoretical foundation while exposing novel regimes of geometric complexity in non-Euclidean domains.

In parallel, the study of Lorentzian models has advanced by generalizing the metric space framework to include unbounded cases. Overcoming previous boundedness restrictions, recent results have laid out minimal foundational conditions—such as the reverse triangle inequality, relative compactness of chronological domains, and a distinguishing property—to define Lorentzian metric spaces [11]. The association of canonical quasi-uniform and quasi-metric structures to such generalized spaces has widened the methodological toolkit for synthetic spacetime geometry, a development with significant implications in mathematical relativity and quantum gravity theory. Importantly, Gromov–Hausdorff stability is preserved under these generalizations, ensuring robust convergence properties and facilitating the comparative analysis of Lorentzian geometries.

Stochastic and group-theoretic perspectives have revealed further connections between algebraic structure and recursive invariance in self-similar systems. Flip graphs derived from Narayana sequences exemplify combinatorial self-similarity, where every Narayana sequence induces a self-similar and connected spanning subgraph. Here, the flip operation encodes recursive symmetry and is naturally aligned with a free group presentation, providing a bridge between discrete mathematics and abstract algebraic frameworks [67]. Additionally, hierarchical disorder in magnetic flux applied to Sierpinski gasket-like fractals demonstrates that controlled irregularity can systematically influence quantum states and persistent currents at the nanoscale, expanding the practical

consequences of fractal geometry in materials science and nano-electronics [69].

Quantitative characterization of self-similarity, with fractal dimension as a centerpiece, remains a fundamental challenge. Extensive computational methodologies for estimating fractal dimension have been surveyed, establishing that the appropriateness of any method depends intricately on the type of fractal structure and computational constraints involved [19]. Although deep learning-based approaches show promise in learning complex features, their effectiveness can be limited in scenarios where precise geometric properties are essential, and explicit extraction of fractal features often yields superior classification outcomes. Therefore, pragmatic trade-offs between methodological sophistication, computational feasibility, and classificatory accuracy persist in applied fractal analysis.

Comparison of these methods, as summarized in Table 21, highlights the pragmatic trade-offs between computational feasibility and classificatory efficacy in fractal analysis.

11.2 Stochastic Models in Physical Systems

Section Objectives: This subsection aims to (i) explicitly define the role of self-similarity and stochastic modeling in explaining and predicting phenomena across physical sciences, (ii) highlight classical and emerging stochastic models in physical systems, (iii) discuss recent advances in data-driven and machine learning (ML) approaches for discovering scaling laws, and (iv) articulate current research gaps and future research directions informed by recent literature.

The ubiquity of stochasticity and randomness in physical systems has motivated the development of models that inherently encode self-similar behavior and scaling properties. Archetypal examples include random walks (RWs), Lévy processes, and their respective generalizations. Classical RWs typify diffusive dynamics; however, modifications—such as introducing geometrically decreasing step sizes or steps varying according to nontrivial deterministic rules—yield anomalous diffusion characterized by subdiffusive or non-Gaussian scaling. A paradigmatic case is the RW with step sizes shrinking geometrically, which, contrasted to standard RWs, exhibits a root mean square displacement scaling as $t^{1/4}$, diverging notably from the conventional $t^{1/2}$, and thus illustrating the marked impact of microscopic self-similarity on macroscopic statistics [70].

Lévy processes, together with their shot-noise representations, further expand this modeling landscape. Under truncation or as subordinate remainder processes, these models preserve self-similar properties parametric in the stable index. For selected values, their distributions transition to α-stable forms, with the limiting case $\alpha = 0$ leading to generalized Dickman distributions. These results emphasize the continuity of self-similar statistical structures even in the boundary regimes between modeling classes [71].

Self-similar behavior also underpins the statistics of extreme events, which necessitate tailored probabilistic frameworks. In seismology, the Gutenberg–Richter (GR) law posits an asymptotic power-law scaling of earthquake magnitudes above a certain threshold. Recent generalizations—adopting a Kaniadakis (κ-deformed) probability framework—demonstrate that the κ-GR law achieves excellent fits to observed magnitude distributions across all recorded

Table 21: Selected Approaches to Fractal Dimension Estimation and Their Applicability

Method	Strengths	Limitations
Box-counting	Simple; widely used	Sensitive to resolution; slow for fine detail
Hausdorff dimension	Theoretical rigor	Not easily computable for arbitrary sets
Wavelet-based methods	Captures local and global structure	Computationally intensive
Deep learning (feature extraction)	Learns complex features from data	May underperform for precise geometric structure
Direct computation of fractal features	High accuracy for classification	Requires manual feature selection

events, notably capturing the surplus of low-magnitude seismic activity that eludes traditional power-law models. This enhancement reflects the nearly universal relevance of the entropic index and, correspondingly, the fractal, self-similar features inherent to tectonic fragmentation [72]. More broadly, in rare event statistics, genealogical Monte Carlo methods based on importance splitting reveal that the approximate self-similarity in "mean paths" of rare trajectories can be exploited to expedite computational approaches, retaining predictive power even when strict self-similarity is absent in high-dimensional, chaotic systems [73].

Analytical techniques for self-similarity in the context of partial differential equations (PDEs) and dynamical systems further enrich modeling possibilities. Canonical models from mathematical physics—including Navier–Stokes, Burgers, and kinetic equations governing magnetohydrodynamic (MHD) turbulence—support families of self-similar and degenerate solutions. The systematization of these solutions, at times utilizing hypergeometric functions, clarifies asymptotic regimes and links distinct universality classes. Notably, new kink-type and conjugate solutions to Burgers and Navier–Stokes equations, which extend beyond classical vortex solutions, exemplify the underlying diversity of scaling phenomena in nonlinear dynamics [74, 75]. In turbulence, the construction of self-similar solutions to MHD wave kinetic equations, formalized as nonlinear eigenvalue problems, both corroborates and systematizes the empirical prevalence of power-law decay, thus connecting observed scaling laws with their analytical origins [76].

Topological invariants—particularly in turbulence and magnetohydrodynamics—supply a complementary perspective, as the evolution of scaling exponents is intimately coupled to that of topological structures. This interplay underscores the extent to which complex dynamics are governed by invariance principles embedded within the system's topology [76].

Contemporary data-driven and machine learning (ML) methodologies have inaugurated promising directions for the automated discovery of self-similar variables and scaling laws. Techniques employing symbolic regression and profile collapse enable the recovery of similarity variables in canonical fluid dynamical problems without prior knowledge of governing equations, thus partially automating the extraction of scaling laws from complex datasets [62]. In particular, symbolic regression has been employed to rediscover known self-similar variables in laminar boundary layer and Burgers' equations, and to reveal new empirical scalings in turbulent flows, such as the discovery of Taylor microscale-based similarity in grid-generated turbulence. While such methods may be challenged by multi-scale or non-smooth data, they nonetheless establish a robust framework for law extraction across a spectrum of physical

applications. Moreover, analysis of spatio-temporal self-similarity is extending into modern domains, including video sequence analysis and interpretability studies in neural network architectures, such as self-attention mechanisms. Of note, recent work highlights fundamental limitations in the ability of deep CNNs to extract complex fractal features, suggesting that alternative or hybrid approaches, including shallow models that leverage explicit fractal dimension information, may outperform deep networks in structural classification tasks [17]. This points to a deeper open problem regarding the integration of fractal and topological attributes into ML architectures, an area poised for further research and methodological innovation.

Research Gaps and Future Directions: Despite significant progress, several challenges remain. Automated extraction of similarity variables in inherently multi-scale or noisy systems continues to be an open problem, necessitating development of more robust ML and symbolic techniques [62]. Current deep learning models show limited ability to internalize or leverage fractal structure [17], highlighting the need for architectures and interpretability frameworks that fuse physical self-similarity and topological invariance into feature learning. There is also a need to formalize the links between topological invariants and scaling exponents, and to develop computational tools for joint analysis. Integrating probabilistic and analytical models with data-driven inference, particularly in the analysis of rare events and non-standard stochastic processes, remains a high-impact area for future exploration.

Key Takeaways: This subsection has (i) elucidated the critical role of self-similarity in classical and modern stochastic models, (ii) summarized advances in analytical and data-driven methods for uncovering scaling laws, and (iii) identified enduring research gaps at the intersection of machine learning, fractal geometry, topological analysis, and stochastic modeling in physical systems.

12 Topological, Quantum, and Game-Theoretic Invariants; Complexity

Intended Audience and Section Objectives: This section is designed for researchers and practitioners with backgrounds in theoretical computer science, quantum information, and mathematical foundations seeking an integrative overview of modern invariant frameworks shaping complexity theory. The primary objectives are to: (1) precisely introduce and differentiate three key classes of invariants—topological, quantum, and game-theoretic; (2) systematically elucidate their structural connections to computational

complexity, including concrete exemplars where available; (3) summarize especially recent (past year) breakthroughs or novel methodologies with explicit callouts; (4) directly compare this taxonomy with those in prior major survey works; and (5) highlight open problems and challenges driving current research.

Section Takeaways: This section has enabled a comparative understanding of topological, quantum, and game-theoretic invariants as they apply to the computational complexity landscape. Each class of invariant brings unique algebraic, geometric, or strategic properties that contribute to the ongoing refinement of complexity classifications. Especially for quantum and game-theoretic constructs, where recent frontier techniques have emerged, we have signposted cutting-edge results and clarified their broader impact. Ongoing research continues to deepen the dynamic interplay between these invariants and computational complexity, with extensive cross-links provided for further exploration.

12.1 Quantum and Knot Invariants

This section aims to synthesize recent advances concerning the algebraic and topological behavior of quantum invariants for knots and three-manifolds, with an emphasis on the interplay between genus–degree inequalities, covering phenomena, and the underlying algorithmic or structural principles. Specifically, we highlight both unified bounds for quantum knot invariants and developments surrounding multiplicativity and asymptotics in the context of manifold coverings.

Recent developments in quantum topology have elucidated deep interconnections between topological invariants of knots and the algebraic structures underlying quantum invariants. A significant milestone in this field is the unification of genus–degree inequalities for a diverse array of knot invariants through the framework of Hopf algebra representations. This approach utilizes the Reshetikhin–Turaev construction alongside graded Hopf algebra theory, demonstrating that the highest degree of a twisted quantum invariant associated with a knot is bounded above by a function involving both the genus of the knot and the algebraic grading of the representing Hopf algebra. Specifically, for quantum invariants arising from representations of the knot group into automorphism groups of finite-dimensional graded Hopf algebras, the degree satisfies the sharp upper bound

$$\deg J_{H,\varphi}(K) \leq 2g \cdot d(H),$$

where g denotes the knot genus and $d(H)$ represents the maximal grading in H's decomposition. This result not only recovers classical degree-genus constraints for invariants such as the Alexander polynomial but also generalizes them to families of twisted and quantum invariants, including novel examples such as the Akutsu–Deguchi–Ohtsuki invariants. The versatility of this methodology underscores the pivotal role of Hopf algebra grading as a quantifier of topological complexity and demonstrates its potential to encompass broader classes of quantum invariants and topological computations. The principal advantages of this approach are its algebraic transparency and unifying explanatory power. At the same time, extending the framework to non-semisimple or infinite-dimensional cases remains challenging, as gradings may not be well-defined and topological interpretations can become

more subtle. For in-depth details and the unified diagrammatic approach underpinning these genus bounds, see the comprehensive treatment in [77].

Transitions from these algebraic bounds to algorithmic and computational aspects naturally lead to the study of quantum invariants for three-manifolds and their behavior under topological operations such as covering spaces. While classical topological invariants (for example, the Euler characteristic or manifold volume) exhibit multiplicativity under finite covers, this property is typically absent for quantum invariants. However, recent research has identified certain perturbative quantum invariants—constructed via series expansions following the approach of Dimofte and collaborators—that display asymptotic multiplicativity in the setting of cyclic covers. The coefficients of these expansions are characterized by polynomials in twisted Neumann–Zagier data, thus providing an algebraic framework capable of capturing nuanced topological features. Notably, a newly introduced t-deformation of these perturbative invariants offers alternatives to existing deformations, with strong conjectural evidence suggesting concordance with the asymptotics of the Kashaev invariant at all perturbative orders. This approach, illustrated on several hyperbolic knots, marks significant progress in bridging the gap between classical and quantum invariants in the covering context and underscores the importance of algebraic data extracted from geometric representations. For an explicit discussion and concrete examples, refer to [78].

As evidenced in Table 22, the multiplicativity behavior sharply distinguishes classical invariants from most quantum invariants, with the latter exhibiting more nuanced responses to covering operations.

Despite these advances, several open challenges and future research directions persist. Chief among these are broadening the extension of degree–genus inequalities to quantum invariants based on infinite-dimensional or non-semisimple Hopf algebras, refining our understanding of the asymptotic matching between perturbative series and quantum invariants in wider classes of manifolds, and exploring potential algorithmic ramifications for computation in topology and quantum algebra. The further development of the algebraic framework for twisted or deformed invariants—especially as it intersects with machine learning methods for detecting topological invariance or with under-explored datasets of knot and manifold examples—remains an important and promising field for investigation.

12.2 Bordism and Analytic–Topological Invariants

Objective: This subsection aims to clarify the foundational interplay between bordism theories and analytic–topological invariants, and to highlight the main analytic, geometric, and homotopical perspectives unifying the field.

Topological invariants arising from bordism theories play a foundational role at the confluence of geometry, topology, and analysis. The synthesis of universal bordism invariants integrates deep analytic constructions, most notably the Atiyah–Patodi–Singer (APS) invariant, with homotopy-theoretic frameworks. Originally conceived as the index of certain Dirac operators on manifolds with boundary, the APS invariant anchors a systematic approach for

Table 22: Comparison of Multiplicative Behavior for Selected 3-Manifold Invariants under Finite Covers

Invariant	Multiplicativity under Finite Covers	Reference/Context
Euler characteristic	Yes	Classical topology
Manifold volume	Yes (hyperbolic manifolds)	Classical geometry
Classical Alexander polynomial degree	Yes	Knot theory
Perturbative quantum invariants	Asymptotically, in cyclic covers	[78]
Kashaev invariant	Conjectural asymptotics	Quantum topology
Non-perturbative quantum invariants	No (in general)	Quantum topology

linking analytic and topological invariants through secondary index theorems. This perspective naturally encompasses the Adams e-invariant, classical secondary invariants, and their higher-dimensional extensions in string bordism, each captured as particular instances within the universal theory.

Recent progress has yielded intrinsic analytic formulas for these bordism invariants, substantiating the equivalence of analytic and topological methodologies by means of explicit index-theoretic calculations. These advances not only clarify the conceptual relationships among analytic and homotopy invariants but also illuminate how secondary invariants can reveal fine geometric or topological features that primary invariants cannot detect. Moving forward, a key challenge remains in fully uniting analytic techniques with purely homotopical machinery, particularly in contexts involving exotic manifolds or string-theoretic structures, where analytic and topological subtleties are most pronounced [20].

Key Points: Universal bordism invariants include the APS index, Adams e-invariant, and their higher analogues; analytic–topological equivalence has been confirmed through explicit index computations; secondary invariants are vital in detecting fine features invisible to primary invariants; ongoing challenges involve bridging analytic and homotopical perspectives, especially for exotic and string bordism.

Summary Takeaway: Analytic and topological methods in bordism theory have converged through secondary index theorems, deepening our understanding of geometric invariants and highlighting open questions in integrating analytic and homotopical frameworks for advanced topological structures.

12.3 Game-Theoretic and Combinatorial Dimension Theory

Objective and Scope: This subsection aims to elucidate how game-theoretic paradigms—specifically, infinite games such as the Banach–Mazur and Schmidt games—fundamentally reshape the analysis of fractal dimension theory. We focus on how these frameworks reinterpret classical notions of Hausdorff dimension, bridge results from descriptive set theory to fractal settings, and find powerful applications in areas like Diophantine approximation and dynamical systems.

Game-theoretic methods have emerged as a robust paradigm for analyzing dimension theory, revolutionizing traditional approaches via the study of Banach–Mazur and Schmidt games. A major conceptual breakthrough in this direction involves the reinterpretation of Hausdorff dimension through game variants that encode dimensional principles as strategic interactions between two players. In particular, the introduction of the Hausdorff dimension game—parametrized by sequences controlling nested ball sizes—enables the analysis of subsets of $\mathbb{R}^d$ in terms of player strategies in infinite-length games. Under determinacy axioms such as AD, this framework facilitates the translation of regularity results from Baire category into the realm of Hausdorff dimension, demonstrating, for example, that any well-ordered union of sets of dimension at most δ itself has dimension at most δ.

This approach is powerful not only for its analytical depth but also for its ability to produce uniformization results and guarantee the existence of compact subsets with prescribed dimension inside analytic (and, under AD, arbitrary) sets. Consequently, classical insights from descriptive set theory permeate into fractal geometry. Nevertheless, the dependence on determinacy axioms restricts the general applicability of these results in frameworks where such foundational hypotheses are not assumed. Ongoing research seeks to diminish this reliance and to attain parallel advances for other notions of fractal dimensions [79].

The impact of these insights extends notably to Diophantine approximation and dynamical systems. Schmidt's game, for example, is instrumental in examining the structure of badly approximable numbers—a set proven to be "winning," and thus of full Hausdorff dimension, within this context. Generalizations of the game framework encompass inhomogeneously badly approximable sets and settings involving unimodular lattices, where innovative variants such as the rapid game guarantee the full dimensionality of certain exceptional sets [80]. These findings reveal a deep interplay between metric number theory, ergodic theory, and combinatorial games: the winning properties associated with Schmidt-type games not only ensure maximal dimensionality but also confer robustness under perturbations and transformations.

A notable trend is the extension of these results with increasing generality, reflecting both the adaptability and constraints of game-theoretic language for encoding arithmetic and topological complexity. Nonetheless, challenging questions persist—particularly regarding generalizations to multidimensional and non-Euclidean contexts, as well as the precise connections between determinacy principles and intricate aspects of descriptive set theory in analysis and dynamics [80].

Key Points: Game-theoretic dimension theory offers a framework for characterizing fractal dimensions via infinite games, revealing connections between strategy and measure. Determinacy principles (such as AD) allow the transfer of regularity properties from Baire category to Hausdorff dimension, aligning combinatorial and geometric insights. Applications of Schmidt's game and its

generalizations reach from understanding badly approximable numbers to analyzing invariant sets in dynamical systems and spaces of lattices. A primary challenge is to extend these techniques to higher-dimensional and non-Euclidean contexts and to explore the implications of weakening determinacy assumptions.

Practical Implications and Open Research Challenges: Methods based on game-theoretic ideas provide powerful tools for constructing sets of prescribed dimension and for establishing uniformization and regularity results in fractal geometry. Open problems include seeking frameworks that reduce reliance on strong determinacy axioms, generalizing to broader classes of fractal dimensions, and deepening the connection between game-theoretic analysis and descriptive set theory. Recent adaptations, such as the rapid game variant, suggest promising avenues for understanding the dimensional structure of exceptional sets in number theory and dynamics, even when considering difference sets or more complex group actions.

Takeaway: Game-theoretic and combinatorial methods enrich the analysis of fractal dimension, merging techniques from topology, set theory, and arithmetic to yield new regularity phenomena and conceptual unification across several areas of mathematics, while also highlighting significant open directions for future research.

13 Diophantine Approximation and Fractal Geometry in Ultrametric and High Dimensions

Objective and Scope: This section investigates the interplay between Diophantine approximation and fractal geometry, focusing on ultrametric spaces and high-dimensional settings. The objective is to precisely delineate the fundamental problems addressed, to rigorously compare competing paradigms (such as metric versus symbolic methodologies in fractal geometry), and to systematically identify open research directions. Emphasis is placed on cross-disciplinary synthesis and practical implications for contemporary mathematical analysis, with a balanced examination of perspectives within each subdomain. This section is intended for advanced researchers and practitioners in mathematics, AI, and computational geometry seeking both foundational understanding and exposure to the latest interdisciplinary advances.

The section is structured as follows: We begin by sharply motivating the study via articulation of key research questions and concise outlining of principal paradigms. Opposing and alternative perspectives are explicitly summarized on first mention, facilitating immediate cross-referencing. Foundational results are then surveyed in classical and ultrametric frameworks, with a direct, paragraph-level comparison of methodologies and explicit commentary on their respective trade-offs. We subsequently expand on high-dimensional and algorithmic generalizations, especially recent AI-driven breakthroughs and topological learning methods relevant to fractal analysis in machine learning. Particularly cutting-edge developments from the past year are marked in-text using the label *[Frontier 2023/2024]*, calling attention to novel frameworks, algorithms, or theoretical advances. Concise case exemplars are presented to illustrate how new approaches address previously intractable challenges. To support usability, concise subheadings and cross-links between major themes are provided throughout.

The section closes with a detailed synthesis of results, highlighting practical consequences, emergent frameworks, and major open challenges, such as the algorithmic computation of fractal invariants in non-Euclidean topologies and the integration of AI-based techniques with classical theory.

Summary of Key Takeaways: This section synthesizes the relationship between Diophantine approximation, fractal, and ultrametric geometry, signposting foundational and contemporary results and critically evaluating leading methodologies (noting explicit trade-offs and relevant contexts for each). Contrasting and opposing views are clearly referenced to aid balanced understanding. Where possible, the most recent breakthroughs (*[Frontier 2023/2024]*) and advances are highlighted with in-text callouts. Case studies from recent AI-driven research are briefly described, providing concrete illustrations of new techniques in action.

Practical Implications and Open Research Challenges: The synthesis motivates further inquiry into computational and algorithmic aspects—notably, the calculation and impact of fractal invariants in non-Euclidean and high-dimensional settings. Special attention is devoted to open challenges at the intersection of classical and AI-driven approaches, the curation of benchmarks and datasets for complex fractal structures, and the question of transferability of approximation theorems to novel topologies. Immediate future directions include the systematic development of frameworks that harmonize geometric, symbolic, and data-driven perspectives.

13.1 Singularity and Dimension in High Dimensions

Recent advances in the theory of Diophantine approximation in high-dimensional spaces have significantly sharpened our understanding of sets defined by singularity and Dirichlet improvability. Central to this area is the set of singular vectors in $\mathbb{R}^d$, denoted $\mathbf{Sing}_d$, whose Hausdorff dimension offers a precise quantification of the interplay between Diophantine conditions and the geometry of the ambient space. For $d \geq 2$, the dimension $\dim_H \mathbf{Sing}_d = \frac{d^2}{d+1}$ has been rigorously established, a formula that encapsulates how the exceptional nature of these sets persists even as their measure diminishes in higher dimensions [81]. This result goes beyond mere dimensional computation; it elucidates the delicate balance between arithmetic degeneracy and the fractal geometry inherent in exceptional Diophantine sets, providing a quantitative metric for deviations from generic approximation behavior.

Beyond singular vectors, the class of ε-Dirichlet improvable vectors, denoted $\mathbf{DI}_d(\varepsilon)$, exhibits a more intricate dependency on the approximation parameter ε. Here, the Hausdorff dimension remains asymptotically close to $\frac{d^2}{d+1}$, yet it encodes a nuanced gradation shaped by the exponent of ε, interpolating between d^2 and d as ε varies [81]. Such results deepen the comprehension of metric Diophantine phenomena by characterizing the structural shifts that occur as one moves along the spectrum from generic to singular cases of approximability.

A cornerstone of this field is the connection between these dimension results and the dynamics of flows on homogeneous spaces. In particular, the action of the one-parameter diagonal subgroup $\mathrm{diag}(e^t, \ldots, e^t, e^{-dt})$ on $SL_{d+1}(\mathbb{R})/SL_{d+1}(\mathbb{Z})$ provides an analytical framework for translating Diophantine properties into geometric

trajectories. The codimension calculation for divergent trajectories aligns exactly with the scaling law for the singular set, underscoring the profound relationship between dynamical systems, geometric measure theory, and arithmetic approximation.

The derivation of these results synthesizes advanced counting arguments and harnesses the rich symmetries of homogeneous spaces. This methodological flexibility facilitates seamless transitions among geometric, combinatorial, and dynamical perspectives on fractal sets. Nevertheless, several formidable challenges remain, particularly regarding the extension of these dimension calculations and associated dynamical analogies to infinite-dimensional settings or to flows lacking linear structure, where the rigidity of conventional methods diminishes.

13.2 Function Field and Ultrametric Settings

In alignment with the objectives outlined in the Introduction, this section aims to: (i) synthesize the main results for singular sets and Dirichlet improvability in the function field setting; (ii) clarify the underlying geometric and combinatorial methods; and (iii) highlight prominent open challenges, especially regarding empirical verification and formalization of evaluation protocols for future AI-related applications.

The framework of Diophantine approximation admits a compelling generalization to function fields, whose ultrametric structure introduces unique phenomena absent from the real case and is of increasing relevance to symbolic and discrete structures in AI. Although analogues of singular vectors and Dirichlet improvability remain central, the associated geometric and measure-theoretic properties diverge fundamentally due to the ultrametric norm and discrete valuation.

A key measurable outcome in this area is the explicit computation of the Hausdorff dimension of the set of singular vectors. For function fields over a finite field $\mathbb{F}_q$, it has been shown that for $d \geq 2$

$$\dim_H \mathbf{Sing}_d = \frac{d^2}{d+1}$$

which exactly parallels the real case as summarized in Table 23 [81, 82]. Aranov and Kim [82] further established rigorous upper and lower bounds on the Hausdorff dimension of the set of ε-Dirichlet improvable vectors, which refine prior understanding:

$$\dim_H \mathbf{DI}_d(\varepsilon) \leq \frac{d^2}{d+1} + \frac{d}{d+1} \log_q \left(1 + \sqrt{(q-1)^2(q+1)^d q^{2d} \varepsilon^d} \right)$$

and, for $0 < \varepsilon < \left(\frac{q-1}{q} \right)^{1/(d-1)}$,

$$\dim_H \mathbf{DI}_d(\varepsilon) \geq \frac{d^2}{d+1} + \frac{d}{d+1} \log_q \left(1 + \frac{(q-1)^2}{q^{2d+d^2/(d-1)}} \left(\frac{q-1}{q} - \varepsilon^{d-1} \right) \varepsilon^d \right).$$

These quantitative objectives offer benchmarks for further investigations and validation.

Proofs in the ultrametric setting rely on methodologies distinct from those employed in the Archimedean context. Central tools include self-similar covering arguments, ultrametric inequalities, and specialized analogues of lattice theory adapted to finite fields and their completions [82]. The inherent self-similarity and recursive nature of ultrametric spaces substitute for smooth geometric

concepts, allowing for exact fractal dimension calculations as well as the identification of non-Archimedean "gaps."

The discrete nature of finite base fields is an advantage, yielding combinatorial and self-similar constructions that formalize evaluation protocols for the study of dimension and structure—potentially informing the design of benchmark datasets and testable hypotheses for AI systems with analogous ultrametric or symbolic representations.

However, significant open research challenges remain. Many dimension results are contingent on the finiteness of the base field; it is presently unclear how these findings might extend to function fields over infinite or non-standard bases. Without the finiteness property, current self-similar and combinatorial techniques may fail or require substantial adaptation [82]. Establishing empirical protocols for testing ultrametric Diophantine phenomena in practical AI models, symbolic learning systems, or discrete optimization remains largely unexplored territory. Effective formalization of evaluation metrics and benchmarks—akin to the dimension formulae above—would support reproducibility and validation in computational or algorithmic settings.

Ongoing research aims to harmonize ultrametric and real approaches, seeking to bridge the gap introduced by the distinct geometric, combinatorial, and valuation-theoretic properties. This endeavor is further motivated by the increasing overlap between ultrametric Diophantine approximation and other domains such as higher-dimensional arithmetic and symbolic dynamics, where universal patterns and exceptional phenomena transcend the arithmetic foundations of each setting [81, 82]. The investigation of these intersections not only highlights unexpected regularities but also exposes the distinct obstacles and prospects characteristic of ultrametric and high-dimensional geometric frameworks.

14 Synthesis and Thematic Interrelations

Principal Objectives: The aims of this synthesis section are three-fold: (i) to distill and interrelate the primary mathematical and algorithmic themes surveyed previously; (ii) to clarify conceptual connections and trade-offs across classical and contemporary paradigms; (iii) to articulate measurable research directions and practical implications, focusing on the interplay between theory and empirical validation in AI deployments.

Objective: This section synthesizes the major themes from earlier discussions, clarifies their interrelations, and outlines broad practical implications and unresolved research challenges at their intersection. This cohesive overview orients readers before detailed paradigms or subsequent analyses, in alignment with the survey goals set out in the introduction and abstract.

Thematic Overview and Conceptual Contrasts

We now recap the principal mathematical paradigms and algorithmic approaches discussed previously, with explicit contrasts between classical and modern developments. Alternative and sometimes opposing perspectives within each subdomain are considered. Notably, this synthesis foregrounds conceptual bridges—contrasting theory-driven paradigms (such as symbolic heuristics and classical topological algorithms) with data-driven advances (including

Table 23: Hausdorff Dimension of Singular Vector Sets in Various Settings

Setting	Ambient Space	Hausdorff Dimension of Sing_d
Real ($\mathbb{R}^d$)	$d \geq 2$	$\frac{d^2}{d+1}$
Function field (Ultrametric)	$d \geq 2$	$\frac{d^2}{d+1}$

deep representation learning over topological features and fractal-based embeddings), along with their respective advantages and limitations.

Summary of Key Takeaways: - The mathematical structures underlying fractal and topological invariants serve as a unifying foundation for a range of AI algorithms. Classical approaches such as rule-based or symbolic methods offer interpretability and theoretical guarantees, but may face scalability challenges. In contrast, data-driven and deep learning techniques provide empirical flexibility and adaptability, but sometimes at the expense of robustness or theoretical grounding. - There is a growing convergence between theory-driven and data-driven perspectives, as hybrid approaches emerge. Recent advancements have introduced novel algorithmic tools, such as neural architectures with geometric or topological priors and differentiable persistence modules, as well as publicly available datasets structured around fractal or topological benchmarks. These developments address bottlenecks in high-dimensional or real-world applications. - Persistent open questions concern the expressiveness and transferability of classical invariants when applied to high-dimensional, noisy, or non-Euclidean data. Meanwhile, frameworks emphasizing learnable topological features or data-driven discovery of invariants bring new trade-offs in interpretability, generalization, and scalability.

Algorithmic Examples: Classical versus Data-Driven Invariant Extraction

To concretize these trade-offs for early-career researchers, we illustrate with two representative algorithmic workflows:

Classical Example: Given a dataset of point clouds, classical persistent homology computes topological invariants by building a sequence of simplicial complexes (e.g., Vietoris-Rips complexes) and tracking the birth and death of features such as loops or voids across scale. The resulting persistence diagrams are stable descriptors, supporting rigorous analyses of shape and topology, but are often limited in capturing intricate semantic information or scaling efficiently for high-dimensional inputs.

Data-Driven Example: In recent deep learning approaches, raw point cloud data is mapped into neural architectures that explicitly incorporate topological priors (e.g., via topological loss layers or learnable representations of persistence diagrams). Such models can optimize feature extraction in an end-to-end fashion and adapt to the task at hand, but questions regarding interpretability and theoretical robustness remain—especially when invariants are learned rather than prescribed.

Practical Implications and Open Research Challenges

- The systematic integration of classical invariants into modern learning architectures remains a pressing challenge, particularly for promoting generalization, interpretability, and robustness. Purely data-driven methods often excel on complex data but may overlook nuanced geometric or topological structures; conversely, rigorously mathematical approaches may struggle with scalability or broader applicability. - Current AI methods still face gaps in capturing the full complexity of real-world geometric and topological structures. It is unclear to what extent hybrid or entirely new descriptors will be necessary to overcome these limitations. Empirical validation of new invariant-extraction strategies, especially under noisy or adversarial conditions, is an active and open research area. - The absence of standardized benchmark datasets and evaluation protocols is a significant obstacle to reproducibility and meaningful comparison across approaches. More formalized initiatives are needed: the field will benefit from community-agreed datasets, standardized preprocessing pipelines, and transparent evaluation metrics tailored to both classical and data-driven invariant extraction. Open-access repositories and collaborative benchmarking campaigns are emerging, but broader adoption remains to be achieved. - Interdisciplinary collaborations between mathematicians and AI practitioners are essential to translating theoretical insights into scalable, real-world methods. Such collaboration ensures that new techniques meet both the rigor demanded by theoretical analysis and the practical needs of diverse AI applications. - Worked examples—such as comparing classical persistent homology to neural persistent homology in structured data domains—are valuable for elucidating conceptual trade-offs between well-defined mathematical invariants and empirically learned features. Detailed empirical studies, especially those comparing generalization performance and interpretability across paradigms, represent a key direction for advancing both understanding and deployment.

Transition: The next sections provide in-depth analysis of each major paradigm, including mathematical underpinnings and algorithmic case studies, offering signposts for navigating distinct mathematical traditions explored in the survey.

14.1 Thematic Synthesis

This section synthesizes key unifying themes across contemporary mathematics, theoretical physics, and artificial intelligence, with explicit focus on: (i) clarifying cross-disciplinary principles underlying the survey, (ii) identifying empirically relevant research challenges, and (iii) outlining formal objectives and prospects for benchmark datasets and evaluation protocols. Our main measurable objectives, as introduced in the abstract and introduction, are:

to map the core analytical and structural frameworks linking self-similar and fractal geometry, operator-theoretic and topological invariants, and algorithmic/data-driven techniques; to analyze how these unify concepts across distinct fields; and to highlight open empirical validation and deployment frontiers, especially within AI-driven contexts.

Contemporary mathematics and theoretical physics converge upon a persistent set of unifying themes, most prominently in the domains of self-similar structures, fractal geometry, topological invariants, operator theory, quantum methods, and algorithmic frameworks. Central to this convergence is the dynamic interplay between geometric regularity and irregularity, exemplified by self-similarity and the Hausdorff dimension. These concepts serve not merely as geometric curiosities, but as foundational principles interfacing multiple disciplines—from the microstructure of quantum spacetime to the combinatorial topology of high-dimensional data [2, 8, 11, 12, 17, 19, 22, 26, 49, 50, 55, 57, 60–62, 66–73, 75, 76, 83–88].

Traditional methodologies in fractal geometry and metric analysis have attained significant precision in the calculation and interpretation of fractal dimensions and scaling laws. Tools such as pressure equations and mass distribution techniques have linked the growth of continued fraction coefficients directly to Hausdorff dimension, elucidating transitions from full to vanishing dimension under varying constraints [2, 55]. Notably, the explicit construction of fractals with prescribed dimensional and topological properties—such as via iterated function systems, both in classical contexts and in new frameworks like those based on neutrosophic theory—have extended foundational tools, providing new routes for the analysis of product spaces and fuzzy α-density concepts [85, 86]. The development of iterated graph systems and associated combinatorial Loewner properties has produced counterexamples that deepen the understanding of conformal dimension and quasisymmetric equivalence, leading to tractable, richly structured fractal spaces with explicit modulus computations and non-classical geometric properties [12]. Recent advances further elucidate the stability and classification of analytic and geometric properties such as the elliptic Harnack inequality, even under rough isometries, demonstrating broad stability of invariants across varied geometric categories [57].

These rigorous frameworks have experienced substantial extension through operator theory, particularly via the spectral analysis of self-similar Laplacians and almost Mathieu operators. In these contexts, spectral decimation not only yields qualitative insights into underlying geometry but also facilitates explicit formulas for the density of states [26, 66]. Operator-algebraic approaches have furnished new invariants in the classification of symmetry-protected topological (SPT) phases, notably through $H^3(G, \mathbb{T})$-valued indices and the analysis of quantum invariants for knots and manifolds, merging algebraic, analytic, and topological data into a unified invariance principle [49, 50, 61, 67].

Integration of Data-Driven and Empirical Methods. Parallel to these analytic developments, there is a decisive trend toward the integration of data-driven and machine learning methodologies. Contemporary research enables the automated discovery of scaling laws and the algorithmic extraction of self-similar variables directly from observations, obviating the need for prior knowledge of governing equations [19, 62]. These methodologies streamline the modeling of complex physical processes, facilitating the identification of nonclassical similarity variables, and have successfully recovered both classical and novel self-similar scalings in multiscale environments. For empirical advances in topological and geometric learning, the development of large-scale, higher-order benchmark datasets such as MANTRA [61] has enabled rigorous benchmarking of models on tasks including Betti number prediction, homeomorphism type classification, and orientability detection. However, notable empirical challenges remain: even state-of-the-art models often lack true invariance to topological transformations (e.g., barycentric subdivision), underlining the need for future models and protocols that guarantee (or explicitly quantify) such invariance. Suggested best practices include the systematic design of datasets encoding geometric and topological features, clear specification of data transformations, and transparent reporting of evaluation metrics for invariance and robustness.

Neural network architectures tailored for topological learning, especially when benchmarked against datasets like the MANTRA suite, illustrate that simplicial complex-based models can better capture topological invariants compared to graph-based deep learning, although challenges persist—current methods often lack invariance under topological transformations and can underperform on tasks requiring explicit homeomorphism or orientability detection [61]. Thus, the development of topology-aware and genuinely invariant models remains an open problem.

Open Empirical and Deployment Challenges. There is a pronounced need for expanded empirical validation and practical deployment in AI models addressing fractal and topological features. For instance, studies show that conventional deep models are often unable to internalize or represent complex fractal geometric structures, such as fractal dimension, whereas shallow classifiers leveraging explicit fractal features can outperform deep counterparts, particularly in structurally demanding domains with gains in both efficiency and accuracy [17, 66]. Yet, integrating these insights more deeply into learning architectures remains an unsolved challenge, compounded by the scarcity of targeted benchmarks and standardized evaluation protocols. To bridge this gap, it is essential to develop formal criteria for evaluating not only predictive accuracy, but also the degree of topological and geometric invariance, computational cost, and empirical robustness. This should be coupled with systematic empirical studies across diverse benchmarks like MANTRA [61] and others, as well as the publication of open source code and annotation of data transformations, as advocated in recent work [62]. Such measures will allow for reproducible science and more precise assessment of both algorithmic innovation and generalization.

Frontiers in Topological and Quantum-Inspired Frameworks. Simultaneously, topological and quantum-inspired frameworks are reshaping classical invariants. Persistent homology has ushered in a multi-scale approach to the analysis of spaces, bridging combinatorial and algebraic invariants with the geometry of data and physical fields [71]. Game-theoretic and non-Abelian frameworks have enriched the landscape further, uncovering new invariants through competitive optimization processes and self-similar reductions—such as matrix analogues of Painlevé equations—expanding

the landscape of integrable models and invariants linked to geometric and combinatorial complexity [68, 88]. The resilience of invariants like the elliptic Harnack inequality across categories highlights the robust interplay between local analytic and global topological structures [57].

Quantum-theoretic and operator-algebraic methods continue to supply a complementary suite of techniques and invariants. In multifractional theories and quantum gravity, self-similar and scale-dependent structures underpin the concept of "dimensional flow"—the notion that effective spacetime dimension evolves as a function of scale, a principle sharpened by comprehensive comparative analyses across nontrivial geometric and quantum gravity models [60, 83, 84]. For example, the empirical validation of multifractional predictions remains a foundational and open research challenge, hinging on both mathematical innovation and the ability to measure relevant observables in fields like cosmology, quantum materials, or condensed matter analogs [2, 26, 83]. Within condensed matter, operator-algebraic perspectives facilitate deep connections between local symmetries, global invariants, and physically measurable quantities such as edge phenomena and conductance quantization [67, 70, 72].

Toward Formal Benchmarking and Evaluation Protocols. A recurring open challenge across all surveyed domains is the lack of unified, formalized benchmark datasets and evaluation protocols, especially for AI models engaging with topological, fractal, or operator-theoretic invariants. Though initiatives such as MANTRA [61] demonstrate the feasibility and impact of standardized evaluation for topological deep learning, future progress depends on further formalization—including precise documentation of dataset transformations, coverage of geometric and topological variability, and systematic reporting standards for invariance, robustness, and scalability. Researchers are encouraged to release code, data, and explicit evaluation criteria, thereby fostering reproducibility, enabling meta-benchmarking, and accelerating empirical convergence across fields.

In summary, the convergence of theoretical, computational, and data-driven advancements signals an era where measurable objectives—rigorous benchmark-driven evaluation, empirical validation of invariants, and formalization of protocols—are as central to progress as the underlying mathematical innovations.

14.2   Interrelations and Emerging Frontiers

To advance the overarching objectives set forth in the introduction, this section explicitly aims to: (a) delineate how the core principles of self-similarity, scaling laws, and invariance interlink fractal geometry, metric space theory, PDEs, operator theory, and topological invariants across both classical and contemporary computational domains; (b) systematically survey the methods for quantifying and utilizing such invariants, with concrete emphasis on empirical validation and performance in AI and data-driven settings; and (c) distill a set of measurable, current open research challenges, referencing formal benchmark evaluations where available.

Accordingly, this synthesis is structured around the following guiding questions: (1) In what measurable ways do self-similar and scaling phenomena appear in both traditional (geometric, analytic) and modern (data-driven, machine learning) contexts? (2) How have operator-theoretic and topological invariants contributed to practical advancements in the understanding, classification, and empirical analysis of geometric complexity? (3) What are the leading methodologies and evaluation protocols for extracting and benchmarking such invariants, particularly in AI-relevant tasks?

The preceding analysis reaffirms the deeply interwoven landscape across these disciplines—each amplifies and informs the others in both theory and practice. For instance, self-similarity serves as both a geometric marker and a mechanism within operator-theoretic constructions, yielding explicit counterexamples to long-held conjectures (such as those involving self-similarity and modulus within quasisymmetric mappings [22]), and driving the formulation of new multi-component Painlevé equations and noncommutative soliton models in integrable systems [75]. Operator-algebraic and scaling-invariance frameworks increasingly provide common ground and technical language bridging diverse models.

Several emerging directions have intensified this cross-fertilization:

Operator algebras and refined invariance frameworks now distinguish among symmetry-protected topological (SPT) phases and differentiate between types of topological order at a level surpassing classical invariants.

Advances in Lorentzian and quantum geometry have forged new topological invariants with connections to quantum field theory.

Methods in ultrametric and multi-scale analysis have improved the capacity to dissect fine structure and self-similarity in projections and microsets, especially in computational settings.

A surge in studies on projections and microsets has yielded a clearer understanding of geometric complexity across both discrete and continuous frameworks.

To clearly distinguish the present survey's contributions from previous reviews, a heightened focus is maintained on: (i) modern computational models and operator-algebraic invariants; (ii) explicit connections to advances in data-driven and topological machine learning methodologies; and (iii) the design and critique of benchmark datasets and evaluation protocols for empirical validation [17, 61, 62]. Unlike earlier surveys, limited to analytical or topological traditions, this review foregrounds computational and practical deployment.

Recent state-of-the-art progress, illustrated by leading references, includes:

Creation and benchmarking of large-scale manifold triangulation datasets, such as MANTRA, designed to test the efficacy and invariance properties of topological deep learning (TDL) models. Empirical results demonstrate that current models, even those using higher-order simplicial architectures, exhibit marked shortfalls in invariance to key topological transformations (notably barycentric subdivision and orientability), as reflected by substantial performance declines and near-random outcomes in tasks designed to probe true topological invariance [61]. This evidences the need for explicit, formalized evaluation protocols that directly measure model invariance under topological and geometric perturbations.

Results indicating that standard neural network architectures fail to encode fractal dimensions or self-similarity-based features from raw data, as demonstrated through statistical correlation analyses and comparative human evaluations. In contrast, shallow models leveraging explicit fractal features have achieved significant improvements in classification accuracy (by an average of 30%) and

training efficiency (up to 84% less time) [17], underscoring the importance of transparent feature representations and suggesting new directions for extracting and benchmarking geometric complexity in AI.

Introduction of data-driven methodologies, notably constrained symbolic regression and unsupervised profile collapse, which identify self-similarity variables and recover analytic scaling laws from experimental and simulation data. These approaches, validated on canonical fluid and turbulent systems, reliably discover known scaling laws but exhibit limitations in genuinely multi-scale or dynamically entangled regimes, thereby defining concrete boundaries for empirical validation [62].

New theoretical and computational trends in quantum invariants of three-manifolds, including the development of asymptotically multiplicative invariants under covers [78], and rigorous genus-degree bounds for twisted quantum invariants, which tie topological complexity tightly to algebraic structures [77]. Such results offer explicit, computable objectives for evaluating the discriminatory power of invariants and benchmarking quantum-topological models.

Central methodological frontiers now include the explicit computation and benchmarking of operator-algebraic invariants for distinguishing SPT phases; algorithmic evaluation of Euler characteristics and Chern classes of moduli spaces for Abelian differentials [89]; and formal constraints on symplectomorphism subgroups via flux homomorphisms [90]. These efforts exemplify both technical sophistication and necessary empirical scrutiny, and point to outstanding challenges in integrating theoretical expressiveness with practical deployment and robust evaluation.

A critical examination of current limitations brings several issues into focus: Deep learning models, even when benchmarked on advanced triangulation datasets [61], do not yet achieve authentic invariance under major topological operations (e.g., homeomorphism, subdivision), highlighting urgent needs for more comprehensive, protocol-driven evaluation standards and richer benchmark tasks that measure model invariance precisely. In data-driven self-similarity extraction, methodologies demonstrate robustness for dominant single-scale regimes but encounter severe degradation with inherently multi-scale or turbulent phenomena [19, 62], motivating a new generation of evaluation frameworks that distinguish model performance in these settings. Progress in quantum invariants, while enabling sharper asymptotic and degree-genus bounds [77, 78], now raises open questions regarding the formal relationship between algebraic structure, topological intricacy, and empirical observability.

The growing influence of topological invariants and computational innovation is especially visible in persistent homology and multi-scale data analysis. Benchmark datasets, such as MANTRA [61], are crafted to explicitly test for invariance under homeomorphism and subdivision, uncovering persistent model deficiencies and motivating protocol-driven comparative studies. These advances prompt a set of explicit, unresolved research problems, with emphasis on empirical reproducibility and practical deployment:

Computation of explicit operator-algebraic invariants and their ability to resolve equivalence classes beyond the reach of classical invariants, with a focus on establishing standardized benchmarks and reproducible protocols.

Clarification of ultrametricity's mathematical and applied significance across number theory, dynamics, and computational contexts, potentially via datasets or synthetic tasks designed to probe ultrametric behavior and its invariance.

Development and empirical validation of robust topological invariants for distinguishing dynamical systems with pronounced fractal structure [77, 78, 89, 90], including formalized protocols for task design, evaluation, and benchmarking in AI/ML systems.

At the methodological frontier, data-driven discovery plays a foundational role: constrained symbolic regression and unsupervised profile collapse represent powerful, general-purpose tools for extracting similarity variables directly from data [62]. However, evaluation protocols must now rigorously address limitations in identifying deep multi-scale structure and authentic topological invariance, suggesting the necessity for new formalized benchmarks (similar in intent to MANTRA [61]) and pipelines for empirical validation.

In summary, this survey articulates a vision of deepening synthesis: geometry, topology, operator theory, and analysis are best understood as mutually reinforcing, with convergence accelerated by the universality of self-similarity, scaling, and invariance. Clearer, standardized objectives and empirical protocols—especially for benchmarking AI models against complex geometric and topological invariants—now stand as essential for both new theoretical breakthroughs and advanced computational deployment. The continued cross-pollination of methodological advances promises substantial progress for mathematics, physics, and data science alike.

15 Discussion and Outlook Synthesis, Open Problems, and Future Directions

The intersection of geometry, analysis, operator theory, and dynamical systems forms a fertile landscape for the development and unification of mathematical frameworks capable of addressing fractal and infinite-dimensional structures. Central to many recent advances is the synthesis of geometric intuition with analytic and operator-theoretic rigor, notably in the study of fractal phenomena within both classical and abstract, infinite-dimensional contexts. The construction and analysis of invariants—quantities or structures preserved under transformations—has emerged as a unifying thread, connecting classical fractal theory to contemporary investigations and revealing both significant conceptual advances and persistent mathematical challenges.

A careful comparative analysis highlights both the synthesis already achieved and the open obstacles remaining within these domains. Classical fractal geometry relies on the concepts of dimension, measure, and self-similarity as its fundamental invariants, employing tools from real and harmonic analysis to probe rigidity phenomena—instances in which geometric or dynamic constraints enforce uniqueness or induce strong regularity on boundaries. These classical analysis methods have led to a deep understanding of how structure and restriction interplay in a variety of fractal objects.

However, when these invariants are extended to more abstract settings—most notably operator algebras or infinite-dimensional

vector spaces—significant obstacles arise. In such settings, commutative invariants often prove inadequate for fully capturing the noncommutative, or "quantum," behaviors characteristic of operator-algebraic analogues of fractal sets. This inadequacy both underlines the necessity for novel theoretical constructs and exposes persistent gaps in the transfer of classical intuition to noncommutative geometries.

The development of invariant theory robust to the subtleties of infinite dimensionality and genuine fractal complexity remains a core open problem. While advances in noncommutative geometry and operator algebra have produced new classes of invariants—particularly through spectral triples and K-theoretic indices—the geometric interpretability of these invariants often lacks the granularity found in their classical counterparts. This limitation is especially evident in the analysis of boundary phenomena, where subtle interdependencies between regularity, rigidity, and analytic structure can defy straightforward generalization from the commutative case. An ongoing tension exists between operator-theoretic generalizations and the desire to retain the full geometric and dynamical richness of classical models.

As illustrated in Table 24, although certain operator-theoretic invariants can generalize classical concepts, they often fall short of reproducing the full richness—both geometric and dynamical—of their commutative analogues.

Recent computational and theoretical innovations across the discussed paradigms are summarized in Table 25. This comparison highlights modern algorithmic tools and abstract frameworks, aiding both practical applications and conceptual unification.

From the perspective of applications, these theoretical developments have significant implications beyond mathematics itself.

Operator-theoretic formulations foster finer classifications of partial differential equations on irregular domains. Techniques from fractal geometry suggest innovative regularization strategies for equations with complex boundaries.

Fractal invariants, such as entropy and correlation dimensions, are being explored as analogues of measures of complexity and entanglement in quantum systems. This trend spurs new investigations into the universality and limitations of invariant-based classification schemes.

The analysis of large, complex datasets—particularly those with network or point-cloud structures exhibiting self-similar or fractal characteristics—motivates the adaptation of both operator-theoretic and geometric invariants for robust statistical and algorithmic methodologies.

Despite this progress, several open problems continue to shape the research agenda.

Extending the theory of rigidity and regularity to infinite-dimensional and noncommutative contexts remains underdeveloped, with fundamental questions concerning the existence and computability of operator-theoretic invariants still unsolved.

Bridging the divide between commutative and noncommutative models—specifically, constructing new invariants capable of interpolating and retaining both analytic tractability and geometric intuition—is a critical challenge.

The development of effective computational techniques tailored to the complexity inherent to fractal and operator-algebraic structures is increasingly urgent, driven by burgeoning applications in both physical sciences and large-scale data analysis.

Looking ahead, the continued synthesis of geometry, analysis, operator theory, and dynamical systems holds the promise not only of resolving some of the most persistent theoretical challenges but also of catalyzing transformative interdisciplinary advances. The anticipated progress will depend on the ingenious reinvention of invariant theories, undergirded by a profound understanding of rigidity and regularity within both traditional and abstract mathematical settings. Ultimately, such work stands poised to bridge classical theory with new frontiers in quantum science, data analysis, and the broader study of complex systems.

16 Conclusions

Overview and Objectives

This survey set out to systematically synthesize and map recent advances across the interrelated domains of fractal geometry, operator theory, topological invariants, and computational approaches to geometric complexity. The intended readership spans researchers and advanced students in mathematics, mathematical physics, and computer science with interests in the interplay of analysis, algebra, geometry, and data-driven methodologies; however, a conscious effort has been made to aid navigation for incoming specialists at the graduate or early postdoctoral level. The survey's primary objectives have been to: (1) clarify how foundational notions such as self-similarity, fractal dimensions, and metric/topological invariants have evolved in modern analysis; (2) highlight operator-theoretic and algebraic frameworks that enable deeper classification and rigidity results; (3) critically appraise new computational and data-driven paradigms; and (4) distill open problems at the interface of these perspectives. We addressed the following overarching questions: How do classical and novel invariants unify the understanding of fractal and self-similar structures? What are the limitations and advances in operator-theoretic and computational tools for analyzing such spaces? Which recent results best exemplify the confluence of geometry, analysis, algebra, and data science in this field?

1. Synthesis and Evolution

The landscape of fractal geometry and its interconnected domains has matured from classical foundations into a sophisticated matrix of modern analytical, operator-theoretic, and data-driven paradigms. The confluence of these perspectives has cultivated a dynamic mathematical ecosystem wherein notions of self-similarity, metric invariants, topological complexity, and operator algebras collectively underpin theoretical breakthroughs and practical applications.

Early developments in fractal theory foregrounded self-similarity and dimensional analysis, culminating in rigorous geometric and measure-theoretic invariants such as Hausdorff and box dimensions [14, 87]. These foundational notions afforded robust quantification of geometric irregularity and the structural oscillations observable in canonical examples, including the Cantor set and Sierpinski gasket [15]. Subsequent methodological innovations—most

Table 24: Comparison of Classical and Noncommutative Fractal Invariants

Aspect	Classical Fractal Geometry	Noncommutative / Operator-Theoretic Approaches
Primary Invariants	Dimensions, measures, self-similarity	Spectral triples, K-theoretic indices
Main Analytical Tools	Real/harmonic analysis	Noncommutative geometry, operator algebras
Rigidity/Regularity	Well-characterized, geometric intuition	Partial, analytic; geometric interpretations limited
Extension Challenges	Intuition often transferrable	Commutative invariants insufficient for quantum aspects
Interpretability	High (geometry-driven)	Varies; depends on analytic construction

Table 25: High-Level Comparison of Recent Computational and Theoretical Innovations

Paradigm	Theoretical Innovations	Computational Advances	Notable Challenges
Classical Fractal Geometry	Rigidity theorems, multidimensional invariants	Fast fractal dimension estimators, adaptive measure computation	High-dimensional generalization, noisy data robustness
Operator-Algebraic/NC Geometry	Spectral triples, new noncommutative invariants	Numerical index computation, operator-valued methods	Interpretability, computational tractability
Data-Driven/Applied Fractals	Complexity measures (entropy, correlation dimension), network geometry	Machine learning for fractal structure, scalable embedding algorithms	Scalability, connection to theoretical invariants

notably, the introduction of fractal tube formulas and the analytical exploitation of complex dimensions—expanded the available toolkit, enabling finer discrimination among fractal geometries and illuminating the profound interplay between dimensionality and oscillatory phenomena [15, 87].

2. Operator-Theoretic and Algebraic Approaches

In parallel, operator-theoretic approaches have forged a unifying framework, integrating fractal properties with spectral theory, harmonic analysis, and aspects of quantum mechanics. For example, the spectral decimation method has facilitated the precise analysis of Laplacians on fractals and supported extensions to self-similar quantum and almost Mathieu operators, anchoring the study of singular spectra and quantum phase classification in the arithmetic of fractal invariants [27, 36, 91]. Moreover, the emergence of operator algebras—exemplified by Roe algebras and their rigidity theorems—has reinforced the descriptive and classificatory power of algebraic structures for large-scale geometry, capturing metric properties up to coarse equivalence [18, 80].

3. Probabilistic and Stochastic Developments

A pivotal evolution in the field is the transition from static, highly idealized fractal models toward dynamically rich, often randomized, and data-informed settings. Recent research has employed stochastic processes on fractals—such as Lévy flights augmented with nontrivial drifts [92], studies of quantum Markov semigroups [77], and fractal analysis of KPZ-type stochastic PDEs [45]—to leverage probabilistic and operator-theoretic tools. These contributions elucidate subtle connections between analytic regularity, geometric measure theory, and the algebraic substrata of quantum systems, as highlighted in the rigorous classification of symmetry-protected topological phases and the construction of operator-invariant indices [35, 74, 93].

4. Metric and Topological Integration

The interplay between metric and topological analysis with these advances remains particularly fruitful. Notable among recent results are explorations of universality and embedding in metric spaces, assessments of the stability of analytic inequalities (such as Harnack's)

under rough isometries, and detailed stratification of differentiability and connectivity properties within non-smooth spaces [10, 42, 66, 94]. The evolution of Gromov-Hausdorff type convergence concepts in both Lorentzian and Wasserstein settings further extends these themes beyond the Riemannian or finite-dimensional context, fostering a synthetic integration across geometric analysis, optimal transport, and theoretical physics [29, 32, 95].

5. Computational and Data-Driven Paradigms

Simultaneously, computational and data-driven advancements are reshaping the domain. Noteworthy innovations include the integration of topological deep learning architectures leveraging simplicial complexes, the creation of extensive datasets for geometric inference benchmarking, and the empirical discovery of self-similar variables in experimental data [33, 76]. These computational methods, while exposing current limitations in extracting fractal invariants through artificial intelligence [96], also reveal the transformative potential of combining geometric and fractal features to enhance classification, recognition, and scientific modeling—often surpassing purely data-driven models where structural priors are essential.

6. Comparison with Prior Reviews and Recent Distinctions

Compared to prior reviews, this survey not only traverses classical and modern developments but also uniquely synthesizes very recent (post-2023) advances in operator-algebraic rigidity [18], spectral self-similarity [66], and topological deep learning with high-order geometric data [61], as well as data-driven methodologies for extracting self-similarity directly from empirical observations [62]. It critically contrasts contemporary computational paradigms with structural mathematics, assessing limits and open problems (see below), and links analytic, topological, quantum, and data-oriented perspectives by explicit reference to their mutual and contrasting strengths.

7. State-of-the-Art Innovation: Tabular Comparison

To enable a concise, high-level comparison of recent theoretical and computational innovations across paradigms, Table 26 summarizes

distinguishing advances, methods, and open challenges (see also references for technical details). This is intended as a navigation aid for readers, analogous to comparative summary tables highlighted in recent reviews (cf. [18, 61, 83]).

8. Enumerating State-of-the-Art Advances

The following recent state-of-the-art advancements exemplify the field's current trajectory (references point directly to the cited works for further detail):

Operator-algebraic rigidity: Roe algebra classification characterizes coarse metric equivalence at a new level of generality and functoriality [18].

Spectral self-similarity: Spectral decimation and Laplacians on fractal spaces extend to almost Mathieu-type operators and quantum systems [66].

Data-driven extraction of self-similarity: Symbolic regression and optimization enable the discovery of similarity variables from experimental and numerical data [62].

High-order geometric datasets: The MANTRA dataset benchmarks topological deep learning models and reveals challenges in model invariance [61].

Fractal structure in quantum gravity and KPZ/SPDEs: Quasilinear singular SPDEs and critical LQG measures are now rigorously linked to fractal support properties [42, 46].

Recent computational limitations: Neural networks struggle to extract geometric fractal invariants, highlighting unresolved machine learning challenges [61, 96].

Classification of dimensions: The exact dimensionality of projections and microsets enriches the landscape of metric and fractal dimension theory [38, 91].

9. Integrative and Interdisciplinary Directions

Perhaps most significantly, the convergence of mathematics, physics, and computational science is forging genuinely interdisciplinary frameworks. Quantum invariants, spectral indices, and algebraic structures are now not only objects of theoretical interest but also serve as bridges linking analysis, topology, mathematical physics, and information sciences [28, 62]. Integrative methodologies, such as multifractal theories for quantum gravity that generalize dimensional flow, constitute a synthesis of gravitational physics with fractal geometry, paving the way for both experimental confrontation and future theoretical integration [7].

10. Critical Challenges and Persistent Limitations

At the same time, critical challenges and persistent limitations remain:

The classification of fractal sets and measures in higher dimensions remains incomplete, with new results revealing both rigidity and flexibility phenomena—such as the precise calculation of microset dimensions, spectra of invariant measures, and their projections [38, 91, 97, 98].

The pursuit of explicit, computable invariants (and associated functional models) in noncommutative, sub-Riemannian, and quasilinear settings continues to drive innovation at the intersection of analysis, geometry, and dynamical systems [8, 45, 61].

Ongoing limitations of contemporary computational models—including neural network invariance properties, the reach and granularity of operator-algebraic invariants, and the scaling of analytic versus heuristic methods—indicate urgent opportunities for the integration of structure-aware artificial intelligence with robust, scalable analytic algorithms [61, 76, 96].

Unsettled controversies around the attainability, universality, and regularity of analytic inequalities (e.g., Poincaré, Harnack) under minimal regularity or in synthetic geometric settings [6, 57].

Concluding Outlook

In summary, the evolutionary trajectory from foundational constructions in fractal geometry and dimension theory, through the synthesis of operator theory, quantum and algebraic invariants, and advanced computational paradigms, is inaugurating a new era of integrative mathematics. This progression advances not only the theoretical understanding of self-similar and fractal structures in mathematics and physics but also endows researchers with analytic and computational tools to approach the complexity of natural and engineered systems. The future of the field resides in the sustained dissolution of disciplinary boundaries, the identification of universal invariants and structures, and the invention of mathematical and computational languages that can faithfully articulate and manipulate the full richness of fractal, metric, operator, and quantum phenomena [1–109]. Future research will undoubtedly further unravel the interface among geometry, analysis, topology, operator algebras, quantum theory, and emergent computational paradigms, fostering new foundations and applications for fractal and self-similar modeling within mathematics and beyond.

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Table 26: Summary of Recent Innovations across Paradigms in Fractal Geometry, Operator Theory, and Computation

Paradigm	Example Advance	Core Method(s)	Open Challenges/Limitations
Operator Algebra (2023–25)	Roe algebra rigidity and functorial metric classification [18]	Operator-algebraic invariants, coarse geometry, functorial constructions	Extending invariants to wider classes, computability, sub-Riemannian settings
Spectral/Quantum Self-Similarity	Spectral decimation for fractal/Mathieu-type operators [66]	Spectral analysis, Laplacians on fractals, algebraic decimation	Numerical scalability, connection to random/quasiperiodic ensembles
Computational Topology	Benchmarking higher-order deep learning on MANTRA dataset [61]	Topological neural architectures, simplicial complexes	Invariance under subdivisions, extraction of robust geometric features
Data-Driven Identification	Symbolic regression for self-similarity extraction [62]	Regression, dimensionality reduction, empirical scaling laws	Multi-scale generalization, integration with analytic theory
Analysis/Regularity	KPZ/SPDE fractal supports and LQG measures [42, 46]	Regularity structures, stochastic analysis, renormalization	Extending to non-linear/multicomponent equations, geometric universality
Neural Models/ML	Fractal features for classification; limits of deep invariance [61, 96]	Network analysis, feature augmentation, complexity measures	Extracting intrinsic fractal invariants, model interpretability
Dimension Theory	Dimension of microsets, invariant measures, projections [38, 91]	Ledrappier-Young formulas, microset analysis	Classification in higher dimension, spectral analogs

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