GOV 1000/2000/2000e/E-2000 09/19/2013

Section 3: Interaction Models, Probability, Random Variables and Distributions

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This Week

Problem Set 2- Due by 1pm on Tuesday to course dropbox.
Problem Set 1 Corrections- Due by 1pm on Tuesday to course dropbox.
Reading Quiz- Due by 9pm on Sunday on course website.
LaTeX Tutorial- Next Friday, September 27, at 4pm. Location TBD.

Probability

Definitions

- **Set:** A set is any well defined collection of elements. If x is an element of S, $x \in S$. We denote sets using $\{\}$.
- **Sample space:** The set of all possible outcomes from some process.
- **Event:** Any subset of the sample space, including the full set itself. Event A $\subset$ S.
- **Empty Set:** A set with no elements. $S = \emptyset$.
- **Union:** The union of two sets A and B, $A \cup B$, is the set containing all of the elements in A *or* B.
- **Intersection:** The intersection of two sets A and B, $A \cap B$, is the set containing all of the elements in both A *and* B.
- **Complement:** If set A is a subset of S, then the complement of A, denoted A^C, is the set containing all of the elements of S that are not in A.
- **Disjointness:** Sets are disjoint when they do not intersection, such that $A \cap B = \{\emptyset\}$.
- **Probability:** A formal model of uncertainty.
- **Marginal Probability:** The probability of an event without including any additional information.
- **Joint Probability:** The probability of some event A and another event B both occurring (P(A$\cap$B) or P(A,B)).

- **Conditional Probability:** The probability of some event A, given that another event B has occurred (P(A|B)).
- **Independence:** If the occurrence or nonoccurrence of either events A and B have no effect on the occurrence or nonoccurrence of other other, then A and B are independent, A $\amalg$ B. If A and B are independent, then P(A$\cap B$) = P(A)P(B).

Notation

Symbol	Meaning	LaTeX Code
$\{\}$	set	`\{\}`
$\subset$	subset	`\subset`
$\emptyset$	empty set	`\emptyset`
$\cup$	union	`\cup`
$\cap$	intersection	`\cap`
$\amalg$	independence	`\amalg`

Axioms of Probability

1. For any event A, $P(A) \geq 0$
2. $P(S) = 1$ where S is the sample space
3. For any sequence of disjoint events (A and B), $P(A \cup B) = P(A) + P(B)$

Rules of Probability

1. $P(\emptyset) = 0$
2. For any event A, $0 \leq P(A) \leq 1$.
3. $P(A^C) = 1 - P(A)$
4. If $A \subset B$, then $P(A) \leq P(B)$.
5. For *any* two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Key Formulas

1. $P(A|B) = \frac{P(A \cap B)}{P(B)}$
2. $P(A \cap B) = P(A|B)P(B)$
3. **Law of Total Probability:** Let S be the sample space and let the disjoint k events $B_1, ..., B_k$ partition S such that P($B_1 \cup ... \cup B_k) = P(S) = 1$. If A is some other event in S, then $P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$
4. **Bayes' Rule:** $P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B)+P(B^C)P(A|B^C)}$

Random Variables and Probability Distributions

Definitions

- **Randomness:** The outcome of some experiment is not deterministic, i.e. there is some probability ($0 < P(A) < 1$) that the event will occur.
- **Random Variable:** A real-valued function mapping the sample space S to the real line.
- **Support:** The support of a random variable is all values for which their is positive probability of occurrence (sample space).
- **Probability Mass Function:** Assigns probabilities to a countable number of distinct values; the probability that some random variable X is equal to some value x; the probability that event x occurs ($P(X = x)$).
- **Probability Density Function:** the function f is a probability density function of some random variable X if $f(x) \geq 0$ and $\int_{-infty}^{\infty} f(x)dx = 1$. **Cumulative Density Function:** The probability that some random variable X is less than or equal to some value x ($P(X \leq x)$).

Properties of a PMF/PDF

1. $0 \leq P(X = x) \leq 1$ or $0 \leq f(x) \leq 1$
2. $\sum_x P(X = x) = 1$ or $\int_{-infty}^{\infty} f(x)dx = 1$

Properties of a CDF

1. $F(x)$ is non-decreasing in x.
2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
3. $F(x)$ is right-continuous.
4. $P(a \leq x \leq b) = \int_a^b f(x)dx$.
5. $f(x) = F'(x) = \frac{dF(x)}{dx}$.

Expectation and Variance

Expectation

- **Expectation of a Discrete Random Variable:** $E[X] = \sum_x xP(X = x)$
- **Expectation of a Continuous Random Variable:** $E[X] = \int_x xf(x)dx$

- **Conditional Expectation of a Discrete Random Variable:** $E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$
- **Conditional Expectation of a Continuous Random Variable:** $E[X|Y=y] = \int_x x \cdot f(x|y)dx$

- **Expectation of a Function with a Discrete Random Variable:** $E[X] = \sum_x g(x)P(X=x)$
- **Expectation of a Function with a Continuous Random Variable:** $E[X] = \int_x g(x)f(x)dx$

- **Sample Mean:** $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Properties of Expectation

1. $E(c) = c$
2. $E[E[X]] = E[X]$
3. $E[cg(X)] = cE[g(X)]$
4. Linearity: $E[g(X_1) + \cdots + g(X_n)] = E[g(X_1)] + \cdots + E[g(X_n)]$, *regardless* of independence
5. $E[XY] = E[X]E[Y]$, if X and Y are independent

Variance

- **Variance:** $Var[X] = E[(X - E(X))^2] = E[X^2] - E[X]^2$
- **Standard Deviation:** $SD[X] = \sqrt{Var[X]}$
- **Covariance:** $Cov[X,Y] = E[(X - E(X))(Y - E(Y))] = E[XY] - E[X]E[Y]$
- **Correlation:** $Corr[X,Y] = \frac{Cov[X,Y]}{SD[X]SD[Y]}$

- **Sample Variance:** $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
- **Sample Covariance:** $\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

Properties of Variance

1. $Var(c) = 0$
2. $Var[cX] = c^2 Var[X]$
3. $Cov[X,X] = Var[X]$
4. $Cov[X,Y] = Cov[Y,X]$

5. $Cov[aX, bY] = abCov[X, Y]$

6. $Cov[X + a, Y] = Cov[X, Y]$

7. $Cov[X + Z, Y + W] = Cov[X, Y] + Cov[X, W] + Cov[Z, Y] + Cov[Z, W]$

8. $Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]$